MAT 2384 – Solutions to suggested Problems – Seperable and homogeneous ODE's

1 (a) $y' = 2\sec(2y) = \dfrac{2}{\cos(2y)}$ since $\sec(2y) = \dfrac{1}{\cos(2y)}$.

$$\frac{dy}{dx} = \frac{2}{\cos(2y)} \Rightarrow \cos(2y)\,dy = 2\,dx$$: seperable Equation.

Integrating both sides gives $\int \cos(2y)\,dy = \int 2\,dx \Rightarrow$

$\frac{1}{2}\sin(2y) = 2x + C \iff \sin(2y) = 4x + A$ [where $A = 2C =$ Constant] $\Rightarrow 2y = \operatorname{Arcsin}(4x + A)$ or

$$y = \frac{1}{2}\operatorname{Arcsin}(4x+A)$$: general solution

(b) $yy' + 25x = 0 \Rightarrow y' = -\dfrac{25x}{y} \Rightarrow \dfrac{dy}{dx} = -\dfrac{25x}{y} \Rightarrow$

$y\,dy = -25x\,dx$: seperable Equation.

$\int y\,dy = \int -25x\,dx \Rightarrow \dfrac{y^2}{2} = -\dfrac{25}{2}x^2 + C \iff \dfrac{25}{2}x^2 + \dfrac{y^2}{2} = C$

(implicite solution) : this is the equation of an ellipse.

(c) $y'\sin(\pi x) = y\cos(\pi x) \Rightarrow \dfrac{dy}{dx}\sin(\pi x) = y\cos(\pi x) \Rightarrow$

$\dfrac{dy}{y} = \dfrac{\cos(\pi x)}{\sin(\pi x)}\,dx$: seperable Equation. Integrating both sides gives:

$$\int \frac{dy}{y} = \int \frac{\cos(\pi x)}{\sin(\pi x)}\,dx \Rightarrow \ln|y| = \int \frac{\cos(\pi x)}{\sin(\pi x)}\,dx$$

For $\int \dfrac{\cos(\pi x)}{\sin(\pi x)}\,dx$, let $u = \sin(\pi x)$, then $\dfrac{du}{dx} = \pi\cos(\pi x) \Rightarrow$

$$dx = \frac{du}{\pi \cos(\pi x)} :$$

$$\int \frac{\cos(\pi x)}{\sin(\pi x)}\, dx = \int \frac{\cos(\pi x)}{\sin(\pi x)} \frac{du}{\pi \cos(\pi x)} = \frac{1}{\pi} \int \frac{1}{u}\, du =$$

$$\frac{1}{\pi} \ln|u| = \frac{1}{\pi} \ln|\sin(\pi x)|.$$

So $\ln|y| = \frac{1}{\pi} \ln|\sin(\pi x)| + C \Rightarrow$

$$\ln|y| = \ln|\sin(\pi x)|^{1/\pi} + C \Leftrightarrow y = A(\sin(\pi x))^{1/\pi}$$

(d) $y' e^{-2x} = y^2+1 \Leftrightarrow \frac{dy}{dx} = (y^2+1)e^{2x} \Rightarrow \frac{dy}{y^2+1} = e^{2x}dx \Rightarrow$

$$\int \frac{dy}{y^2+1} = \int e^{2x}dx \Rightarrow \text{Arctan}(y) = \frac{1}{2}e^{2x} + C \Rightarrow y = \tan\left[\frac{1}{2}e^{2x}+C\right]$$

(e) $(x^3+y^3)\,dx - 3xy^2dy = 0$: this is Not a seperable DE, but it is homogeneous of degree 3;

$$M(x,y) = x^3+y^3 \Rightarrow M(\lambda x, \lambda y) = \lambda^3x^3+\lambda^3y^3 = \lambda^3(x^3+y^3)$$
$$= \lambda^3 M(x,y)$$

$$N(x,y) = -3xy^2 \Rightarrow N(\lambda x, \lambda y) = -3(\lambda x)(\lambda y)^2 = -3\lambda^3xy^2$$
$$= \lambda^3 N(x,y)$$

