MAT 2784 A - DEVOIR #6 Solutions

1. $\mathcal{L}\{e^{2t}\cos(3t)\} = F(s-2)$ $\quad\left(F(s) = \mathcal{L}\{\cos(3t)\} = s/s^2+9\right)$

$$= \boxed{\frac{s-2}{(s-2)^2+9}} = \boxed{\frac{s-2}{s^2-4s+13}}$$

2. $\mathcal{L}\{u(t-1)(t^2+3t-4)\}$ $\quad\left(f(t-1) = t^2+3t-4, \quad f(t) = (t+1)^2+3(t+1)-4,\right.$

$$= e^{-s}\,\mathcal{L}\{t^2+5t\}$$

$$= \boxed{e^{-s}\left(\frac{2}{s^3}+\frac{5}{s^2}\right)}$$

3. $\mathcal{L}\{te^{-t}\sin(2t)\} = -\frac{d}{ds}\mathcal{L}\{e^{-t}\sin(2t)\}$

$$= -\frac{d}{ds}\left(\frac{2}{(s+1)^2+4}\right) = \boxed{\frac{4(s+1)}{((s+1)^2+4)^2}} = \boxed{\frac{4s+4}{(s^2+2s+5)^2}}$$

4. $\mathcal{L}^{-1}\left\{\frac{13}{s^2-4s+13}\right\} = \mathcal{L}^{-1}\left\{\frac{13}{(s-2)^2+9}\right\} = \boxed{\frac{13}{3}e^{2t}\sin(3t)}$

5. $\mathcal{L}^{-1}\left\{e^{-3s}\frac{6s+2}{s^2-s-6}\right\}$

$$\frac{6s+2}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2}$$

$A+B = 6 \qquad A=4$

$2A-3B = 2 \qquad B=2$

$\Rightarrow f(t) = 4e^{3t} + 2e^{-2t}$

$$= u(t-3)\,f(t-3)$$

$$= \boxed{u(t-3)\left(4e^{3(t-3)} + 2e^{-2(t-3)}\right)}$$

6. $y'' - y' - 6y = \begin{cases} 0 & 0<t<2 \\ e^t & t>2 \end{cases} = u(t-2)e^t \qquad y(0)=3, \quad y'(0)=4$

Soit $Y(s) = \mathcal{L}\{y(t)\}$. On transforme les 2 côtés :

$\mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 6\mathcal{L}\{y\} = \mathcal{L}\{u(t-2)e^t\}$ $\qquad f(t-2) = e^t, \quad f(t) = e^{t+2} = e$

$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 6Y(s) = e^{-2s}\mathcal{L}\{e^2e^t\}$

$(s^2-s-6)Y(s) - 3s - 4 + 3 = e^{-2s}\frac{e^2}{s-1}$

$$\Rightarrow Y(s) = \frac{3s+1}{s^2-s-6} + e^{-2s}e^2\frac{1}{(s-1)(s^2-s-6)}$$

$$\Leftrightarrow \quad Y(s) = \frac{2}{s-3} + \frac{1}{s+2} + e^{-2s} e^2 \left(\frac{a}{s-1} + \frac{b}{s+2} + \frac{c}{s-3} \right)$$

$a(s+2)(s-3) + b(s-1)(s-3) + c(s-1)(s+2) = 1$

- $s = 1 \Longrightarrow -6a = 1 \Leftrightarrow a = -1/6$
- $s = -2 \Longrightarrow 10b = 1 \Leftrightarrow b = 1/15$
- $s = 3 \Longrightarrow 10c = 1 \Leftrightarrow c = 1/10$

$$Y(s) = \frac{2}{s-3} + \frac{1}{s+2} + e^2 e^{-2s} \left(-\frac{1}{6}\frac{1}{s-1} + \frac{1}{15}\frac{1}{s+2} + \frac{1}{10}\frac{1}{s-3} \right)$$

La solution unique est donc:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \boxed{2e^{3t} + e^{-2t} + e^2 u(t-2)\left[-\frac{1}{6} e^{t-2} + \frac{1}{15} e^{-2(t-2)} + \frac{1}{10} e^{3(t-2)}\right]}$$

7: $y'' + 9y = \delta(t-\pi) \qquad y(0) = -1, \; y'(0) = 2$

Soit $Y(s) = \mathcal{L}\{y(t)\}$ puis on transforme les deux côtés:

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\delta(t-\pi)\}$$

$$s^2 Y(s) - sy(0) - y'(0) + 9Y(s) = e^{-\pi s}$$

$$(s^2+9)Y(s) + s - 2 = e^{-\pi s}$$

$$Y(s) = \frac{-s+2}{s^2+9} + \frac{e^{-\pi s}}{s^2+9}$$

La solution unique est donc

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{3}\sin(3t) - \cos(3t) + \frac{1}{3}u(t-\pi)\sin(3(t-\pi))$$

$$= \frac{2}{3}\sin(3t) - \cos(3t) + \frac{1}{3}u(t-\pi)\sin(3t-3\pi)$$

$$= \boxed{\frac{2}{3}\sin(3t) - \cos(3t) - \frac{1}{3}u(t-\pi)\sin(3t)}$$

ou

$$\boxed{y(t) = \begin{cases} \frac{2}{3}\sin(3t) - \cos(3t) & 0 < t < \pi \\ \frac{1}{3}\sin(3t) - \cos(3t) & t > \pi \end{cases}}$$

8.. $y' = -2xy \qquad y(0) = 3, \qquad h = 0.2 \Rightarrow x_1 = 0.2, \; x_2 = 0.4, \ldots$

et on doit faire 5 étapes.

