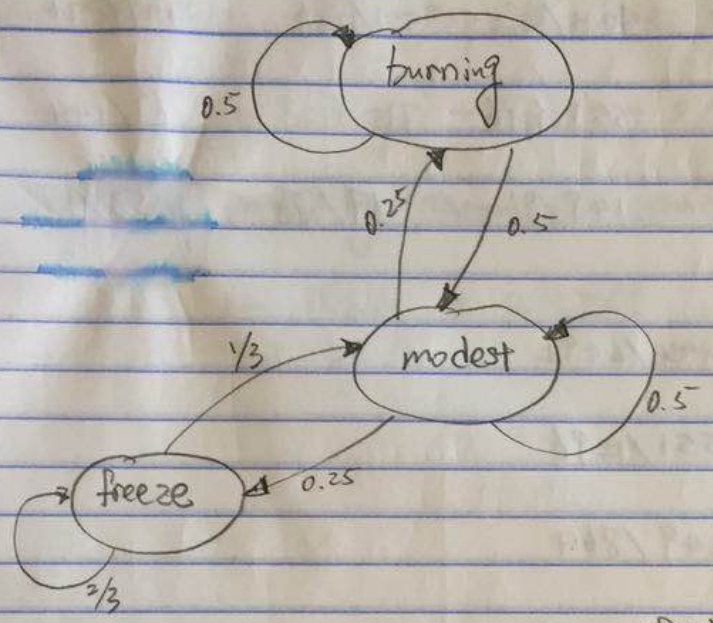
Homework #1 Due 1/30/2018

problem #1

1(a)



$$P = \begin{array}{cc|ccc} & & & ① = i & \\ & & \text{freeze} & \text{modest} & \text{burning} \\ & \text{freeze} & 2/3 & 1/4 & 0 \\ ② = j & \text{modest} & 1/3 & 1/2 & 1/2 \\ & \text{burning} & 0 & 1/4 & 1/2 \end{array}$$

1(b) For example, consider freeze

period may be $\{1, 2, 3 \cdots\}$ $\Rightarrow$ it is aperiodic with

$r = \gcd = 1$ the chain is irreducible, periodicity is contagious

(c)

$$P_{new} = \overset{x_4}{P}\ \overset{x_3}{P}\ \overset{x_2}{P}\ \overset{x_1}{P} \cdot \overset{x_0}{\text{freeze}}$$

$$= \begin{array}{ccc} f & m & b \\ 1043/2592 & 551/1728 & 149/576 \\ 551/1296 & 775/1728 & 67/144 \\ 149/864 & 67/288 & 53/192 \end{array}$$

$P_{f \to f} = 1043/2592$

$P_{f \to m} = 551/1296$

$P_{f \to b} = 149/864$

(d)

$$\begin{array}{cccccc} & x_0 & x_1 & x_2 & x_3 & x_4 \\ ① & f & \longrightarrow & b & & \\ ② & f & \longrightarrow & & b & \\ ② & f & \longrightarrow & & & b \end{array}$$

in these middle, no b exist

① $x_2 = b$ $\quad P^{(2)}_{fb} = [P \cdot P]_{fb}\, x_0$

$\Rightarrow P_{f \to b} = \boxed{1/12}$

② $x_3 = b$ $\quad P^{3}_{fb} = PPP\, x_0 \quad - \quad P_{bb} P^{(2)}_{fb}$

$\Rightarrow P_{f \to b} = 5/36 - \frac{1}{2} \times 1/12 = \boxed{7/72}$

③ $x_4 = b$ $\quad P_{fb}^{4} = PPPP\, x_0 - \cancel{P^3 \cdot P_{ij}^{(1)}} - [P_{ab}]^2 \cdot P^{(2)} - P_{ab} \cdot P^{(3)}$

$$\Rightarrow P_{f\to b} = 149/864 - \tfrac{3}{8} \times 1/12 - \tfrac{1}{2} \cdot 7/72$$

$$= \boxed{5/54}$$

the probability the ship can leave is

$$p = 1/12 + 7/72 + 5/54 = 59/216$$

Thus, the probability the ship cannot leave is

$$p = 1 - 59/216 = 0.7269$$

problem # 2

1a)

$$\pi^* = \min_{\pi} J_0^{\pi}(x_0) = \min E_{x_{1:T}} \left[g_T(x_T) + \sum_{t=0}^{T-1} g_t(x_t, \pi_t(x_t)) \,\middle|\, x_0 \right]$$

