U O W

Software Requirements, Specifications and Formal Methods

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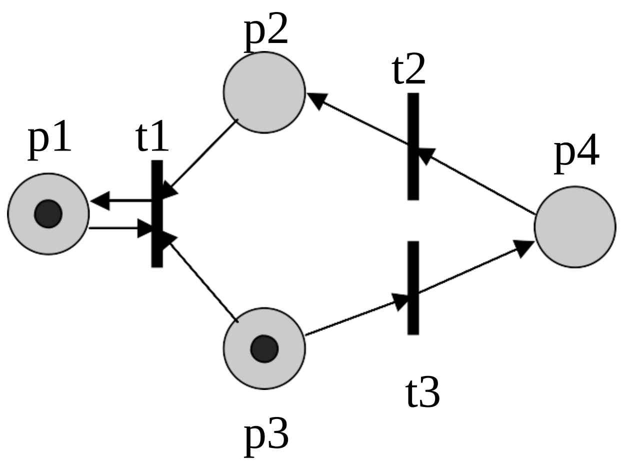
Petri net analysis

- Petri net reachability
- Petri net liveness
- Petri net soundness
- Petri net safeness
- Petri net conservation
- Petri net conflict
- Structural Analysis: P-invariants and T-invariants
- Petri Nets Modelling

Reachability

Marking M is **reachable** from marking M_0 if there exists a sequence of firings $\sigma = t_1\ t_2 \ldots$ (i.e., $M_0\ t_1\ M_1\ t_2\ M_2 \ldots M$) that transforms M_0 to M.

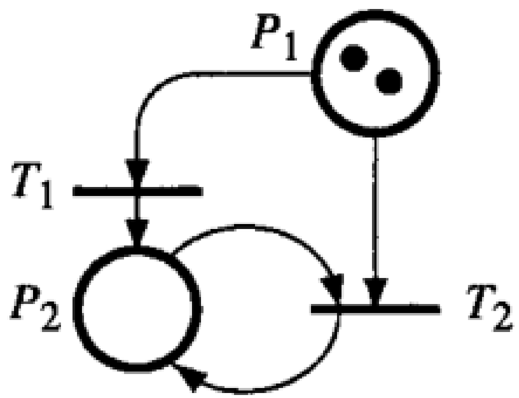


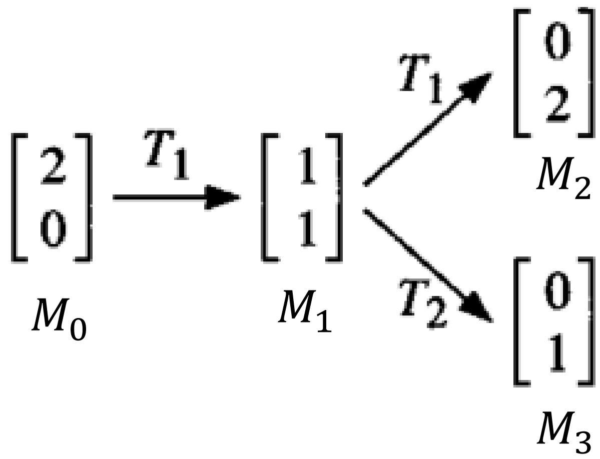
$M_0 = (1,0,1,0)$
$M = (1,1,0,0)$

$$\begin{array}{c} M_0 = (1,0,1,0) \\ \big\downarrow t3 \\ M_1 = (1,0,0,1) \\ \big\downarrow t2 \\ M = (1,1,0,0) \end{array}$$

Reachability graph

Reachability graph is made up of vertices which correspond to reachable markings and of arcs corresponding to firing of transitions resulting in the passing from one marking to another.



