MAT 2384 _ Solutions to suggested Problems.
Linear ODEs - Bernoulli Equation

1) $x^2y' + 3xy = \frac{1}{x}$; $y(1) = -1$.

Rewrite the equation: $y' + \frac{3}{x}y = \frac{1}{x^3}$: linear ODE with $f(x) = \frac{3}{x}$ and $r(x) = \frac{1}{x^3}$. Thus

$$\mu(x) = e^{\int f(x)dx} = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3.$$

The solution is $y(x) = \dfrac{\int \mu(x) r(x) dx + C}{\mu(x)} = \dfrac{\int (x^3 \cdot \frac{1}{x^3}) dx + C}{x^3}$

$$= \frac{\int 1 \cdot dx + C}{x^3} = \frac{x + C}{x^3} = \frac{1}{x^2} + \frac{C}{x^3}$$: general solution

$y(1) = -1 \Rightarrow -1 = 1 + C \Rightarrow C = -2$ and the particular solution is $y(x) = \frac{1}{x^2} - \frac{2}{x^3}$.

2) $y' + ky = e^{2kx}$; k is a constant.

This is a linear ODE with $f(x) = k$ and $r(x) = e^{2kx}$

$\mu(x) = e^{\int k dx} = e^{kx}$. The general solution is

$$y(x) = \frac{\int \mu(x) r(x) dx + C}{\mu(x)} = \frac{\int e^{kx} \cdot e^{2kx} dx + C}{e^{kx}} = \frac{\int e^{3kx} dx + C}{e^{kx}}$$

$$= e^{-kx}\left[\frac{1}{3k} e^{3kx} + C\right] = \frac{1}{3k} e^{2kx} + C e^{-kx}$$

3) $y' + 2y\sin(2x) = 2e^{\cos(2x)}$, $y(0) = 0$

This is a linear ODE with $f(x) = 2\sin(2x)$, $r(x) = 2e^{\cos(2x)}$

$$\mu(x) = e^{\int f(x)dx} = e^{\int 2\sin(2x)\,dx} = e^{-\cos(2x)}.$$ The general solution is $$y(x) = \frac{\int \mu(x) r(x)\,dx + C}{\mu(x)} = \frac{\int e^{-\cos(2x)}\, 2e^{\cos(2x)}\,dx + C}{e^{-\cos(2x)}}$$

$$= e^{\cos(2x)}\left[2x + C\right] = 2x e^{\cos(2x)} + C e^{\cos(2x)}$$

$y(0) = 0 \Rightarrow 0 = C \Rightarrow y = 2x e^{\cos(2x)}$ is the particular solution.

4) $y' + 4y\cot(2x) = 6\cos(2x)$, $y(\pi/4) = 2$.

This is a linear ODE with $f(x) = 4\cot(2x)$ and $r(x) = 6\cos(2x)$

$$\mu(x) = e^{\int f(x)dx} = e^{\int 4\cot(2x)\,dx} = e^{4\int \cot(2x)\,dx} =$$

$$e^{2\ln|\sin(2x)|} = \sin^2(2x) \quad \text{since } \int \cot(2x)\,dx = \frac{1}{2}\ln|\sin(2x)|$$

The general solution is thus

$$y(x) = \frac{\int \mu(x) r(x)\,dx + C}{\mu(x)} = \frac{\int \sin^2(2x)\, 6\cos(2x)\,dx + C}{\sin^2(2x)}$$

$$= \frac{6\int \sin^2(2x)\cos(2x)\,dx}{\sin^2(2x)} + \frac{C}{\sin^2(2x)}$$

Let $t = \sin(2x) \Rightarrow dt = 2\cos(2x)\,dx$

$$\int \sin^2(2x)\cos(2x)\,dx = \frac{1}{2}\int t^2\,dt = \frac{1}{6}t^3 = \frac{1}{6}\sin^3(2x)$$

Therefore the general solution is

$$y(x) = \frac{\sin^3(2x)}{\sin^2(2x)} + \frac{C}{\sin^2(2x)} = \sin(2x) + \frac{C}{\sin^2(2x)}$$

$$y(\pi/4) = 2 \Rightarrow 2 = 1 + \frac{C}{1} \Rightarrow C = 1.$$

The particular solution is

$$y(x) = \sin(2x) + \frac{1}{\sin^2(2x)}.$$

5) $y' + y/x^2 = 2x e^{1/x}$, $y(1) = 13.86$.

Linear ODE with $f(x) = \frac{1}{x^2}$, $r(x) = 2x e^{1/x}$

$\mu(x) = e^{\int \frac{1}{x^2} dx} = e^{-1/x}$ and the general solution is

$$y(x) = \frac{\int \mu(x) r(x) dx + C}{\mu(x)} = \frac{\int e^{-1/x} 2x e^{1/x} dx + C}{e^{-1/x}}$$

$$= e^{1/x} \left[\int 2x dx + C \right] = x^2 e^{1/x} + C e^{1/x}$$

$y(1) = 13.86 \Rightarrow 13.86 = e + Ce \Rightarrow 13.86 = e(C+1) \Rightarrow$

$C+1 = \frac{13.86}{e}$ and $C = \frac{13.86}{e} - 1 \simeq 4.1$

The particular solution is $y(x) = x^2 e^{1/x} + 4.1 e^{1/x}$

6) $y' + y = y^2, \quad y(0) = -1$

This is a Bernoulli equation with $p(x) = 1$, $g(x) = 1$ and $a = 2$.