where $T = 3$, $x_0 = 2$, $g_T(x_T) = x_T^2$, $g_t(x_t, u_t) = x_t^2 + u_t^2$

$$\Rightarrow \pi^* = \min_{\pi} J_0^{\pi}(x_0) = \min E_{x_{1:3}} \left[g_3(x_3) + \sum_{t=0}^{2} g_t(x_t, \pi_t(x_t)) \,\middle|\, x_0 = 2 \right]$$

$$= \min_{\pi} E \left[x_3^2 + g_0(x_0, \pi_0) + g_1(x_1, \pi_1) + g_2(x_2, \pi_2) \,\middle|\, x_0 \right]$$

$$= \min_{\pi} E \left[x_3^2 + x_0^2 + u_0^2 + x_1^2 + u_1^2 + x_2^2 + u_2^2 \,\middle|\, x_0 \right]$$

(*)

$x_1 = x_0 + u_0$; $x_2 = x_1 + u_1$; $x_3 = x_2 + u_2$

$= x_0 + u_0 + u_1$ $\quad = x_0 + u_0 + u_1 + u_2$

$$\Rightarrow (*) = (x_0 + u_0 + u_1 + u_2)^2 + x_0^2 + u_0^2 + (x_0 + u_0)^2 + u_1^2 + (x_0 + u_0 + u_1)^2 + u_2^2$$

$$\Rightarrow \arg\min_u \left\{ (x_0 + u_0 + u_1 + u_2)^2 + (x_0 + u_0)^2 + (x_0 + u_0 + u_1)^2 \right\}$$

$$= \arg\min_u \left\{ (2 + u_0 + u_1 + u_2)^2 + (2 + u_0)^2 + (2 + u_0 + u_1)^2 \right.$$

As $u_0 = -1$ $\quad u_1 = -1$

$$= \arg\min_u \; u_2^2 + 1$$ $\quad u_2$ can either be -1 or $+1$

$$\Rightarrow \boxed{\begin{array}{lll} \pi^{(1)} = u_0 = -1 & u_1 = -1 & u_2 = -1 \\ \text{or} & & \\ \pi^{(2)} = u_0 = -1 & u_1 = -1 & u_2 = 1 \end{array}}$$ is the optimal policy

use Above policy

$$(*) = (2 - 1 - 1 \pm 1)^2 + 4 + 1 + 1 + 1 + 0 + 1$$

$$= 1 + 8 = 9$$

$$\Rightarrow \boxed{J_0^*(x_0) = J_0^{\pi^{(1)}}(x_0) = J_0^{\pi^{(2)}}(x_0) = 9}$$

$\to \int p(x, u \mid x_0)\, dx\, du$

1b) $J_0^{\pi_0}(x_0) = E_{x_{1:T}, w_{0:T}}\left[g_T(x_T) + \sum_{t=0}^{T-1} g'(x_t, \pi_t, w_t) \mid x_0 \right]$

$= E_{x_{1:T}, w_{0:T}}\left[x_3^2 + g'(x_0, \pi_0, w_0) + g'(x_1, \pi_1, w_1) + g'(x_2, \pi_2, w_2) \mid x_0 \right]$

$= E_{x_{1:T}}\left[x_3^2 + g(x_0, \pi_0) + g(x_1, \pi_1) + g(x_2, \pi_2) \mid x_0 \right]$

$g(x_0, \pi_0) = x_0^2 + \pi_0^2 \longrightarrow x_1 = x_0 + \pi_0 + w_0$

$g(x_1, \pi_1) = x_1^2 + \pi_1^2 = (x_0 + \pi_0 + w_0)^2 + \pi_1^2 \longrightarrow x_2 = x_1 + \pi_1 + w_1$

$g(x_2, \pi_2) = x_2^2 + \pi_2^2 = (x_1 + \pi_1 + w_1)^2 + \pi_2^2 \longrightarrow x_3 = x_2 + \pi_2 + w_2$

$g(x_3) = x_3^2 = (x_2 + \pi_2 + w_2)^2$

$\Rightarrow g(x_0, \pi_1) = x_0^2 + \pi_0^2 + w_0^2 + 2x_0\pi_0 + 2x_0 w_0 + 2\pi_0 w_0 + \pi_1^2$

$\int w_0^2 p_w\, dw$

$E_w\left[g(x_1, \pi_1)\right] = \left\{ x_0^2 + \pi_0^2 + 2x_0\pi_0 + \frac{1}{2} + \frac{1}{2} \right\} + \pi_1^2$