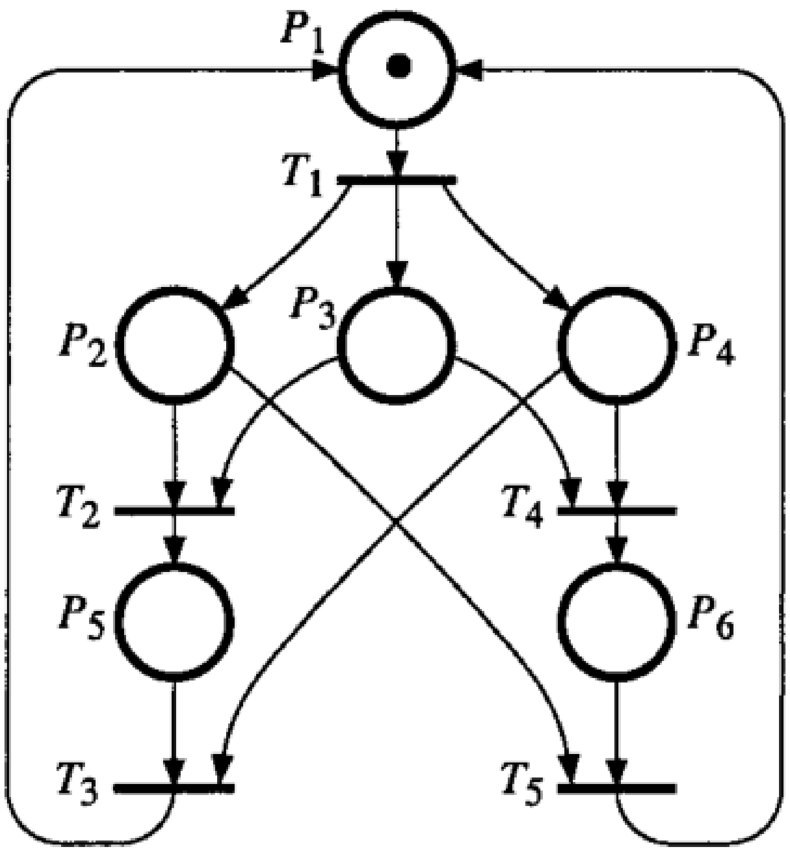


Firing sequence: T_1T_2

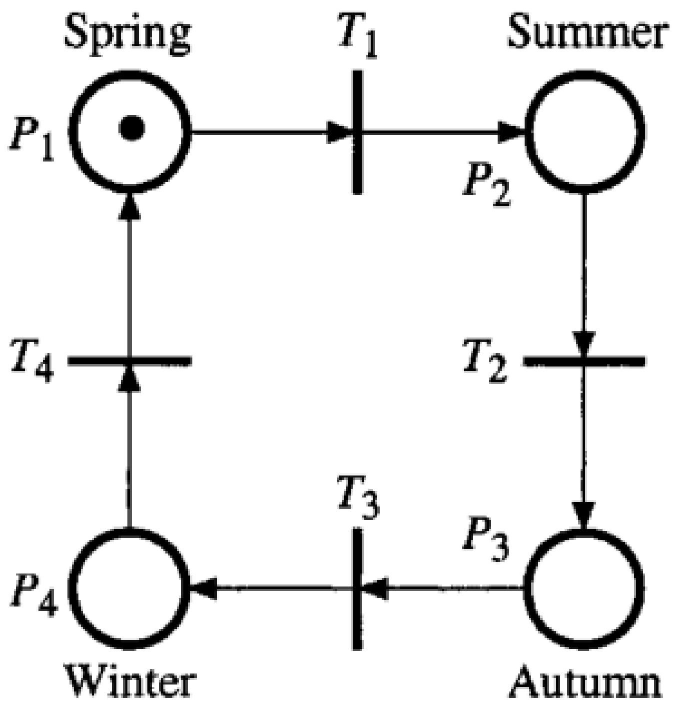
$$M_0 \xrightarrow{T_1T_2} M_3$$

Exercise: Reachability graph

Please provide the reachability graph of the following Petri net.



