Due oct /25/2017

problem # 1

1a) $f_U(u) = \begin{cases} 1 , & 0<u<1 \\ 0 , & \text{otherwise} \end{cases} \Rightarrow F_U(u) = \begin{cases} 0 , & u<0 \\ u , & 0\le u \le 1 \\ 1 , & u>1 \end{cases}$

$f_V(v) = \begin{cases} 1 , & 0< v<1 \\ 0 , & \text{otherwise} \end{cases} \Rightarrow F_V(v) = \begin{cases} 0 , & v<0 \\ v , & 0\le v \le 1 \\ 1 , & v>1 \end{cases}$

it seems like, log has the base of 10, and ln has the base of e, so I use base 10 to do following problem, but if i should use base e, the deduction way should be same

$$F_R(r) = P(R\le r) = P(\sqrt{-2\log(1-u)} < r)$$

$$= P(u < 1-10^{\frac{r^2}{-2}})$$

$$= \int_{-\infty}^{1-10^{\frac{r^2}{-2}}} f_U(u)\,du = 1-10^{\frac{r^2}{-2}}$$

$\boxed{R \in [0, +\infty]}$

$$F_\Theta(\theta) = P(\Theta \le \theta) = P(2\pi v \le \theta)$$

$$= P(v \le \frac{\theta}{2\pi})$$

$$= \int_0^{\frac{\theta}{2\pi}} f_V(v)\,dv = \frac{\theta}{2\pi}$$

$\boxed{\Theta \in [0, 2\pi]}$

$$\begin{cases} f_R(r) = +10^{\frac{r^2}{-2}}(\ln 10)\cdot r \\ f_\Theta(\theta) = \frac{1}{2\pi} \end{cases}$$

Summary $F_R(r) = \begin{cases} 0 & r<0 \\ 1-10^{-\frac{r^2}{2}} & 0 \le r \le +\infty \end{cases}$

$$f_R(r) = \begin{cases} 10^{-\frac{r^2}{2}} (\ln 10) \cdot r & , 0 \le r < +\infty \\ 0 & , \text{otherwise} \end{cases}$$

$$F_\Theta(\theta) = \begin{cases} 0 & , \text{otherwise} \\ \frac{\theta}{2\pi} & , 0 \le \theta \le 2\pi \end{cases}$$

$$f_\Theta(\theta) = \begin{cases} 0 & , \text{otherwise} \\ \frac{1}{2\pi} & , 0 \le \theta \le 2\pi \end{cases}$$

(b) $V := (R, \Theta)$ $\qquad f_V(r, \theta) = f_R(r) \cdot f_\Theta(\theta)$

$$\begin{cases} \Theta \in [0, 2\pi] \\ R \in [0, +\infty] \end{cases}$$

$T := (X, Y) = g(V) := (R\cos\Theta, R\sin\Theta)$

$$\det\left|\frac{dg}{dv}\right| = \det\begin{vmatrix} \cos\Theta & -R\sin\Theta \\ \sin\Theta & R\cos\Theta \end{vmatrix}$$

$$= R\cos^2\theta + R\sin^2\theta = R$$

$$f_T(x, y) = \begin{cases} \frac{1}{R} f_V\left(\sqrt{x^2+y^2}, \tan^{-1}\frac{x}{y}\right) & \left(\begin{matrix} 0<X<R \\ 0<Y<R \end{matrix}\right) \\ 0 & , \text{otherwise} \end{cases}$$

$$f_T(x,y) = \frac{1}{R} \, 10^{\frac{x^2+y^2}{-2}} \ln(10) \cdot \frac{\sqrt{x^2+y^2}}{y} \times \frac{1}{2\pi}$$

$$F_T(x,y) = \begin{cases} \int_{-\infty}^{y}\int_{-\infty}^{x} \frac{1}{R} \cdot 10^{\frac{x^2+y^2}{-2}} \ln(10) \frac{\sqrt{x^2+y^2}}{2\pi} \, dx \, dy & \left(\begin{matrix} 0<x<R \\ 0<y<R \end{matrix}\right) \\ 1 & R<x, \quad R<y \\ 0 & x<0, \quad y<0 \end{cases}$$

