MAT 2784 A
ÉQUATIONS DIFFÉRENTIELLES ET MÉTHODES NUMÉRIQUES
Examen de pratique
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Il y aura environ 7 questions.
Voici une composition possible.

(A)

Question 1 (6 marks) Solve the initial value problems / Résoudre les problèmes aux valeurs initiales:

(a) $e^{2y}\, y' = 2\cos x, \quad y(0) = 0$

the DE is separable $\quad e^{2y}\, dy = 2\cos x\, dx$

integrate on both sides $\quad \int e^{2y}\, dy = \int 2\cos x\, dx + C$

to get $\quad \frac{1}{2} e^{2y} = 2\sin x + C$

or $\quad e^{2y} = 4\sin x + C$

or $\quad 2y = \ln|4\sin x + C|$

and so $\quad y = \frac{1}{2}\ln|4\sin x + C|$ (general solution)

$y(0) = 0 \Rightarrow 0 = \frac{1}{2}\ln(0 + c) \Rightarrow c = 1$

$\therefore$ the unique solution is $\boxed{y = \frac{1}{2}\ln|4\sin x + 1|}$

(b) $y' + \dfrac{y}{x} = \dfrac{2x}{y}, \quad y(1) = 2$

this is a Bernoulli equation with $p(x) = \frac{1}{x}$, $g(x) = 2x$ and a

then we let $u = y^{1-a} = y^2$ and the DE will become

$u' + (1-a)\,p(x)\,u = (1-a)\,g(x)$ or $u' + \frac{2}{x} u = 4x$

the integrating factor is $\mu(x) = e^{\int \frac{2}{x} dx} = e^{2\ln x} = x^2$

and then $\quad u = \frac{1}{x^2}\left[\int (x^2)(4x)\, dx + C\right]$

$= x^{-2}\left(\int 4x^3\, dx + C\right)$

$= x^{-2}(x^4 + C) = x^2 + Cx^{-2}$

and so $\quad y^2 = x^2 + Cx^{-2}$ (general solution)

let $y(1) = 2 \Rightarrow (2)^2 = (1)^2 + C(1)^{-2} \Rightarrow C = 3$

$\therefore$ the unique solution is $\boxed{y = \sqrt{x^2 + 3x^{-2}}}$ $\quad (y > 0)$

(A)

Question 2 (6 marks) Solve the initial value problem / Résoudre le problème à valeur initiale:

$$(2e^x y^2 + 6xy + 7y^3)\,dx + (8e^x y + 9x^2 + 35xy^2)\,dy = 0, \quad y(0) = 1.$$ (not separable)

$M(x,y) = 2e^x y^2 + 6xy + 7y^3 \Rightarrow M_y = 4e^x y + 6x + 21y^2$

$N(x,y) = 8e^x y + 9x^2 + 35xy^2 \Rightarrow N_x = 8e^x y + 18x + 35y^2$

$M_y \neq N_x$, DE not exact

$$\frac{M_y - N_x}{M} = \frac{-4e^x y - 12x - 14y^2}{2e^x y^2 + 6xy + 7y^3} = \frac{-2(2e^x y + 6x + 7y^2)}{y(2e^x y + 6x + 7y^2)} = \frac{-2}{y} \text{ (function of } y\text{)}$$

integrating factor is $\mu(y) = e^{-\int \frac{-2}{y}\,dy} = e^{2\ln y} = y^2$ and the DE becomes

$$(2e^x y^4 + 6xy^3 + 7y^5)\,dx + (8e^x y^3 + 9x^2 y^2 + 35xy^4)\,dy = 0$$

then $M^*(x,y) = 2e^x y^4 + 6xy^3 + 7y^5 \Rightarrow M^*_y = 8e^x y^3 + 18xy^2 + 35y^4$

$N^*(x,y) = 8e^x y^3 + 9x^2 y^2 + 35xy^4 \Rightarrow N^*_x = 8e^x y^3 + 18xy^2 + 35y^4$

