Homework#3

problem#1

$$\chi = \{(a, b, c)\},\ a, b \in \{\overset{0}{0}, \overset{15}{1}, \overset{30}{2}, \overset{40}{3}, 4\} \quad c \in \{1, 2\}$$

Also, the states $\{(3,4,c), (4,3,c), (3,3,c), (4,4,c)\} = \chi^E$

should be excluded.

Basically, the first a, b represent the score of the player and opponent, and c represents whether it is the first serve or the second serve

$\pi_t(x_t) \in U = \{B, T\}$

1) $u = B$, $x_t = (m, n, 1)$

$$P_f(x_{t+1} | x_t, B) = \begin{cases} P_B q_B & x_{t+1} = (m+1, n, 1) \\ P_B(1-q_B) & x_{t+1} = (m, n+1, 1) \\ 1-P_B & x_{t+1} = (m, n, 2) \end{cases}$$

2) $u = B$, $x_t (m, n, 2)$

$$P_f(x_{t+1} | x_t, B) = \begin{cases} P_B q_B & x_{t+1} = (m+1, n, 1) \\ 1-P_B q_B & x_{t+1} = (n, n+1, 1) \end{cases}$$

3) $u = T$, $x_t (m, n, 2)$

$$P_f(x_{t+1} | x_t, T) = \begin{cases} P_T q_T & x_{t+1} = (m+1, n, 1) \\ P_T(1-q_T) & x_{t+1} = (m, n+1, 1) \\ 1-P_T & x_{t+1} = (m, n, 2) \end{cases}$$

4) $u = T$, $x_t = (m, n, 2)$

$$Pf\,(x_{t+1} \mid x_t, T) = \begin{cases} P_T q_{\&T} & x_{t+1} = (m+1, n, 1) \\ 1 - P_T q_{\&T} & x_{t+1} = (m, n+1, 1) \end{cases}$$

for the condition that x_{t+1} in X^E, the two variables a, b should be decreased one.

The terminal stage is $\{ m, n, (1,2) \}$ $m = 4$ or $n = 4$

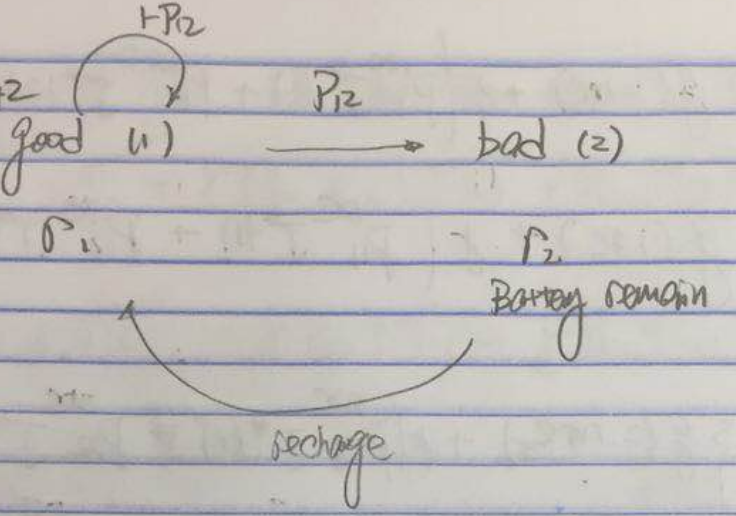
Also, the stage cost is defined $g(x_t, u)$