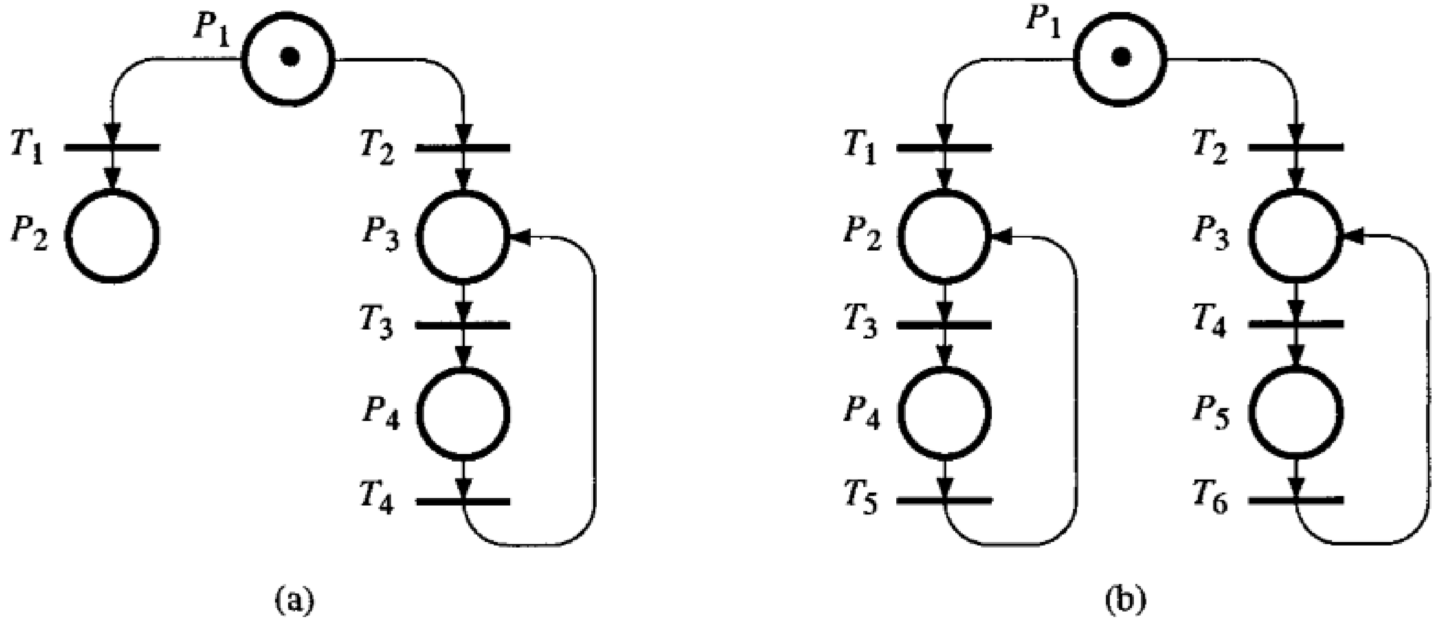
Reachability graph (cont.)



- Possible firing sequences from the initial marking M_0: T_1, T_1T_2, $T_1\ T_2T_3$, $T_1T_2T_3T_4$.
- Firing sequence $T_1T_2T_3T_4$ makes $M_0 \xrightarrow{T_1T_2T_3T_4} M_0$
- The sequence causes a return to the initial state, and this is a **repetitive sequence**.
- A repetitive sequence which contains all the transitions (each at least once) is a **complete repetitive sequence**.

Deadlock

- A deadlock (or sink state) is a marking such that no transition is enable.
- A Petri net is deadlock-free for an initial marking M_0 if no reachable marking M_i is deadlock.



(a) With deadlock (a) Deadlock free

Liveness

- Liveness: **from any marking any transition can become fireable**
- Closely related to the complete absence of deadlocks in operating systems.
- Liveness implies deadlock freedom, not vice versa
- A **live** Petri net guarantees **deadlock-free** operation
- Different levels of liveness are defined.

Liveness (cont.)

