# MAT 2384 – Solutions to Suggested Problems

## Exact ODEs – Integrating factors

1) $(e^y - ye^x)\,dx + (xe^y - e^x)\,dy = 0$

$M(x,y) = e^y - ye^x$, $N(x,y) = xe^y - e^x$

$\frac{\partial M}{\partial y} = e^y - e^x$, $\frac{\partial N}{\partial x} = e^y - e^x$. The equation is exact since

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. We look for a function $F(x,y)$ such that

$\frac{\partial F}{\partial x} = M(x,y) = e^y - ye^x$ and $\frac{\partial F}{\partial y} = N(x,y) = xe^y - e^x$

$\frac{\partial F}{\partial x} = e^y - ye^x \Rightarrow F(x,y) = \int (e^y - ye^x)\,dx + g(y)$ where

$g(y)$ is a function of $y$ only.

$F(x,y) = xe^y - ye^x + g(y)$ which gives $\frac{\partial F}{\partial y} = xe^y - e^x + g'(y)$

Now $\frac{\partial F}{\partial y} = xe^y - e^x \Rightarrow g'(y) = 0 \Rightarrow g(y) = A =$ Constant.

$F(x,y) = xe^y - ye^x + A$. The solution is $F(x,y) = B \Rightarrow$

$$xe^y - ye^x = C$$

Let us check via implicit differentiation:

$e^y + xe^y y' - y'e^x - ye^x = 0 \Rightarrow (e^y - ye^x) + (xe^y - e^x)y' = 0$

$\Rightarrow (e^y - ye^x)\,dx + (xe^y - e^x)\,dy = 0$ by rewriting $y'$ as $\frac{dy}{dx}$.

2) $-\pi \sin(\pi x) \sinh y \, dx + \cos(\pi x) \cosh(y) \, dy = 0$

$$\frac{\partial M}{\partial y} = -\pi \sin(\pi x) \cosh y \quad \text{and} \quad \frac{\partial N}{\partial x} = -\pi \sin(\pi x) \cosh(y)$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is then exact and the solution is of the form $F(x,y) = C$ where $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$

$$\frac{\partial F}{\partial x} = -\pi \sin(\pi x) \cosh y \Rightarrow F = \int -\pi \sin(\pi x) \cosh(y) \, dx + g(y)$$

$$\Rightarrow F(x,y) = \cos(\pi x) \cosh(y) + g(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = \cos(\pi x) \sinh(y) + g'(y)$$

Comparing with $\frac{\partial F}{\partial y} = N = \cos(\pi x) \cosh(y)$, we get $g'(y) = 0$

$\Rightarrow g(y) = A =$ Constant.

So $F(x,y) = \cos(\pi x) \cosh(y) + A$ and the solution is

$\cos(\pi x) \cosh(y) = C$. Let us check by implicit Differentiation

$$-\pi \sin(\pi x) \cosh(y) + \cos(\pi x) \sinh(y) \frac{dy}{dx} = 0 \Leftrightarrow$$

$$-\pi \sin(\pi x) \cosh(y) \, dx + \cos(\pi x) \sinh(y) \, dy = 0.$$

3) $9x \, dx + 4y \, dy = 0$ ; Check for exactness:

$\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = 0$ ; Exact ODE. Look for $F(x,y)$ such

that $\frac{\partial F}{\partial x} = 9x$ and $\frac{\partial F}{\partial y} = 4y$

$$F = \int 9x \, dx + g(y) = \frac{9}{2} x^2 + g(y) \Rightarrow \frac{\partial F}{\partial y} = g'(y)$$

$\Rightarrow g'(y) = 4y \Rightarrow g(y) = 2y^2 + A$. The solution is then

$$F(x,y) = C \Rightarrow \frac{9}{2}x^2 + 2y^2 = C \quad \text{or} \quad 9x^2 + 4y^2 = K;$$

Let us check: $18x + 8y\frac{dy}{dx} = 0 \Rightarrow 18x\,dx + 8y\,dy = 0 \Rightarrow$

$9x\,dx + 4y\,dy = 0$

4) $e^{-2\theta}dr - 2re^{-2\theta}d\theta = 0 \Leftrightarrow$

$\frac{\partial M}{\partial \theta} = -2e^{-2\theta}$, $\frac{\partial N}{\partial r} = -2e^{-2\theta}$; Exact equation.

we look for $F(r,\theta)$ such that $\frac{\partial F}{\partial r} = e^{-2\theta}$ and $\frac{\partial F}{\partial \theta} = -2re^{-2\theta}$

$\frac{\partial F}{\partial r} = e^{-2\theta} \Rightarrow F(r,\theta) = \int e^{-2\theta}dr + g(\theta) = re^{-2\theta} + g(\theta)$

$\Rightarrow \frac{\partial F}{\partial \theta} = -2re^{-2\theta} + g'(\theta)$. Comparing with $\frac{\partial F}{\partial \theta} = -2re^{-2\theta}$,

we get that $g'(\theta) = 0 \Rightarrow g(\theta) = A =$ Constant.

