MAT 2384 _ Solutions to Suggested Problems

Second-order linear ODEs with Constant Coefficients

1) $4y'' - 20y' + 25y = 0$, $y(0) = 1$, $y'(0) = 2$

The Characteristic equation is $4\lambda^2 - 20\lambda + 25 = 0$

$b^2 - 4ac = (-20)^2 - 4(4)(25) = 0$, The equation has a unique solution

$\lambda = \frac{20}{8} = \frac{5}{2}$. The general solution has the form

$y(x) = C_1 e^{5/2 x} + C_2 x e^{5/2 x}$, let us check:

$y' = \frac{5}{2} C_1 e^{5/2 x} + C_2 e^{5/2 x} + \frac{5}{2} C_2 x e^{5/2 x}$

$y'' = \frac{25}{4} C_1 e^{5/2 x} + \frac{5}{2} C_2 e^{5/2 x} + \frac{5}{2} C_2 e^{5/2 x} + \frac{25}{4} C_2 x e^{5/2 x}$

$4y'' - 20y' + 25y = 25C_1 e^{5/2 x} + 10C_2 e^{5/2 x} + 10C_2 e^{5/2 x} + 25C_2 x e^{5/2 x}$
$- 50C_1 e^{5/2 x} - 20C_2 e^{5/2 x} - 50C_2 x e^{5/2 x} + 25C_1 e^{5/2 x} + 25C_2 x e^{5/2 x} = 0$

Let us now find the particular solution:

$y(0) = 1 \Rightarrow C_1 = 1$

$y'(0) = 2 \Rightarrow \frac{5}{2} C_1 + C_2 = 2 \Rightarrow C_2 = -1/2$

The particular solution is $y(x) = e^{5/2 x} - \frac{1}{2} x e^{5/2 x}$

2) $y'' + 4\pi y' + 4\pi^2 y = 0$. The characteristic equation is

$\lambda^2 + 4\pi\lambda + 4\pi^2 = 0$: $b^2 - 4ac = (4\pi)^2 - 4(1)(4\pi^2) = 0$,

the equation has a unique solution $\lambda = -\frac{4\pi}{2} = -2\pi$

The general solution has the form

$y(x) = C_1 e^{-2\pi x} + C_2 x e^{-2\pi x}$; C_1, C_2 are constants.

3) $y'' - y' + \frac{5}{2}y = 0$. The characteristic equation is

$\lambda^2 - \lambda + 5/2 = 0$. $b^2 - 4ac = (-1)^2 - 4(1)(5/2) = -9 < 0$. The equation has 2 complex roots $\lambda_{1,2} = \frac{1 \pm \sqrt{-9}}{2} = \frac{1 \pm 3i}{2} = \frac{1}{2} \pm \frac{3}{2}i$

The general solution is $y(x) = C_1 e^{\frac{x}{2}} \cos(\frac{3}{2}x) + C_2 e^{\frac{x}{2}} \sin(\frac{3}{2}x)$

4) $y'' - 2y' - 5.25y = 0$. The characteristic equation is

$\lambda^2 - 2\lambda - 5.25 = 0$; $b^2 - 4ac = (-2)^2 - 4(1)(-5.25) = 25 > 0$. The equation has 2 distinct real roots $\lambda_1 = \frac{2-5}{2} = -\frac{3}{2}$,

$\lambda_2 = \frac{2+5}{2} = 7/2$

The general solution is $y(x) = C_1 e^{-\frac{3}{2}x} + C_2 e^{7/2 x}$

Let us check $y' = -\frac{3}{2}C_1 e^{-\frac{3}{2}x} + \frac{7}{2}C_2 e^{7/2 x}$ and $y'' = \frac{9}{4}C_1 e^{-\frac{3}{2}x} + \frac{49}{4}C_2 e^{7/2 x}$

$y'' - 2y' - 5.25y = \frac{9}{4}C_1 e^{-\frac{3}{2}x} + \frac{49}{4}C_2 e^{7/2x} + 3C_1 e^{-\frac{3}{2}x} - 7C_2 e^{7/2x} - 5.25C_1 e^{-3/2x}$
$- 5.25 C_2 e^{7/2 x} = 0$

5) $y'' + 4\pi^2 y = 0$. The characteristic equation is $\lambda^2 + 4\pi^2 = 0$

$\Rightarrow \lambda^2 = -4\pi^2 = 4\pi^2 i^2 \Rightarrow \lambda = \pm 2\pi i$. The general solution

is $y(x) = C_1 \cos(2\pi x) + C_2 \sin(2\pi x)$. Let us check:

$y' = -2\pi C_1 \sin(2\pi x) + 2\pi C_2 \cos(2\pi x)$, $y'' = -4\pi^2 C_1 \cos(2\pi x) - 4\pi^2 C_2 \sin(2\pi x)$

$y'' + 4\pi^2 y = -4\pi^2 C_1 \cos(2\pi x) - 4\pi^2 C_2 \sin(2\pi x) + 4\pi^2 C_1 \cos(2\pi x) +$
$4\pi^2 C_2 \sin(2\pi x) = 0$