$\Rightarrow$ $n = 4$ $\quad g(x_t, u) = -1$ opponent wins

otherwise $\quad g(x_t, u) = 0$

$$J^*(i) = \min_{u \in U(i)} \left[g(i, u) + \sum_{j=1}^{n} Pf\,(j \mid i, u) \cdot J^*(j) \right)$$

problem#2



(1) State space $x = \{1, 2\}$, Control space $\pi(x_t) \in U(x_t)$

$= \{$ recharging (rc), not recharging (nrc) $\}$

$g(1, nrc) = -r_1$, $g(1, rc) = -(r_1 - c)$

$g(2, nrc) = -r_2$, $g(2, rc) = -(r_2 - c)$

$P(2|1, nrc) = P_{12}$ $\quad P(1|1, nrc) = 1 - P_{12}$

$P(1|1, rc) = 1$ $\quad P(2|2, nrc) = 1$

$P(1|2, rc) = 1$

$P_{ij}^u = P(i|j, u)$

with Discounted Infinite horizon problem

$$J^*(x) = \min_{\pi} J^{\pi}(x) = E\left[\sum_{t=0}^{\infty} \gamma^t g(x_t, \pi(x_t)) \,\middle|\, x_0 = x\right]$$

And the Bellman equation

$$J^*(i) = \min_{u \in U(i)} \left(g(i,u) + \gamma \sum_{j=1}^{n} P_{ij}^u J^*(j) \right)$$

prove $J^*(2) - J^*(1) \geq 0$

(b) $J^*(1) = \min \{ g(1, nrc) + \gamma (P_{11}^{nrc} J^*(1) + P_{12}^{nrc} J^*(2)),$

$g(1, rc) + \gamma (P_{11}^{rc} J^*(1) + P_{12}^{rc} J^*(2)) \}$

$J^*(2) = \min \{ g(2, nrc) + \gamma (P_{21}^{nrc} J^*(1) + P_{22}^{nrc} J^*(2)),$

$g(2, rc) + \gamma (P_{21}^{rc} J^*(1) + P_{22}^{rc} J^*(2)) \}$

$\Rightarrow J^*(1) = \min \{ -r_1 + \gamma [(1-P_{12}) J^*(1) + P_{12} J^*(2)],$

$c - r_1 + \gamma J^*(1) \}$

$J^*(2) = \min \{ -r_2 + \gamma J^*(2), c - r_2 + \gamma J^*(1) \}$

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① in $J^*(2)$ if the first term > second term

$-r_2 + \gamma J^*(2) \geq c - r_2 + \gamma J^*(1)$

$\Rightarrow J^*(2) - J^*(1) \geq \frac{c}{\gamma} \geq 0$

② in $J^*(2)$ if the second term > first term

$J^*(2) = -r_2 + \gamma J^*(2) \Rightarrow J^*(2) = \frac{r_2}{\gamma - 1}$

i> For $J^*(1)$ if the first term > second term

$$\cancel{-r_1} + \gamma\left[(1-P_{12})J^*(1) + P_{12}J^*(2)\right] \geq C \cancel{-r_1} + \gamma J^*(1)$$

$$\cancel{\gamma J^*(1)} - \gamma P_{12}J^*(1) + \gamma P_{12}J^*(2) \geq C + \cancel{\gamma J^*(1)}$$

$$J^*(2) - J^*(1) \geq \frac{C}{\gamma P_{12}} \geq 0$$

ii> For $J^*(1)$, if the second term > first term

$$J^*(1) = -r_1 + \gamma\left[(1-P_{12})J^*(1) + P_{12}J^*(2)\right]$$

Subsitute $\boxed{J^*(2) = \frac{r_2}{\gamma - 1}}$

$$\Rightarrow J^*(1) = -r_1 + \gamma\left[(1-P_{12})J^*(1) + P_{12}\frac{r_2}{\gamma-1}\right]$$

$$\boxed{J^*(1) = \frac{-r_1}{1-\gamma(1-P_{12})} + \frac{\gamma P_{12} \cdot r_2}{\gamma - 1} \cdot \frac{1}{1-\gamma(1-P_{12})}}$$

check $$J^*(2) - J^*(1) = \frac{r_2}{\gamma-1}\left[1 - \frac{\gamma P_{12}}{1-\gamma+\gamma P_{12}}\right] + \frac{r_1}{1-\gamma+\gamma P_{12}}$$

