MAT 2384 - Solutions to Suggested Problems

Fixed-Point iteration - Newton's Method, Secant Method

1) $\sin x - \frac{x}{1.4} = 0 \Rightarrow x = 1.4 \sin x$. Let $g(x) = 1.4 \sin x$

Let us start by noting that $\sin(1) - \frac{1}{1.4} = 0.127 > 0$ and $\sin(\pi/2) - \frac{\pi/2}{1.4} = -0.1219 < 0$. Therefore the function $\sin x - \frac{x}{1.4}$ has a zero in the interval $[1, \frac{\pi}{2}]$. In other words, $g(x) = 1.4 \sin x$ has a fixed point in $[1, \pi/2]$

On the other hand $g'(x) = 1.4 \cos x$ is continuous on $[1, \pi/2]$ and its derivative is $g''(x) = -1.4 \sin x < 0$ on $[1, \pi/2]$. This means that $g'(x)$ is decreasing on $[1, \pi/2]$ and so $0 < g'(x) < g'(1) \simeq 0.75$ for any $x \in [1, \pi/2]$. We conclude that the iteration sequence $x_{n+1} = g(x_n)$ will converge to the fixed point in $[1, \pi/2]$.

$x_1 = g(x_0) = g(1.4) = 1.4 \sin(1.4) = 1.37963$

$x_2 = g(x_1) = (1.4) \sin(1.37963) = 1.37450$

$x_3 = g(x_2) = (1.4) \sin(1.37450) = 1.37311$

$x_4 = g(x_3) = (1.4) \sin(1.37311) = 1.37273$

$x_5 = g(x_4) = 1.4 \sin(1.37273) = 1.37263$

$x_6 = g(x_5) = 1.4 \sin(1.37263) = 1.37260$

So $x = 1.3726$ is a solution in $[1, \pi/2]$ to 4 decimal places.

2) $x^4 - x + 0.2 = 0 \iff x = x^4 + 0.2$. Let $g(x) = x^4 + 0.2$ ②

First note that if $f(x) = x^4 - x + 0.2$, then $f(0) = 0.2 > 0$ and $f(0.25) = -0.046 < 0$, so the function f has a zero in the interval $[0, 0.25]$ and so $g(x)$ has a fixed point in this interval.

On the other hand, $|g'(x)| = 4|x^3| \le 4 \cdot (0.25)^3 = 0.0625 < 1 \Rightarrow$ The iteration sequence will converge in $[0, 0.25]$ to the fixed point.

$x_1 = g(x_0) = 0^4 + 0.2 = 0.2$

$x_2 = g(x_1) = 0.2^4 + 0.2 = 0.20160$

$x_3 = g(x_2) = 0.20165$

$x_4 = g(x_3) = 0.20165^4 + 0.2 = 0.20165$

so $x = 0.20165$ is a root near 0 to 5 decimal places

3) $\sin x = e^{-0.5x} \Rightarrow \sin x - e^{-0.5x} = 0$

Let $f(x) = \sin x - e^{-0.5x} \Rightarrow f(1) = 0.234 > 0$ and $f(0.1) = -0.85 < 0$

$\Rightarrow f$ has a zero in the interval $[0.1, 1]$

$\sin x = e^{-0.5x} \Rightarrow x = \arcsin(e^{-0.5x})$. Let $g(x) = \arcsin(e^{-0.5x})$

$$\Rightarrow g'(x) = \frac{1}{\sqrt{1-(e^{-0.5x})^2}}(-0.5e^{-0.5x}) = \frac{-0.5}{e^{0.5x}\sqrt{1-e^{-x}}}$$

$$0.1 < x < 1 \Rightarrow -1 < -x < -0.1 \Rightarrow e^{-1} < e^{-x} < e^{-0.1} \Rightarrow$$

$$-e^{-0.1} < -e^{-x} < -e^{-1} \Rightarrow 1-e^{-0.1} < 1-e^{-x} < 1-e^{-1} \Rightarrow$$

$$\sqrt{1-e^{-0.1}} < \sqrt{1-e^{-x}} < \sqrt{1-e^{-1}} \Rightarrow \frac{1}{\sqrt{1-e^{-x}}} \le \frac{1}{\sqrt{1-e^{-0.1}}} \sim 0.308$$

On the other hand, $e^{0.5x} > e^{0.5(0.1)} = e^{0.05}$ on $[0.1, 1] \Rightarrow$

$$\frac{1}{e^{0.5x}} < \frac{1}{e^{0.05}} = 0.9512 \text{ on } [0.1, 1].$$

Therefore, $|g'(x)| = \dfrac{0.5}{e^{0.5}\sqrt{1-e^{-x}}} < 0.5(0.9512)(0.308) \approx$

$0.146 < 1$ on $[0.1, 1]$. The iteration sequence will converge to the fixed point in $[0.1, 1]$.