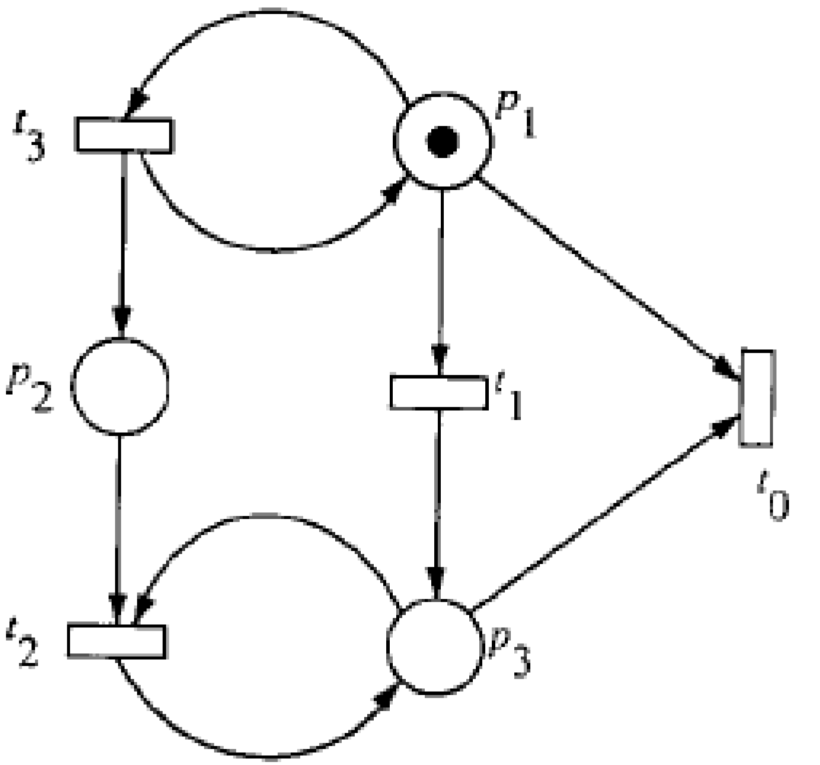
A transition t in a Petri net (N, M_0) is said to be:

- **Dead (L0-live)** if t can never be fired in any firing sequence in $L(M_0)$
- **L1-live** if t can be fired at least once in some firing sequence $L(M_0)$
- **L2-live** if given any positive integer k, t can be fired at least k times in some firing sequence in $L(M_0)$
- **L3-live** if t appears infinitely, often in some firing sequence in $L(M_0)$
- **L4-live** or **live** if t is L1-live for every marking M in $L(M_0)$

L4-live is the strongest and corresponds to the **liveness** defined earlier.

Exercise 2

Please analyse the liveness of the transitions in the Petri net.

