Homework #3

1. $x_{t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + w_t \qquad w_t \sim N(0, diag(1/10, 1))$

motion model

$z_t = [1, 0] x_t + v_t \qquad v_t \sim N(0, 10^2)$

observation model

a) kalman filter

$x_{t+1} = Ax_t + Bu + w_t$

$z_{t+1} = Hx_{t+1} + v_t$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad H = [1, 0]$

$w = \begin{bmatrix} 1/100 & 0 \\ 0 & 1 \end{bmatrix} \qquad V = 10^2$

prediction

$\mu_{t+1|t} = A\mu_{t|t} + Bu_t$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mu_{t|t}$

$\Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W$

$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/100 & 0 \\ 0 & 1 \end{bmatrix}$

kalman Gain

$k_{t+1|t} = \Sigma_{t+1|t} H^T (H\Sigma_{t+1|t} H^T + V)^{-1}$

$$= \Sigma_{t+1|t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left([1,0] \, \Sigma_{t+1|t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 10^2 \right)^{-1}$$

update

$$\mu_{t+1|t+1} = \mu_{t+1|t} + K_{t+1|t} \left(z_{t+1} - H \mu_{t+1|t} \right)$$

$$z_{t+1} = H x_{t+1} + V_{t+1}$$

$$= [1,0] \, x_{t+1} + V_{t+1}$$

$$= [1,0] \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x_t + w_t \right\} + V_{t+1}$$

$$\Rightarrow \mu_{t+1|t+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mu_{t|t} + K_{t+1|t} \left[z_{t+1} - [1,0] \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mu_{t|t} \right]$$

$$\Sigma_{t+1|t+1} = \left(I - K_{t+1|t} H \right) \Sigma_{t+1|t}$$

$$= \Sigma_{t+1|t} - K_{t+1|t} \cdot H \Sigma_{t+1|t}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/100 & 0 \\ 0 & 1 \end{bmatrix} - K_{t+1|t} [1,0] \, *$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \Sigma_{t|t} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1/100 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x_t \sim (\mu_{t|t}, \Sigma_{t|t}) \longrightarrow x_{t+1|t} \sim (\mu_{t+1|t}, \Sigma_{t+1|t})$$

$$x_{t+1|t+1} \sim (\mu_{t+1|t+1}, \Sigma_{t+1|t+1})$$

1(b) $P_h(z_{t+1}|x)$: $z_{t+1} \sim N([1,0]x_t, 10^2)$

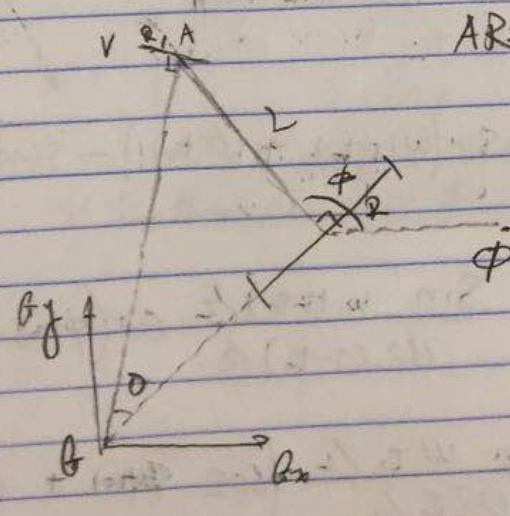
$P_a(x|u_{t+1|t}, u_t)$: $x_{t+1|t} \sim N\left(\begin{bmatrix}1&1\\0&1\end{bmatrix} u_{t+1|t}, \begin{bmatrix}1/100&0\\0&1\end{bmatrix}\right)$