$x_1 = g(x_0) = \arcsin(e^{-0.5}) = 0.57002$

$x_2 = g(x_1) = \arcsin(e^{-0.5(0.57002)}) \approx 0.85110$

$x_3 = g(x_2) = \arcsin(e^{-0.5(0.85110)}) = 0.77002$

$x_4 = g(x_3) = \arcsin[e^{-0.5(0.77002)}] = 0.74837$

$x_5 = g(x_4) = \arcsin[e^{-0.5(0.74837)}] = 0.75852$

$x_6 = g(x_5) = \arcsin(e^{-0.5(0.75852)}) = 0.75374$

$x_7 = g(x_6) = \arcsin(e^{-0.5(0.75374)}) = 0.75598$

$x_8 = g(x_7) = \arcsin(e^{-0.5(0.75598)}) = 0.75493$

$x_9 = g(x_8) = \arcsin(e^{-0.5(0.75493)}) = 0.75542$

$x_{10} = g(x_9) = \arcsin(e^{-0.5(0.75542)}) = 0.75519$

$x_{11} = g(x_{10}) = \arcsin(e^{-0.5(0.75519)}) = 0.75530$

$x_{12} = g(x_{11}) = \arcsin(e^{-0.5(0.75530)}) = 0.75525$

$x_{13} = g(x_{12}) = \arcsin(e^{-0.5(0.75525)}) = 0.75527$

$x_{14} = g(x_{13}) = \arcsin(e^{-0.5(0.75527)}) = 0.75526$

$x_{15} = g(x_{14}) = \arcsin\left(e^{-0.5(0.75526)}\right) = 0.75526$

so the root is $x = 0.75526$ to 5 decimal places

Check $\sin(0.75526) - e^{-0.5(0.75526)}$

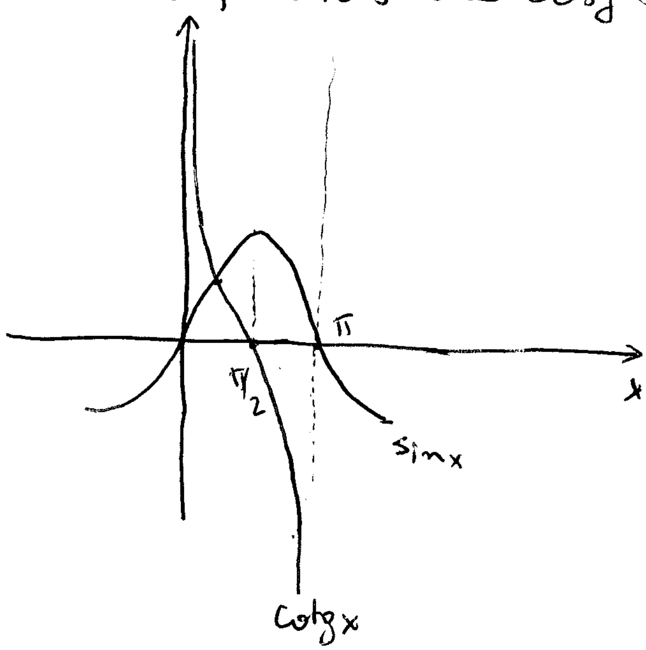
$= -0.000006$

Note that since $\sin x = e^{-0.5x} \Rightarrow \frac{e^{-0.5x}}{\sin x} = 1$ and therefore $x = x.1$

$= x\,\frac{e^{-0.5x}}{\sin x} \Rightarrow x = \frac{x}{e^{0.5x} \cdot \sin x}$. One can then use the function

$g(x) = \frac{x}{e^{0.5x} \sin x}$ for the iteration.

4) $\sin x = \cot x$. Both functions are easy to sketch:



From the graph, we see that there is a solution in the interval $[0, \pi/2]$

Let $f(x) = \sin x - \cot x = \frac{\sin^2 x - \cos x}{\sin x}$

$f'(x) = \frac{2\sin^2 x \cos x + \sin^2 x - \sin^2 x \cos x + \cos^2 x}{\sin^2 x} = \cos x + 1 + \cot^2(x)$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\sin(x_n) - \cot(x_n)}{\cos(x_n) + 1 + \cot^2(x_n)}$$

$$x_0 = 1 \Rightarrow x_1 = 1 - \frac{\sin(1) - \cot(1)}{\cos(1) + 1 + \cot^2(1)} = 0.897890$$

$$x_2 = 0.897890 - \frac{\sin(0.897890) - \cot(0.897890)}{\cos(0.897890) + 1 + \cot^2(0.897890)} = 0.904524$$

$$x_3 = 0.904524 - \frac{\sin(0.904524) - \cot(0.904524)}{\cos(0.904524) + 1 + \cot^2(0.904524)} = 0.904557$$

