

AGENDA

- ▶ Derivatives
- ▶ Chain rule
- ▶ Gradient
- ▶ Directional Derivatives
- ▶ Integrals

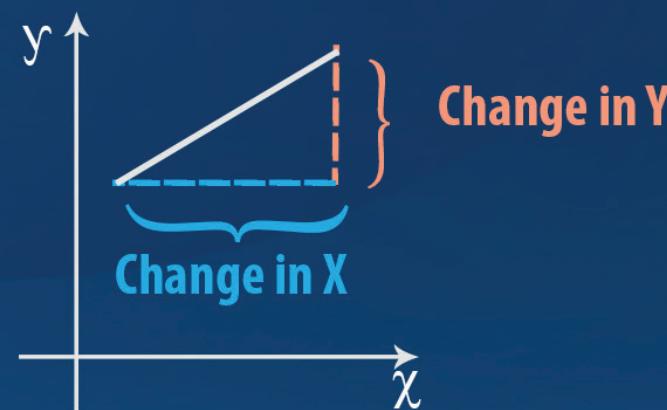
DERIVATIVES

Definition

- ▶ Instantaneous rate of change - From Physics
- ▶ Slope of a line at a point - From Geometry

Geometry view

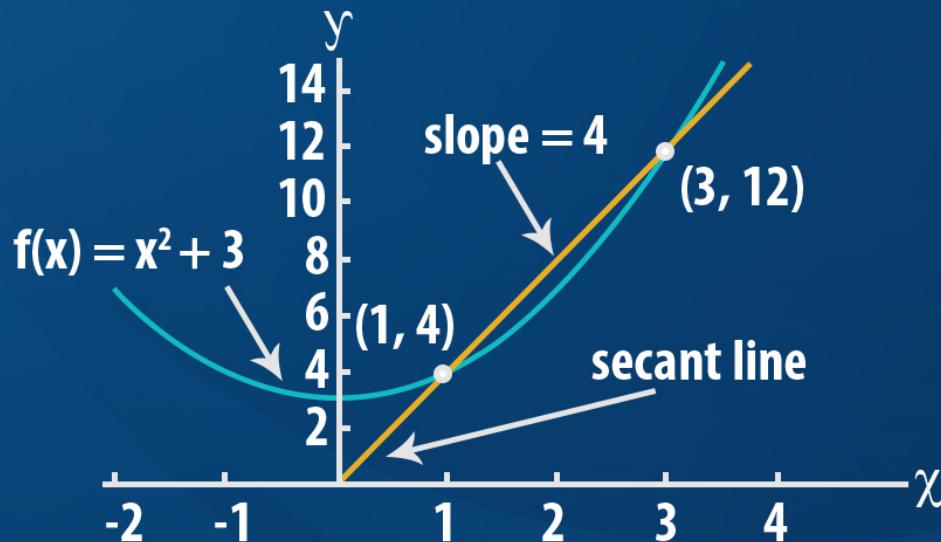
$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}}$$



DERIVATIVES

Example

Consider the graph below, where $f(x) = x^2 + 3$



The slope between $(1, 4)$ and $(3, 12)$ would be:

$$\text{slope} = \frac{y^2 - y^1}{x^2 - x^1} = \frac{12 - 4}{3 - 1} = 4$$

What would be the slope at a specified point?
Hint: We would use derivatives!

TAKING THE DERIVATIVE

Logic

Calculating the derivative is the same as calculating normal slope, however in this case, we calculate the slope between our point and a point infinitesimally close to it.

Step 1 Define function $f(x) = x^2$

Step 2 Increment x by an infinitesimally small value h , ($h = \Delta x$)

Step 3 Compute slope $\frac{f(x + h) - f(x)}{h} = 2x + h$

Step 3 Apply limit on h $\lim_0 h(2x + h) = 2x$

Thus, we have the slope formula,

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

CHAIN RULE

The chain rule is a formula for calculating the derivatives of composite functions.
Composite functions are functions composed of functions inside another function(s).

Generic rule

Composite function derivative = outer function derivative * inner function derivative

Consider the following case,

$$f(x) = A(B(C(x)))$$

Where say: $A(x) = \sin(x)$

$$B(x) = x^2$$

$$C(x) = 4x$$

Chain rule applied here,
would mean,

$$\frac{df}{dx} = \frac{dA}{dB} \frac{dB}{dC} \frac{dC}{dx}$$

Individual

$$A'(x) = \cos(x)$$

$$B'(x) = 2x$$

$$C'(x) = 4$$

The derivatives then
maybe computed as,

$$f'(x) = A'((4x)^2) \cdot B'(4x) \cdot C'(x)$$

Putting them all together!

$$\begin{aligned} f'(x) &= \cos((4x)^2) \cdot 2(4x) \cdot 4 \\ &= \cos(16x^2) \cdot 8x \cdot 4 \\ &= \cos(16x^2)32x \end{aligned}$$

Gradient

A gradient is a vector that stores the partial derivatives of multivariable functions. It helps us calculate the slope at a specific point on a curve for functions with multiple independent variables.

