MAT 2384 – Solutions to Suggested Problems on Interpolation

1) $x_0 = 1.01,\ f_0 = 1,\ x_1 = 1.02,\ f_1 = 0.9888,\ x_2 = 1.04,\ f_2 = 0.9784$

$$P_2(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1.02)(x-1.04)}{(1.01-1.02)(1.01-1.04)} = 3333.3333x^2 - 6866.6667x + 3436$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-1.01)(x-1.04)}{(1.02-1.01)(1.02-1.04)} = -5000x^2 + 10250x - 5252$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-1.01)(x-1.02)}{(1.04-1.01)(1.04-1.02)} = 1666.6667x^2 - 3383.3333x + 1717$$

$$P_2(x) = 1.\left[3333.3333x^2 - 6866.6667x + 3536\right] + 0.9888\left[-5000x^2 + 10250x - 5252\right] + 0.9784\left[1666.6667x^2 - 3383.3333x + 1717\right]$$

$$= 20x^2 - 41.72x + 22.7352$$

Using $P_2(x)$, we can estimate $f(1.035)$:

$$f(1.035) \simeq P_2(1.035) = 20(1.035)^2 - 41.72(1.035) + 22.7352$$

$$= 0.9795 \quad \text{and}$$

$$f(1.055) \simeq P_2(1.055) = 20(1.055)^2 - 41.72(1.055) + 22.7352$$

$$= 0.9811$$

Using the error estimate for Lagrange polynomial, we have $\varepsilon_2(x) = (x-x_0)(x-x_1)\dfrac{f'''(t)}{3!}$ for some $t \in [1.01, 1.04]$

Therefor $|\mathcal{E}_2(1.035)| = \left|(1.035-1.01)(1.035-1.02)\dfrac{f'''(t)}{6}\right| =$

$0.0000625\,|f'''(t)|$. Since $0.251 \le |f'''(t)| \le 0.45 \Rightarrow$

$0.0000156\,88 \le |\mathcal{E}_2(1.035)| \le 0.000028125$

The minimum value of $\mathcal{E}_2(1.035)$ is 0.000015688 and the Maximum value is 0.000028125.

2) $\operatorname{erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-x^2}\,dx$

Here $x_0 = 0.25$, $f_0 = 0.27633$, $x_1 = 0.5$, $f_1 = 0.52050$ and $x_2 = 1$, $f_2 = 0.84270$

The Lagrange polynomial of degree 2 is given by

$$P_2(x) = L_0(x)\,f_0 + L_1(x)\,f_1 + L_2(x)\,f_2$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.5)(x-1)}{(0.25-0.5)(0.25-1)} = 5.33333x^2 - 8x + 2.66667$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.25)(x-1)}{(0.5-0.25)(0.5-1)} = -8x^2 + 10x - 2$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.25)(x-0.5)}{(1-0.25)(1-0.5)} = 2.66667x^2 - 2x + 0.33333$$

So $P_2(x) = 0.27633\,L_0(x) + 0.52050\,L_1(x) + 0.84270\,L_2(x)$

$= 0.27633[5.33333x^2 - 8x + 2.66667] + 0.52050[-8x^2 + 10x - 2] +$
$0.84270[2.66667x^2 - 2x + 0.33333] = -0.44304x^2 + 1.30846x - 0.02322$

using $f(x) \simeq P_2(x)$, we get

$$erf(0.75) \simeq -0.44304(0.75)^2 + 1.30896(0.75) - 0.02322 = 0.70929.$$

For the error bounds on $erf(0.75)$, we have

$$erf(0.75) - P_2(0.75) = (0.75-0.25)(0.75-0.5)(0.75-1)\frac{[erf(t)]'''}{6}$$

for a certain $t \in [0.25, 1] \Rightarrow$

$$erf(0.75) - P_2(0.75) = -0.005208\,[erf(t)]'''. \qquad (*)$$

We now use the following fact from Calculus I:

If $f(x) = \int_a^x g(t)\,dt$, then $f'(x) = g(x)$.

