SOLUTION OF THE SUMMER 2023 MIDTERM EXAMINATION

QUESTION 1

A system has the following state space representation

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ -1 & -3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \mathbf{x}$$

(a) (6 points) Draw the signal flow graph that corresponds to this state space representation

(b) (9 points) Use Mason's rule to find the transfer function of the system

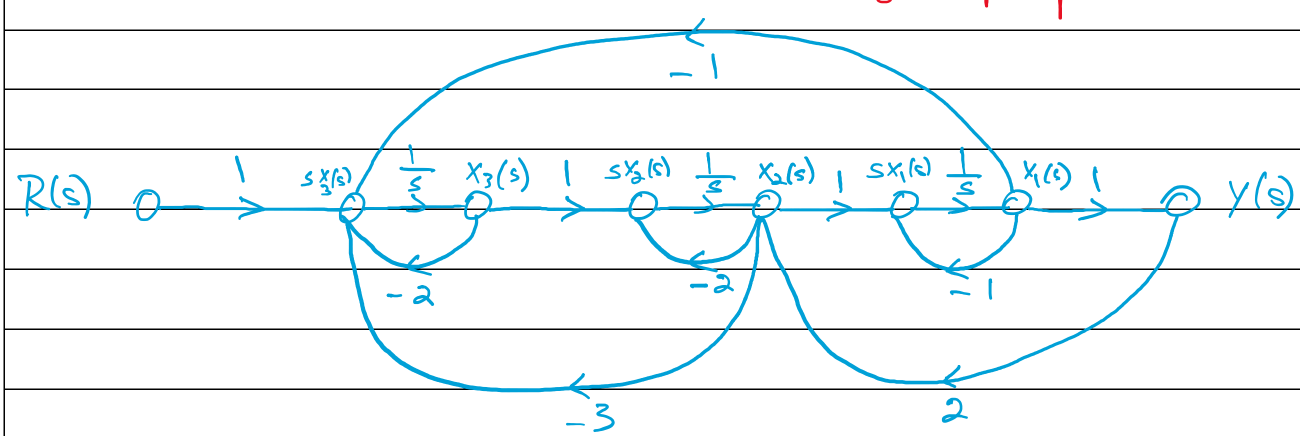
a) $\dot{x}_1 = -x_1 + x_2$

$\dot{x}_2 = -2x_2 + x_3$

$\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + r(t)$

$y(t) = x_1 + 2x_2$ * 1.5 points if 3 integrators are shown

* 4.5 points for proper connections



b) Loop gains:

$L_1 = -\frac{2}{s}$, $L_2 = -\frac{2}{s}$, $L_3 = -\frac{1}{s}$, $L_4 = -\frac{3}{s^2}$

$L_5 = -\frac{1}{s^3}$ (0.5 point for each loop gain)

Forward paths: $T_1 = \frac{1}{s^3}$, $T_2 = \frac{2}{s^2}$ (0.5 point for each forward path)

Non-touching loops taken 2 at a time: $\frac{4}{s^2}$, $\frac{2}{s^2}$, $\frac{2}{s^2}$, $\frac{3}{s^3}$ (0.5 point for each)

Non-touching loops taken 3 at a time: $-\frac{4}{s^3}$ (0.5 point)

$$\Delta = 1 - \left(-\frac{2}{s} - \frac{2}{s} - \frac{1}{s} - \frac{3}{s^2} - \frac{1}{s^3}\right) + \left(\frac{4}{s^2} + \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s^2}\right) - \left(-\frac{4}{s^3}\right)$$

(1 point)

$$\Delta = \frac{5}{s} + \frac{11}{s^2} + \frac{8}{s^3} + 1 \Rightarrow \Delta = \frac{s^3 + 5s^2 + 11s + 8}{s^3}$$

$\Delta_1 = 1$ (0.5) $\Delta_2 = 1 + \frac{1}{s}$ (0.5)

$$T(s) = \frac{T_1 \Delta_1}{\Delta} + \frac{T_2 \Delta_2}{\Delta} = \frac{\frac{1}{s^3} + \left(1 + \frac{1}{s}\right)}{\frac{s^3 + 5s^2 + 11s + 8}{s^3}}$$

$$\Rightarrow \boxed{T(s) = \frac{2s + 3}{s^3 + 5s^2 + 11s + 8}}$$

(1 point)