PARTIAL DERIVATIVES

Consider the multi-variable function,

$$f(x, z) = 2z^3x^2$$

How do we calculate partial derivative say wrt x?

- ① Swap $2z^3$ with a constant 'b'

$$f(x, z) = bx^2$$

- ② Calculate the derivative with 'b' constant

$$\frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{b(x+h)^2 - b(x^2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{b((x+h)(x+h)) - bx^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{b((x^2 + xh + hx + h^2)) - bx^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{bx^2 + 2bxh + bh^2 - bx^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2bxh + bh^2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2bxh + bh^2}{h} \\
 &= \lim_{h \rightarrow 0} 2bx + bh
 \end{aligned}$$

As $h \rightarrow 0 \dots$
 $2bx + 0$

Plug back for 'b'

$$\begin{aligned}
 \frac{df}{dx}(x, z) &= 2(2z^3)x \\
 &= 4z^3x
 \end{aligned}$$

Similarly the derivative for z would give keeping x constant

$$\frac{df}{dz}(x, z) = 6x^2z^2$$

GRADIENT AS A VECTOR AND DIRECTIONAL DERIVATIVES

Storing the partial derivatives in a gradient,

$$\nabla f(x, z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4z^3x \\ 6x^2z^2 \end{bmatrix}$$

In cartesian system, gradient is given as in 3 Dimensions,

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k},$$

GRADIENT AS A VECTOR AND DIRECTIONAL DERIVATIVES

The directional derivative is computed by taking the dot product of the gradient of f and a unit vector \vec{v} representing the direction. The unit vector describes the proportions we want to move in each direction.

Say we want to compute the directional derivative of $f(x,y,z)$ along the following vector

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

We have the dot product thus,

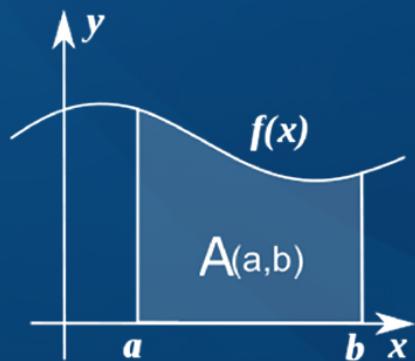
$$\left[\begin{array}{c} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{array} \right] \cdot \left[\begin{array}{c} 2 \\ 3 \\ -1 \end{array} \right] \quad \nabla_v f = 2 \frac{\partial f}{\partial x} + 3 \frac{\partial f}{\partial y} - 1 \frac{\partial f}{\partial z}$$

INTEGRALS

The integral of $f(x)$ corresponds to the computation of the area under the graph of $f(x)$. The area under $f(x)$ between the points $x = a$ and $x = b$ is denoted as follows:

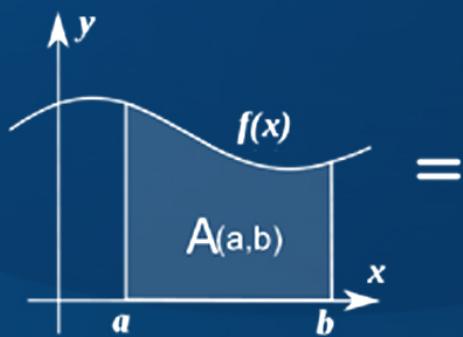
Area under curve

$$A(a, b) = \int_a^b f(x) dx.$$



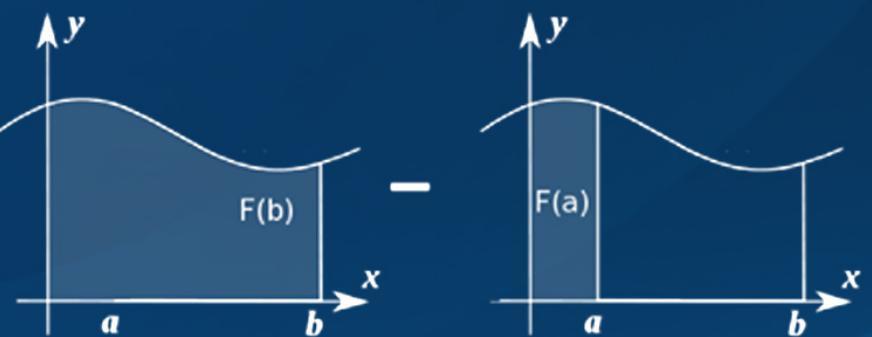
Integral function

$$F(c) \equiv \int_0^c f(x) dx$$



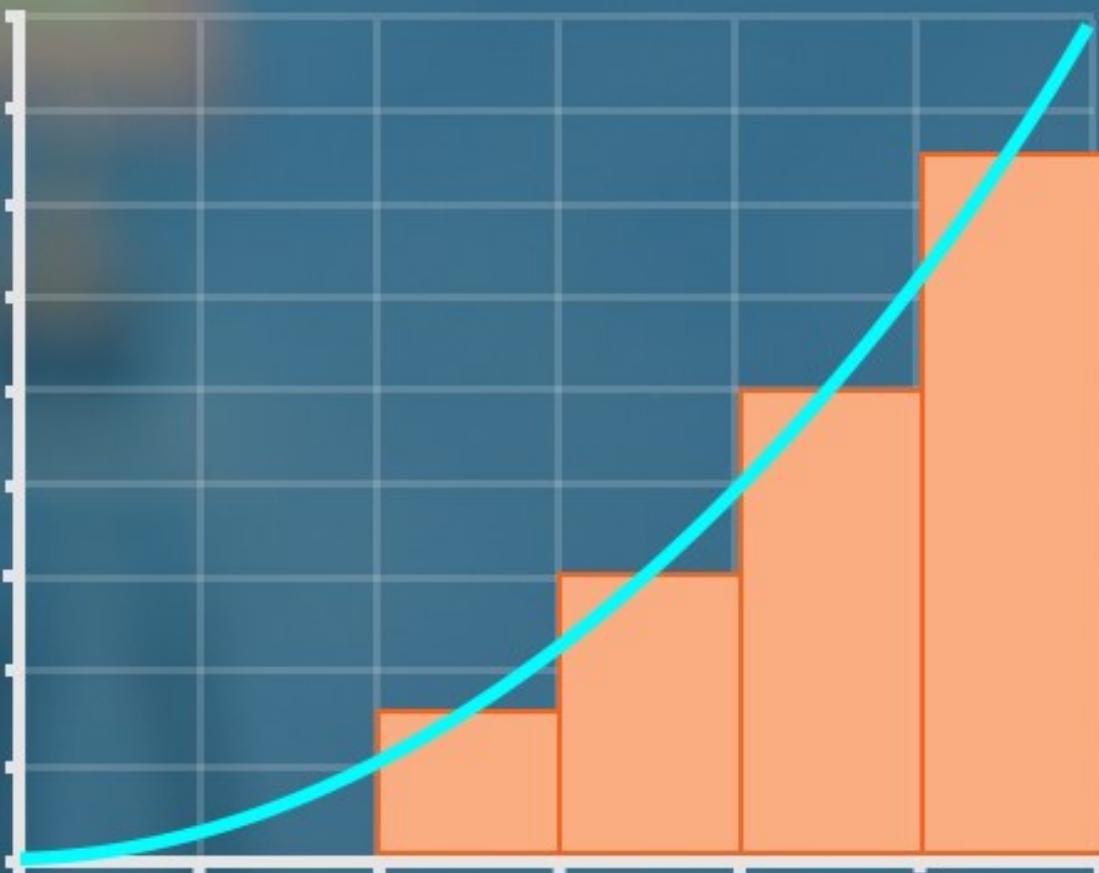
Area under curve as change in integral function

$$A(a, b) = \int_a^b f(x) dx = F(b) - F(a).$$



COMPUTING INTEGRALS

Approximate Area under curve



We can approximate the total area under the function $f(x)$ between $x=a$ and $x=b$ by splitting the region into tiny vertical strips of width h , then adding up the areas of the rectangular strips.

COMPUTING INTEGRALS

Finding integral functions

$$F(x) = \int f(x) dx \quad \downarrow$$

Equivalent problem: Find a function whose derivative is the function we are trying to integrate!

$$F'(x) = f(x)$$

COMPUTING INTEGRALS

Example

$\int x^2$ We note its quadratic in nature:
thus a cubic function is what we
are looking for!

$$F(x) = cx^3 \rightarrow F'(x) = 3cx^2 = x^2$$

Solving for c; we finally have:

$$F(x) = \int x^2 dx = \frac{1}{3} x^3 + C.$$

$$\text{Slope} = \frac{\text{Change in } Y}{\text{Change in } X}$$