Make the substitution $y = ux$ for some variable u, then $dy = x\,du + u\,dx$. The Equation can then be written as

$$(x^3 + x^3u^3)\,dx - 3xu^2x^2(x\,du+u\,dx) = 0 \Leftrightarrow$$

$$[x^3(1+u^3) - 3x^3u^3]\,dx - 3x^4u^2du = 0 \Rightarrow$$

$(x^3-2x^3u^3)\,dx-3x^4u^2\,du=0 \Rightarrow x^3\left[(1-2u^3)\,dx-3xu^2\,du\right]=0$

$\Rightarrow (1-2u^3)\,dx-3xu^2\,du=0 \Rightarrow (1-2u^3)\,dx=3xu^2\,du \Rightarrow$

$\frac{dx}{3x}=\frac{u^2}{1-2u^3}\,du$: seperable ODE. Integrating both sides gives $\int\frac{dx}{3x}=\int\frac{u^2}{1-2u^3}\,du \Rightarrow \frac{1}{3}\ln|x|=\int\frac{u^2}{1-2u^3}\,du.$

Use the substitution $t=1-2u^3 \Rightarrow \frac{dt}{du}=-6u^2 \Rightarrow du=\frac{dt}{-6u^2}$

$\int\frac{u^2}{1-2u^3}\,du=\int\frac{u^2}{t}\,\frac{dt}{-6u^2}=-\frac{1}{6}\ln|t|=\ln|t|^{-1/6}=\ln\left(\frac{1}{|t|^{1/6}}\right)$

Therefore: $\frac{1}{3}\ln|x|=\ln\left(\frac{1}{|t|^{1/6}}\right)+C \Rightarrow \ln|x|=3\ln\left[\frac{1}{|t|^{1/6}}\right]+C$

$=\ln\frac{1}{|t|^{1/2}}+C \Rightarrow x=\frac{A}{\sqrt{|t|}}$ or $x^2t=C$

$\Longleftrightarrow x^2(1-2u^3)=C$. Replacing u by $\frac{y}{x}$ gives $x^2\left(1-2\frac{y^3}{x^3}\right)=C$ or $x^3-2y^3=Cx$. Solving for y we get $y=\frac{1}{2}\sqrt[3]{x^3-Cx}$.

(f) $-(x^2+3y^2)\,dx+2xy\,dy=0$; this is clearly not a seperable equation, but it is homogeneous of degree 2:

$M(x,y)=-(x^2+3y^2) \Rightarrow M(\lambda x,\lambda y)=-\left[(\lambda x)^2+3(\lambda y)^2\right]$

$\lambda^2\left[-(x^2+3y^2)\right]$

$=\lambda^2 M(x,y)$

$$N(x,y) = -2xy \Rightarrow N(\lambda x, \lambda y) = -2(\lambda x)(\lambda y) =$$
$$-2\lambda^2 xy = \lambda^2[-2xy] = \lambda^2 N(x,y)$$

Make the substitution $y = xu$ for some function u, then

$dy = x\,du + u\,dx$ and the DE becomes

$$-[x^2 + 3x^2u^2]dx + 2x(xu)[x\,du + u\,dx] = 0 \Rightarrow$$
$$x^2[-dx - 3u^2dx + 2xu\,du + 2u^2dx] = 0 \Rightarrow$$

$-(1+u^2)\,dx + 2xu\,du = 0$ which is now seperable:

$(1+u^2)\,dx = 2xu\,du \Rightarrow \frac{dx}{x} = \frac{2u}{1+u^2}\,du$. Integrating

both sides gives: $\ln|x| = \ln(1+u^2) + C \Rightarrow x = A(1+u^2)$

or $x = A\left[1 + \left(\frac{y}{x}\right)^2\right]$. Multiplying through with x^2 gives:

$x^3 = A(x^2+y^2) \Leftrightarrow x^2 + y^2 - Cx^3 = 0$; this is the general

solution.