QUESTION 2

(8 points) For the system of Figure 1, find the values of K_1 and K_2 to yield a peak time of 1.5 second and a settling time of 3.2 seconds for the closed-loop system's step response

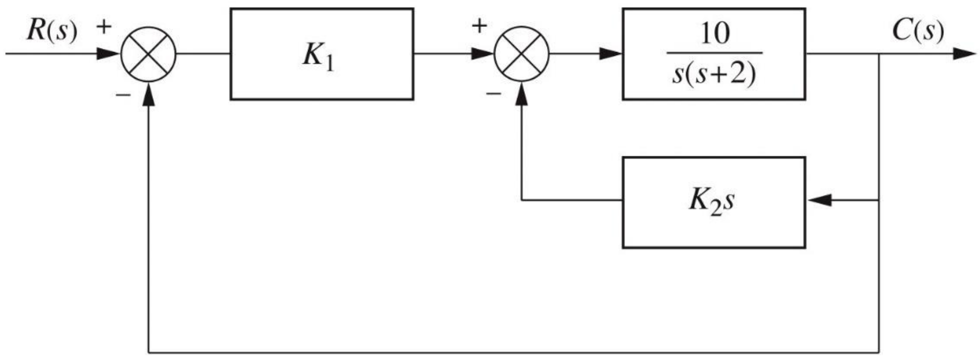


Figure 1: An interconnected system

We need to simplify the diagram to a single transfer function.

- the feedback part can be reduced to

$$\frac{10/s(s+2)}{1+\frac{K_2 s \times 10}{s(s+2)}} = \frac{10/s(s+2)}{\frac{s(s+2)+10K_2 s}{s(s+2)}} = \frac{10}{s^2+2s+10K_2 s}$$

$$= \frac{10}{s^2+s(2+10K_2)}$$ (2 points)

Now, the equivalent T.F is

$$T(s) = \frac{10K_1/s^2+s(2+10K_2)}{1+10K_1/s^2+s(2+10K_2)}$$

$$T(s) = \frac{10K_1}{s^2+s(10K_2+2)+10K_1}$$ (2 points)

$$T_p = 1.5\,s = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow 2.25 = \frac{\pi^2}{\omega_n^2(1-\zeta^2)}$$

$$\Rightarrow \omega_n^2(1-\zeta^2) = \frac{\pi^2}{2.25} \quad (1)$$ (1 point)

$$T_s = 3.2\,s = \frac{4}{\zeta\omega_n} \Rightarrow \zeta\omega_n = 1.25 \quad (2)$$ (1 point)

plugging (2) in (1) $\Rightarrow \omega_n^2 - (1.25)^2 = \dfrac{\pi^2}{2.25}$

$$\Rightarrow \omega_n = \sqrt{\frac{\pi^2}{2.25} + (1.25)^2}$$

$\omega_n = 2.44$ rad/s (0.5 point)

$$\zeta = \frac{1.25}{2.44} \Rightarrow \zeta = 0.513$$ (0.5 point)

$$10K_1 = \omega_n^2 \Rightarrow K_1 = \frac{\omega_n^2}{10} = \frac{(2.44)^2}{10} \Rightarrow \boxed{K_1 = 0.595}$$ (0.5 point)

$$10K_2 + 2 = 2\zeta\omega_n = 2(1.25) \Rightarrow K_2 = \frac{2.5-2}{10}$$

$$\Rightarrow \boxed{K_2 = 0.05}$$ (0.5 point)