The solution is then $re^{-2\theta} = C$ or $r = Ce^{2\theta}$

check $e^{-2\theta} - 2re^{-2\theta}\frac{d\theta}{dr} = 0 \Leftrightarrow e^{-2\theta}dr - 2re^{-2\theta}d\theta = 0$

5) $\left[-\frac{y}{x^2} + 2\cos 2x\right]dx + \left[\frac{1}{x} - 2\sin(2y)\right]dy = 0$

$\frac{\partial M}{\partial y} = -\frac{1}{x^2}$, $\frac{\partial N}{\partial x} = -\frac{1}{x^2}$; Exact Equation.

Look for $F(x,y)$ such that $\frac{\partial F}{\partial x} = -y/x^2 + 2\cos(2x)$

$$\frac{\partial F}{\partial y} = \frac{1}{x} - 2\sin(2y)$$

using the second equation, we get

$F = \int \left[\frac{1}{x} - 2\sin(2y)\right] dy + g(x) = y/x + \cos(2y) + g(x)$

$\Rightarrow \frac{\partial F}{\partial x} = -y/x^2 + g'(x)$. Comparing with $\frac{\partial F}{\partial x} = -y/x^2 + 2\cos 2x$

we get that $g'(x) = 2\cos(2x) \Rightarrow g(x) = \int 2\cos(2x)\, dx = \sin(2x) + K$

so $F(x,y) = y/x + \cos(2y) + \sin(2x) + K$ and the solution

is of the form $\frac{y}{x} + \cos(2y) + \sin(2x) = C$

Check by implicit differentiation:

$$\frac{y'x - y}{x^2} - 2y'\sin(2y) + 2\cos(2x) = 0 \Rightarrow$$

$$\frac{y'}{x} - \frac{y}{x^2} - 2y'\sin(2y) + 2\cos(2x) = 0 \Rightarrow$$

$$\left[-y/x^2 + 2\cos(2x)\right] + \left[\frac{1}{x} - 2\sin(2y)\right] y' = 0 \Leftrightarrow$$

$\left[-y/x^2 + 2\cos(2x)\right] dx + \left[\frac{1}{x} - 2\sin(2y)\right] dy = 0$ by rewriting

$y'$ as $\frac{dy}{dx}$.

6) $-2xy\sin(x^2)\,dx + \cos(x^2)\,dy = 0$

$M_y = -2x\sin(x^2)$, $N_x = -2x\sin(x^2)$ : Exact Equation.

Look for $F(x,y)$ such that $\frac{\partial F}{\partial x} = -2xy\sin(x^2)$

$$\frac{\partial F}{\partial y} = \cos(x^2)$$

using $\frac{\partial F}{\partial y} = \cos(x^2)$, we get $F = \int \cos(x^2)\, dy + g(x) \Rightarrow$

$F(x,y) = y\cos(x^2) + g(x) \Rightarrow \frac{\partial F}{\partial x} = -2xy\sin(x^2) + g'(x)$.

Comparing with $\frac{\partial F}{\partial x} = -2xy\sin(x^2)$, we get that $g'(x) = 0 \Rightarrow$

$g(x) = A$ and $F(x,y) = y\cos(x^2) + A$. The solution looks like:

$$y\cos(x^2) = C \quad \text{or} \quad y = \frac{C}{\cos(x^2)}.$$

Check $y'\cos(x^2) - 2xy\sin(x^2) = 0 \Leftrightarrow -2xy\sin(x^2)dx + \cos(x^2)dy = 0$

7) $-y\,dx + x\,dy = 0 \Rightarrow M_y = -1,\ N_x = 1$ : Not exact.

$$\frac{M_y - N_x}{N} = -\frac{2}{x} = f(x), \text{ so } \mu(x) = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx}$$

$$= e^{-2\ln|x|} = e^{\ln(1/x^2)} = \frac{1}{x^2} \text{ is an integrating factor.}$$