$\boxed{2}$ (a) $\frac{dy}{dx} = -2xy$, $y(0) = 2$

$$\frac{dy}{y} = -2x\,dx \Rightarrow \int \frac{dy}{y} = \int -2x\,dx \Rightarrow \ln|y| = -x^2 + C$$

$\Rightarrow y = Ae^{-x^2}$. use the initial condition to find A:

$y(0) = 2 \Rightarrow 2 = Ae^{-0^2} = A \Rightarrow y = 2e^{-x^2}$; this is

the particular solution.

(b) $L \dfrac{dw}{dt} + Rw = 0$, $w(0) = w_0$

$$L\frac{dw}{dt} = -Rw \Rightarrow \frac{dw}{w} = -\frac{R}{L}dt \Rightarrow \int \frac{dw}{w} = \int -\frac{R}{L}\,dt \Rightarrow$$

$\ln|w| = -\dfrac{R}{L}t + C \Rightarrow w = A e^{-(R/L)t}$ where A is a

Constant. Let us now use the initial condition: $t=0$, $w = w_0$:

$w_0 = A e^{(-R/L)(0)} \Rightarrow A = w_0$. The particular solution is

$$w = w_0 e^{-(R/L)t}$$

(c) $y' = 2(x+2)y^3 e^{-2x}$, $y(0) = \dfrac{1}{\sqrt{5}}$

$$\frac{dy}{dx} = 2(x+2)y^3 e^{-2x} \Rightarrow \frac{dy}{y^3} = 2(x+2)e^{-2x}dx \Rightarrow$$

$$\int \frac{dy}{y^3} = \int 2(x+2)e^{-2x}dx \qquad (*)$$

For $\int 2(x+2)e^{-2x}$, we proceed by parts:

$u = 2(x+2)$, $v' = e^{-2x}$

$u' = 2$ $\qquad v = -\frac{1}{2}e^{-2x}$

$$\int 2(x+2)e^{-2x}dx = uv - \int u'v\,dx = -(x+2)e^{-2x} - \int -e^{-2x}dx$$

$$= -(x+2)e^{-2x} - \frac{1}{2}e^{-2x} = -(x+5/2)e^{-2x}$$

Relation $(*)$ can then be written as:

$$-\frac{1}{2y^2} = -(x+5/2)e^{-2x} + C$$

$y(0) = \dfrac{1}{\sqrt{5}} \Rightarrow -\dfrac{5}{2} = -\dfrac{5}{2} + C \Rightarrow C = 0$

So, $-\frac{1}{2y^2} = -(x+5/2)e^{-2x} = -\left(\frac{2x+5}{2}\right)e^{-2x} \Rightarrow y^2 = \frac{1}{(2x+5)e^{-2x}}$

$= \frac{e^{2x}}{(2x+5)} \Rightarrow y = \frac{e^x}{\sqrt{2x+5}}$ (we take $+\sqrt{\frac{e^{2x}}{2x+5}}$ since $y(0) = \frac{1}{\sqrt{5}} > 0$).

(d) $y' x\ln(x) = y$, $y(3) = \ln(81)$

$\frac{dy}{dx} x\ln(x) = y \Rightarrow \frac{dy}{y} = \frac{dx}{x\ln x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x\ln x}$ (*)

For $\int \frac{dx}{x\ln x}$, let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

$\int \frac{dx}{x\ln x} = \int \frac{x\,du}{x\,u} = \int \frac{1}{u}\,du = \ln|u| = \ln|\ln|x||$.

So (*) becomes: $\ln|y| = \ln|\ln|x|| + C \Rightarrow$

$y = A\ln|x|$. For $x = 3$, $y = \ln(81) \Rightarrow$

$\ln(81) = A\ln 3 \Rightarrow \ln 3^4 = A\ln 3 \Rightarrow A = \frac{\ln 3^4}{\ln 3} = \frac{4\ln 3}{\ln 3} = 4$

The particular solution is $y = 4\ln|x|$.