$$J^*(2) - J^*(1) = \frac{r_1 - r_2}{1-\gamma(1-P_{12})} > 0$$

as $r_2 < r_1$, & $\gamma < 1$, $(1-P_{12}) < 1$

1(c)

i> check optimal to recharge at 1 → optimal to recharge at 2

when optimal recharge at 1

$\Rightarrow \quad (c - \cancel{r_1} + \gamma J^*(1)) < -\cancel{r_1} + \gamma[(1-P_{12})J^*(1) + P_{12}J^*(2)]$

$$\frac{c}{\gamma P_{12}} < J^*(2) - J^*(1) \quad \text{(shown previously)}$$

$0 < P_{12} < 1$

$$\Longrightarrow \quad \frac{c}{\gamma} < J^*(2) - J^*(1)$$

$$\Rightarrow \quad c + \gamma J^*(1) < \gamma J^*(2)$$

$$\Rightarrow \quad c - r_2 + \gamma J^*(1) < -r_2 + \gamma J^*(2)$$

which is the conclusion of ① second term < first term

recharge. not recharge

∴ it is correct.

ii> check not optimal recharge at 1 → not optimal recharge at 1

$$J(\text{recharge} \mid 2) > J(\text{not recharge} \mid 2)$$

$$c - r_2 + \gamma J^*(1) > -r_2 + \gamma J^*(2)$$

$$\Rightarrow \gamma\left(J^*(1) - J^*(2)\right) > -C$$

$$\Rightarrow J^*(2) - J^*(1) < \frac{C}{\gamma} < \frac{C}{\gamma P_{12}}$$

not optimal to rechage at 1, to show

$$\Rightarrow \overbrace{C - r_1 + \gamma J^*(1)}^{RHS} > \overbrace{-r_1 + \gamma\left[(1-P_{12})J^*(1) + P_{12} J^*(2)\right]}^{LHS}$$

check

$$-r_1 + \gamma\left((1-P_{12})J^*(1) + P_{12}J^*(2)\right) - (C - r_1 + \gamma J^*(1))$$

$$= \gamma P_{12}\left(J^*(2) - J^*(1)\right) - C < \gamma P_{12} \cdot \frac{C}{\gamma P_{12}} - C = 0$$

$$\Rightarrow LHS - RHS < 0 \Rightarrow \boxed{LHS < RHS}$$

therefore it is correct.

(d) $P_{ij} \Rightarrow$ transition from $i \rightarrow j$

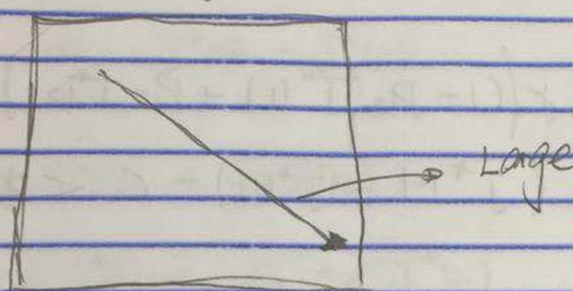
$r_i \downarrow$ with $i \uparrow$

more likely to get worse

$\sum_{j=1}^{n} P_{ij} \le \sum_{j=1}^{n} P_{2j}$

j

$P_{ij} = i$



$$J^*(i) = \min_{u \in U} \left\{ g(i,u) + \gamma \sum_{j}^{n} P_{ij} J^*(j) \right\}$$

$$= \min_{u \in U} \begin{cases} -r_i & \gamma \sum_{j \ge i}^{n} P_{ij} J^*(j), \quad u=0 \\ C - r_i & \gamma J(1), \quad u = C \end{cases}$$

if $u^* = 0$

$$J^*(n-1) = -r_{n-1} + \gamma\left(P_{n-1,n-1} J^*(n-1) + P_{n-1,n} J^*(n)\right)$$

$$\Rightarrow \left(1 - \gamma P_{n-1,n-1}\right) J^*(n-1) = -r_{n-1} + \gamma P_{n-1,n} J^*(n)$$