$= x_0^2 + \pi_0^2 + 2x_0\pi_0 + 1 + \pi_1^2$

$\Rightarrow E_{x_{1:T}}\Big[\overbrace{x_0^2 + \pi_0^2}^{x_1^2} + \overbrace{\left\{ x_0^2 + \pi_0^2 + 2x_0\pi_0 + 1 \right\}}^{x_1^2} + \pi_1^2 \longrightarrow g_1$

$+ \overbrace{\left\{ x_1^2 + \pi_1^2 + 2x_1\pi_1 + 1 \right\}}^{x_2^2} + \pi_2^2 \longrightarrow g_2$

$+ \overbrace{\left\{ x_2^2 + \pi_2^2 + 2x_2\pi_2 + 1 \right\}}^{x_3^2} \longrightarrow g_3$

$\Leftrightarrow \arg\min_{\pi_0} E\left\{ 2x_0\pi_0 + 2x_1\pi_1 + 2x_2\pi_2 + x_1^2 + x_2^2 \right\}$

$= E\left\{ 2x_0\pi_0 + 2x_1\pi_1 + 2x_2\pi_2 + \left[x_0^2 + \pi_0^2 + 2x_0\pi_0 + 1 \right] + \left[x_1^2 + \pi_1^2 + 2x_1\pi_1 + 1 \right] \right\}$

$$\underset{u}{\arg\min}\, E\left(6x_0u_0 + 4x_1u_1 + 2x_2u_2\right)$$

$$E\left(6x_0u_0 + 4(x_0 + u_0 + w_0)u_1 + 2(x_1 + u_1 + w_1)u_2\right)$$

$$= E\left(6x_0u_0 + 4x_0u_1 + 4u_0u_1 + 2x_1u_2 + 2u_1u_2\right)$$

$$E\left(2(x_0 + u_0 + w_0)u_2\right)$$

$$\parallel$$

$$E\left(2x_0u_2 + 2u_0u_2\right)$$

$$\Rightarrow \underset{u}{\arg\min}\; 6x_0u_0 + 4x_0u_1 + 2x_0u_2 + 4u_0u_1 + 2u_0u_2 + 2u_1u_2$$

$$= 12u_0 + 8u_1 + 4u_2 + 4u_0u_1 + 2u_0u_2 + 2u_1u_2$$

$\Rightarrow$ $u_0 = -1$ $\quad u_1 = -1 \quad u_2 = 1$

or

$u_0 = -1 \quad u_2 = -1 \quad u_2 = -1$

$x_0 = 2$, $x_1 = 1 + w_0$, $x_2 = w_0 + w_1$, $x_3 = w_0 + w_1 + w_2 \pm 1$

$$J(x) = E\left\{x_0^2 + x_1^2 + x_2^2 + x_3^2 + u_0^2 + u_1^2 + u_2^2\right\}$$

$$J_0(x) = E\left\{(w_0 + w_1 + w_2 + 1)^2 + 3 + 4 + (1 + w_0)^2 + (w_0 + w_1)^2\right\}$$

$$= 1 + 1 + 1 + 1 + 3 + 4 + 2 + 2 \boxed{= 15}$$

1c) considering noise model b

1) calculate $\pi_2(x_2)$

$V_3(x_3) = g_3(x_3) = x_3^2 \qquad x_3 = x_2 + u_2 + w_2$

At $t=2$

$Q_2(x,u) = g_2(x,u) + E_{x_3 \sim x_2}[V_3(x_3)]$

$= g_2(x,u) + \frac{1}{2}(x_2+u_2+1)^2 + \frac{1}{2}(x_2+u_2-1)^2$

$= x_2^2 + u_2^2 + \frac{1}{2}\left(x_2^2+u_2^2+1+2x_2u_2+\cancel{2x_2}+\cancel{2u_2} + x_2^2+u_2^2+1+2x_2u_2-\cancel{2x_2}-\cancel{2u_2}\right)$