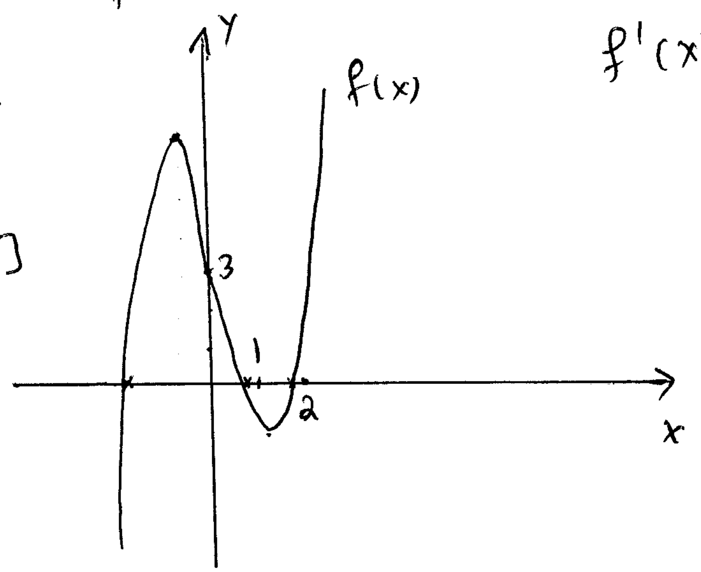
$$x_4 = 0.904557 - \frac{\sin(0.904557) - \cot(0.904557)}{\cos(0.904557) + 1 + \cot^2(0.904557)} = 0.904557$$

So the root is 0.904557 to 6 decimal places.

Check $\sin(0.904557) = 0.786151443$

$\cot(0.904557) = 0.786151206$

5) $x^3 - 5x + 3 = 0$. Let $f(x) = x^3 - 5x + 3$, the graph looks like:



$f'(x) = 3x^2 - 5 = 0$

$\Rightarrow x^2 = 5/3$

$x = \pm\sqrt{5/3}$

We notice from the graph that f has one root in [0,1] and another in [1,2].

$$f'(x) = 3x^2 - 5$$

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 3}{3x_n^2 - 5} = \frac{2x_n^3 - 3}{3x_n^2 - 5}$$

$x_0 = 2 \Rightarrow x_1 = \dfrac{2(2)^3 - 3}{3(2^2) - 5} = 1.857143$

$x_2 = \dfrac{2(1.857143)^3 - 3}{3(1.857143)^2 - 5} = 1.834787$

$x_3 = \dfrac{2(1.834787)^3 - 3}{3(1.834787)^2 - 5} = 1.834244$

$x_5 = \dfrac{2(1.834244)^3 - 3}{3(1.834244)^2 - 5} = 1.834243$

$x_6 = \dfrac{2(1.834243)^3 - 3}{2(1.834243)^2 - 5} = 1.834243$

So the solution is $x = 1.834243$ to 6 decimal places.

Check $(1.834243)^3 - 5(1.834243) + 3 = -0.000000939$.

6) Let $x = \sqrt[5]{2}$, then $x^5 = 2 \iff x^5 - 2 = 0$. Let $f(x) = x^5 - 2 \Rightarrow$

$f'(x) = 5x^4$ is Continuous

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5 - 2}{5x_n^4} = \frac{4x_n^5 + 2}{5x_n^4}$$

$x_0 = 1 \Rightarrow x_1 = \dfrac{4(1^5) + 2}{5(1^4)} = \dfrac{6}{5} = 1.2$

$x_2 = \dfrac{4(1.2)^5 + 2}{5(1.2)^4} = 1.152901$

$x_3 = \dfrac{4(1.152901)^5 + 2}{5(1.152901)^4} = 1.148729$

$x_4 = \dfrac{4(1.148729)^5 + 2}{5(1.148729)^4} = 1.148698$

$x_5 = \dfrac{4(1.148698)^5 + 2}{5(1.148698)^4} = 1.148698$

So $\sqrt[5]{2} \simeq 1.148698$ to 6 decimal places.

7) $e^{-x} - \tan x = 0$. Let $f(x) = e^{-x} - \tan(x)$.

Use the secant method with $x_0 = 1$, $x_1 = 0.7$

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

$$x_2 = x_1 - f(x_1) \frac{x_1 - x_0}{f(x_1) - f(x_0)} = 0.7 - f(0.7) \frac{(0.7 - 1)}{f(0.7) - f(1)} = 0.57709$$

$$x_3 = x_2 - f(x_2) \frac{x_2 - x_1}{f(x_2) - f(x_1)} = 0.57709 - f(0.57709) \frac{(0.57709 - 0.7)}{f(0.57709) - f(0.7)} = 0.53416$$

$$x_4 = x_3 - f(x_3) \frac{x_3 - x_2}{f(x_3) - f(x_2)} = 0.53143$$

$$x_5 = x_4 - f(x_4) \frac{x_4 - x_3}{f(x_4) - f(x_3)} = 0.53139$$

$$x_6 = x_5 - f(x_5) \frac{x_5 - x_4}{f(x_5 - x_4)} = 0.53139$$

Therefore $x = 0.53139$ is the solution to the equation

$e^{-x} - \tan x = 0$ to 5 decimal places.

<u>Check</u> $e^{-0.53139} - \tan(0.53139) = 0.0000016$