problem # 2

$$var(x) = E(x^2) - E(x)^2$$

(a) $E(X^T - X'^T)(X - X')$

$$= E(X^TX - X'^TX - X^TX' + X'^TX')$$

$$= E(X^TX) - 2E(X^TX') + E(X'^TX')$$

$$= \sum_{i=1}^{m} E(x_i^2) - 0 + \sum_{j=1}^{m} E(x_j'^2)$$

$$= \mu_1^2 + m\sigma^2 + \mu_1^2 + m\sigma^2 = 2(\mu_1^T\mu_1 + m\sigma^2)$$

(b) $E(X^TX) - 2E(X^TX') + E(X'^TX')$

$$= \mu_1^T\mu_1 + m\sigma^2 - 2\mu_1^T\mu_2 + \mu_2^T\mu_2 + m\sigma^2$$

$$= 2m\sigma^2 + \mu_1^T\mu_1 - 2\mu_1^T\mu_2 + \mu_2^T\mu_2$$

$$\text{Ratio} = \frac{2\mu_1^T\mu_1 + 2m\sigma^2}{2m\sigma^2 + \mu_1^T\mu_1 + \mu_2^T\mu_2 - 2\mu_1^T\mu_2}$$

$$= \frac{2(\mu_1^T\mu_1 - \mu_{11}\mu_{11} + \mu_{11}\mu_{11}) + 2m\sigma^2}{2m\sigma^2 + (\mu_1^T\mu_1 - \mu_{11}\mu_{11} + \mu_{11}\mu_{11}) + (\mu_2^T\mu_2 - \mu_{21}\mu_{21} + \mu_{21}\mu_{21}) - 2(\mu_1^T\mu_2 - \mu_{11}\mu_{21} + \mu_{11}\mu_{21})}$$

Assum. $\mu_{1j} = \mu_{2j} = \mu$

$$\Rightarrow \frac{2((m-1)\mu^2 + \mu_{11}\mu_{11}) + 2m\sigma^2}{2m\sigma^2 + (m-1)\mu^2 + \mu_{11}\mu_{11} + (m-1)\mu^2 + \mu_{21}\mu_{21} - 2((m-1)\mu^2 + \mu_{11}\mu_{21})}$$

$$\Rightarrow \text{Ratio} = \frac{2m\mu^2 - 2\mu^2 + 2\mu_{11}\mu_{11} + 2m\sigma^2}{2m\sigma^2 + \cancel{m\mu^2} \cancel{-\mu^2} + \mu_{11}\mu_{11} + \cancel{m\mu^2} \cancel{-\mu^2} + \mu_{21}\mu_{21} + \cancel{2m\mu^2} + \cancel{2\mu^2} + 2\mu_{11}\mu_{21}}$$

$$= \frac{2m(\mu^2 + \sigma^2) + 2\mu_{11}^2 - 2\mu^2}{2m\sigma^2 + \mu_{11}^2 + \mu_{21}^2 + 2\mu_{11}\mu_{21}}$$

As m increases, the ratio $\rightarrow 1$, which meens for a given point the distance to inner point is nearly equal to the inter class poin. Thus, in such case it is hard to classify the point.

$$= \frac{\lambda^{\sum_{j=1}^{n} x_j + \alpha - 1} e^{-\lambda(n+\beta)}}{\prod_{j=1}^{n} x_j!} \cdot \left[\frac{\beta^{\alpha}}{\Gamma(\alpha)}\right] \quad (*)$$