$M^*_y = N^*_x$, DE exact

$$F(x,y) = \int M^*(x,y)\,dx + g(y) = \int (2e^x y^4 + 6xy^3 + 7y^5)\,dx + g(y)$$

$$= 2e^x y^4 + 3x^2 y^3 + 7xy^5 + g(y)$$

then $\frac{\partial F}{\partial y} = 8e^x y^3 + 9x^2 y^2 + 35xy^4 + g'(y) = N^*(x,y) = 8e^x y^3 + 9x^2 y^2 + 35xy^4$

and so $g'(y) = 0 \Rightarrow$ we take $g(y) = 0$

thus $F(x,y) = 2e^x y^4 + 3x^2 y^3 + 7xy^5$

and the general solution is $2e^x y^4 + 3x^2 y^3 + 7xy^5 = C$

$y(0) = 1 \Rightarrow 2e^0 (1)^4 + 3(0)^2 (1)^3 + 7(0)(1)^5 = C \Rightarrow C = 2$

$\therefore$ the unique solution is $\boxed{2e^x y^4 + 3x^2 y^3 + 7xy^5 = 2}$

(A)

Question 3 (6 marks) Solve the initial value problems / Résoudre les problèmes aux valeurs initiales:

(a) $y'' + 8y' + 16y = 0, \quad y(0) = 0, \quad y'(0) = 2$

the characteristic equation is $\lambda^2 + 8\lambda + 16 = (\lambda+4)^2 = 0$
so the roots are $\lambda_1 = \lambda_2 = -4$ and the general solution is $y(x) = C_1 e^{-4x} + C_2 x e^{-4x}$

$y(0) = 0 \Rightarrow 0 = C_1 e^0 + C_2 (0) e^0 \Rightarrow C_1 = 0$

$y'(x) = -4C_1 e^{-4x} + C_2 e^{-4x} - 4C_2 x e^{-4x}$

$y'(0) = 2 \Rightarrow 2 = -4C_1 e^0 + C_2 e^0 - 4C_2 (0) e^0 \Rightarrow C_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2x e^{-4x}}$

(b) $y'' - 2y' + 10y = 0, \quad y(0) = 2, \quad y'(0) = 8$

the char. eq. is $\lambda^2 - 2\lambda + 10 = 0$
the roots are $\lambda_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4(10)}}{2} = 1 \pm 3i$

the general solution is $y(x) = C_1 e^x \cos(3x) + C_2 e^x \sin(3x)$

$y(0) = 2 \Rightarrow 2 = C_1 e^0 \cos(0) + C_2 e^0 \sin(0) \Rightarrow C_1 = 2$

$y'(x) = C_1 e^x \cos(3x) - 3C_1 e^x \sin(3x) + C_2 e^x \sin(3x) + 3C_2 e^x \cos(3x)$

$y'(0) = 8 \Rightarrow 8 = C_1 e^0 \cos(0) - 3C_1 e^0 \sin(0) + C_2 e^0 \sin(0) + 3C_2 e^0 \cos(0)$
$\Rightarrow C_1 + 3C_2 = 8 \Rightarrow C_2 = 2$

$\therefore$ the unique solution is $\boxed{y(x) = 2e^x \cos(3x) + 2e^x \sin(3x)}$

Question 4: Résoudre

EX

- $xy' = y + x\sec(y/x), \quad y(1) = \pi/2.$

$\Longleftrightarrow \underbrace{\left(y + x\sec(y/x)\right)}_{M} dx \underbrace{- x}_{N} dy = 0$. On a : $M(\lambda x, \lambda y) = \lambda M(x,y)$ et $N(\lambda x, \lambda y) = \lambda N(x,y)$.

Donc M et N sont homogènes de degré 1 en x et y.

On pose $y = ux \Rightarrow dy = u\,dx + x\,du$ et l'équa devient

$(ux + x\sec(u))dx - xu\,dx - x^2 du = 0 \Longleftrightarrow x\sec u\, dx - x^2 du = 0$

$\Longleftrightarrow \frac{1}{\sec u} du = \frac{x}{x^2} dx \Longleftrightarrow \boxed{\cos u\, du = \frac{1}{x} dx}$ Séparable.

