Homework#2 Nov / 10 / 2017

(a)

$$T(t) = \begin{bmatrix} \cos t\pi/3 & 0 & -\sin t\pi/3 & t \\ 0 & 1 & 0 & 0 \\ \sin t\pi/3 & 0 & \cos t\pi/3 & 2t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T(1) = \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 & 1 \\ 0 & 1 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1/2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Around } y \text{ Axis}$$

$$\theta = \|\omega\| = \arccos\left(\frac{tr(R) - 1}{2}\right)$$

$$= \arccos\left(\frac{\frac{1}{2} + 1 + \frac{1}{2} - 1}{2}\right) \Rightarrow \theta = 60^\circ \text{ or } -60^\circ$$

$$\xi = \frac{\omega}{\|\omega\|} = \frac{1}{2\sin(\theta)} \begin{bmatrix} 0 \\ \sqrt{3} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

in this case choose $-\pi/3$

$$\Rightarrow \vec{\theta} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{-\pi}{3} \right\}$$

b) $q = \left[\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\, \xi \right]$ where, θ, ξ are Above

$$= \left[\frac{\sqrt{3}}{2}, 0, \frac{1}{2}, 0 \right]$$

c) $\hat{\omega}(t) = \dot{R}(t)\, R^T(t)$

$$= \begin{bmatrix} -\sin\frac{\pi}{3}t * \frac{\pi}{3} & 0 & -\cos\frac{\pi}{3}t * \frac{\pi}{3} \\ 0 & 0 & 0 \\ \cos\frac{\pi}{3}t * \frac{\pi}{3} & 0 & -\sin\frac{\pi}{3}t * \frac{\pi}{3} \end{bmatrix}_{t=1} \cdot R(1)^T$$

$$= \begin{bmatrix} -\frac{\sqrt{3}}{2} * \frac{\pi}{3} & 0 & -\frac{1}{2} * \frac{\pi}{3} \\ 0 & 0 & 0 \\ \frac{1}{2} * \frac{\pi}{3} & 0 & -\frac{\sqrt{3}}{2} * \frac{\pi}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ -\sqrt{3}/2 & 0 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{\sqrt{3}}{4} \times \frac{\pi}{3} + \frac{\sqrt{3}}{4} \times \frac{\pi}{3} & 0 & -\frac{3}{4} \times \frac{\pi}{3} - \frac{1}{4} \times \frac{\pi}{3} \\ 0 & 0 & 0 \\ \frac{1}{4} \times \frac{\pi}{3} + \frac{3}{4} \times \frac{\pi}{3} & 0 & \frac{\sqrt{3}}{4} \times \frac{\pi}{3} - \frac{\sqrt{3}}{4} \times \frac{\pi}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -\frac{\pi}{3} \\ 0 & 0 & 0 \\ \frac{\pi}{3} & 0 & 0 \end{bmatrix} \qquad \omega_2 = \frac{-\pi}{3}$$

$$\hat{\omega}(t) \xrightarrow{\vee} \omega = \begin{bmatrix} 0 \\ -\pi/3 \\ 0 \end{bmatrix}$$

$$V(t) = \dot{p}(t) - \hat{\omega}(t) \cdot p(t)$$

$$\dot{p}(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad p(t) = \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix}$$

$$V(t) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{\omega}(t) \\ 0 \end{bmatrix} \cdot \begin{bmatrix} t \\ 0 \\ 2t \end{bmatrix}_{t=1}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -\pi/3 \\ 0 \\ \pi/3 \end{bmatrix} = \begin{bmatrix} 1 + \frac{2\pi}{3} \\ 0 \\ 2 - \frac{\pi}{3} \end{bmatrix}$$

(d) $$W_{orld}T(t)_{Body} = \begin{bmatrix} \cos t\pi/3 & 0 & -\sin t\pi/3 & t \\ 0 & 1 & 0 & 0 \\ \sin t\pi/3 & 0 & \cos t\pi/3 & 2t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\omega = \begin{bmatrix} 0 \\ -\pi/3 \\ 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the Angular velocity is independent from choice origin, it is same in body frame, and world frame.