Let $u = y^{1-a} = y^{1-2} = \frac{1}{y}$, then $u' = -\frac{1}{y^2} y' \Rightarrow y' = -y^2 u'$

$= -\frac{1}{u^2} u'$. The DE becomes:

$-\frac{1}{u^2} u' + \frac{1}{u} = \frac{1}{u^2}$. Multiplying with u^2: $-u' + u = 1 \Leftrightarrow$

$u' - u = -1$: this is a linear ODE with $f(x) = -1, r(x) = -1$.

$\mu(x) = e^{\int f(x)dx} = e^{\int (-1)dx} = e^{-x}$. The general solution is

$u(x) = \dfrac{\int \mu(x) r(x) dx + C}{\mu(x)} = \dfrac{\int e^{-x}(-1)dx + C}{e^{-x}} = e^{x}\left[-\int e^{-x} dx + C\right]$

$= e^{x}\left[e^{-x} + C\right] = 1 + Ce^{-x} \Rightarrow y(x) = \frac{1}{u} = \dfrac{1}{1+Ce^{-x}}$

$y(0) = -1 \Rightarrow -1 = \dfrac{1}{1+C} \Rightarrow 1 + C = -1 \Rightarrow C = -2$.

The particular solution is $y(x) = \dfrac{1}{1-2e^{-x}}$.

7) $y' + (x+1) y = e^{x^2} y^3, \quad y(0) = \frac{1}{2}$.

This is a Bernoulli equation with $p(x) = x+1$, $g(x) = e^{x^2}$, $a = 3$.

Setting $u(x) = [y(x)]^{1-a} = \frac{1}{y^2}$, the DE becomes

$u' + (1-a)p(x)\, u = (1-a)\, g(x)$ or $u' - 2(x+1)u = -2e^{x^2}$

which is now a linear first-order ODE:

$\mu(x) = e^{\int f(x)dx} = e^{\int -2(x+1)dx} = e^{-(x+1)^2}$ and the general solution is $u(x) = \dfrac{\int \mu(x) r(x)\,dx + C}{\mu(x)} = \dfrac{\int e^{-(x+1)^2}(-2e^{x^2})\,dx + C}{e^{-(x+1)^2}}$ ⑤

$= e^{(x+1)^2}\left[-2\int e^{(-x^2-2x-1+x^2)}\,dx + C\right] = e^{(x+1)^2}\left[-2\int e^{(-2x-1)}\,dx + C\right]$

$= e^{(x+1)^2}\left[e^{-2x-1} + C\right]$ (by making the substitution $t = -2x-1$

in the integral) $= e^{(x+1)^2 - 2x - 1} + Ce^{(x+1)^2} = e^{x^2} + Ce^{(x+1)^2}$

$\Rightarrow u(x) = e^{x^2}\left(1 + Ce^{(2x+1)}\right) = e^{x^2}\left(1 + Ce^{2x}\right)$ since

$e^{2x+1} = e^{2x}\cdot e = Ce^{2x}$ where $C = e$

$y^2 = \dfrac{1}{e^{x^2}(1+Ce^{2x})} \Rightarrow y = \dfrac{1}{\sqrt{e^{x^2}(1+Ce^{2x})}}$

$y(0) = \frac{1}{2} \Rightarrow \frac{1}{2} = \dfrac{1}{\sqrt{1+C}} \Rightarrow 1 + C = 4 \Rightarrow C = 3.$

The unique (or particular) solution is $y(x) = \dfrac{1}{\sqrt{e^{x^2}(1+3e^{2x})}}$

8) $2yy' + y^2\sin(x) = \sin(x)$; $y(0) = \sqrt{2}$.

Rewrite the equation: $y' + \frac{1}{2}y\sin(x) = \frac{1}{2}\sin(x)\,y^{-1}$;

this is a Bernoulli equation with $p(x) = \frac{1}{2}\sin(x)$, $g(x) = \frac{1}{2}\sin(x)$

and $a = -1$. Setting $u(x) = y^{1-a} = y^2$, the DE becomes

$u' + (1-a)\,p(x)\,u = (1-a)\,g(x)$ or $u' + \sin(x)\,u = \sin(x)$ which

is now linear with $f(x) = \sin x$, $r(x) = \sin x$

$\mu(x) = e^{\int f(x)\,dx} = e^{\int \sin x\,dx} = e^{-\cos x}$. The general solution is

$$u(x) = \frac{\int \mu(x) r(x)\,dx + C}{\mu(x)} = \frac{\int e^{(-\cos x)} \sin x\,dx + C}{e^{-\cos x}} =$$ (6)

$e^{\cos x}\left[\int \sin x\, e^{(-\cos x)}\,dx + C\right]$. Let $t = -\cos x \Rightarrow \frac{dt}{dx} = \sin x$

and $\int \sin x\, e^{(-\cos x)}\,dx = \int e^t\,dt = e^t = e^{-\cos x}$. Therefore

$u(x) = e^{\cos x}\left[e^{-\cos x} + C\right] = 1 + Ce^{(\cos x)} \Rightarrow y = \sqrt{u(x)} = \sqrt{1 + Ce^{\cos x}}$

$y(0) = \sqrt{2} \Rightarrow \sqrt{2} = \sqrt{1 + Ce} \Rightarrow 1 + Ce = 2 \Rightarrow C = \frac{1}{e}$. The particular solution is $y(x) = \sqrt{1 + e^{(\cos x - 1)}}$.

<u>Note</u> The equation $2yy' + y^2 \sin x = -\sin x$ is a seperable one and can be solved by the seperation of variables techniques.