Homework#4 Due Dec/11/2017

problem#1

1a) $P(Y_5 = S_3) = 1 - P(Y_5 = S_2) - P(Y_5 = S_1)$

| | t=1 | t=2 | t=3 | t=4 | t=5 | |
|---|---|---|---|---|---|---|
| $P(Y_5 = S_2) =$ | 1 | 1 | 1 | 1 | 2 | |
| | $\pi(1)$ $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | | $= \frac{1}{32}$ |
| | 1 | 1 | 1 | 2 | 2 | |
| | $\pi(1)$ $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | | $= \frac{1}{32}$ |
| | 1 | 1 | 2 | 2 | 2 | |
| | $\pi(1)$ $\frac{1}{2}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | | $= \frac{1}{32}$ |
| | 1 | 2 | 2 | 2 | 2 | |
| | $\pi(1)$ $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | | $= \frac{1}{32}$ |
| | 1 | 1 | 1 | 1 | 1 | |
| | $\pi(1)$ $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | | $= \frac{1}{16} = \frac{2}{32}$ |

$\therefore P(Y_5 = S_3) = 1 - \frac{4}{32} - \frac{2}{32} = \frac{26}{32}$

$x = 5$
$S_3$

1b) $P(Y_5 = S_3 \mid x_{1:7} = AABCABC)$

$$= \frac{P(Y_5 = S_3,\ x_{1:7} = AABC\overset{x_5}{A}BC)}{P(x_{1:7} = AABCABC)} = \pi(Y_1) \prod_{t=1}^{5} B(x_t, Y_t) \prod_{t=1}^{5} K(Y_t, Y_{t-1})$$

+

other Combinators

Since $x_5 = A$ $\quad B(A, S_3) = 0$

$B(A, S_3)\ T(S_3, (S_1, S_2, S_3)) = 0$ At $t=5$

for $t=6$ update

initialization: $\alpha_1(1) = P(X = A, Y = 1) = B(A, 1) \cdot \pi(1)$

k) $\alpha_{t+1}(1) = P(y_{0:t+1}, x_{t+1} = 1)$

$= B(\{A, B, C\}, 1)\left[T(1,1)\alpha_t(1) + T(1,2)\alpha_t(2) + T(1,3)\alpha_t(3)\right]$

$= B((A, B, C), 1) * \frac{1}{2}\alpha_t(1)$

$\alpha_{t+1}(2) = P(y_{0:t+1}, x_{t+1} = 2)$

$= B(\{A, B, C\}, 2)\left[T(2,1)\alpha_t(1) + T(2,2)\alpha_t(2) + T(2,3)\alpha_t(3)\right]$

$= B(\{A, B, C\}, 2)\left[\frac{1}{4}\alpha_t(1) + \frac{1}{2}\alpha_t(2)\right]$

$\alpha_{t+1}(3) = P(y_{0:t+1}, x_{t+1} = 3)$

$= B(\{A, B, C\}, 3)\left[T(3,1)\alpha_t(1) + T(3,2)\alpha_t(2) + T(3,3)\alpha_t(3)\right]$

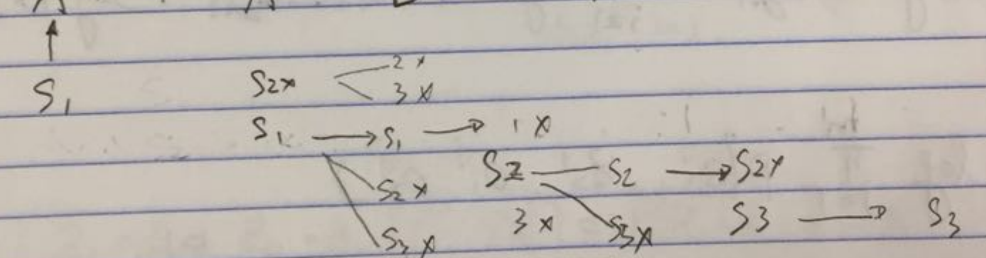
$= B(\{A, B, C\}, 3)\left[\frac{1}{4}\alpha_t(1) + \frac{1}{2}\alpha_t(2) + \alpha_t(3)\right]$

