MAT 2384 – Solutions to suggested Problems

Homogeneous Euler – Cauchy Equations

1) $x^2y'' - 6y = 0$. The characteristic equation is $m^2 + (0-1)m - 6 = 0$
$\Leftrightarrow \quad m^2 - m - 6 = 0 \Leftrightarrow (m-3)(m+2) = 0 \Leftrightarrow m = -2,\ m = 3.$
The general solution is $Y(x) = C_1 X^{-2} + C_2 X^3$.

2) $x^2y'' - 7xy' + 16y = 0$. The characteristic equation is $m^2 - 8m + 16 = 0$
$\Leftrightarrow (m-4)^2 = 0 \Leftrightarrow m_1 = m_2 = 4$. The general solution is
$Y(x) = C_1 X^4 + C_2 (\ln x)\, X^4$. Let us check:
$Y'(x) = 4C_1 X^3 + C_2 X^3 + 4C_2 (\ln x) X^3$ and
$Y''(x) = 12C_1 X^2 + 3C_2 X^2 + 4C_2 X^2 + 12 C_2 (\ln x) X^2 = 12C_1X^2 + 7C_2X^2 +$
$12 C_2 X^2 \ln x$. Therefore:
$x^2y'' - 7xy' + 16y = 12C_1X^4 + 7C_2X^4 + 12C_2X^4\ln x - 28C_1X^4 - 7C_2X^4 -$
$28 C_2 X^4 \ln X + 16 C_1 X^4 + 16 C_2 X^4 \ln x = 0$

3) $x^2y'' - xy' + 2y = 0$. The characteristic equation is
$m^2 + (a-1)m + b = 0 \Leftrightarrow m^2 - 2m + 2 = 0$; $b^2 - 4ac = (-2)^2 - 4(1)(2)$
$= -4 < 0$; there are 2 complex roots: $m_1 = \dfrac{2-2i}{2} = 1 - i$
and $m_2 = 1+i$. The general solution is

$Y(x) = C_1\, x \cos(\ln x) + C_2\, X \sin(\ln x) \quad ; \quad x > 0$

<u>Check</u> $y' = C_1 \cos(\ln x) - C_1 \sin(\ln x) + C_2 \sin(\ln x) + C_2 \cos(\ln x)$
$y'' = -\dfrac{C_1}{X}\sin(\ln x) - \dfrac{C_1}{X}\cos(\ln x) + \dfrac{C_2}{X}\cos(\ln x) - \dfrac{C_2}{X}\sin(\ln x)$
$x^2y'' - xy' + 2y = -C_1 x \sin(\ln x) - C_1 x \cos(\ln x) + C_2 x \cos(\ln x) -$
$C_2 X \sin(\ln x) - C_1 X \cos(\ln x) + C_1 x \sin(\ln x) - C_2 X \sin(\ln x) - C_2 X \cos(\ln x)$

$+2C_1 x \cos(\ln x) + 2C_2 x \sin(\ln x) = 0$

4) $2x^2y'' + 4xy' + 5y = 0 \iff x^2y'' + 2xy' + \frac{5}{2}y = 0$. The characteristic equation is $m^2 + (a-1)m + b = 0 \Rightarrow m^2 + m + 5/2 = 0$.

$b^2 - 4ac = 1^2 - 4(1)(5/2) = 1 - 10 = -9 = 9i^2$, There are 2 complex roots: $m_1 = \frac{-1-3i}{2} = -\frac{1}{2} - \frac{3}{2}i$ and $m_2 = -\frac{1}{2} + \frac{3}{2}i$. The general solution is $y(x) = C_1 x^{-1/2} \cos(3/2 \ln x) + C_2 x^{-1/2} \sin(3/2 \ln x)$

$$= \frac{C_1 \cos(3/2 \ln x) + C_2 \sin(3/2 \ln x)}{\sqrt{x}} \quad ; \ x > 0$$

5) $10x^2y'' - 20xy' + 22.4y = 0 \iff x^2y'' - 2xy' + 2.24 = 0$. The characteristic equation is $m^2 - 3m + 2.24 = 0$.

