

MAT 2784 A Assignment # 5 Solutions

1. $y'' + y = \sec x$, $y(0) = y'(0) = 1$

The corresponding homog DE is $y'' + y = 0 \Rightarrow y_1(x) = C_1 \cos x + C_2 \sin x$
 i.e. $y_1(x) = \cos x$, $y_2(x) = \sin x$

$r(x) = \sec x \Rightarrow$ we must use Variation of Parameters

we need to solve $u_1' y_1 + u_2' y_2 = 0 \quad \text{or} \quad u_1' \cos x + u_2' \sin x = 0 \quad (1)$

$$u_1' y_1' + u_2' y_2' = r \quad -u_1' \sin x + u_2' \cos x = \sec x \quad (2)$$

then $(1) \times \sin x \quad u_1' \sin x \cos x + u_2' \sin^2 x = 0 \quad (3) \quad (3) + (4)$ gives $u_2' = 1$

$$(2) \times \cos x \quad -u_1' \sin x \cos x + u_2' \cos^2 x = 1 \quad (4) \quad \text{so } u_2(x) = x$$

and thus $u_1' = -u_2' \tan x = -\tan x \Rightarrow u_1(x) = \ln |\cos x|$

so $y_p(x) = u_1 \cos x + u_2 \sin x = \cos x \ln |\cos x| + x \sin x$

and the general solution is $y_p(x) = y_h(x) + y_p(x)$

$$= C_1 \cos x + C_2 \sin x + \cos x \ln |\cos x| + x \sin x$$

$$y(0) = 1 \Rightarrow 1 = C_1 \cos(0) + C_2 \sin(0) + \cos(0) \ln |\cos(0)| + (0) \sin(0) \Rightarrow C_1 = 1$$

$$y_p'(x) = -C_1 \sin x + C_2 \cos x - \sin x \ln |\cos x| - \sin x + \sin x + x \cos x$$

$$= -C_1 \sin x + C_2 \cos x - \sin x \ln |\cos x| + x \cos x$$

$$y'(0) = 1 \Rightarrow 1 = -C_1 \sin(0) + C_2 \cos(0) - \sin(0) \ln |\cos(0)| + (0) \cos(0) \Rightarrow C_2 = 1$$

∴ the unique solution is

$$y(x) = \cos x + \sin x + \cos x \ln |\cos x| + x \sin x$$

2. $x^2 y'' - 2xy' + 2y = x^2$, $x > 0$, $y(1) = 2$, $y'(1) = 4$

The correspond. homog. DE is $x^2 y'' - 2xy' + 2y = 0$, which has characteristic equation $m(m-1) - 2m + 2 = m^2 - 3m + 2 = (m-1)(m-2) = 0$

i.e. $m_1 = 1$, $m_2 = 2$ and $y_h(x) = C_1 x + C_2 x^2$ or $y_h(x) = x$, $y_2(x) = x^2$

The DE is E-C, so must use Var. of Para, where $r(x) = 1$

we solve $u_1' x + u_2' x^2 = 0 \quad \text{or} \quad u_1' + u_2' x = 0 \quad (1)$

$$u_1' + 2u_2' x = 1 \quad u_1' + 2u_2' x = 1 \quad (2)$$

(2) - (1) $\Rightarrow u_2' x = 1$, so $u_2' x = 1/x$, then $u_2(x) = \ln x$

and thus (1) tells us $u_1' = -u_2' x = -1/x \Rightarrow u_1(x) = -x$

and so $y_p(x) = (-x)(x) + (x^2)(\ln x) = -x^2 + x^2 \ln x$

but x^2 appears in $y_2(x)$, so we can take $y_{p1}(x) = x^2 \ln x$
 and so the general solution is $y_2(x) = C_1 x + C_2 x^2 + x^2 \ln x$

$$y(1) = 2 \Rightarrow 2 = C_1(1) + C_2(1)^2 + (1)^2 \ln(1) \Rightarrow C_1 + C_2 = 2$$

$$y_2'(x) = C_1 + 2C_2 x + 2x \ln x + x \quad \left. \begin{array}{l} \\ C_1 = C_2 = 1 \end{array} \right\}$$

$$y'(1) = 4 \Rightarrow 4 = C_1 + 2C_2(1) + 2(1)\ln(1) + (1) \Rightarrow C_1 + 2C_2 = 3$$