$\alpha_1(1) = P(X = A, Y = 1) = B(A, 1) \cdot \pi(1)$

$= \frac{1}{2}$

| ( A | t | $\alpha_t(1)$ | $\alpha_t(2)$ | $\alpha_t(3)$ |
|---|---|---|---|---|
| | 1 | 1/2 | 0 | 0 |
| B(A, i) | | (1/2) | (1/2) | (0) |
| | 2 | 1/8 | 1/16 | 0 |
| B(B, i) | | (1/2) | (0) | (1/2) |
| | 3 | 1/32 | 0 | 1/32 |
| B(C, i) | | (0) | (1/2) | (1/2) |
| | 4 | 0 | 1/256 | 5/256 |
| B(A, i) | | (1/2) | (1/2) | (0) |
| | 5 | 0 | 1/1024 | 0 |
| B(B, i) | | (1/2) | (0) | (1/2) |
| | 6 | 0 | 0 | 1/4096 |
| B(C, i) | | (0) | (1/2) | (1/2) |
| | 7 | 0 | 0 | 1/8192 |

| | t=1 | t=2 | t=3 | t=4 | t=5 | t=6 | t=7 |
|---|---|---|---|---|---|---|---|
| (d) | A | A | B | C | A | B | C |




| t=1 | t=2 | t=3 | t=4 | t=5 | t=6 | t=7 |
|---|---|---|---|---|---|---|
| $S_1$ | $S_1$ | $S_1$ | $S_2$ | $S_2$ | $S_3$ | $S_3$ |

$\Rightarrow$ $S_1\ S_1\ S_1\ S_2\ S_2\ S_3\ S_3$

$$P(x_{0:T}, z_{0:T}) = P(S_1 S_1 S_1 S_2 S_2 S_3 S_3,\ AA\ BCABC)$$

$$= \pi(1)\cdot B(A,1)\ T(1,1)\cdot B(A,1)\cdot T(1,1)\cdot B(B,1)\ T(2,1)\ B(C,2)\ T(2,2)\ B(A,2)$$

t=1 fixed, t=2 fixed, t=3, T=4, t=5

$$T(3,2)\cdot B(B,3)\ T(3,3)\cdot B(C,3)$$

$= (1 \times \frac{1}{2}) \times (\frac{1}{2} \times \frac{1}{2}) \times (\frac{1}{2} \times \frac{1}{2}) \times (\frac{1}{4} \times \frac{1}{2}) \times (\frac{1}{2} \times \frac{1}{2})$

$\times (\frac{1}{2} \times \frac{1}{2}) \times (1 \times \frac{1}{2}) = (\frac{1}{4})^{13}$

problem#2

1a) $\log P(z_1 z_2, O_1 O_2) = \pi(z_1) \left[\prod_1^2 B(O_i; z_i)\right] T(z_2, z_1)$

$= \log \{ p(z_1 = z_1) \cdot P(O_1 = O_1 | z_1 = z_1) P(O_2 = O_2 | z_2 = z_2)$

$P(z_2 = z_2 | z_1 = z_1) \}$

$= \log \left[ \theta_{z_1 = z_1} \cdot \prod_{i=1}^{2} \theta_{O_i = O_i | z_i = z_i} \cdot \theta_{z_2 = z_2 | z_1 = z_1} \right]$

$= \log \theta_{z_1 = z_1} + \sum_{i=1}^{2} \log \theta_{O_i = O_i | z_i = z_i} + \log \theta_{z_2 = z_2 | z_1 = z_1}$

1b) $\log \prod_{j=1}^{m} P(z_1^j, z_2^j, O_1^j O_2^j)$

$= \log \prod_{j=1}^{m} \left[ \theta^{(j)}_{z_1 = z_1} \cdot \prod_{j=1}^{2} \theta^{(j)}_{O_i = O_i | z_i = z_i} \cdot \theta^{(j)}_{z_2 = z_2 | z_1 = z_1} \right]$ (*)