Multiplying our DE with $\frac{1}{x^2}$, we get

$$\underbrace{-y/x^2}_{M^*}dx + \underbrace{\frac{1}{x}}_{N^*}dx = 0 \quad ! \quad M^*_y = -\frac{1}{x^2},\ N^*_x = -\frac{1}{x^2} \text{ which}$$

is now exact. Look for $F(x,y)$ such that

$\frac{\partial F}{\partial x} = -y/x^2$, $\frac{\partial F}{\partial y} = \frac{1}{x}$. The second equation gives

$$F(x,y) = \int \frac{1}{x}dy + g(x) = y/x + g(x) \Rightarrow \frac{\partial F}{\partial x} = -y/x^2 + g'(x)$$

Comparing with $\frac{\partial F}{\partial x} = -y/x^2$, we get that $g'(x) = 0 \Rightarrow g(x) = A$.

So the solution looks like $y/x = C$ or

$$y = kx$$

8) $(x^4+y^2)\,dx - xy\,dy = 0$, $y(2)=1$

$M_y = 2y$, $N_x = -y$ ; $M_y \neq N_x$, the equation is not exact.

Since $\frac{M_y - N_x}{N} = \frac{3y}{-xy} = -\frac{3}{x} = f(x)$, an integrating factor

is $e^{\int f(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|} = e^{\ln\left|\frac{1}{x^3}\right|} = \frac{1}{x^3}$.

Multiplying our DE with $\frac{1}{x^3}$, we get:

$$\underbrace{\left(x + \frac{y^2}{x^3}\right)}_{M^*} dx - \underbrace{\frac{y}{x^2}}_{N^*} dy = 0 \;;$$

$M^*_y = \frac{2y}{x^3}$, $N^*_x = \frac{2y}{x^3}$ the new equation is exact.

Look for $F(x,y)$ such that $\frac{\partial F}{\partial x} = x + \frac{y^2}{x^3}$, $\frac{\partial F}{\partial y} = \frac{-y}{x^2}$

$\frac{\partial F}{\partial y} = \frac{-y}{x^2} \Rightarrow F(x,y) = -\int \frac{y}{x^2}\,dy + g(x) = \frac{-y^2}{2x^2} + g(x)$

$\Rightarrow \frac{\partial F}{\partial x} = \frac{y^2}{x^3} + g'(x)$. Comparing with $\frac{\partial F}{\partial x} = x + \frac{y^2}{x^3}$ gives

$g'(x) = x \Rightarrow g(x) = \frac{1}{2}x^2$. The solution looks like

$-\frac{y^2}{2x^2} + \frac{1}{2}x^2 = C$. Multiplying with 2 gives

$-\frac{y^2}{x^2} + x^2 = K$. We look for the value of $K$

using the initial condition $y(2)=1$ :

$-\frac{1}{4} + 4 = K \Rightarrow K = \frac{15}{4} = 3{,}75$. The particular

solution is $-\frac{y^2}{x^2} + x^2 = 3{,}75$.

9) $-y\sin(xy)\,dx - x\sin(xy)\,dy = 0, \quad y(1) = \pi$

$M_y = -\sin(xy) - xy\cos(xy)$, $\quad N_x = -\sin(xy) - xy\cos(xy)$ :

Exact equation. Look for $F(x,y)$ satisfying:

$$\frac{\partial F}{\partial x} = -y\sin(xy) \text{ and } \frac{\partial F}{\partial y} = -x\sin(xy) :$$
(1) (2)

$$F(x,y) = \int -y\sin(xy)\,dx = \cos(xy) + g(y)$$

$\frac{\partial F}{\partial y} = -x\sin(xy) + g'(y)$. Comparing with $\frac{\partial F}{\partial y} = -x\sin(xy)$

gives $g'(y) = 0 \Rightarrow g(y) = A = \text{constant}$.

The general solution looks like $\cos(xy) = C$

Check $\quad -\sin(xy)\,[y + xy'] = 0 \Leftrightarrow -y\sin(xy)dx - x\sin(xy)dy = 0.$

Using the condition $y(1) = \pi$ ;

$$\cos(\pi) = C \Rightarrow C = -1$$

The particular solution is $\cos(xy) = -1$

10) $[\cos(2x) + 2\sin(2x)]\,dx + e^x dy = 0, \quad y(0) = 1$

$M_y = 0$, $N_x = e^x$ : the equation is not exact.

