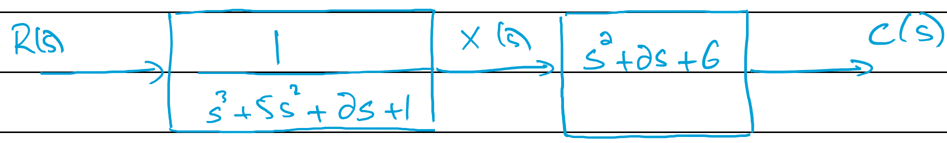
Solution Midterm Examination (Summer 2022)

QUESTION 1 (10 points)

Represent the system below in state space in phase-variable form

$$G(s) = \frac{s^2 + 2s + 6}{s^3 + 5s^2 + 2s + 1}$$



$$\frac{X(s)}{R(s)} = \frac{1}{s^3 + 5s^2 + 2s + 1} \Rightarrow s^3X(s) + 5s^2X(s) + 2sX(s) + X(s) = R(s)$$

$$\Rightarrow \frac{d^3x}{dt^3} + 5\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = r$$

let x_1, x_2 and x_3 be the state variables.

$$x_2 = \frac{dx_1}{dt} = \dot{x}_1$$

$$x_3 = \frac{dx_2}{dt} = \dot{x}_2$$

and $\dot{x}_3 = r - 5x_3 - 2x_2 - x_1$

on the other hand,

$$\frac{C(s)}{X(s)} = s^2 + 2s + 6 \Rightarrow s^2 X(s) + 2s X(s) + 6 X(s) = C(s)$$

$$\Rightarrow x_3 + 2x_2 + 6x_1 = c(t)$$

thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t)$$

$$y = c(t) = \begin{bmatrix} 6 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

QUESTION 2 (6 points)

Use the block diagram reduction method to find the equivalent transfer function for the system shown in Figure 1.