$= \sum_{j=1}^{m} \log (*)$

$= \sum_{j=1}^{m} \left[ \log \theta^{(j)}_{z_1 = z_1} + \sum_{j=1}^{2} \log \theta^{(j)}_{O_i = O_i | z_i = z_i} + \log \theta^{(j)}_{z_2 = z_2 | z_1 = z_1} \right]$

1c) $Q_{z_2 = z_2 | z_1 = z_1} = T(z_2, z_1)$

$$= \frac{\sum_{j=1}^{m} \sum_{t=1}^{2} \mathbb{1}\{z_{t+1}^{(j)} = z_2^{j}, z_t^{(j)} = z_1\}}{\sum_{j=1}^{m} \sum_{t=1}^{2} \mathbb{1}\{z_t^{(j)} = z_1\}}$$

$$= \frac{\sum_{j=1}^{m} \mathbb{1}\{z_2^{(j)} = z_2, z_1^{(j)} = z_1\}}{\sum_{j=1}^{m} \mathbb{1}\{z_2^{(j)} = z_1\}}$$

1d) $$B(i,j) = Q_{o_i = o_i | z_i = z_i} = \frac{\sum_{j=1}^{m} \sum_{i=1}^{2} \mathbb{1}\{o_i^{(j)} = o_i, z_i^{(j)} = z_i\}}{\sum_{j=1}^{m} \sum_{i=1}^{2} \mathbb{1}\{z_i^{j} = z_i\}}$$

1e) $$Q(z_1 = z_1, z_2 = z_2 | o_1^{j}, o_2^{j}) = \frac{P(z_1 = z_1, z_2 = z_2, o_1 = o_1, o_2 = o_2)}{\sum_{z_1, z_2} P(z_1 = z_1, z_2 = z_2, o_1 = o_1, o_2 = o_2)}$$

$\Uparrow$

$\underbrace{P_Q(z_{0:T} | O_{0:T})}$

Q - Function

$$\Rightarrow z = z^{*} = P_Q(z_{0:T} | O_{0:T})$$

$$= P_Q(z_1, z_2 | O_1, O_2)$$

$$// \quad (*)$$

$$Q = \frac{\left[ Q_{z_1 = z_1} \cdot \prod_{i=1}^{2} Q_{O_i = o_i \mid z_i = z_i} \cdot Q_{z_2 = z_2 \mid z_1 = z_1} \right]}{\left[ \sum_{z_1, z_2} (*) \right]}$$

f) $\gamma_t(i) = P(z_t = i \mid O_{0:T}) = P(z_1 = z_1 \mid O_2 = o_2) = \sum_{z} P(z_1 = z_1, z_2 = z_2 \mid O_1, O_2)$

$$T(ij) = \frac{\sum_{k=1}^{N} \sum_{t=1}^{T} \xi^k (i,j)}{\sum_{k=1}^{N} \sum_{t=1}^{T} \gamma^k (i,j)}$$

$\gamma = P(z_t = i \mid O_{0:T})$

$= P(z_2 \mid O_1, O_2) = \sum_{z_1} P(z_1, z_2 \mid O_1 O_2)$

$= \sum_{z_1} (Q)$

$\xi(ij) = P(z_1 z_2 \mid O_1 O_2) = Q$

(g) $Q_{O_1=O_1 | S_i = S_i} = B(i,j)$

$$= \frac{\sum_{k=1}^{N} \sum_{t=1}^{T} \mathbb{I}(O_t^{(k)} = 1)\, \gamma_t^{(k)}(1)}{\sum_{k=1}^{N} \sum_{t=1}^{T} \gamma_t^{(k)}(1)}$$

$$\gamma_t^{(k)}(S) = \sum_{S_i} P(S_1, S_2 | O_1, O_2) = \sum_{S_i} Q$$

$$\gamma(S_1) = P(S_1 | O_1, O_2) = \sum_{S_2} Q$$

$$\gamma(S_2) = P(S_2 | O_1 O_2) = \sum_{S_1} Q$$