$= x_2^2 + u_2^2 + x_2^2 + u_2^2 + 1 + 2x_2u_2$

$= 2x_2^2 + 2u_2^2 + 2x_2u_2 + 1$

$V_2(x) = \min Q_2(x,u) = 2x_2^2 + 3 - 2|x_2|$

$\pi_2(x) = \begin{cases} -1 & x_2 > 0 \\ 1 & x_2 < 0 \end{cases}$

2) calculate $\pi_1(x_1)$

$V_2(x_2) = 2x_2^2 + 3 - 2|x_2|, \quad x_2 = x_1 + u_1 + w_1$

At $t=1$

$Q_1(x,u) = g_1(x,u) + E_{x_2 \sim x_1}[V_2(x_2)]$

$E[V_2(x_2)] = \frac{1}{2}\left(2(x_1+u_1-1)^2 + 3 - 2|x_1+u_1-1|\right)$

$$\frac{1}{2}\left(2(x_1+u_1+1)^2+3-2|x_1+u_1+1|\right)$$

$$\Rightarrow Q_1(x,u) = x_1^2+u_1^2+(x_1+u_1-1)^2+(x_1+u_1+1)^2+3-|x_1+u_1+1|-|x_1+u_1-1|$$

$$= 3x_1^2+3u_1^2+2+4x_1u_1+3-|x_1+u_1+1|-|x_1+u_1-1|$$

$$= 3x_1^2+3u_1^2+5+4x_1u_1-|x_1+u_1+1|-|x_1+u_1-1|$$

$\pi = \arg Q(x,u)$

$u_1=1 \rightarrow Q(x,u_1=1) = 3x_1^2+8+4x_1-|x_1+2|-|x_1|$ $\quad x<0$

$u_1=-1 \rightarrow Q(x,u_1=-1) = 3x_1^2+8-4x_1-|x_1|-|x_1-2|$ $\quad x>0$

$Q(x,u_1=1)-Q(x,u_1=-1) = (*)$

$= 8x_1-|x_1+2|+|x_1-2|$

As $x_1<0$ $\quad (*)<0$; $x_1>0$ $\quad (*)>0$

$$\Rightarrow \pi_1(x_1)\begin{cases} 1 & x_1<0 \\ -1 & x_1>0 \end{cases}$$

$$V_1 = \min Q_1(x,u) = 3x_1^2+8-5|x_1|-\begin{cases}|x_1+2| & (u_1=1) \\ |x_1-2| & (u_2=-1)\end{cases}$$

3) calculate $\pi_0(x_0)$

$V_1(x_1)$ is calculated above

At $t=0$ $\quad x_1 = x_0+u_0+w_0$

$Q_0(x,u) = g_0(x,u) + E_{x,u,x_0}[V_1(x_1)]$

$$E_{x_1 u_1}[v_1(x_1)] = \frac{1}{2}\left\{3(x_0+u_0+1)^2+8-5|x_0+u_0+1|\right\}$$

$$+\frac{1}{2}\left\{3(x_0+u_0-1)^2+8-5|x_0+u_0-1|\right\}$$

$$= 3(x_0^2+u_0^2+1+2x_0u_0)+8-\frac{5}{2}|x_0+u_0-1|-\frac{5}{2}|x_0+u_0+1|$$

$$Q_0(x,u_0) = 4x_0^2+4u_0^2+3+6u_0x_0+8-5|x_0+u_0-1|$$

$$= 4x_0^2+15+6u_0x_0-\frac{5}{2}|x_0+u_0-1|-\frac{5}{2}(x_0+u_0+1)$$

$$+\text{Cal}\begin{cases} -\left\{\frac{1}{2}|x_0+u_0-1+2|+\frac{1}{2}|x_0+u_0+1+2|\right\} \\ = -\left\{\frac{1}{2}|x_0+u_0+1|+\frac{1}{2}|x_0+u_0+3|\right\} \quad ① \\ -\left\{\frac{1}{2}|x_0+u_0-1-2|+\frac{1}{2}|x_0+u_0+1-2|\right\} \\ = -\left\{\frac{1}{2}|x_0+u_0-3|+\frac{1}{2}|x_0+u_0-1|\right\} \quad ② \end{cases}$$

2 ± 1 $\quad$ 2 ± 1

1 ± 1

Since $x_0=2$

$$① = -\left\{\frac{1}{2}(x_0+u_0+1)+\frac{1}{2}(x_0+u_0+3)\right\}$$

$$= -(4+u_0)$$

$$② = -\left\{-\frac{1}{2}(x_0+u_0-3)+\frac{1}{2}(x_0+u_0-1)\right\}$$

$$= -1$$

$$\Rightarrow Q_0(x,u) = 4x_0^2 + 15 + 12u_0 - \frac{5}{2}(x_0 + u_0 - 1) + 5 \quad ①$$