$\Rightarrow \sin u = \ln|x| + C \Longleftrightarrow u = \frac{y}{x} = \arcsin(\ln|x| + C)$

Donc la S.G. est : $\boxed{y = x\arcsin(\ln|x| + C)}$.

- S.U. : $y(1) = \frac{\pi}{2} \Rightarrow C = 1 \Rightarrow$ S.U : $\boxed{y = x\arcsin(\ln|x| + 1)}$

- $x^2y'' - xy' + y = 0.$

  – C'est une equation d'Euler-Cauchy.
  – Equation carateristique: $m^2 - 2m + 1 = (m-1)^2 = 0$. Donc on a la racine double $m_1 = m_2 = 1$.
  – Solution generale: $y = C_1x + C_2x\ln(x)$.

(A)

Question 5 (6 marks)
(a) Consider $f(x) = x^3 + 6x - 5$. If you were to use Fixed Point Iteration to find the root of this function in the interval $[0, 1]$, what would you use for $g(x)$ and why?

Soit $f(x) = x^3 + 6x - 5$. Si on aimerait trouver la racine de la fonction sur l'intervalle $[0, 1]$ par <<Fixed Point Iteration>>, quelle fonction utiliserait-on pour $g(x)$ et pourquoi?

$f(x) = 0 \Rightarrow x^3 + 6x - 5 = 0 \Rightarrow 6x = 5 - x^3 \Rightarrow x = \frac{5 - x^3}{6}$

so $\boxed{g(x) = \frac{5 - x^3}{6}}$ (this will satisfy fixed point condition)

and $\boxed{|g'(x)| = \frac{1}{2}x^2 < 1 \text{ on } [0, 1]}$

so this $g(x)$ will generate a convergent sequence

(b) Use Newton's Method to find the root of this $f(x)$ to 4 decimal places. Start with $x_0 = 1$. Verify your answer.

Utilisez la Méthode de Newton pour trouver la racine de même $f(x)$ à 4 décimales. Commencez avec $x_0 = 1$. Vérifiez la réponse.

$f(x) = x^3 + 6x - 5 \Rightarrow f'(x) = 3x^2 + 6$

Newton's Method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \boxed{x_n - \frac{x_n^3 + 6x_n - 5}{3x_n^2 + 6} = \frac{2x_n^3 + 5}{3x_n^2 + 6}}$

$x_0 = 1$, $x_1 = \frac{7}{9} = 0.7778$, $x_2 = 0.7602$, $x_3 = 0.7601 = x_4$ (S

$f(0.7601) = (0.7601)^3 + 6(0.7601) - 5 \approx \boxed{-2.5 \times 10^{-4}}$ okay

$\therefore$ the root is $\boxed{0.7601}$

(c) If we were going to find the interpolating polynomial $p_n(x)$ for 5 data points, what is the maximum possible degree that the polynomial could have?

Si on aimerait trouver le polynôme d'interpolation $p_n(x)$ pour 5 pointes, quel est le degré maximal que le polynôme puisse avoir?

(A) 2 (B) 3 (C) 4 (D) 5 (E) 6 (F) impossible to answer

$(n + 1 = 5 \Rightarrow n = 4)$

(d) Considérez les points suivants ((x_j, f_j) collectés lors d'une expérience, où $f_j = f(x_j)$): $(1.2, 2.3)$, $(1.9, 3.5)$ and $(2.3, 4.9)$.
(i) Trouvez le polynôme de Lagrange $p_2(x)$ (avec coefficients à 4 décimales près).
(ii) Interpolez la valeur au point $x = 2$.
(iii) Étant donné que $0.5 \leq |f'''(x)| \leq 2$ sur $[1, 3]$, donnez l'erreur maximale et l'erreur minimal lors de cette interpolation.
(Voir notes de cours pour la solution.)