QUESTION 3

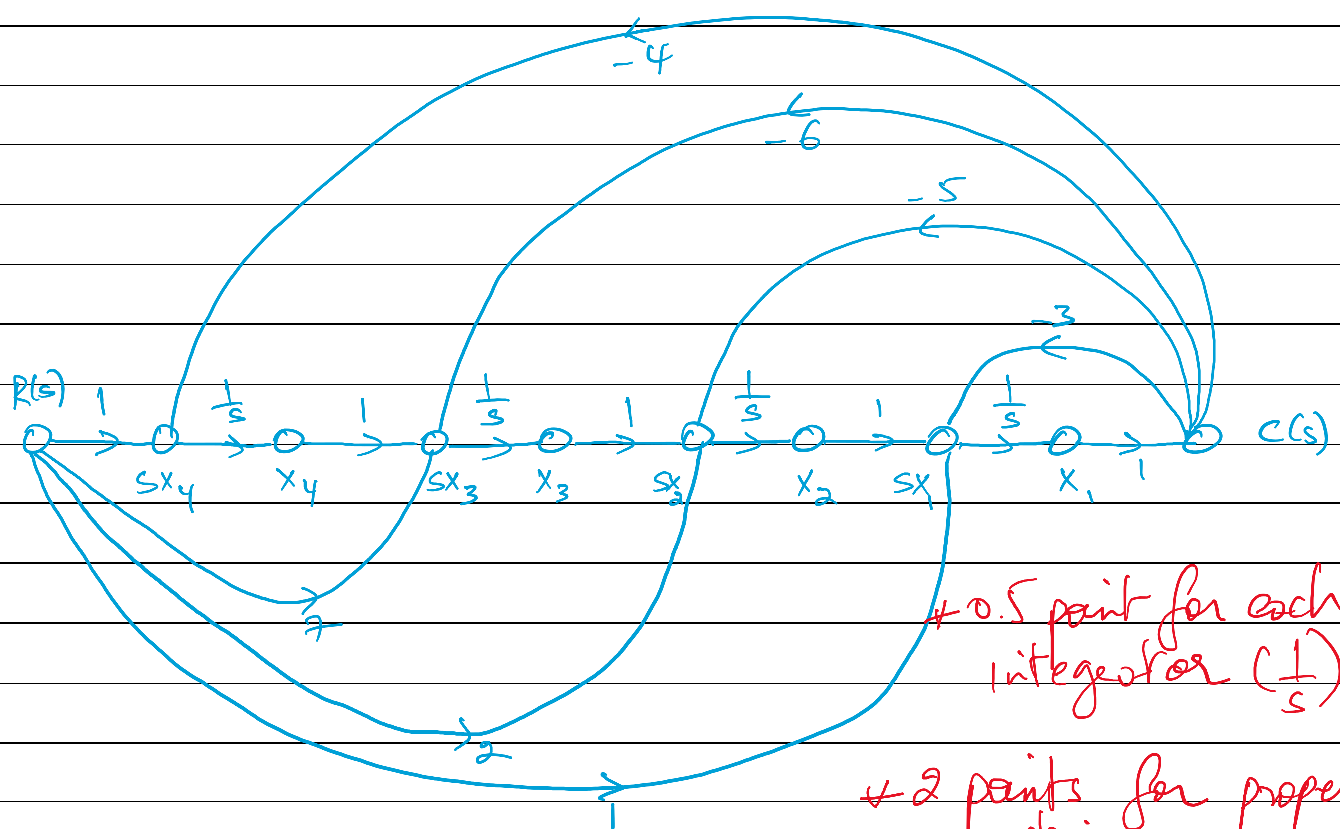
(10 points) Represent the following transfer function in the observer canonical state-space representation form

$$G(s) = \frac{s^3 + 2s^2 + 7s + 1}{s^4 + 3s^3 + 5s^2 + 6s + 4}$$

$$G(s) = \frac{s^4\left(\frac{1}{s} + \frac{2}{s^2} + \frac{7}{s^3} + \frac{1}{s^4}\right)}{s^4\left(1 + \frac{3}{s} + \frac{5}{s^2} + \frac{6}{s^3} + \frac{4}{s^4}\right)} = \frac{C(s)}{R(s)}$$ (0.5 point)

$$C(s)\left(1 + \frac{3}{s} + \frac{5}{s^2} + \frac{6}{s^3} + \frac{4}{s^4}\right) = R(s)\left(\frac{1}{s} + \frac{2}{s^2} + \frac{7}{s^3} + \frac{1}{s^4}\right)$$

$$C(s) = \frac{1}{s}\left(R(s) - 3C(s)\right) + \frac{1}{s^2}\left(2R(s) - 5C(s)\right) + \frac{1}{s^3}\left(7R(s) - 6C(s)\right) + \frac{1}{s^4}\left(R(s) - 4C(s)\right)$$ (2 points)