6) $y'' - 2y = 0$. The characteristic equation is $\lambda^2 - 2 = 0 \Rightarrow$
$\lambda = \pm\sqrt{2}$: two distinct real roots. The general solution is
$y(x) = C_1 e^{-\sqrt{2}x} + C_2 e^{\sqrt{2}x}$. Let us check:

$$y' = -\sqrt{2}C_1 e^{-\sqrt{2}x} + \sqrt{2}C_2 e^{\sqrt{2}x} \Rightarrow y'' = 2C_1 e^{-\sqrt{2}x} + 2C_2 e^{\sqrt{2}x}$$

$$y'' - 2y = 2C_1 e^{-\sqrt{2}x} + 2C_2 e^{\sqrt{2}x} - 2C_1 e^{-\sqrt{2}x} - 2C_2 e^{\sqrt{2}x} = 0.$$

7) $y'' - 2y' - 3y = 0$, $y(0) = 2$, $y'(0) = 14$
The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda+1)(\lambda-3) = 0$
$\lambda_1 = -1$, $\lambda_2 = 3$. The general equation is

$$y(x) = C_1 e^{-x} + C_2 e^{3x}.$$ Let us check

$$y' = -C_1 e^{-x} + 3C_2 e^{3x} \Rightarrow y'' = C_1 e^{-x} + 9C_2 e^{3x}$$

$$y'' - 2y' - 3y = C_1 e^{-x} + 9C_2 e^{3x} + 2C_1 e^{-x} - 6C_2 e^{3x} - 3C_1 e^{-x} - 3C_2 e^{3x} = 0$$

Let us now find the unique solution

$$y(0) = C_1 + C_2 = 2 \quad (1), \quad y'(0) = -C_1 + 3C_2 = 14 \quad (2)$$

$(1) + (2) \Rightarrow 4C_2 = 16 \Rightarrow C_2 = 4 \Rightarrow C_1 = 2 - 4 = -2$

The unique solution is $y(x) = -2e^{-x} + 4e^{3x}$

8) $y'' + 4y' + 5y = 0$, $y(0) = 2$, $y'(0) = -5$

The characteristic equation is $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow b^2 - 4ac =$
$16 - 4(1)(5) = -4 < 0$. The equation has 2 complex roots:

$$\lambda_1 = \frac{-4-2i}{2} = -2 - i, \quad \lambda_2 = -2 + i$$

The general solution is $y(x) = C_1 e^{-2x} \cos(x) + C_2 e^{-2x} \sin(x)$

$y(0) = 2 \Rightarrow C_1 = 2$

$y'(x) = -2C_1 e^{-2x} \cos(x) - C_1 e^{-2x} \sin(x) - 2C_2 e^{-2x} \sin x + C_2 e^{-2x} \cos x$

$y'(0) = -5 \Rightarrow -2C_1 + C_2 = -5 \Rightarrow C_2 = 2C_1 - 5 = 4 - 5 = -1.$

The particular solution is $y(x) = 2e^{-2x} \cos x - e^{-2x} \sin x$

9) $20y'' + 4y' + y = 0$, $y(0) = 3.2$, $y'(0) = 0$.

The characteristic equation is $20\lambda^2 + 4\lambda + 1 = 0$

$b^2 - 4ac = 16 - 4(20)(1) = 16 - 80 = -64$. The roots are

$\lambda_1 = \frac{-4 - 8i}{40} = -\frac{1}{10} - \frac{1}{5} i$ and $\lambda_2 = -\frac{1}{10} + \frac{1}{5} i$

The general solution is $y(x) = C_1 e^{-0.1x} \cos(0.2x) + C_2 e^{-0.1x} \sin(0.2x)$

$y(0) = C_1 = 3.2$

$y'(x) = -0.1 C_1 e^{-0.1x} \cos(0.2x) - 0.2 C_1 e^{-0.1x} \sin(0.2x) - 0.1 C_2 e^{-0.1x} \sin(0.2x)$

$+ 0.2 C_2 e^{-0.1x} \cos(0.2x)$

$y'(0) = -0.1 C_1 + 0.2 C_2 = 0 \Rightarrow C_2 = \frac{1}{2} C_1 = 1.6$

The particular solution is

$$y(x) = 3.2 e^{-0.1x} \cos(0.2x) + 1.6 e^{-0.1x} \sin(0.2x)$$

10) $y'' - 25y = 0$, $y(0) = 0$, $y'(0) = 40$. The characteristic equation is $\lambda^2 - 25 = 0 \Rightarrow \lambda = \pm 5$. The general solution is $y(x) = C_1 e^{-5x} + C_2 e^{5x}$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$ ①

$y'(x) = -5c_1 e^{-5x} + 5c_2 e^{5x} \Rightarrow y'(0) = -5c_1 + 5c_2 = 40$

$\Rightarrow -c_1 + c_2 = 8$ ②

① + ② $2c_2 = 8 \Rightarrow c_2 = 4 \Rightarrow c_1 = -c_2 = -4$

The particular solution is

$y(x) = -4e^{-5x} + 4e^{5x}$