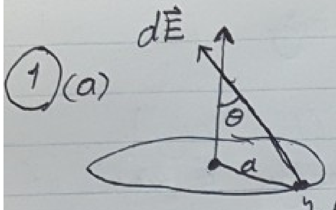
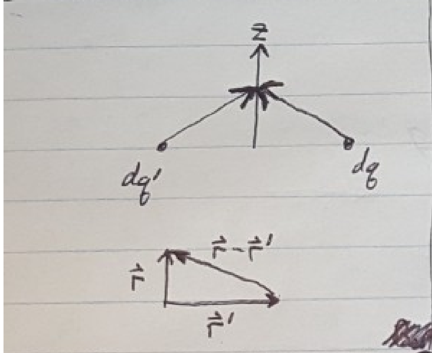
PHY2323 DGD#2 Solutions

1 (a)



Consider $\vec{E}$ field due to small line element of ring; $dq = \lambda dl$

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq}{|\vec{r}-\vec{r}'|^2} \underbrace{\left(\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}\right)}_{\text{direction of field}}$$



Let θ be angle between $\vec{E}$ field of dq & z-axis. Then direction vector is

$$\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} = \cancel{-\sin\theta\,\hat{\rho}} + \cos\theta\,\hat{z}$$

$\to 0$, due to symmetry

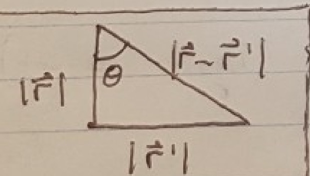
$\hat{\rho}$ component canceled by dq' on opposite side of circle. (see figure to left)

$$\therefore\ d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{dq\cos\theta}{|\vec{r}-\vec{r}'|^2}\,\hat{z}$$

Now, $dq = \lambda dl$ and $dl = a\,d\varphi$, integrating around circle.

So,

$$\vec{E} = \frac{a\lambda}{4\pi\epsilon_0} \int_0^{2\pi} d\varphi \frac{\cos\theta}{|\vec{r}-\vec{r}'|^2}\,\hat{z}$$



Use trig to figure out:

$|\vec{r}| = z$, $|\vec{r}'| = a \Rightarrow |\vec{r}-\vec{r}'| = (a^2+z^2)^{1/2}$

and $\cos\theta = \dfrac{|\vec{r}|}{|\vec{r}-\vec{r}'|} = \dfrac{z}{(a^2+z^2)^{1/2}}$

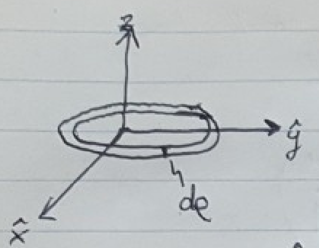
So,

$$\vec{E} = \frac{a\lambda}{4\pi\epsilon_0} \frac{z}{(a^2+z^2)^{3/2}} \int_0^{2\pi} d\varphi\,\hat{z}$$

$$\boxed{\vec{E} = \frac{a\lambda}{2\epsilon_0} \frac{z}{(a^2+z^2)^{3/2}}\,\hat{z}}$$

Field due to ring of radius a.

(b) We know from part (a) the $\vec{E}$-field of a ring.



Consider $\vec{E}$ field of annulus (thin ring).
If σ is surface charge density, $\vec{E}$ field of annulus equal to $\vec{E}$ field of ring with

$$\lambda \to \sigma \, d\rho$$

$$\therefore \; d\vec{E} = \frac{\sigma}{2\epsilon_0} \frac{z \rho \, d\rho}{(\rho^2+z^2)^{3/2}}$$

Integration over entire disk $(0 \le \rho \le a)$ gives

$$\vec{E} = \frac{\sigma z}{2\epsilon_0} \int_0^a \frac{\rho}{(\rho^2+z^2)^{3/2}} \, \hat{z}$$

$$= \frac{\sigma z}{2\epsilon_0} \left[\frac{1}{z} - \frac{1}{(a^2+z^2)^{1/2}} \right] \hat{z}$$

i) $a \gg z$, ignore 2nd term.

$$\therefore \; \vec{E} = \frac{\sigma}{2\epsilon_0} \hat{z}$$

ii) $z \gg a$, approximate: $\displaystyle \frac{1}{(a^2+z^2)^{1/2}} = \frac{1}{z} \Big(\frac{1}{1+\underbrace{a^2/z^2}_{\text{small}}} \Big)^{1/2}$

$$\simeq \frac{1}{z} \left(1 - \frac{a^2}{2z^2} \right) \quad \text{(Taylor series)}$$

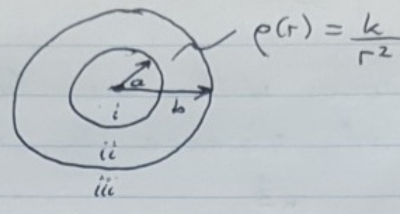
$$\therefore \; \vec{E} = \frac{\sigma z}{2\epsilon_0} \left[\frac{a^2}{2z^2} \right] = \frac{\sigma a^2}{4\epsilon_0 z^2}$$