Since $\frac{M_y - N_x}{N} = \frac{0 - e^x}{e^x} = -1 = f(x)$, on integrating

factor is $e^{\int(-1)dx} = e^{-x}$. Multiplying the ODE by $e^{-x}$ gives

$$\underbrace{e^{-x}[\cos(2x) + 2\sin(2x)]}_{M^*}\,dx + \underbrace{1}_{N^*}\,dy = 0$$

$M_y^* = 0 = N_x^*$ ; Now exact. Look for $F(x,y)$ such that

$\frac{\partial F}{\partial x} = e^{-x}[\cos(2x) + 2\sin(2x)]$ and $\frac{\partial F}{\partial y} = 1$. The second equation

gives $F = \int 1\,dy + f(x) = y + f(x)$

$$\frac{\partial F}{\partial x} = f'(x) = e^{-x}\cos(2x) + 2e^{-x}\sin(2x) \Rightarrow f(x) = \int e^{-x}\cos(2x)\,dx$$
$$+ 2\int e^{-x}\sin(2x)\,dx$$

For $\int e^{-x}\cos(2x)dx$, use integration by parts:

$$u = e^{-x}, \quad v' = \cos(2x)$$
$$u' = -e^{-x}, \quad v = \frac{1}{2}\sin(2x)$$

$$\int e^{-x}\cos(2x)\,dx = uv - \int u'v\,dx = \frac{1}{2}e^{-x}\sin(2x) - \int(-e^{-x})\frac{1}{2}\sin(2x)dx$$
$$= \frac{1}{2}e^{-x}\sin(2x) + \frac{1}{2}\int e^{-x}\sin(2x)\,dx. \quad (*)$$

For $\int e^{-x}\sin(2x)\,dx$, use integration by parts again:

$$u = e^{-x}, \quad v' = \sin(2x)$$
$$u' = -e^{-x}, \quad v = -\frac{1}{2}\cos(2x)$$

$$\int e^{-x}\sin(2x)\,dx = -\frac{1}{2}e^{-x}\cos(2x) - \int(-e^{-x})\left(-\frac{1}{2}\cos 2x\right)dx$$
$$= -\frac{1}{2}e^{-x}\cos(2x) - \frac{1}{2}\int e^{-x}\cos 2x\,dx$$

Going back to line (*) we get

$$\int e^{-x}\cos(2x)\,dx = \frac{1}{2}e^{-x}\sin(2x) - \frac{1}{4}e^{-x}\cos(2x) - \frac{1}{4}\int e^{-x}\cos(2x)\,dx$$

$$\Rightarrow \frac{5}{4}\int e^{-x}\cos(2x)\,dx = \frac{1}{2}e^{-x}\sin(2x) - \frac{1}{4}e^{-x}\cos(2x) \text{ giving}$$

$$\int e^{-x}\cos(2x) = \frac{1}{5}e^{-x}\left[2\sin(2x) - \cos(2x)\right].$$

In a similar way, we get $\int e^{-x}\sin(2x)\,dx = -\frac{e^{-x}}{5}\left[\sin(2x) - 2\cos(2x)\right]$

Therefore $f(x) = \int e^{-x}\cos(2x)\,dx + 2\int e^{-x}\sin(2x)\,dx = e^{-x}\cos(2x)$

and $F(x,y) = y + f(x) = y + e^{-x}\cos(2x)$

The general solution is $y + e^{-x}\cos(2x) = C$

<u>Check</u> $y' - e^{-x}\cos(2x) - 2e^{-x}\sin(2x) = 0 \Rightarrow$

$$-e^{-x}\left(\cos(2x) + 2\sin(2x)\right)dx + dy = 0 \quad \text{or}$$

$$\left[\cos(2x) + 2\sin(2x)\right]dx + e^{x}\,dy = 0$$

$y(0) = 1 \Rightarrow \quad 1 + 1 = C \quad \Rightarrow C = 2$

The particular solution is $y + e^{-x}\cos(2x) = 2$

or $y = 2 - e^{-x}\cos(2x)$

11) $y \cos(x+y)\,dx + [3\sin(x+y) + y\cos(x+y)]\,dy = 0, \quad y(0) = \pi/2$

$M_y = \cos(x+y) - y\sin(x+y)$ ; $N_x = 3\cos(x+y) - y\sin(x+y)$

$M_y \neq N_x$ , the equation is NOT exact

Now $\dfrac{M_y - N_x}{M} = \dfrac{-2\cos(x+y)}{y\cos(x+y)} = -\dfrac{2}{y} = g(y)$ , an integrating factor

is given by $e^{-\int g(y)\,dy} = e^{-\int -\frac{2}{y}\,dy} = e^{2\ln|y|} = y^2$