+0.5 point for each integrator ($\frac{1}{s}$)

+2 points for proper connections

$\dot{x}_1 = -3c(t) + r(t) + x_2 = -3x_1 + x_2 + r(t)$ (0.5)

$\dot{x}_2 = -5c(t) + 2r(t) + x_3 = -5x_1 + x_3 + 2r(t)$ (0.5)

$\dot{x}_3 = -6c(t) + 7r(t) + x_4 = -6x_1 + x_4 + 7r(t)$ (0.5)

$\dot{x}_4 = -4c(t) + r(t) = -4x_1 + r(t)$ (0.5)

$c(t) = x_1$ (0.5)

thus, the state-space representation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -3 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \\ -4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 7 \\ 1 \end{bmatrix} r(t)$$

$$y = c(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

1 point for matrix representation

QUESTION 4

(7 points) Use the block diagram reduction method to find the equivalent transfer function for the system shown in Figure 2.

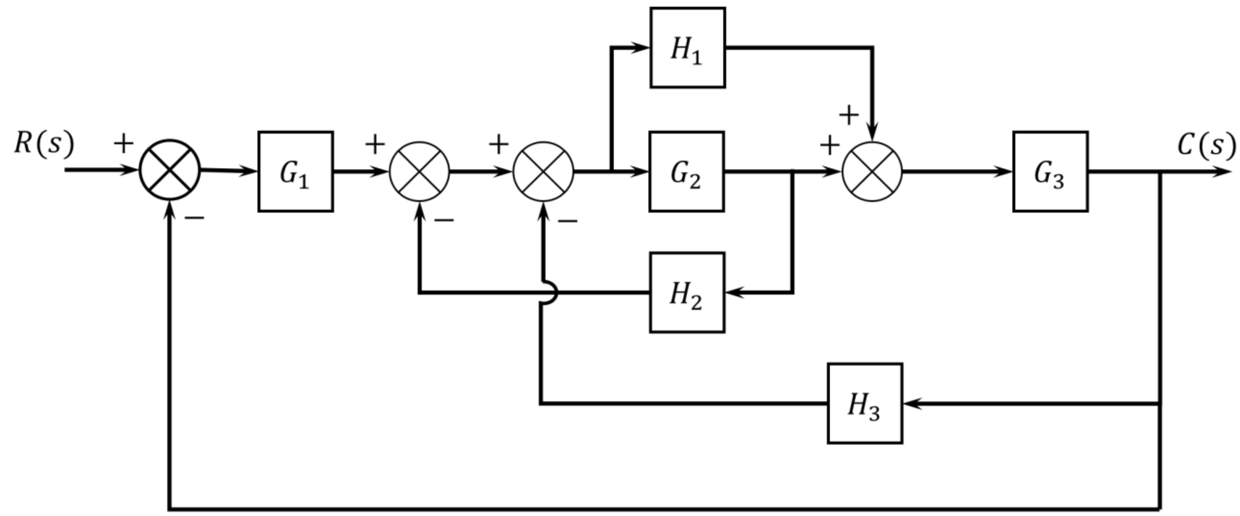
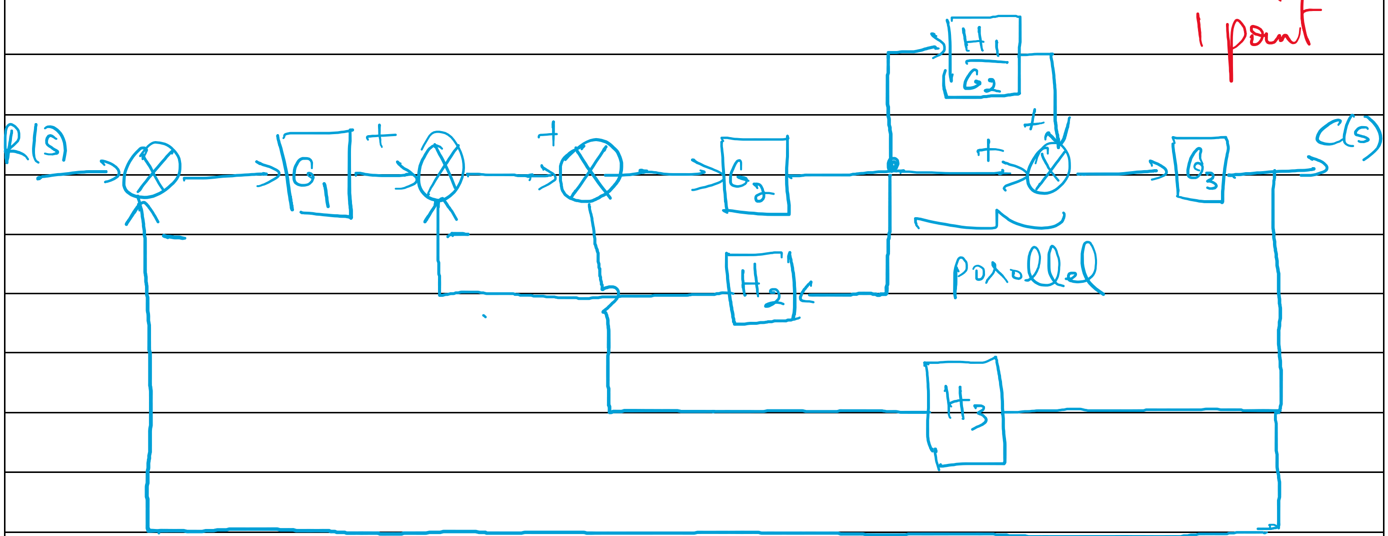