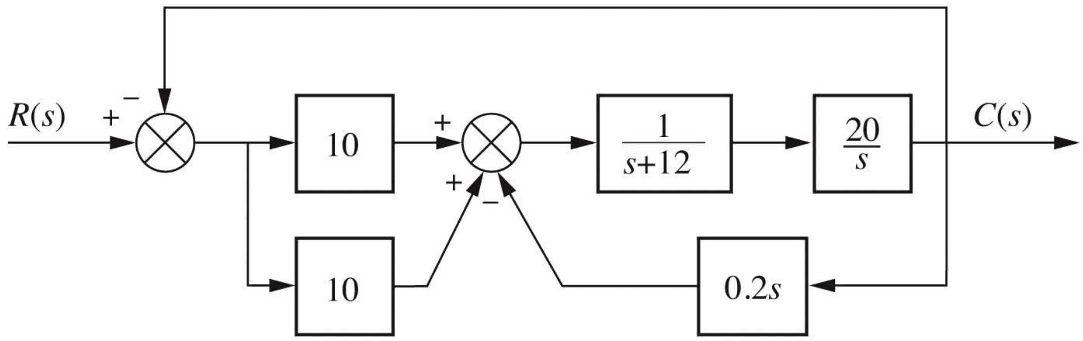
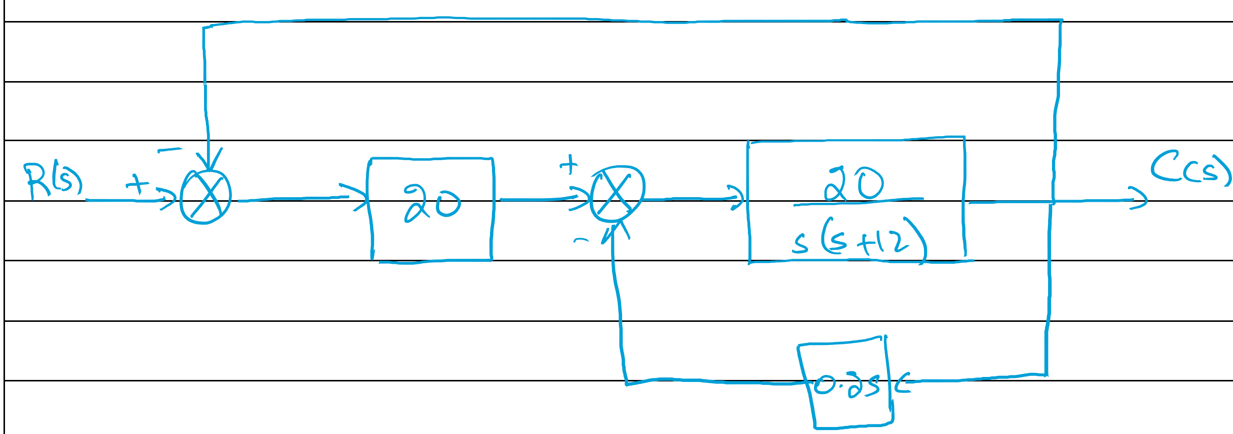
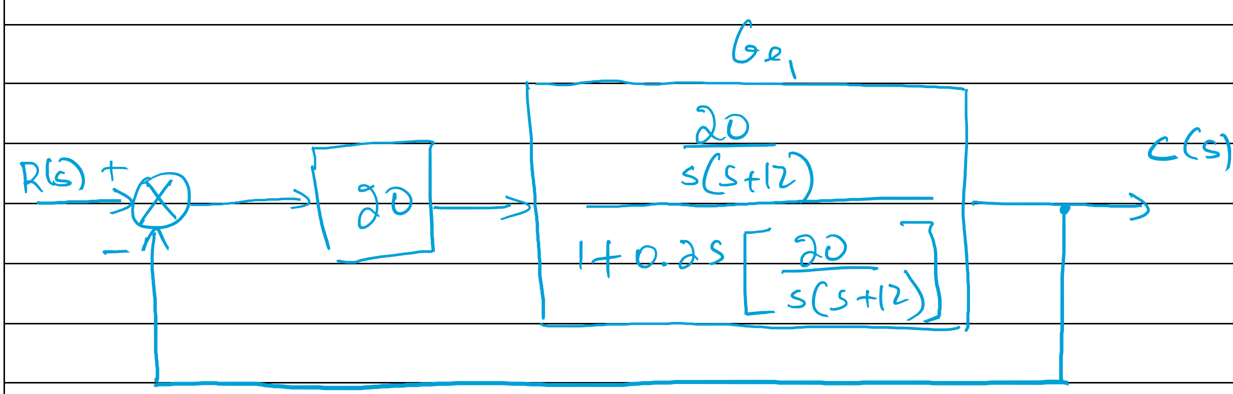


Figure 1: A closed-loop control system





$$G_{e_1} = \frac{20}{s(s+12+4)} = \frac{20}{s^2+16s}$$

Finally, the equivalent T.F is

$$G_E(s) = \frac{\frac{400}{s^2+16s}}{1+\frac{400}{s^2+16s}}$$

$$\Rightarrow \boxed{G_E(s) = \frac{400}{s^2+16s+400}}$$

QUESTION 3 (18 points)

Figure 3 shows an RLC network with an input voltage $v_i(t)$ and an output current $i(t)$.

$R_1 = 1\,\Omega$.
$R_2 = 3\,\Omega$.
$L = 1\,H$
$C_1 = C_2 = 1\,F$.

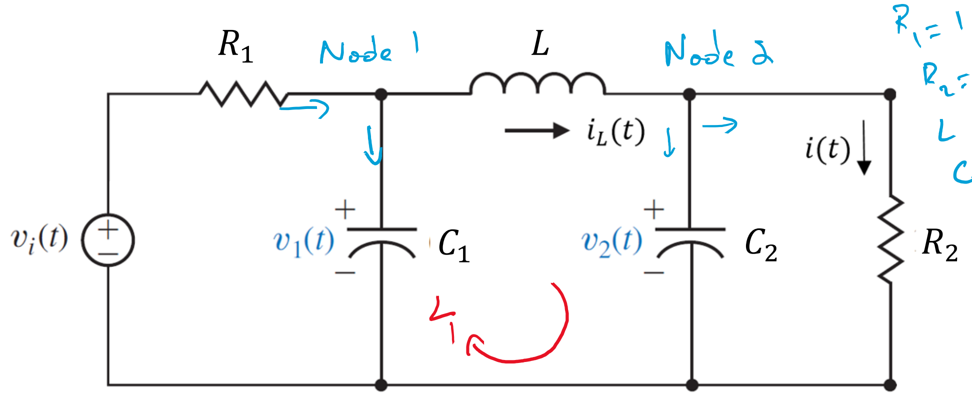


Figure 3: An RLC circuit

(a) (6 points) Represent the network in state-space model.