$$- \frac{5}{2}(x_0 + u_0 + 1)$$

$$= \cancel{16} + \cancel{15} + 12u_0 - \cancel{10} - 5u_0 + 5 \quad ②$$

$$= 21 + 7u_0 + 5 - 4 - u_0$$

$$-1$$

$$= 17 + 6u_0 =$$

$$20 + 7u_0$$

$$\Rightarrow \boxed{u_0 = -1}$$

$$\min Q(x,u) = 11$$

Choose tail #1

Thus in $\tau >$ choose V_{min} with $(u_1 = 1)$

Stationary policy

noise model 1b 1

$T = 3 \quad J_3(x_3) = x_3^2$

$t=2$ $\begin{cases} J_2(6) = \min\left\{ (J_3(6) + J_3(6))/2,\ (J_3(6) + J_3(4))/2 \right\} + 36 + 1 = 63 \\ \pi_2^*(6) = -1 \end{cases}$

$\begin{cases} J_2(4) = \min\left\{ (J_3(6) + J_3(4))/2,\ (J_3(4) + J_3(2))/2 \right\} + 16 + 1 = 27 \\ \pi_2^*(4) = -1 \end{cases}$

$\begin{cases} J_2(2) = \min\left[(J_3(4) + J_3(2))/2,\ (J_3(2) + J_3(0))/2 \right] + 4 + 1 = 7 \\ \pi_2^*(2) = -1 \end{cases}$

$\begin{cases} J_2(0) = \min\left[(J_3(2) + J_3(0))/2,\ (J_3(0) + J_3(-2))/2 \right] + 0 + 1 = 3 \\ \pi_2^*(0) = \pm 1 \end{cases}$

$\begin{cases} J_2(-2) = \min\left[(J_3(0) + J_3(-2))/2,\ (J_3(2) + J_3(4))/2 \right] + 4 + 1 = 7 \\ \pi_2^*(-2) = 1 \end{cases}$

$t=1$ $\begin{cases} J_1(4) = \min\left[(J_2(6) + J_2(4))/2,\ (J_2(4) + J_2(2))/2 \right] + 16 + 1 = 34 \\ \pi_1^*(4) = -1 \end{cases}$

$\begin{cases} J_1(2) = \min\left[(J_2(4) + J_2(2))/2,\ (J_2(2) + J_2(0))/2 \right] + 4 + 1 = 10 \\ \pi_1^*(2) = -1 \end{cases}$

$\begin{cases} J_1(0) = \min\left[(J_2(2) + J_2(0))/2,\ (J_2(0) + J_2(-2))/2 \right] + 0 + 1 = 6 \\ \pi_1^*(0) = \pm 1 \end{cases}$

$t=0$ $J_0(2) = \min\left[(J_2(4) + J_2(2))/2,\ (J_2(2) + J_2(0))/2 \right] + 4 + 1 = 13$

stationary policy

Noise model (a)

$T=3 \qquad J_3(x_3) = x_3^2$

$T=2$

$$\begin{cases} J_2(4) = \min\left(J_3(5), J_3(3)\right) + 16 + 1 = 26 \\ \pi_2^*(4) = -1 \end{cases}$$

$$\begin{cases} J_2(2) = \min\left(J_3(5), J_3(1)\right) + 4 + 1 = 6 \\ \pi_2^*(2) = -1 \end{cases}$$

$$\begin{cases} J_2(0) = \min\left(J_3(1), J_3(-1)\right) + 0 + 1 = 2 \\ \pi_2^*(0) = \pm 1 \end{cases}$$

$T=1$

$$\begin{cases} J_1(3) = \min\left(J_2(4), J_2(2)\right) + 9 + 1 = 16 \\ \pi_1^*(3) = -1 \end{cases}$$

$$\begin{cases} J_1(1) = \min\left(J_2(2), J_2(0)\right) + 1 + 1 = 4 \\ \pi_1^*(1) = -1 \end{cases}$$

$t=0$

$$\begin{cases} J_0(2) = \min\left(J_1(3), J_1(1)\right) + 4 + 1 = 9 \\ \pi_0^*(2) = -1 \end{cases}$$

policies are stationary

(d) in the noise model (a), the cost for close Loop and open loop are same, as the model is deterministic.