Figure 2: A closed-loop control system

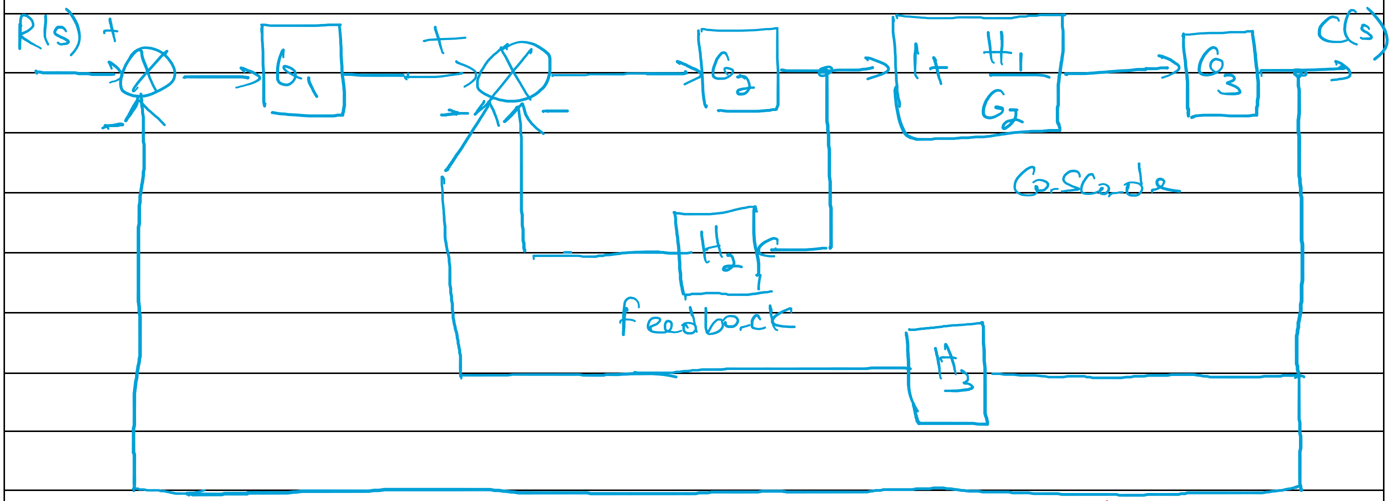
step 1: Move H_1 after the pickoff point