\therefore the unique solution is

$$y(x) = x + x^2 + x^2 \ln x$$

3. $y_1' = 4y_1 - 2y_2 - 2x - 5, \quad y_1(0) = 2$

$$y_2' = 3y_1 - y_2 - 2x - 3, \quad y_2(0) = 2$$

so we have $\vec{y}' = A\vec{y} + \vec{F}$ where $A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}, \quad \vec{F}(x) = \begin{bmatrix} -2x-5 \\ -2x-3 \end{bmatrix}$
 and $\vec{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

The corresponding homogeneous system is $\vec{y}' = A\vec{y}$

$$\det(A - \lambda I) = \left| \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{vmatrix}$$

$$= (4-\lambda)(-1-\lambda) + 6 = \lambda^2 - 3\lambda - 4 + 6 = \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0$$

so the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$

$$\text{for } \lambda_1 = 1, \quad (A - \lambda_1 I)\vec{v}_1 = \vec{0} \text{ is } \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \vec{v}_1 = \vec{0} \Rightarrow \text{take } \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{for } \lambda_2 = 2, \quad (A - \lambda_2 I)\vec{v}_2 = \vec{0} \text{ is } \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \vec{v}_2 = \vec{0} \Rightarrow \text{take } \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and so $\vec{y}_{h}(x) = C_1 e^x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{f}(x) = \begin{bmatrix} -2x-5 \\ -2x-3 \end{bmatrix} \Rightarrow \vec{y}_p(x) = \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix}, \quad \text{so } \vec{y}_p'(x) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$A\vec{y}_p + \vec{f} = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} a_1 x + b_1 \\ a_2 x + b_2 \end{bmatrix} + \begin{bmatrix} -2x-5 \\ -2x-3 \end{bmatrix} = \begin{bmatrix} (4a_1 - 2a_2 - 2)x + 4b_1 - 2b_2 - 5 \\ (3a_1 - a_2 - 2)x + 3b_1 - b_2 - 3 \end{bmatrix}$$

the $\vec{y}_p' = A\vec{y}_p + \vec{f}$ means $(4a_1 - 2a_2 - 2)x + 4b_1 - 2b_2 - 5 = a_1$
 $(3a_1 - a_2 - 2)x + 3b_1 - b_2 - 3 = a_2$

$$\text{so } \begin{cases} 4a_1 - 2a_2 - 2 = 0 \\ 3a_1 - a_2 - 2 = 0 \end{cases} \quad \begin{cases} 4a_1 - 2a_2 = 2 \\ 3a_1 - a_2 = 2 \end{cases} \quad a_1 = a_2 = 1$$

$$\text{then } \begin{cases} 4b_1 - 2b_2 - 5 = a_1 \\ 3b_1 - b_2 - 3 = a_2 \end{cases} \quad \begin{cases} 4b_1 - 2b_2 = 1 \\ 3b_1 - b_2 = 4 \end{cases} \quad \begin{matrix} b_1 = 1 \\ b_2 = -1 \end{matrix}$$

and the particular solution is $\tilde{y}_p(x) = \begin{bmatrix} x+1 \\ x-1 \end{bmatrix}$

The general solution is $\tilde{y}_g(x) = \tilde{y}_h(x) + \tilde{y}_p(x) = c_1 e^{2x} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} x+1 \\ x-1 \end{bmatrix}$

$$\tilde{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \begin{cases} 2c_1 + c_2 + 1 = 2 \\ 3c_1 + c_2 - 1 = 2 \end{cases} \quad \begin{cases} 2c_1 + c_2 = 1 \\ 3c_1 + c_2 = 3 \end{cases} \quad \begin{matrix} c_1 = 2 \\ c_2 = -3 \end{matrix}$$