$b^2 - 4ac = (-3)^2 - 4(1)(2.24) = 0.04 > 0$. There are 2 distinct real roots $m_1 = \frac{3+0.2}{2} = 1.6$ and $m_2 = \frac{3-0.2}{2} = 1.4$.

The general solution is $y(x) = C_1 x^{1.4} + C_2 x^{1.6}$

6) $100x^2y'' + 9y = 0 \Rightarrow x^2y'' + \frac{9}{100}y = 0$, The characteristic equation is

$m^2 + (0-1)m + \frac{9}{100} = 0 \Rightarrow 100m^2 - 100m + 9 = 0$

$b^2 - 4ac = (-100)^2 - 4(100)(9) = 6400$; there are 2 real roots:

$m_1 = \frac{100+80}{200} = \frac{180}{200} = 0.9$ and $m_2 = \frac{100-80}{200} = 0.1$

The general solution is $y(x) = C_1 x^{0.1} + C_2 x^{0.9}$.

Let us check: $y'(x) = 0.1 C_1 x^{-0.9} + 0.9 C_2 x^{-0.1}$

$y''(x) = -0.09 C_1 x^{-1.9} - 0.09 C_2 x^{-1.1}$

$100x^2y'' + 9y = -9C_1x^{0.1} - 9C_2x^{0.9} + 9C_1x^{0.1} + 9C_2x^{0.9} = 0$

7) $x^2y'' - 4xy' + 6y = 0$, $y(1) = 1$, $y'(1) = 0$. The characteristic equation is $m^2 - 5m + 6 = 0 \iff (m-2)(m-3) = 0 \Rightarrow m_1 = 2, m_2 = 3$.

The general solution is $y(x) = C_1x^2 + C_2x^3$.

$y(1) = 1 \Rightarrow 1 = C_1 + C_2$ ①

$y'(x) = 2C_1x + 3C_2x^2$, $y'(1) = 0 \Rightarrow 2C_1 + 3C_2 = 0$ ②

-2 ① $+$ ② $\Rightarrow C_2 = -2$ and $C_1 = 1 - C_2 = 3$. The particular solution is $y(x) = 3x^2 - 2x^3$.

9) $x^2y'' - 2xy' + 2.25y = 0$, $y(1) = 2.2$, $y'(1) = 2.5$. The characteristic equation is $m^2 - 3m + 2.25 = 0$.

$b^2 - 4ac = (-3)^2 - 4(1)(2.25) = 0$: A unique real root

$m = \frac{3}{2} = 1.5$. The general solution is

$y(x) = C_1x^{1.5} + C_2x^{1.5}\ln(x)$; $x > 0$.

$y'(x) = 1.5C_1x^{0.5} + 1.5C_2x^{0.5}\ln(x) + C_2x^{0.5}$

$y(1) = 2.2 \Rightarrow C_1 = 2.2$

$y'(1) = 2.5 \Rightarrow 1.5C_1 + C_2 = 2.5 \Rightarrow C_2 = 2.5 - 1.5(2.2) = -0.8$

The particular solution is $y(x) = 2.2x^{1.5} - 0.8x^{1.5}\ln(x)$; $x > 0$.

10) $xy'' + 4y = 0$, $y(1) = 12$, $y'(1) = -6$.

Multiplying the equation by x we get $x^2y'' + 4xy = 0$.

The characteristic equation is $m^2 + 3m = 0 \Rightarrow m = 0$, $m = -3$

The general solution is $y(x) = C_1 x^0 + C_2 x^{-3} = C_1 + \frac{C_2}{x^3}$

$y(1) = 12 \Rightarrow C_1 + C_2 = 12$

$y'(x) = -3C_2 x^{-4}$; $y'(1) = -6 \Rightarrow -3C_2 = -6 \Rightarrow C_2 = 2$

$\Rightarrow C_1 = 10$

The particular solution is $y(x) = 10 + \frac{2}{x^3}$.