1c) $\hat{\lambda} = \arg\max_{\lambda} p(\lambda|x) = \arg\max_{\lambda} (*)$

$l(\lambda) = \log(*) = \sum_{j=1}^{n} x_j + \alpha - 1 \cdot \log\lambda + [-\lambda(n+\beta)] + \log \text{const} - \log \prod_{j=1}^{n} x_j!$

$$\frac{d\, l(\lambda)}{d\lambda} = \frac{\sum_{j=1}^{n} x_j + \alpha - 1}{\lambda} - (n+\beta)$$

$$\sum_{j=1}^{n} x_j + \alpha - 1 - \lambda(n+\beta) = 0$$

$$\Rightarrow \lambda = \frac{\alpha - 1 + \sum_{j=1}^{n} x_j}{n+\beta}$$

1d) in this case $\lambda = \frac{\ln\eta}{-2}$ $\qquad p(x|\eta) = \frac{\left(\frac{\ln\eta}{-2}\right)^{x} e^{\frac{\ln\eta}{2}}}{x!}$

$$MLE(\eta) = \arg\max_{\eta} p(x|\eta)$$

$$\Rightarrow \log MLE) = x \ln(\ln\eta) - x\ln(-2) + \frac{\ln\eta}{2} - \ln x!$$

$$\frac{d\, l(\theta)}{d\eta} = \frac{x}{\ln\eta} \cdot \frac{1}{\eta} + \frac{1}{2}\frac{1}{\eta} = 0$$

$$-2x = \ln\eta \quad \Rightarrow \quad \hat{\eta} = e^{-2x}$$

problem #3

(a)

1) $L(\lambda ; X) = L(\lambda ; x_1 \cdots x_n)$

$$= \prod_{j=1}^{n} P(x_j ; \lambda)$$

$$= \prod_{j=1}^{n} \exp(-\lambda) \frac{1}{x_j!} \lambda^{x_j}$$

$$l(\lambda ; x) = \sum_{j=1}^{n} (-\lambda + x_j \ln\lambda - \ln x_j!)$$

$$= -n\lambda + \ln\lambda \sum_{j=1}^{n} x_j - \sum_{j=1}^{n} \ln x_j!$$

$$l' = -n + \frac{1}{\lambda} \sum_{j=1}^{n} x_j = 0$$

$$\hat{\lambda} = \frac{1}{n} \sum_{j=1}^{n} x_j$$

2) $E(\hat{\lambda}) = \frac{1}{n} \sum_{j=1}^{n} E(x_j) = \frac{1}{n} \cdot n\lambda = \lambda$

1b) $P(\lambda | x) = P(x|\lambda) \cdot P(\lambda | \alpha, \beta)$

$$= \prod_{j=1}^{n} \frac{\lambda^{x_j} e^{-\lambda}}{x_j!} \cdot \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \right)$$

$$= \frac{\lambda^{\sum_{j=1}^{n} x_j} e^{-n\lambda} \cdot e^{-\beta\lambda} \cdot \lambda^{\alpha-1}}{\prod_{j=1}^{n} x_j!} \cdot \left(\frac{\beta^\alpha}{\Gamma(\alpha)} \right)$$

(e) Bias $\hat{\eta} = E(\hat{\eta}) - \eta$ $\qquad \eta = e^{-2\lambda}$

$$E[\hat{\eta}] = \int_{-\infty}^{+\infty} \hat{\eta}\, f(x) \cdot dx$$

$$= \int_{-\infty}^{+\infty} e^{-2x} \cdot \frac{\lambda^x e^{-\lambda}}{x!}\, dx$$

$$= e^{-\lambda} \sum_{x=0}^{+\infty} \left(\frac{\lambda}{e^2}\right)^x \cdot \frac{1}{x!}$$

$$= e^{-\lambda} e^{\lambda/e^2} \qquad \text{Taylor series}$$

$$= e^{\lambda\left(\frac{1}{e^2} - 1\right)}$$

$$\Rightarrow \text{Bias}[\hat{\eta}] = e^{\lambda\left(\frac{1}{e^2}-1\right)} - e^{-2\lambda}$$

(f) $\hat{\eta} = (-1)^x$

$$E[\hat{\eta}] = \sum_{-\infty}^{+\infty} (-1)^x \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= e^{-\lambda} \sum_{-\infty}^{+\infty} \frac{(-\lambda)^x e^{-\lambda}}{x!}$$

$$= e^{-2\lambda} \sum_{-\infty}^{+\infty} \frac{(-\lambda)^x}{x!} = e^{-2\lambda}$$

$(-1)^x$ is a bad estimate as it fluctuates between +1 and -1, even if it is unbias