Here $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \Rightarrow [erf(x)]' = \frac{2}{\sqrt{\pi}} e^{-x^2}$

$$[erf(x)]'' = -\frac{4x}{\sqrt{\pi}}e^{-x^2} \Rightarrow [erf(x)]''' = \frac{8x^2}{\sqrt{\pi}}e^{-x^2} - \frac{4e^{-x^2}}{\sqrt{\pi}} = \frac{4(2x^2-1)}{\sqrt{\pi}\,e^{x^2}}.$$

We would like to know if there are any bounds for $[erf(x)]''' = \frac{4(2x^2-1)}{\sqrt{\pi}\,e^{x^2}}$ on $[0.25, 1]$ in order to use $(*)$.

Differentiating $[erf(x)]'''$ we get $[erf(x)]^{(4)} = -\frac{8x}{\sqrt{\pi}}(-3+2x^2)e^{-x^2}$.

$[erf(x)]^{(4)} = 0 \Rightarrow x = 0,\ x = \pm\sqrt{\frac{3}{2}}$ which are not in $[0.25, 1]$.

This means that the function $[erf(x)]'''$ is monotonic on that interval. In fact $[erf(x)]^{(4)} > 0$ on $[0.25, 1]$ which implies that $[erf(x)]'''$ is strictly increasing on $[0.25, 1]$ which means that the extrema of $[erf(x)]'''$ occur at the ends of the interval

$[0.25, 1] \Rightarrow -1.85502 \le [erf(x)]''' \le 0.83021$

$(**) \Rightarrow -0.00432 \le erf(0.75) - P_2(0.75) \le 0.00967 \Rightarrow$

$0.70497 \le erf(0.75) \le 0.71896$

3) $f(0) = 0, \quad f(1) = 0.9461, \quad f(2) = 1.6054$

Here $h = 1$ in the Gregory-Newton forword formula

$P_2(x) = \sum_{k=0}^{2} \binom{r}{k} \Delta^k f_0$ where $r = \frac{1.5 - 0}{1} = 1.5$

$$P_2(x) = \binom{1.5}{0} f_0 + \binom{1.5}{1} \Delta f_0 + \binom{1.5}{2} \Delta^2 f_0$$

$$= f_0 + \frac{1.5}{1!} \Delta f_0 + \frac{1.5(1.5-1)}{2!} \Delta^2 f_0$$

$$= f_0 + (1.5) \Delta f_0 + 0.375 \Delta^2 f_0$$

$$= f_0 + 1.5 (f_1 - f_0) + 0.375 (\Delta f_1 - \Delta f_0)$$

$$= f_0 + 1.5 f_1 - 1.5 f_0 + 0.375 (f_2 - f_1) - 0.375 (f_1 - f_0)$$

$$= -0.125 f_0 + 0.75 f_1 + 0.375 f_2$$

$$= -0.125(0) + 0.75(0.9461) + 0.375(1.6054) = 1.3116$$

4) $f(0.5) = 0.479, \quad f(1) = 0.841, \quad f(2) = 0.909$

Using Newton's divided difference polynomial, we have

$$P_2(x) = f_0 + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{0.841 - 0.479}{1 - 0.5} = 0.724$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0}$$

$$f[x_1,x_2] = \frac{f_2 - f_1}{x_2 - x_1} = \frac{0.909 - 0.841}{2 - 1} = 0.068$$

$$\Rightarrow f[x_0,x_1,x_2] = \frac{0.068 - 0.724}{2 - 0.5} = -0.4373$$

So $P_2(x) = 0.479 + (x-0.5)(0.724) + (x-0.5)(x-1)(-0.4373)$

$$= -0.4373x^2 + 1.3800x - 0.1017$$

$f(0.8) \simeq P_2(0.8) = -0.4373(0.8)^2 + 1.38(0.8) - 0.1017 = 0.7224$

$f(0.9) \simeq P_2(0.9) = -0.4373(0.9)^2 + 1.38(0.9) - 0.1017 = 0.7861$

5) $(1, -3.02)$, $(2, 1.25)$, $(3, 3.1487)$, $(4, -2.546)$

(a) Using Newton's divided difference polynomial of degree 3:

$$P_3(x) = f_0 + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]$$

$f[x_0,x_1] = \frac{f_1 - f_0}{x_1 - x_0} = 4.27$, $f[x_1,x_2] = \frac{f_2 - f_1}{x_2 - x_1} = 1.899$