(e) Given $_WT_R$ (+)

now to find $P_R = {}_RT_W \, P_W$ (+)

$\Rightarrow$

$$T^{-1}(1) = \begin{bmatrix} 1/2 & 0 & -\sqrt{3}/2 & 1 \\ 0 & 1 & 0 & 0 \\ \sqrt{3}/2 & 0 & 1/2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$\therefore \quad P_R = T^{-1}(1) \cdot \begin{bmatrix} 9 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.768 \\ 0 \\ -7.93 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad P_R = \begin{bmatrix} 2.768 \\ 0 \\ -7.93 \end{bmatrix}$$

problem #2

(a) $w_t \sim N(0,1)$ $\quad V_t \sim N(0, 0.5)$ $\quad x_0 \sim N(0,1)$

$$x_{t+1} = a(x_t, u_t, w_t) = -0.1 x_t + \cos(x_t) + w_t$$

$$z_{t+1} = h(x_{t+1}, v_{t+1}) = x_t^2 + v_t$$

motion model

i) $x_t \sim N(\mu_{t|t}, \Sigma_{t|t})$ which is Linearize around $\mu_{t|t}$

$$x_{t+1|t} = a(x_t, u_t, w_t) \overset{u}{=} a(\mu_{t|t}, u_t, 0) + \left[\frac{da}{dx}(\mu_{t|t}, u_t, 0)\right]^T (x_t - \mu_{t|t})$$

$$+ \left[\frac{da}{dw}(\mu_{t|t}, u_t, 0)\right]^T (w_t - 0)$$

A_t $\quad Q_t$

ii) to Linearize it around $x_t^{norm} = 1$

$$\Rightarrow a(x_t, u_t, w_t) \overset{u}{=} a(x_t^{norm}, u_t, 0) + \left[\frac{da}{dx}(x_t^{norm}, u_t, 0)\right]^T (x_t - x_t^{norm})$$

$$+ \left[\frac{da}{dw}(x_t^{norm}, u_t, 0)\right]^T (w_t - 0)$$

A_t' $\quad Q_t'$

$$\Rightarrow a(x_t, u_t, w_t) \overset{u}{=} a(x_t^{norm}, u_t, 0) + A_t'(x_t - x_t^{norm}) + Q_t'(w_t - 0)$$

prediction

$$\mu_{t+1|t} = \iint \overset{const}{a(x_t, u_t, w_t)} \cdot \phi(x_t; \mu_{t|t}, \Sigma_{t|t})\, \phi(w_t; 0, W)\, dx_t\, dw_t$$

$$= a(x_t^{norm}, u_t, 0) + A_t'\left[\int x_t \phi(x; \mu_{t|t}, \Sigma_{t|t})\, dx_t - x_t^{norm}\right]$$

$$+ Q_t' \underbrace{\int w_t \phi(w_t; 0, W)\, dw_t}_{=0}$$

$$= a(x_t^{norm}, u_t, 0) + A_t'[\mu_{t|t} - x_t^{norm}]$$

now Assum $x_t \sim N(x_t^{norm}, \Sigma_{t|t})$ $\quad \boxed{x_t^{norm} = 1}$

$$\Rightarrow \mu_{t+1|t} = a(x_t^{norm}, u_t, 0) = \boxed{-0.1 + \cos(1)}$$

$cov = E(x^2) - E^2(x)$

$$\Sigma_{t+1|t} = \Big[\underbrace{a(x_t^{norm}, h_t, 0) + A_t'(x_t - x_t^{norm}) + Q_t' w_t}_{=(*)}\Big]\Big[(*)\Big]^T$$

$$\phi(x; \mu_{t|t}, \Sigma_{t|t})\,\phi(w; 0, W)\,dx_t\,dw_t - \mu_{t+1|t}\,\mu_{t+1|t}^T$$

$$= A_t' \Big(\int (x_t - x_t^{norm})\,\phi(x; \mu_{t|t}, \Sigma_{t|t})\,dx\Big) \cdot a^T(x_t^{norm}, u_t, 0)$$

$$a^T(x_t^{norm}, u_t, 0)\Big(\int (x_t - x_t^{norm})^T \phi(x; \mu_{t|t}, \Sigma_{t|t})\,dx\Big) \cdot A_t'^T$$