1 point



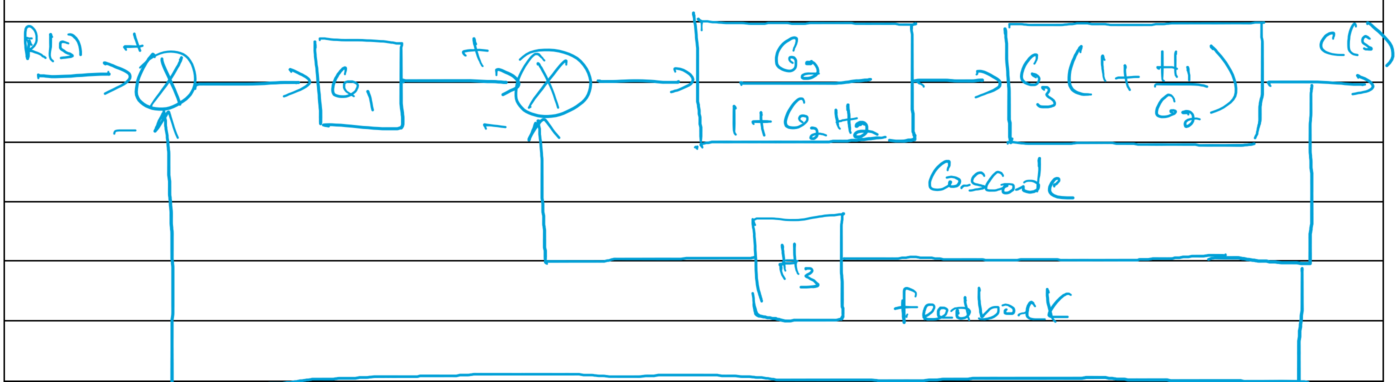
step 2: Replace the parallel block by

$1+\frac{H_1}{G_2}$ and replace the two summers in series by one



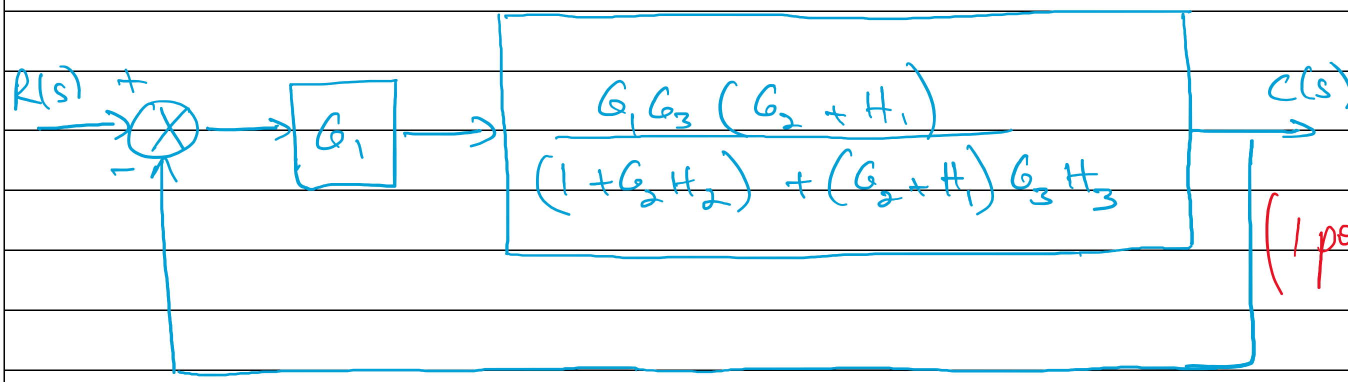
2 points

step3: solve for the feedback and the cascade blocks



2 points

step4: Multiply the two blocks in cascade and solve for the feedback



(1 point)

step 5: Multiply the two blocks in cascade and solve for the feedback

$$\boxed{T(s) = \frac{G_1 G_2 G_3 + G_1 G_3 H_1}{1 + G_2 H_2 + G_2 G_3 H_3 + G_3 H_1 H_3 + G_1 G_2 G_3 + G_1 G_3 H_1}}$$

(1 point)