$$\begin{cases} P_h(z_{t+1}|x) = \text{Const}\ \exp\left[\frac{-1}{2}\frac{(z_{t+1}-[1,0]x_t)^T(z-[1,0]x_t)}{100}\right] \\ P_a(x|u_{t+1|t}, u_t) = \text{Const}\ \exp\left[\frac{-1}{2}\left(x-\begin{bmatrix}1&1\\0&1\end{bmatrix}u_{t+1|t}\right)^T\begin{bmatrix}1/100&0\\0&1\end{bmatrix}^{-1}\left(x-\begin{bmatrix}1&1\\0&1\end{bmatrix}u_{t+1|t}\right)\right] \end{cases}$$

$$\Rightarrow \pi^k(x) = P_h(z_{t+1}|x)\, P_a(x|u_{t+1|t}, u_t)$$

$$= \frac{1}{\eta_{t+1}} \exp\left[\frac{-1}{2}\frac{(z_{t+1}-[1,0]x_t)^T(z-[1,0]x_t)}{100}\right]\exp\left[\frac{-1}{2}\left(x-\begin{bmatrix}1&1\\0&1\end{bmatrix}u^{(k)}_{t+1|t}\right)^T\begin{bmatrix}1/100&0\\0&1\end{bmatrix}^{-1}\left(x-\begin{bmatrix}1&1\\0&1\end{bmatrix}u^{(k)}_{t+1|t}\right)\right]$$

2.



$AR = L$, $RG = L/\tan\theta$, $GA = L/\sin\theta$.

$$\omega_R = \omega_A = \dot\phi$$

$$\dot\phi = \omega_A = \frac{V_A}{GA} = \frac{V}{L/\sin\theta} = \frac{V\sin\theta}{L}$$

$$V_R = \omega_A \cdot RG = \frac{V\sin\theta}{L}\,\frac{L}{\tan\theta}$$

$$= V\cos\theta$$

$\dot{x} = V_x = -\sin(\phi - 90^\circ) V_R$

$\dot{y} = V_y = \cos(\phi - 90^\circ) V_R$

$\dot{\phi} = V \sin\theta / L$

$\cos\phi$

$\Rightarrow \quad \dot{x} = [-\sin(\phi - 90^\circ)] \cdot V \cos\theta \quad = \cos\phi \cdot V \cos\theta$

$\dot{y} = [\cos(\phi - 90^\circ)] \cdot V \cos(\theta) \quad = \sin\phi \, V \cos\theta$

$\dot{\phi} = w = V \sin\theta / L$

where θ is $\theta(t)$, V is $V(t)$

1b) i) $\phi(t) = \phi(t_0) + \int_{t_0}^{t} \dot{\phi} \, ds = \phi(t_0) + \dot{\phi}(t - t_0)$

$= \phi(t_0) + V \sin\theta (t - t_0) / L$

$\boxed{w = \frac{V\sin\theta}{L}}$

ii) $x(t) = x(t_0) + \int_{t_0}^{t} \cos\theta \cdot V \cos\phi(t) \, ds$

$= x(t_0) + \frac{V\cos\theta}{w} \left(\sin(w(t - t_0) + \phi(t_0)) - \sin\phi(t_0) \right)$

$= x(t_0) + \cos\theta * V(t - t_0) \frac{\sin w(t-t_0)/2}{w(t-t_0)/2} \cos\left(\phi(t_0) + w(t - t_0)/2\right)$

$= x(t_0) + \cos\theta * V(\tau) \frac{\sin w\tau/2}{w\tau/2} \left(\cos \phi(t_0) + w\tau/2\right)$

$$y(t) = y(t_0) + \int_{t_0}^{t} \sin\phi(s)\, v\cos\theta \quad ds$$

$$= y(t_0) - \frac{v}{\omega}\left[\cos\phi(t_0) - \cos\left(\omega(t-t_0) + \phi(t_0)\right)\right]\cos\theta$$

$$= y(t_0) + v(t-t_0)\,\frac{\sin\left(\omega(t-t_0)/2\right)}{\omega(t-t_0)/2}\,\sin\left(\phi(t_0) + \omega(t-t_0)/2\right)\cos\theta$$