$$A_t'\Big(\int (x - x_t^{norm})(x - x_t^{norm})^T \phi(x; \mu_{t|t}, \Sigma_{t|t})\,dx\Big) A_t'^T$$

$$Q_t \Big(\int w \cdot w^T \phi(w; 0, W)\,dw\Big) Q_t'^T$$

with $x_t \sim N(x_t^{norm}, \Sigma_{t|t})$

$$\Sigma_{t+1|t} \Rightarrow \boxed{A_t' \Sigma_{t|t} A_t'^T + Q_t' W Q_t'^T}$$

$$Q_t' = \left[\frac{da}{dw}(x_t^{norm}, u_t, 0)\right]^T = 0$$

$$A_t' = \left[\frac{da}{dx}(x_t^{norm}, u_t, 0)\right]^T = 0.1 - \sin x_t \Big|_{x_t = x_t^{norm} = 1}$$

$$= 0.1 - \sin(1)$$

$$\Rightarrow \Sigma_{t+1|t} = (0.1 - \sin(1)) * (\Sigma_{t|t}) * (0.1 - \sin(1)) \quad * \; A_t \; t = norm$$

A_t x_1 $\Sigma_{1|0} = [0.1 - \sin(1)]^2$

observation model

i) linearize around $\mu_{t+1|t}$

$$z_{t+1} = h(x_{t+1}, v_{t+1}) \overset{u}{=} h(\mu_{t+1|t}, 0) + \left[\frac{dh}{dx}(\mu_{t+1|t}, 0)\right]^T (x_{t+1} - \mu_{t+1})$$

$$+ \frac{dh}{dv}(\mu_{t+1|t}, 0)(v_{t+1} - 0)$$

$$H_{t+1} = \left[\frac{dh}{dx_{t+1|t}}(\mu_{t+1|t}, 0)\right]^T$$

$$R_{t+1} = \left[\frac{dh}{dv}(\mu_{t+1|t}, 0)\right]^T$$

$$\boxed{\text{normally}: \begin{pmatrix} x_{t+1} \\ z_{t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} \mu_{t+1|t} \\ m_{t+1|t} \end{pmatrix}, \begin{bmatrix} \Sigma_{t+1|t}, & C_{t+1|t} \\ C_{t+1|t}, & S_{t+1|t} \end{bmatrix}\right)}$$

update:

① $m_{t+1|t} = \iint h(x,v)\,\phi(x;\mu_{t+1|t}, \Sigma_{t+1|t})\,\phi(v;0,V) \overset{u}{=} h(\mu_{t+1|t}, 0)$

$\therefore \mu_{t+1|t} = h(-0.1+\cos(1), 0)$

$= [-0.1+\cos(1)]^2$

② $S_{t+1|t} = \iint (h(x,v) - m_{t+1|t})(h(x,v) - m_{t+1|t})^T \phi(x;\mu_{t+1|t}, \Sigma_{t+1|t})\,\phi(v;0,V)$

$\overset{u}{=} H_{t+1}\Sigma_{t+1|t}H_{t+1}^T + \cancel{R_{t+1}VR_{t+1}^T}$ 0

$H_{t+1} \overset{u}{=} 2x_{t+1|t}\big|_{x_{t+1|t}=\mu_{t+1|t}}$

$= 2\times[-0.1+\cos(1)]$

$\therefore S_{t+1|t} = \{2\times[-0.1+\cos(1)]\times(0.1-\sin(1))\}^2 + \Sigma_{t+1|t}$

③ $C_{t+1|t} = \iint (x-\mu_{t+1|t})(h(x,v) - m_{t+1|t})^T \phi(x;\mu_{t+1|t}, \Sigma_{t+1|t})\,\phi(v;0,V)$

$\overset{u}{=} \Sigma_{t+1|t}H_{t+1}^T$

$= \Sigma_{t+1|t}(0.1-\sin(1))^2 * 2*(-0.1+\cos(1))$

Then use the kalman gain to update $\mu_{t+1|t}$, $\Sigma_{t+1|t}$ to

$\mu_{t+1|t+1}$, $\Sigma_{t+1|t+1}$ with $K_{t+1|t} = C_{t+1|t} \cdot S^{-1}_{t+1|t}$