Define "effective charge" $Q = \pi a^2 \sigma$, then

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 z^2} \quad \text{(field due to point charge)}$$

(2)

~~[illegible]~~ Spherical shell



Region i: $Q_{enc} = 0$ so $\vec{E} = 0$

Region ii: $\oint \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon_0} Q_{enc}$

Symmetry $\rightarrow$ $\vec{E} = E\,\vec{a}_r$, E will depend on r

$\therefore \oint \vec{E} \cdot \vec{ds} = E \cdot 4\pi r^2$ (Gaussian sphere of radius r)

$Q_{enc} = \int_V \rho(r)\,dV$

↳ volume enclosed by Gaussian surface

$= \int_a^r dr \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \left(\frac{k}{r^2}\right) r^2 \sin\theta$

↳ inner bound

$= \frac{4\pi k}{\epsilon_0} \int_a^r dr$

$= \frac{4\pi k (r-a)}{\epsilon_0}$ $\therefore$ $E = \frac{k}{\epsilon_0} \left(\frac{r-a}{r^2}\right) \vec{a}_r$

Region iii)

Similar, just change bounds of radial integral

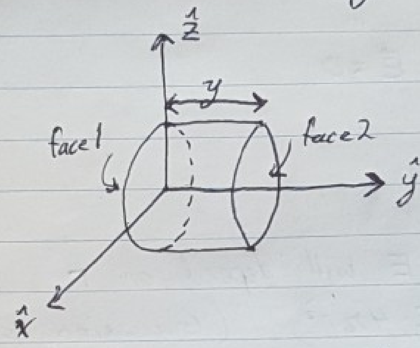
$Q_{enc} = \frac{4\pi k}{\epsilon_0} \int_a^b dr = \frac{4\pi k}{\epsilon_0}(b-a)$

so, $\vec{E} = \frac{k}{\epsilon_0} \left(\frac{b-a}{r^2}\right) \vec{a}_r$

③ $\vec{E} = E\,\vec{a}_y$ by symmetry. Also, $\vec{E} = 0$ on xz plane $(y=0)$

E will depend on y

$\therefore$ Use Gaussian pillbox with one face on xz plane, and other face at distance y from xz plane.



Each face has arbitrary area A. (Shape of face and value of A unimportant; A cancels).

Volume enclosed $= yA$

$\therefore$ $Q_{enc} = yA\rho$ (y inside slab. If y outside slab, $Q_{enc} = dA\rho$)

Since face 1 is at $y=0$, only face 2 contributes to flux:

$\therefore$ $\oint \vec{E}\cdot\vec{ds} = E\cdot A$

Gauss's Law $\quad \oint \vec{E}\cdot\vec{ds} = \dfrac{Q_{enc}}{\epsilon_0}$

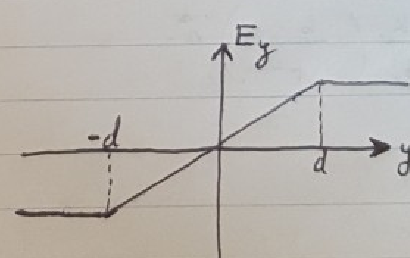
Inside slab $(|y| < d)$

$$E\cdot A = \frac{yA\rho}{\epsilon_0} \Rightarrow E = \frac{\rho y}{\epsilon_0}$$

or $\vec{E} = \dfrac{\rho y}{\epsilon_0}\,\vec{a}_y$

Outside slab: $(|y| > d)$

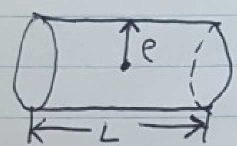
$$\vec{E} = \frac{\rho d}{\epsilon_0}\,\vec{a}_y$$



(4)

By symmetry, $\vec{E} = E\,\vec{a}_\varrho$; E will depend on ϱ

Use coaxial cylinder of radius ϱ & length L as surface



surface area $= 2\pi\varrho L$

volume $= \pi\varrho^2 L$

Region i): $\oint \vec{E}\cdot\vec{ds} = \frac{1}{\epsilon_0} Q_{enc}$

$$\left.\begin{aligned} \oint \vec{E}\cdot\vec{ds} &= E\cdot 2\pi\varrho L \\ Q_{enc} &= \bar{\rho}\,\varrho^2\pi L \end{aligned}\right\} \Rightarrow \vec{E} = \bar{\rho}\frac{\varrho}{2\epsilon_0}\,\vec{a}_\varrho$$

Region ii: $\oint \vec{E}\cdot\vec{ds}$ is unchanged

$Q_{enc} = \rho a^2 \pi L$

$\therefore\ \vec{E} = \frac{\bar{\rho} a^2}{2\epsilon_0 \varrho}\,\vec{a}_\varrho$

Region iii: Question states σ is chosen to cancel $\bar{\rho}$.

$\therefore$ Charge outside radius b is zero

i.e. $Q_{enc} = 0$

$\therefore\ \vec{E} = 0$