$$= y(t_0) + v\tau\,\frac{\sin(\omega\tau/2)}{\omega\tau/2}\,\sin\left[\phi(t_0) + \omega\tau/2\right]\cos\theta$$

$$\boxed{\omega = v\sin\theta/L}$$

$$s_{t+1} = g(s_t, u_t) = s_t + \tau\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\phi}\end{pmatrix}$$

$$= s_t + \tau\begin{pmatrix} v_t\,\operatorname{sinc}\left(\frac{\omega_t\tau}{2}\right)\cos\left(\phi_t + \frac{\omega_t\tau}{2}\right)\cos\theta \\ v_t\,\operatorname{sinc}\left(\frac{\omega_t\tau}{2}\right)\sin\left(\phi_t + \frac{\omega_t\tau}{2}\right)\cos\theta \\ \omega_t \end{pmatrix}$$

$$\omega_t = v\sin\theta_t/L = \dot{\phi}$$

$\dot{\phi} = w$

(c) Motion model

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \\ \phi_{t+1} \end{pmatrix} = \begin{pmatrix} v_t \, \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \cos\left(\phi_t + \frac{w_t \tau}{2}\right) \cos\theta \\ v_t \, \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \sin\left(\phi_t + \frac{w_t \tau}{2}\right) \cos\theta \\ w_t = \left(\frac{v \sin\theta_t}{2}\right) \end{pmatrix} \tau + \begin{pmatrix} x_t \\ y_t \\ \phi_t \end{pmatrix}$$

$= k_1$, $= k_2$

$a(x, u, w)$

$$A_t = \frac{da_t}{dx} = \begin{bmatrix} 1 & 0 & -\tau v_t \, \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \sin\left(\phi_t + \frac{w_t \tau}{2}\right) \cos\theta \\ 0 & 1 & \tau v_t \, \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \cos\left(\phi_t + \frac{w_t \tau}{2}\right) \cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$x \quad y \quad \phi$

$$\boxed{\dot{\phi} = w = v \sin\theta / L}$$

$$\boxed{\begin{aligned} v_t &= (v + w_v) \\ \theta_t &= (\theta + w_\theta) \end{aligned}}$$ noise Added

$$Q_t = \frac{da}{dw} = \left[\frac{da}{dw_v}, \frac{da}{dw_\theta}\right]$$

$$= \tau \begin{bmatrix} \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \cos\left(\phi_t + \frac{w_t \tau}{2}\right) \cos\theta & g_1 = dk_1/dw_\theta \\ \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \sin\left(\phi_t + \frac{w_t \tau}{2}\right) \cos\theta & g_2 = dk_2/dw_\theta \\ \frac{\sin(\theta + w_\theta)}{L} & \frac{(v + w_v)\cos(\theta + w_\theta)}{L} \end{bmatrix}$$

$$g_1 = \frac{d\left[(V_t' + W_v)\,\mathrm{sinc}\left[\overset{=\delta}{\frac{(V_t' + W_v)\sin(\theta_t + W_\theta)\,\tau}{2L}} \right] \cos\left[\phi_t + \overset{=\delta}{\frac{(V_t' + W_v)\sin(\theta_t + W_\theta)}{2L}} \right] \cos(\theta_t' + W_\theta) \right]}{d W_\theta}$$

$$= \Big(-\cos(\theta + W_\theta)\sin(\theta + W_\theta)\sin(\delta) + (V + W_v)\cos(\theta + W_\theta) \, * \, \frac{\delta\cos(\delta) - \sin(\delta)}{\delta^2} \, \frac{\tau\,(V + W_v)\cos(\theta + W_\theta)}{2L} \Big) \cos(\phi_t + \delta)$$

$$- (V + W_v)\cos(\theta + W_\theta)\sin(\delta)\sin(\phi_t + \delta)\, \tau \frac{(V + W_v)\cos(\theta + W_\theta)}{2L}$$