$\mu_{t+1|t} + K_{t+1|t}(z_{t+1} - m_{t+1|t})$

$$z_{t+1} = h(x_{t+1|t}, v_{t+1}) \qquad x_{t+1|t} \sim N(\mu_{t+1|t}, \Sigma_{t+1|t})$$
$$= (x_{t+1|t})^2 + v_{t+1} \qquad v_{t+1} \sim N(0, 0.5)$$

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} - K_{t+1|t} S_{t+1|t} K^T_{t+1|t}$$

Since the update is based on prediction, I only linearize at x_t^{norm} for prediction. if wanne to linearize for update just change $\mu_{t+1|t}$ to x_t^{norm} And do same procedure above.

(b) UKF equation At $x_t^{norm} \equiv 1$; $w_t \sim N(0,1)$ $v_t \sim N(0, 0.5)$

motion model $x_{t+1|t} = a(x_t, u_t, w_t)$

$$\begin{pmatrix} x^{(0)}_{t|t} \\ w^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_{t|t} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x^{(i)}_{t|t} \\ w^{(i)} \end{pmatrix} = \begin{pmatrix} \mu_{t|t} \\ 0 \end{pmatrix} \pm \sqrt{d+dw} \begin{bmatrix} \sqrt{\Sigma_{t|t}} & 0 \\ 0 & \sqrt{w} \end{bmatrix}_i$$

in this case the x is linearized at $x_t^{norm} = 1 = \mu_{t|t}$

$$\Rightarrow \begin{pmatrix} x^{(0)}_{t|t} \\ w^{(0)} \end{pmatrix} = \begin{pmatrix} x_t^{norm} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \#①$$

i from 1 → 4

$$\begin{pmatrix} x^{(i)}_{t|t} \\ w^{(i)} \end{pmatrix} = \begin{pmatrix} x_t^{norm} \\ 0 \end{pmatrix} \pm \sqrt{d+dw} \begin{bmatrix} \overset{①}{\sqrt{\Sigma_{t|t}}} & \overset{①}{0} \\ 0 & \sqrt{w} \end{bmatrix}_i \quad \#②④$$

i and i+d+dw

i and i+2

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \sqrt{2} \begin{pmatrix} \sqrt{\Sigma_{t|t}} & 0 \\ 0 & \sqrt{1} \end{pmatrix}_i$$

prediction

$$\mu_{t+1|t} = \sum_{i=0}^{2(d+d_w)} w_i^m \cdot a(x_{t+1|t}^{(i)}, u_t, w^{(i)})$$

$$a(x_{t+1|t}^{(i)}, u_t, w^{(i)}) = -0.1 * x_{t+1|t}^{(i)} + \cos(x_{t+1|t}^{(i)}) + w^{(i)}$$

$$w_0^m = \frac{\lambda}{d+\lambda} = \frac{\lambda}{1+\lambda} \qquad w_i^m = \frac{1}{2(d+\lambda)} = \frac{1}{2(1+\lambda)} \quad i>0$$

$$\Rightarrow \mu_{t+1|t} = \sum_{i=0}^{4} w_i^{(m)} \cdot \left[-0.1\, x_{t+1|t}^{(i)} + \cos(x_{t+1|t}^{(i)}) + w^{(i)}\right]$$

$$\Sigma_{t+1|t} = \sum_{i=0}^{2(d+d_w)} w_i^{(c)} \left(a(x_{t+1|t}^{(i)}, u_t, w^{(i)}) - \mu_{t+1|t}\right)\left(a(x_{t+1|t}^{(i)}, u_t, w^{(i)}) - \mu_{t+1|t}\right)^T$$

$$a(x_{t+1|t}^{(i)}, u_t, w^{(i)}) = -0.1 * x_t^{(i)} + \cos(x_{t+1|t}^{(i)}) + w^{(i)}$$

$$w_0^{(c)} = \frac{\lambda}{d+\lambda} + (1-\alpha^2+\beta) = \frac{\lambda}{1+\lambda} + (1-\alpha^2+\beta)$$