(e) $yy'e^{y^2} = (x-1)$, $y(0) = 1$

$y\frac{dy}{dx}e^{y^2} = (x-1) \Rightarrow ye^{y^2}dy = (x-1)\,dx \Rightarrow \int ye^{y^2}dy = \int (x-1)\,dx$

Let $t = y^2 \Rightarrow \frac{dt}{dy} = 2y \Rightarrow dy = \frac{dt}{2y}$, so

$$\int y e^{y^2} dy = \int y e^t \frac{dt}{2y} = \frac{1}{2}\int e^t dt = \frac{1}{2}e^t = \frac{1}{2}e^{y^2}.$$

So $\int y e^{y^2} dy = \int (x-1)\,dx$ implies: $\frac{1}{2}e^{y^2} = \frac{x^2}{2} - x + C$

or $e^{y^2} = x^2 - 2x + A \Rightarrow y^2 = \ln|x^2 - 2x + A|$ and

$$y = \sqrt{\ln|x^2 - 2x + A|} \qquad [\text{since } y(0) = 1 > 0]$$

$y(0) = 1 \Rightarrow 1 = \sqrt{\ln A} \Rightarrow \ln A = 1 \Rightarrow A = e$

$y = \sqrt{\ln(x^2 - 2x + e)}$ is the particular solution.

(f) $(2x + 3y)\,dx + (y - x)\,dy = 0$, $y(1) = 0$

This is clearly a homogeneous DE of degree 1.

$y = xu \Rightarrow dy = x\,du + u\,dx$

$(2x + 3xu)\,dx + (xu - x)[x\,du + u\,dx] = 0 \Rightarrow$

$(u^2 + 2u + 2)\,dx + x(u-1)\,du = 0 \Rightarrow \int \frac{dx}{x} = -\int \frac{(u-1)}{u^2+2u+2}\,du$

Be careful how to compute $\int \frac{u-1}{u^2+2u+2}\,du$:

$$\int \frac{u-1}{u^2+2u+2}\,du = \frac{1}{2}\int \frac{2u-2}{u^2+2u+2}\,du = \frac{1}{2}\int \frac{2u+2-4}{u^2+2u+2}\,du$$

$$= \frac{1}{2}\int \frac{2u+2}{u^2+2u+2}\,du - 2\int \frac{1}{u^2+2u+2}\,du$$

$$= \frac{1}{2}\ln|u^2+2u+2| - 2\int \frac{1}{(u+1)^2+1}\,du$$

(by substitution) (by completing the square)

$$= \frac{1}{2}\ln(u^2+2u+2) - 2\,\text{Arctan}(u+1)$$

The equation $\int \frac{dx}{x} = -\int \frac{u-1}{u^2+2u+2}\,du$ becomes:

$$\ln|x| = -\frac{1}{2}\ln(u^2+2u+2) + 2\,\text{Arctan}(u+1) + C$$ or

$$\ln|x| = -\frac{1}{2}\ln\left[\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) + 2\right] + 2\,\text{Arctan}\left(\frac{y}{x}+1\right) + C$$

$$y(1) = 0 \Rightarrow \ln 1 = -\frac{1}{2}\ln 2 + 2\,\text{Arctan}(1) + C \Rightarrow$$

$$-\frac{1}{2}\ln 2 + 2(\pi/4) + C = 0 \Rightarrow C = \frac{1}{2}\ln 2 - \pi/2 = \ln\sqrt{2} - \pi/2$$

The particular solution is given by:

$$\ln x = -\frac{1}{2}\ln\left[\left(\frac{y}{x}\right)^2 + 2(y/x) + 2\right] + 2\,\text{Arctan}\left[(y/x)+1\right] + \ln\sqrt{2} - \pi/2.$$