\therefore the unique solution is

$$\boxed{\tilde{y}(x) = 2e^{2x} \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 3e^{2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} x+1 \\ x-1 \end{bmatrix}}$$

$$\text{or } \boxed{y_1(x) = 4e^x - 3e^{2x} + x + 1, \quad y_2(x) = 6e^x - 3e^{2x} + x - 1}$$

$$4. \quad y_1' = 2y_1 - 4y_2 + 10x - 2x^2, \quad y_1(0) = 2$$

$$y_2' = 4y_1 - 6y_2 + 2 + 12x - 4x^2, \quad y_2(0) = 1$$

$$A = \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}, \quad \vec{F}(x) = \begin{bmatrix} 10x - 2x^2 \\ 2 + 12x - 4x^2 \end{bmatrix}, \quad \tilde{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -4 \\ 4 & -6-\lambda \end{vmatrix} = (2-\lambda)(-6-\lambda) + 16 = \lambda^2 + 4\lambda - 12 + 16 = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0$$

so we have a repeated eigenvalue $\lambda_1 = \lambda_2 = -2$

$$(A - \lambda I)\vec{v} = \vec{0} \text{ is } \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\vec{v} = \vec{0} \Rightarrow \text{take } \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda I)\vec{u} = \vec{v} \text{ is } \begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix}\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \text{take } \vec{u} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

$$\text{and so } \tilde{y}_h(x) = c_1 e^{-2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2x} \left(x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right)$$

$$\vec{F}(x) = \begin{bmatrix} 10x - 2x^2 \\ 2 + 12x - 4x^2 \end{bmatrix} \Rightarrow \tilde{y}_p(x) = \begin{bmatrix} 0, n^2 + b_1 n + d_1 \\ c_3 n^2 + b_2 n + d_2 \end{bmatrix} \Rightarrow \tilde{y}_p'(x) = \begin{bmatrix} 2a_1 n + b_1 \\ 2a_2 n + b_2 \end{bmatrix}$$

$$\begin{aligned}
 A\bar{y}_p + \bar{F} &= \begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} a_1 x^2 + b_1 x + d_1 \\ a_2 x^2 + b_2 x + d_2 \end{bmatrix} + \begin{bmatrix} 10x - 2x^2 \\ 2 + 12x - 4x^2 \end{bmatrix} \\
 &= \begin{bmatrix} (2a_1 - 4a_2 - 2)x^2 + (2b_1 - 4b_2 + 10)x + 2d_1 - 4d_2 \\ (4a_1 - 6a_2 - 4)x^2 + (4b_1 - 6b_2 + 12)x + 4d_1 - 6d_2 + 2 \end{bmatrix}
 \end{aligned}$$

thus $\bar{y}_p = A\bar{y}_p + \bar{F}$ means $(2a_1 - 4a_2 - 2)x^2 + (2b_1 - 4b_2 + 10)x + 2d_1 - 4d_2 = 2a_1 x + b_1$,
 $(4a_1 - 6a_2 - 4)x^2 + (4b_1 - 6b_2 + 12)x + 4d_1 - 6d_2 + 2 = 2a_2 x + b_2$

$$\begin{aligned}
 \text{so } \left. \begin{array}{l} 2a_1 - 4a_2 - 2 = 0 \\ 4a_1 - 6a_2 - 4 = 0 \end{array} \right\} \quad \left. \begin{array}{l} 2a_1 - 4a_2 = 2 \\ 4a_1 - 6a_2 = 4 \end{array} \right\} \quad \left. \begin{array}{l} a_1 - 2a_2 = 1 \\ 2a_1 - 3a_2 = 2 \end{array} \right\} \quad \left. \begin{array}{l} a_1 = 1 \\ a_2 = 0 \end{array} \right.
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \left. \begin{array}{l} 2b_1 - 4b_2 + 10 = 2a_1 \\ 4b_1 - 6b_2 + 12 = 2a_2 \end{array} \right\} \quad \left. \begin{array}{l} 2b_1 - 4b_2 = -8 \\ 4b_1 - 6b_2 = -12 \end{array} \right\} \quad \left. \begin{array}{l} b_1 - 2b_2 = -4 \\ 2b_1 - 3b_2 = -6 \end{array} \right\} \quad \left. \begin{array}{l} b_1 = 0 \\ b_2 = 2 \end{array} \right.
 \end{aligned}$$

$$\begin{aligned}
 \text{and also } \left. \begin{array}{l} 2d_1 - 4d_2 = b_1 \\ 4d_1 - 6d_2 + 2 = b_2 \end{array} \right\} \quad \left. \begin{array}{l} 2d_1 - 4d_2 = 0 \\ 4d_1 - 6d_2 = 0 \end{array} \right\} \quad \left. \begin{array}{l} d_1 - 2d_2 = 0 \\ 2d_1 - 3d_2 = 0 \end{array} \right\} \quad d_1 = d_2 = 0
 \end{aligned}$$

and the particular solution is $\bar{y}_p(x) = \begin{bmatrix} x^2 \\ 2x \end{bmatrix}$

the general solution is $\bar{y}(x) = C_1 e^{-2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2x} \left(x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} x^2 \\ 2x \end{bmatrix}$

$$\bar{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} C_1 + 1/4 C_2 = 2 \\ C_1 = 1 \end{array} \Rightarrow C_2 = 4$$