$$w_i^{(c)} = \frac{1}{2(d+\lambda)} = \frac{1}{2(1+\lambda)}$$

$$\Rightarrow \Sigma_{t+1|t} = \sum_{i=0}^{4} w_i^{(c)} \left[-0.1\, x_t^{(i)} + \cos(x_{t+1|t}^{(i)}) + \overset{=(*)}{w^{(i)}} - \mu_{t+1|t}\right] * \left[(*)\right]^T$$

observation model $z_{t+1} = h(x_{t+1}, v_{t+1})$, $v_{t+1} \sim N(0, V)$

$$\begin{pmatrix} x_{t+1|t}^{(0)} \\ v^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_{t+1|t}^{(i)} \\ w^{(i)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix} \pm \sqrt{d+d_w} \begin{pmatrix} \sqrt{\Sigma_{t+1|t}} & 0 \\ 0 & \sqrt{V} \end{pmatrix}_i$$

$$\Rightarrow \begin{pmatrix} x_{t+1}^{(0)} \\ v^{(0)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} x_{t+1|t}^{(i)} \\ w^{(i)} \end{pmatrix} = \begin{pmatrix} \mu_{t+1|t} \\ 0 \end{pmatrix} \pm \sqrt{2} \begin{pmatrix} \sqrt{\Sigma_{t+1|t}} & 0 \\ 0 & \sqrt{0.5} \end{pmatrix}_i \quad i=1,2,3,4$$

$h(x_{t+1}, v_{t+1})$

innovation

update

$$\mu_{t+1|t+1} = \mu_{t+1|t} + k_{t+1|t}(z_{t+1} - m_{t+1|t})$$

$$\Sigma_{t+1|t+1} = \Sigma_{t+1|t} + k_{t+1|t} S_{t+1|t} k_{t+1|t}^T$$

where

$$z_{t+1} = x_{t+1}^2 + v_{t+1} \qquad x_{t+1|t} \sim N(\mu_{t+1|t}, \Sigma_{t+1|t})$$

$$v_{t+1} \sim N(0, 0.5)$$

$$\text{K-gain} = k_{t+1|t} = C_{t+1|t} S_{t+1|t}^{-1}$$

$$\Rightarrow m_{t+1|t} = \sum_{i=0}^{2(d_x+d_v)} w_i^{(m)} \left[h(x_{t+1|t}, v^{(i)}) \right]$$

$$= \sum_{i=0}^{4} w_i^{(m)} \left((x_{t+1|t}^{(i)})^2 + v^{(i)} \right)$$

$$S_{t+1|t} = \sum_{i=0}^{2(d_x+d_v)} w_i^{(c)} \left(h(x_{t+1|t}^{(i)}, v^{(i)}) - m_{t+1|t} \right) \cdot \left(\quad \right)^T$$

$$= \sum_{i=0}^{4} w_i^{(c)} \left[(x_{t+1|t}^{(i)})^2 + v^{(i)} - m_{t+1|t} \right] \cdot \left[\cdots \right]^T$$

sigma point

$$C_{t+1|t} = \sum_{i=0}^{2(d_x+d_v)} w_i^{(c)} \left(x_{t+1|t}^{(i)} - \mu_{t+1|t} \right) \left[h(x_{t+1|t}^{(i)}, v^{(i)}) - m_{t+1|t} \right]^T$$

$$z_{t+1}^{(i)}$$

$$= \sum_{i=0}^{4} w_i^{(c)} \left(x_{t+1|t}^{(i)} - \mu_{t+1} \right) \left[(x_{t+1|t}^{(i)})^2 + v^{(i)} - m_{t+1|t} \right]^T$$

Now: $\begin{pmatrix} x_{t+1} \\ z_{t+1} \end{pmatrix} \sim \begin{pmatrix} \mu_{t+1|t} \\ m_{t+1|t} \end{pmatrix}, \begin{bmatrix} \Sigma_{t+1|t} & C_{t+1|t} \\ C_{t+1|t}^T & S_{t+1|t} \end{bmatrix}$

$\Rightarrow$ Then do the update step