La solution réelle: $y' = -2xy \Rightarrow \frac{dy}{y} = -2x$

$$\Longleftrightarrow \int \frac{dy}{y} = \int -2x\,dx + C \quad \Rightarrow \quad \ln y = -x^2 + C$$

$$\Longleftrightarrow y = Ke^{-x^2} \quad , \text{mais } y(0) = 3 \quad \Rightarrow \quad K = 3$$

$$\therefore \; y = \underline{3e^{-x^2}}$$

• Avec Euler amélioré:

$$y_{n+1}^{p} = y_n^{c} + h f(x_n, y_n^{c}) = y_n^{c}(1 - 0.4x_n)$$

$$y_{n+1}^{c} = y_n^{c} + \tfrac{1}{2}h\left[f(x_n, y_n^{c}) + f(x_{n+1}, y_{n+1}^{p})\right]$$

$$= y_n^{c} + \tfrac{1}{2}(0.2)\left(-2x_n y_n^{c} - 2x_{n+1}y_{n+1}^{p}\right)$$

$$= y_n^{c}(1 - 0.2x_n) - 0.2x_{n+1}y_{n+1}^{p}$$

$$y_1^{p} = y_0(1 - 0.4x_0) = 3(1 - 0.4(0)) = 3$$

$$y_1^{c} = y_0(1 - 0.2x_0) - 0.2x_1 y_1^{p} = 3(1 - 0.2(0)) - (0.2)(0.2)(3) = 2.88$$

$$y_2^{p} = y_1^{c}(1 - 0.4x_1) = 2.88(1 - 0.4(0.2)) = 2.6496$$

$$y_2^{c} = y_1^{c}(1 - 0.2x_1) - 0.2x_2 y_2^{p} = 2.88(1 - 0.2(0.2)) - 0.2(0.4)(2.6496) = 2.5528$$

$$y_3^{p} = y_2^{c}(1 - 0.4x_2) = 2.5528(1 - 0.4(0.4)) = 2.1444$$

$$y_3^{c} = y_2^{c}(1 - 0.2x_2) - 0.2x_3 y_3^{p} = 2.5528(1 - 0.2(0.4)) - (0.2)(0.6)(2.1444) = 2.0912$$

$$y_4^{p} = y_3^{c}(1 - 0.4x_3) = 2.0912(1 - 0.4(0.6)) = 1.5893$$

$$y_4^{c} = y_3^{c}(1 - 0.2x_3) - 0.2x_4 y_4^{p} = 2.0912(1 - 0.2(0.6)) - 0.2(0.8)(1.5893) = 1.5860$$

$$y_5^{p} = y_4^{c}(1 - 0.4x_4) = 1.5860(1 - 0.4(0.8)) = 1.0785$$

$$y_5^{c} = y_4^{c}(1 - 0.2x_4) - 0.2x_5 y_5^{p} = 1.5860(1 - 0.2(0.8)) - 0.2(1)(1.0785) = 1.1165$$

n	x_n	y_n^{c}	$y(x_n)$	erreur	erreur relative
1	0.2	2.88	2.8824	0.0024	0.0833
2	0.4	2.5528	2.5564	0.0036	0.1408
3	0.6	2.0912	2.0930	0.0018	0.0860
4	0.8	1.5860	1.5819	0.0041	0.2592
5	1.0	1.1165	1.1036	0.0129	1.1689

Bonnes comme approximations.

9. $y' = -2xy \Longrightarrow f(x,y) = -2xy$.

$y(0) = 3 \Longrightarrow x_0 = 0,\ y_0 = 3$; $h = 0.25 \Longrightarrow x_1 = 0.25,\ x_2 = 0.5, \ldots$

L'intervalle est $[0,1] \Longrightarrow$ on doit faire $n = \frac{1-0}{h} = 4$ étapes.

• $n = 0$

$$\begin{cases} k_1 = f(x_0, y_0) = -2x_0 y_0 = 0 \\ k_2 = f\left(x_0 + \frac{h}{2},\ y_0 + \frac{hk_1}{2}\right) = -0.750000 \\ k_3 = f\left(x_0 + \frac{h}{2},\ y_0 + \frac{hk_2}{2}\right) = -0.726564 \\ k_4 = f(x_0 + h,\ y_0 + hk_3) = -1.409180 \\ y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 2.818237 \end{cases}$$

• On fait de même pour $n = 1, 2, 3, 4$. On a le tableau suivant :

n	x_n	y_n	$y(x_n)$	Erreur	Erreur relative
0	0	3	3	0	0
1	0.25	2.818237	2.818239	0.000002	7.097×10^{-5}
2	0.50	2.336390	2.336402	0.000012	5.136×10^{-4}
3	0.75	1.709357	1.709349	0.000008	4.680×10^{-4}
4	1	1.103803	1.103638	0.000165	1.495×10^{-2}

Meilleurs approximations !