However, in noise model (b), the solution of close loop is better than ~~op~~ open loop, as the policy in close loop is determined in time.

generally, the close loop solution is Always better than open loop.

problem # 3

(a) ① Induction prove

Assume $t=k$

$J_k^*(x_k)$ is the minimal solution of 2^{T-k} positive define quadratic function

At $t=k-1$

$$J_{k-1}^*(x_{k-1}) = x_{k-1}^T Q x_{k-1} + \min\left[J_k^*(U_1 x_{k-1}), J_k^*(U_2 x_{k-1})\right]$$

As $J_k^*(U_1 x_{k-1})$ and $J_k^*(U_2 x_{k-1})$ are minimum of 2^{T-k} positive define quadratic functions

$\Rightarrow$ $J_k^*(U_j x_{k-1}) + x_{k-1}^T Q x_{k-1}$ is pd quadratic function

$\Rightarrow$ $= x_{k-1}^T U_j^T Q U_j x_{k-1} + x_{k-1}^T Q x_{k-1}$

$J_{k-1}^*(x_{k-1}) = x_{k-1}^T [U_i^T Q U_i + Q] x_{k-1}$ → mix terms of the original terms

$\Rightarrow$ $J_{k-1}^*(x_{k-1})$ is the minimal of $(2^{T-k}) \times 2 = 2^{T-(k-1)}$

As $t=T$ $2^0=1$

$J_T^*(x_T)$ is ALSO the minimal of positive define quadratic function

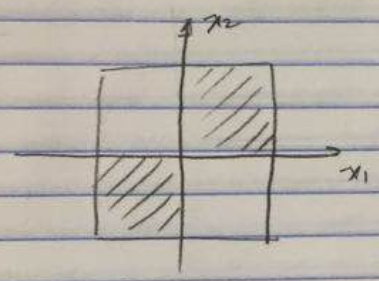
$\Rightarrow$ $J_t^*(x_t)$ is the minimum of 2^{T-t} positive define functions

Problem #3

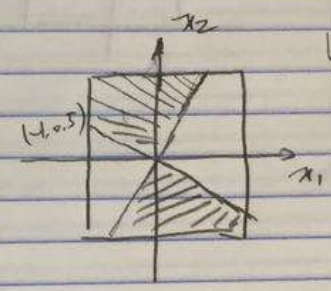
1b)

$$x_t = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x_{t+1} = \begin{cases} \begin{bmatrix} 0.5x_1 - x_2 \\ x_1 + 0.5x_2 \end{bmatrix} & u_1 \\ \begin{bmatrix} x_1 + 0.5x_2 \\ 0.5x_1 + 0.5x_2 \end{bmatrix} & u_2 \end{cases}$$

$t=0$



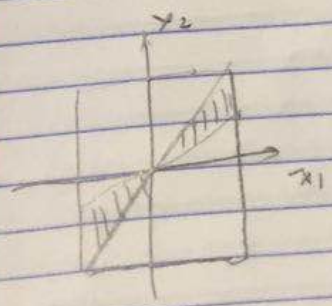
$t=1$ use u_1



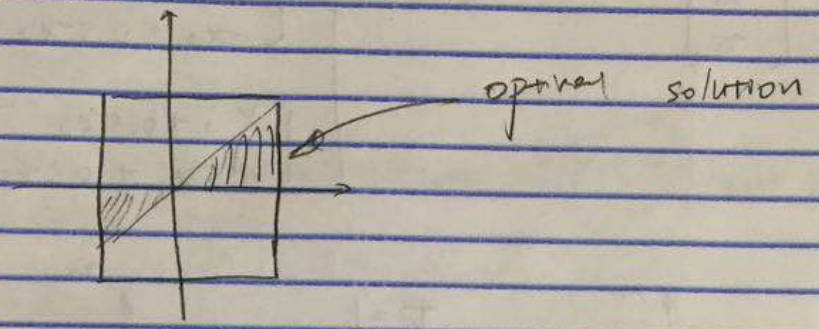
$$\begin{cases} x_1 = 0 & x_2 = 1 \\ & \Downarrow u_1 \\ x_1 = -1 & x_2 = 0.5 \end{cases}$$

$$\begin{cases} x_1 = 1 & x_2 = 0 \\ & \Downarrow u_1 \\ x_1 = 0.5 & x_2 = 1 \end{cases}$$

$t=1$ use u_2



$$\begin{cases} x_1 = 0 & x_2 = 1 \\ & \Downarrow u_2 \\ x_1 = 0.5 & x_2 = 0.5 \end{cases}$$

$$\begin{cases} x_1 = 1 & x_2 = 0 \\ & \Downarrow u_2 \\ x_1 = 1 & x_2 = 0.5 \end{cases}$$

For $t=3$