: The Solution

Problem 1

$$SPS^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$$

Problem 2

(a)
$$SS^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ S is orthogonal

(b)
$$B = SAS^T$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\cos^2\alpha + 2a_{12}\sin\alpha\cos\alpha + a_{22}\sin^2\alpha & (a_{22}-a_{11})\sin\alpha\cos\alpha - \cdots \\ a_{12}(\cos^2\alpha - \sin^2\alpha) + (a_{22}-a_{11})\sin\alpha\cos\alpha & a_{11}\sin^2\alpha + a_{22}\cos^2\alpha - 2a_{12}\sin\alpha\cos\alpha \end{bmatrix}$$

from the title $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$$= \begin{bmatrix} a_{11}\cos^2\alpha + 2\sin^2\alpha + a_{22}\sin^2\alpha & 0 \\ 0 & (a_{11}-2)\sin^2\alpha + a_{22}\cos^2\alpha \end{bmatrix}$$

$\therefore$ B is diagonal

(c) $Tr(B) = (a_{11} + a_{22})\cos^2\alpha + (a_{11} + a_{22})\sin^2\alpha$

$= a_{11} + a_{22} = Tr(A)$

Problem 3

Event A: pick fair coin

Event B: pick two-head coin

Event C: toss coin twice and two heads

$P(A) = P(B) = \frac{1}{2}$ Full probability

$$P(A|C) = \frac{P(C|B)P(A)}{P(C)} = \frac{P(C|A)P(A)}{P(A)P(C|A) + P(B)P(C|B)}$$

$$= \frac{\frac{1}{4} \times \frac{1}{2}}{\frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times 1} = \frac{\frac{1}{8}}{\frac{5}{8}} = \frac{1}{5}$$

$\therefore$ $P(A|C) = \frac{1}{5}$

Problem 4

Event A: select from Box A bulb
Event B: select from Box B bulb
Event C: selected bulb is defective.

(a) $P(A) = P(B) = \frac{1}{2}$

$P(C) = P(A)P(C|A) + P(B)P(C|B)$

$= \frac{1}{2} \times \frac{1}{10} + \frac{1}{2} \times \frac{1}{20} = \frac{3}{40}$

Event D: both bulbs are defective.

$P(D) = P(C) \cdot P(C) = \frac{9}{1600} \approx 0.00563$

(b) Event E: both bulbs are from A.

$$P(E|D) = \frac{P(D|E) \cdot P(E)}{P(D)} = \frac{\frac{1}{10} \times \frac{1}{10} \times \frac{1}{4}}{\frac{9}{1600}}$$

$$= \frac{1600}{3600} \approx 0.444$$