(b) (2 points) Draw a signal-flow graph for the state and output equations found in part (a)

(c) (10 points) For this part, assume that $R_1 = 1\,\Omega$, $R_2 = 3\,\Omega$, $C_1 = C_2 = 1$ F and $L = 1$ H. Use Mason's rule to find the equivalent transfer function of the system.

a) $v_1(t)$, $v_2(t)$ and $i_L(t)$ are state variables

at Node 1: $\dfrac{v_i(t) - v_1(t)}{R_1} = i_L(t) + C_1 \dfrac{dv_1(t)}{dt}$

$$\boxed{\frac{dv_1(t)}{dt} = \frac{1}{RC_1} v_i(t) - \frac{1}{RC_1} v_1(t) - \frac{1}{C_1} i_L(t)}$$

at Node 2: $i_L(t) = C_2 \dfrac{dv_2(t)}{dt} + i(t)$ or $i(t) = \dfrac{v_2(t)}{R_2}$

$$\Rightarrow \boxed{\frac{dv_2(t)}{dt} = \frac{1}{C_2} i_L(t) - \frac{1}{R_2 C_2} v_2(t)}$$

$L_1 \Rightarrow -v_1(t) + L \dfrac{di_L(t)}{dt} + v_2(t) = 0$

$$\Rightarrow \boxed{\frac{di_L(t)}{dt} = \frac{1}{L} v_1(t) - \frac{1}{L} v_2(t)}$$

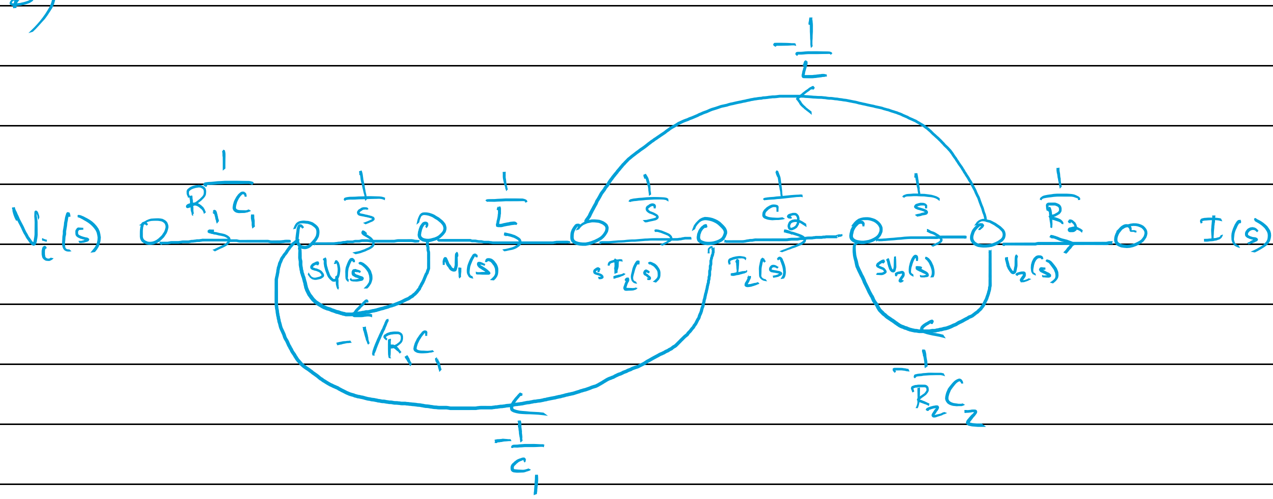
$y = i(t) = \dfrac{v_2(t)}{R_2}$

In matrix form,

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{i}_L(t) \end{bmatrix} = \begin{bmatrix} -1/R_1C_1 & 0 & -1/C_1 \\ 0 & -1/R_2C_2 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} 1/R_1C_1 \\ 0 \\ 0 \end{bmatrix} v_i(t)$$

$$y = \begin{bmatrix} 0 & \frac{1}{R_2} & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ i_L(t) \end{bmatrix}$$

b)



c) Loop gains: (1) $-\frac{1}{R_1C_1s}$ (2) $-\frac{1}{C_1Ls^2}$ (3) $-\frac{1}{LC_2s^2}$ (4) $-\frac{1}{R_2C_2s}$