$f[x_2,x_3] = \frac{f_3 - f_2}{x_3 - x_2} = -5.695$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_0} = \frac{1.899 - 4.27}{3 - 1} = -1.1855$$

$$f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3] - f[x_0,x_1,x_2]}{x_3 - x_0}$$

$$f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1} = \frac{-5.695-1.899}{4-2} = -3.797$$

So $$f[x_0,x_1,x_2,x_3] = \frac{-3.797-(-1.1855)}{4-1} = -0.8705$$

So $$P_3(x) = -3.02 + (x-1)(4.27) + (x-1)(x-2)(-1.1855) + (x-1)(x-2)(x-3)(-0.8705)$$

$$P_3(x) = -0.8705x^3 + 4.0375x^2 - 1.7440x - 4.4380$$

$$f(2.5) \simeq P_3(2.5) = 2.82231$$

$$f(3.5) \simeq P_3(3.5) \simeq 1.57719$$

2) Since the nodes are equidistant, one can use the Newton's forward formula with $h=1$.

For $x = 2.5$, $r = \frac{x-x_0}{h} = \frac{2.5-1}{1} = 1.5$

$$P_3(2.5) = \sum_{k=0}^{3} \binom{1.5}{k} \Delta^k f_0 = \binom{1.5}{0} f_0 + \binom{1.5}{1} \Delta f_0 + \binom{1.5}{2} \Delta^2 f_0 + \binom{1.5}{3} \Delta^3 f_0$$

$$\binom{1.5}{0} = 1, \quad \binom{1.5}{1} = \frac{1.5}{1!} = 1.5, \quad \binom{1.5}{2} = \frac{(1.5)(1.5-1)}{2!} = 0.375$$

and $$\binom{1.5}{3} = \frac{(1.5)(1.5-1)(1.5-2)}{3!} = -0.0625$$

On the other hand,

$$\Delta f_0 = f_1 - f_0 = 1.25 - (-3.02) = 4.27$$

$$\Delta^2 f_0 = \Delta f_1 - \Delta f_0 = (f_2 - f_1) - (f_1 - f_0) = f_2 - 2f_1 + f_0 =$$

$$3.1487 - 2(1.25) - 3.02 = -2.3713$$

$$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0 = \Delta f_2 - \Delta f_1 - (\Delta f_1 - \Delta f_0) =$$

$$\Delta f_2 - 2\Delta f_1 + \Delta f_0 = f_3 - f_2 - 2(f_2 - f_1) + (f_1 - f_0)$$

$$= f_3 - 3f_2 + 3f_1 - f_0 = -2.546 - 3(3.1487) + 3(1.25) - (-3.02)$$

$$= -5.2221 \text{ . Thus}$$

$$P_3(2.5) = 1.(-3.02) + 1.5(4.27) + 0.375(-2.3713) - 0.0625(-5.2221)$$

$$= 2.8221$$

For $x = 3.5$, $r = \frac{x - x_0}{h} = \frac{3.5-1}{1} = 2.5$

$$P_3(3.5) = f_0 + \binom{2.5}{1}\Delta f_0 + \binom{2.5}{2}\Delta^2 f_0 + \binom{2.5}{3}\Delta^3 f_0$$

$$\binom{2.5}{1} = \frac{2.5}{1!} = 2.5; \quad \binom{2.5}{2} = \frac{(2.5)(2.5-1)}{2!} = 1.875,$$

$$\binom{2.5}{3} = \frac{(2.5)(2.5-1)(2.5-2)}{3!} = 0.3125$$

Thus $P_3(3.5) = -3.02 + 2.5(4.27) + 1.875(-2.3713) + 0.3125(-5.2221)$

$$= 1.5769$$

(c) by the error formula with any polynomial interpolation, we have:

$$\mathcal{E}_3(2.5) = (2.5-1)(2.5-2)(2.5-3)(2.5-4)\frac{f^{(4)}(t)}{4!} = 0.0234375\, f^{(4)}(t)$$

$$\Rightarrow \quad 0.0234375 \le |\mathcal{E}_3(2.5)| \le 0.046875 \text{ since } 1 \le |f^{(4)}(t)| \le 2$$

$$\mathcal{E}_3(3.5) = (3.5-1)(3.5-2)(3.5-3)(3.5-4)\frac{f^{(4)}(t)}{4!} = -0.0390625$$

$$\Rightarrow \quad 0.0390625 \le |\mathcal{E}_3(3.5)| \le 0.078125$$