Homework#4 Due March/24/2018

problem#1

$$\dot{x}(t) = f(x, u) = -x(t)\,u(t) \quad , \quad x(0) = 1$$

$$\text{minimize } J^{\pi}(x(0), u(0)) = \underbrace{\int_0^1 [x(t)u(t)]^2 dt}_{\text{running cost}, \; g(x,u)} + \underbrace{x^2(1)}_{\text{terminal cost}, \; g_T(x)}$$

this is deterministic continuous-time optimal control problem

(a) HJB-PDE

$$J^*(T, x) = g_T(x) = x^2(1)$$

$$-\frac{\partial J^*}{\partial t} = \min_u \; g(x,u) + \nabla_x J^*(t,x)^T f(x,u)$$

$$= \min_u \; (x(t)u(t))^2 + \nabla_x J^*(t,x)^T \underbrace{[-x(t)\cdot u(t)]}_{= f(x)}$$

(b) $$\frac{\partial f(x)}{\partial u} = 2x^2(t)\,u(t) + \nabla_x J^*(t,x)^T [-x(t)] = 0$$

$$\Rightarrow u^*(t) = \frac{\nabla_x J^*(t,x)^T}{2x(t)}$$

Therefore $$-\frac{\partial J^*}{\partial t} = \left[x(t)\cdot \frac{\nabla_x J^*(t,x)^T}{2x(t)}\right]^2 + \nabla_x J^*(t,x)^T \cdot \left[\frac{-x(t)\nabla_x J^*(t,x)^T}{2x(t)}\right]$$

$$\Rightarrow -\frac{\partial J^*}{\partial t} = \left(\frac{\nabla_x J^*(t,x)^T}{2}\right)^2 - \left(\frac{\nabla_x J^*(t,x)^T}{2}\right)^2$$

$$\Rightarrow \begin{cases} -\dfrac{\partial J^*(x,t)}{\partial t} = \dfrac{-1}{4}\left(\nabla_x J^*(x,u)^T\right)^2 \\ J^*(x,T) = g_T(x(T)) = x^2(T) \end{cases}$$

(c) $$\frac{\partial J^*(x,t)}{\partial t} = \frac{1}{4}\left(\nabla_x J^*(x,u)^T\right)^2$$

Assume $J^*(x,t) = V(x,t)$

$$\Rightarrow \begin{cases} \frac{1}{2}\dot{\alpha}(t)\,x^2 + \dot{\beta}(t)\,x + \dot{\gamma}(t) = \dfrac{\partial V(x,t)}{\partial t} \\ \frac{1}{4}\left[\alpha(t)\,x + \beta(t)\right]^2 = \dfrac{\partial V(x,t)}{\partial t} \end{cases}$$

$$\Rightarrow \boxed{\begin{aligned} &\frac{1}{4}\alpha^2(t)\,x^2 + \frac{1}{4}\beta^2(t) + \frac{1}{2}\alpha(t)\beta(t)\,x \\ &= \frac{1}{2}\dot{\alpha}(t)\,x^2 + \dot{\beta}(t)\,x + \dot{\gamma}(t) \end{aligned}}$$

$\Rightarrow$ ① $\frac{1}{4}\alpha^2(t)\,x^2 = \frac{1}{2}\dot{\alpha}(t)\,x^2$

$$\int \frac{1}{2}\,dt = \int \frac{d\alpha(t)}{\alpha^2(t)}$$

$$t/2 = -\alpha^{-1}(t) + C_1$$

$$\alpha(t) = \frac{-1}{[t/2 - C_1]}$$

② $\frac{1}{2}\ \alpha(t) \cdot \beta(t) = \frac{d\beta(t)}{dt}$

$$\int \frac{1}{2}\ \alpha(t) \cdot dt = \ln|\beta(t)|$$

$$= \int \frac{1}{t-2C_1}\, dt = \ln|\beta(t)| - C_2$$

$$= -\ln|t-2C_1| = \ln|\beta(t)| - C_2$$

$$= -\ln|t-2C_1| = \ln|\beta(t)| - \ln C_2$$

$$= C_2 / |t-2C_1| = |\beta(t)|$$

③ $\frac{1}{4}\ \beta^2(t) = \frac{d\gamma}{dt} \Rightarrow \frac{1}{4}\ \frac{C_2^2}{(t-2C_1)^2} = \frac{d\gamma}{dt}$

$$\Rightarrow \frac{1}{4}\int \frac{C_2^2}{(t-2C_1)^2}\, dt = \gamma(t) - C_3$$

$$\frac{1}{4} \cdot \frac{C_2^2}{(t-2C_1)} = \gamma(t) - C_3$$

$$\gamma(t) = \frac{1}{4}\ \frac{C_2^2}{(t-2C_1)} + C_3$$

Assume $x(t) \neq 0$

$$V(1, x(1)) = \frac{1}{2}\alpha(1)x^2(1) + \beta(1)\cdot x(1) + \gamma(1) = x^2(1)$$

$$\Rightarrow \begin{cases} \frac{1}{2}\alpha(1) = 1 \\ \beta(1) = 0 \\ \gamma(1) = 0 \end{cases} \Rightarrow \begin{cases} \alpha(1) = 2 \\ \beta(1) = 0 \\ \gamma(1) = 0 \end{cases}$$

①

$$\Rightarrow \alpha(1) = \frac{-1}{\frac{1}{2} - C_1} = 2$$

$$\Rightarrow C_1 = 1$$

$$\therefore \alpha(t) = -1/(\tfrac{t}{2} - 1)$$

②

$$\frac{C_2}{|t-2|} = |\beta(t)|$$

$$\left|\frac{C_2}{-1}\right| = \beta(1) = 0$$

$$\Rightarrow C_2 = 0 \Rightarrow \beta(t) = 0$$

③ $\gamma(t) = 0$

problem # d

$$V(t,x) = \left[-1/(t-2)\right] x^2(t)$$

$$u^*(t) = \frac{\nabla_x J^*(t,x)}{2x(t)}$$

$$= \left(\frac{-1}{t-2}\right) \cdot 2x(t) \Big/ 2x(t)$$

$$= \left(\frac{-1}{t-2}\right)$$