\therefore the unique solution is

$$\boxed{\bar{y}(x) = e^{-2x} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4e^{-2x} \left(x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} x^2 \\ 2x \end{bmatrix}}$$

$$\underline{\underline{y_1(x) = (4x+2)e^{-2x} + x^2, \quad y_2(x) = (4x+1)e^{-2x} + 2x}}$$

$$5. \int_0^2 \frac{2x}{1+x^2} dx \text{ with Simpson's Rule with } 2n=8$$

$$\begin{aligned}
 \text{thus } h &= \frac{2-0}{8} = \frac{1}{4} \Rightarrow x_0 = 0, x_1 = 0.25, x_2 = 0.50, x_3 = 0.75, \\
 &\quad x_4 = 1, x_5 = 1.25, x_6 = 1.50, x_7 = 1.75, x_8 = 2
 \end{aligned}$$

$$\int_0^2 \frac{2x}{1+x^2} dx \approx \frac{h}{3} \sum_{j=0}^3 (f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2}))$$

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AS (5)

$$= \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) \left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8) \right]$$

$$\begin{aligned} &= \frac{1}{12} \left[\frac{2(0)}{1+(0)^2} + 4 \left(\frac{2(0.25)}{1+(0.25)^2} \right) + 2 \left(\frac{2(0.5)}{1+(0.5)^2} \right) + 4 \left(\frac{2(0.75)}{1+(0.75)^2} \right) \right. \\ &\quad + 2 \left(\frac{2(1)}{1+1^2} \right) + 4 \left(\frac{2(1.25)}{1+(1.25)^2} \right) + 2 \left(\frac{2(1.5)}{1+(1.5)^2} \right) \\ &\quad \left. + 4 \left(\frac{2(1.75)}{1+(1.75)^2} \right) + \frac{2(2)}{1+2^2} \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{12} [0 + 1.8823529 + 1.6 + 3.84 + 2 + 3.902439 \\ &\quad + 1.8461538 + 3.4461538 + 0.8] \\ &\approx \boxed{1.609758} \end{aligned}$$

the true value is $\int_0^{2\pi} \frac{2x}{1+x^2} dx = \ln(1+x^2)|_0^2 = \ln 5 = \boxed{1.609438}$

so the error is $E = 0.000320$

6. $\int_0^2 \frac{2x}{1+x^2} dx$ with GQ with $n=4$

change of variable : $x = \frac{1}{2}((0)(1-t) + (2)(t+1)) = t+1$, $dx = dt$

$$\begin{aligned} \int_0^2 \frac{2x}{1+x^2} dx &= \int_{-1}^1 \frac{2(t+1)}{1+(t+1)^2} dt \\ &\approx \sum_{j=1}^4 f(t_j) A_j \end{aligned}$$

$$= A_1 \frac{2(t_1+1)}{1+(t_1+1)^2} + A_2 \frac{2(t_2+1)}{1+(t_2+1)^2} + A_3 \frac{2(t_3+1)}{1+(t_3+1)^2} + A_4 \frac{2(t_4+1)}{1+(t_4+1)^2}$$

$$= (0.3478548451) \left(\frac{2(-0.8011363116+1)}{1+(-0.8011363116+1)^2} \right)$$

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A5 (6)

$$\begin{aligned} & + (0.6521451549) \left(\frac{2(-0.3399810436+1)}{1+(-0.3399810436+1)^2} \right) \\ & + (0.6521451549) \left(\frac{2(0.3399810436+1)}{1+(0.3399810436+1)^2} \right) \\ & + (0.3478548451) \left(\frac{2(0.8611363116+1)}{1+(0.8611363116+1)^2} \right) \\ = & 0.0947811 + 0.5996386 + 0.6251809 + 0.2900672 \\ = & \boxed{1.609668} \end{aligned}$$

The error is $E = \boxed{0.000230}$