Forward path: $\frac{1}{R_1C_1LC_2R_2s^3} = T_1$

non-touching loops taken 2 at a time

(1) $\frac{1}{R_1C_1C_2Ls^3}$ (2) $\frac{1}{R_1R_2C_1C_2s^2}$ (3) $\frac{1}{C_1C_2R_2Ls^3}$

$$G(s) = \frac{T_1 \Delta_1}{\Delta}$$

$$= 1 - \left(-\frac{1}{R_1 C_1 s} - \frac{1}{C_1 L s^2} - \frac{1}{L C_2 s^2} - \frac{1}{R_2 C_2 s} \right)$$

$$+ \left(\frac{1}{R_1 C_1 C_2 L s^3} + \frac{1}{R_1 R_2 C_1 C_2 s^2} + \frac{1}{C_1 C_2 R_2 L s^3} \right)$$

$$= 1 + \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s} + \frac{1}{3s} + \frac{1}{s^3} + \frac{1}{3s^2} + \frac{1}{3s^3}$$

$$= 1 + \frac{4}{3s} + \frac{7}{3s^2} + \frac{1}{s^3} = \frac{3s^3 + 4s^2 + 7s + 4}{3s^3}$$

$$\Delta_1 = 1$$

$$G(s) = \frac{1}{3s^3 \left(\frac{3s^3 + 4s^2 + 7s + 4}{3s^3} \right)} \Rightarrow \boxed{G(s) = \frac{1}{3s^3 + 4s^2 + 7s + 4}}$$

QUESTION 4 (6 points)

A system is represented by the following state-space representation model

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & \dfrac{-1}{C} \\ \dfrac{1}{L} & \dfrac{-R}{L} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \dfrac{1}{C} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & R \end{bmatrix} \mathbf{x}$$

Design the system so that it yields a damping factor of 0.591 and a peak time of 1 sec. Assume that $R = 1\ K\Omega$.

$$A = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{1}{L} \end{bmatrix} \quad B = \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = 0$$

$$G(s) = C(sI - A)^{-1}B + D$$

$$(sI - A) = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{1}{L} \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\left(s\left(s + \frac{1}{L}\right) + \frac{1}{CL}\right)} \begin{bmatrix} s + \frac{1}{L} & -\frac{1}{C} \\ +\frac{1}{L} & s \end{bmatrix}$$

Now, $$G(s) = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s + \frac{1}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}}{s^2 + \frac{s}{L} + \frac{1}{CL}}$$

$$G(s) = \frac{\begin{bmatrix} \frac{1}{L} & s \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}}{s^2 + \frac{s}{L} + \frac{1}{CL}} = \frac{\frac{1}{LC}}{s^2 + \frac{s}{L} + \frac{1}{LC}} = \frac{1}{LCs^2 + Cs + 1}$$

$$\boxed{G(s) = \frac{1}{LCs^2 + Cs + 1}}$$

$$\zeta = 0.591 \qquad T_p = \frac{\pi}{w_n\sqrt{1-\zeta^2}} \Rightarrow$$

$$w_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} = \frac{\pi}{\sqrt{1-(0.591)^2}} \Rightarrow w_n = 3.89 \text{ rad/sec}$$

We now need to compare $G(s)$ to $\dfrac{w_n^2}{s^2+2\zeta w_n s + w_n^2}$

$$G(s) = \frac{\frac{1}{LC}}{s^2 + \frac{1}{L}s + \frac{1}{LC}}$$

$$2\zeta w_n = \frac{1}{L} \Rightarrow L = \frac{1}{2\zeta w_n} = \frac{1}{2(0.591)(3.89)} \Rightarrow$$

$$\boxed{L = 0.217 \text{ H}}$$

$$\frac{1}{LC} = w_n^2 \Rightarrow C = \frac{1}{L w_n^2} = \frac{1}{(0.217)(3.89)^2} \Rightarrow \boxed{C = 0.3 \text{ F}}$$