Multiplying with $y^2$ gives:

$$\underbrace{y^3\cos(x+y)}_{M^*}\,dx + \underbrace{[3y^2\sin(x+y) + y^3\cos(x+y)]}_{N^*}\,dy = 0$$

$$\left.\begin{aligned} M^*_y &= 3y^2\cos(x+y) - y^3\sin(x+y) \\ N^*_x &= 3y^2\cos(x+y) - y^3\sin(x+y) \end{aligned}\right\} \quad M^*_y = N^*_x \text{ . The new DE is Exact .}$$

Look for $F(x,y)$ such that $\dfrac{\partial F}{\partial x} = y^3\cos(x+y)$ ①

$$\frac{\partial F}{\partial y} = 3y^2\sin(x+y) + y^3\cos(x+y) \quad ②$$

① $\Rightarrow F(x,y) = \int y^3\cos(x+y)\,dx + h(y) = y^3\sin(x+y) + h(y)$

$\Rightarrow \dfrac{\partial F}{\partial y} = 3y^2\sin(x+y) + y^3\cos(x+y) + h'(y)$ . Comparing with ②

gives $h'(y) = 0 \Rightarrow h(y) = k$ . The general solution is

$$y^3\sin(x+y) = C$$

$y(0) = \pi/2 \Rightarrow (\pi/2)^3\sin(0+\pi/2) = C \Rightarrow C = \dfrac{\pi^3}{8}$

The particular solution is $y^3 \sin(x+y) = \frac{\pi^3}{8}$.

12) $[\sin(y)\cos(y) + x\cos^2 y]\,dx + x\,dy = 0$

$M_y = \cos^2 y - \sin^2 y - 2x\cos y \sin y$ , $N_x = 1$ : Not exact.

$M_y - N_x = \cos^2 y - \sin^2 y - 2x\cos y\sin y - 1 = -2\sin^2 y - 2x\cos y \sin y$

since $\cos^2 y - 1 = -\sin^2 y$.

$$\frac{M_y - N_x}{M} = -\frac{2\sin y(\sin y + x\cos y)}{\cos y[\sin y + x\cos y]} = -2\tan y = g(y).$$

An integrating factor is $e^{-\int g(y)dy} = e^{-\int -2\tan y\,dy} = e^{2\int \tan y\,dy}$

$= e^{-2\ln|\cos y|} = e^{\ln\left(\frac{1}{\cos^2 y}\right)} = \frac{1}{\cos^2 y}$ since $\int \tan y\,dy = -\ln|\cos x|$.

Multiplying by $\frac{1}{\cos^2 y}$ gives:

$$\underbrace{(\tan y + x)}_{M^*}\,dx + \underbrace{\frac{x}{\cos^2 y}}_{N^*}\,dy = 0$$

$M^*_y = \frac{1}{\cos^2 y}$ , $N_x = \frac{1}{\cos^2 y}$ . The new equation is Exact.

Find $F(x,y)$ such that $\frac{\partial F}{\partial x} = \tan y + x$ and $\frac{\partial F}{\partial y} = \frac{x}{\cos^2 y}$

$F = \int (\tan y + x)\,dx + g(y) \Rightarrow F(x,y) = x\tan y + \frac{1}{2}x^2 + g(y)$

$\Rightarrow \frac{\partial F}{\partial y} = \frac{x}{\cos^2 y} + g'(y)$ which gives $g'(y) = 0$ and $g(y) = A$

The general solution is

$$x \tan y + \frac{1}{2} x^2 = C$$

13) $(Ax+By)\,dx + (Cx+Dy)\,dy = 0$

$M_y = B$ , $N_x = C$ . The equation is Exact if $M_y = N_x \Leftrightarrow B = C$.

In this case, we look for $F(x,y)$; $\frac{\partial F}{\partial x} = Ax + By$ , $\frac{\partial F}{\partial y} = Bx + Dy$

$F = \frac{A}{2} x^2 + Bxy + g(y) \Rightarrow \frac{\partial F}{\partial y} = Bx + g'(y) = Bx + Dy \Rightarrow$

$g'(y) = Dy \Rightarrow g(y) = \frac{1}{2} D y^2$. The general solution is

$$\boxed{\frac{1}{2} A x^2 + Bxy + \frac{1}{2} D y^2 = C}$$