$$g_2 = \Big(-\cos(\theta + W_\theta)\sin(\theta + W_\theta)\sin(\delta) + (V + W_v)\cos(\theta + W_\theta) \, * \, \frac{\delta\cos(\delta) - \sin(\delta)}{\delta^2} \, \frac{\tau\,(V + W_v)\cos(\theta + W_\theta)}{2L} \Big) \sin(\phi_t + \delta)$$

$$+ (V + W_v)\cos(\theta + W_\theta)\sin(\delta)\cos(\phi_t + \delta) \frac{\tau\,(V + W_v)\cos(\theta + W_\theta)}{2L}$$

$$g_1 = v_t \, \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \cos\left(\frac{\phi_t + w_t \tau}{2}\right)\left(-\sin\theta\right)$$

$$g_2 = v_t \, \mathrm{sinc}\left(\frac{w_t \tau}{2}\right) \sin\left(\frac{\phi_t + w_t \tau}{2}\right)\left(-\sin\theta\right)$$

state is $\begin{pmatrix} x \\ y \\ \phi \end{pmatrix}$ control input is $\begin{pmatrix} \theta \\ v \end{pmatrix}$

(μ_x, μ_y, μ_ϕ)

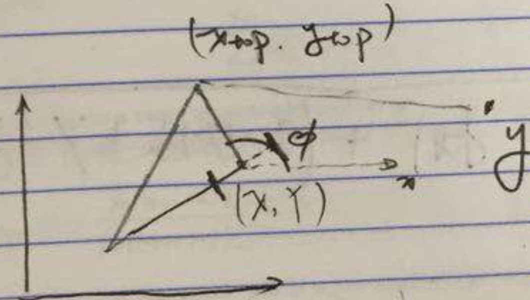
Q is motion model

$$\mu_{t+1|t} = Q(\mu_{t|t}, u_t, 0)$$

$$W = \begin{bmatrix} \alpha_\theta^2 & 0 \\ 0 & \alpha_v^2 \end{bmatrix}$$

$$\Sigma_{t+1|t} = A_t \Sigma_{t|t} A_t^T + Q_t W Q_t^T$$

(d)



Any point y for measurement Device

$$x_{top} = \cos(\phi) L + x$$

$$y_{top} = \sin(\phi) L + Y$$

measurement model

$$D = \sqrt{(x_{top} - y_x)^2 + (y_{top} - y_y)^2}$$

ii)

$$H = \frac{dD}{dS} = \left[\frac{dD}{dx} \quad \frac{dD}{dy} \quad \frac{dD}{d\phi} \right]$$

$$\frac{dD}{dx} = \sqrt{(\cos(\phi)L + x - y_x)^2 + \text{cons}}$$

$$= \frac{1}{2}\left[(x_{top} - y_x)^2 + (y_{top} - y_y)^2\right]^{-\frac{1}{2}} \times 2(\cos(\phi)L + x - y_x)$$

$$= \left[(x_{top} - y_x)^2 + (y_{top} - y_y)^2\right]^{-\frac{1}{2}} (\cos(\phi)L + x - y_x)$$

$$\frac{dD}{dy} = \left[(x_{top} - y_x)^2 + (y_{top} - y_y)^2\right]^{-\frac{1}{2}} \times (\sin(\phi)L + y - y_y)$$

$$\frac{dD}{d\phi} = \sqrt{(\cos(\phi)L + x - y_x)^2 + (\sin(\phi)L + y - y_y)^2}$$

$$= \frac{1}{2}\left[(\cos(\phi)L + x - y_x)^2 + (\sin(\phi)L + y - y_y)^2\right]^{-\frac{1}{2}} *$$

$$\cdot \{-2(\cos(\phi)L + x - y_x) \cdot L \cdot \sin(\phi) + 2(\sin(\phi)L + y - y_y) L \cdot \cos(\phi)\}$$