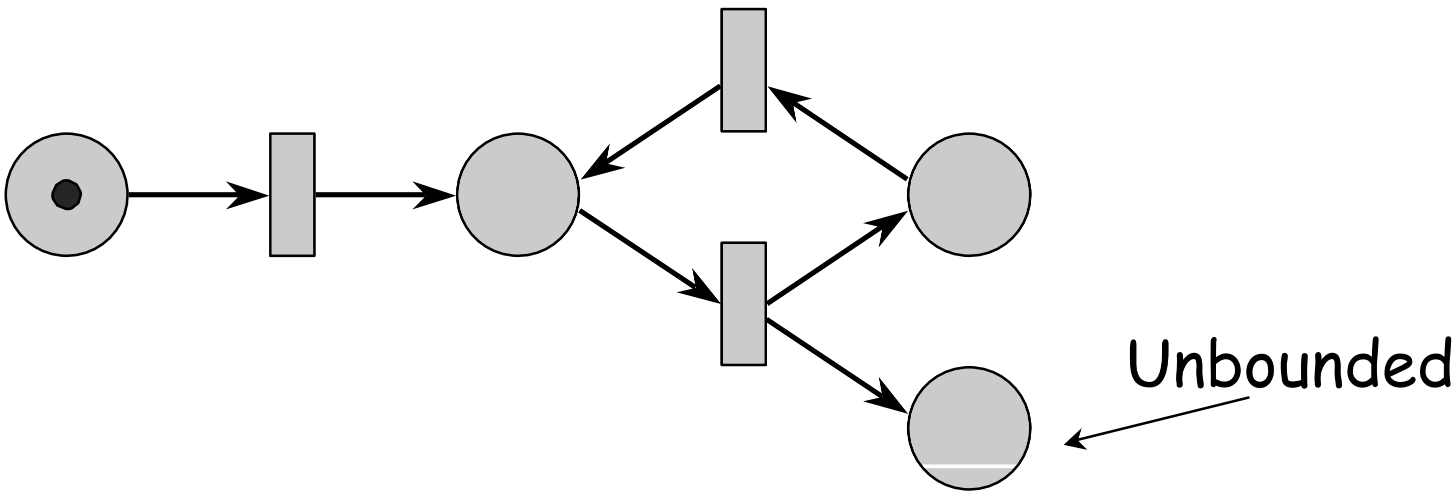


Boundedness

- A **place P_i is said to be bounded** for an initial marking M_0 if there is a natural integer k such that, for all markings reachable from M_0, the number of tokens in P_i is not greater than k (P_i is said to be k-bounded).
- A **Petri net is bounded** for an initial marking M_0 if all the places are bounded for M_0 (the Petri net is k-bounded if all the places are k-bounded).

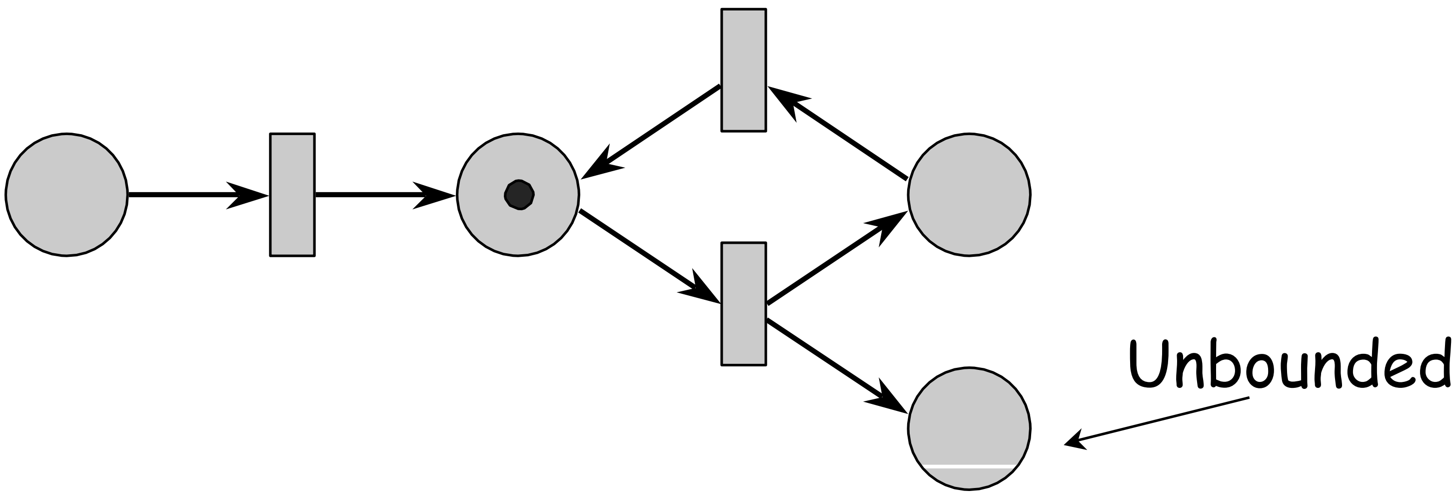
Boundedness (cont.)

Boundedness: the number of tokens in any place cannot grow indefinitely



Boundedness (cont.)

Boundedness: the number of tokens in any place cannot grow indefinitely



Boundedness (cont.)

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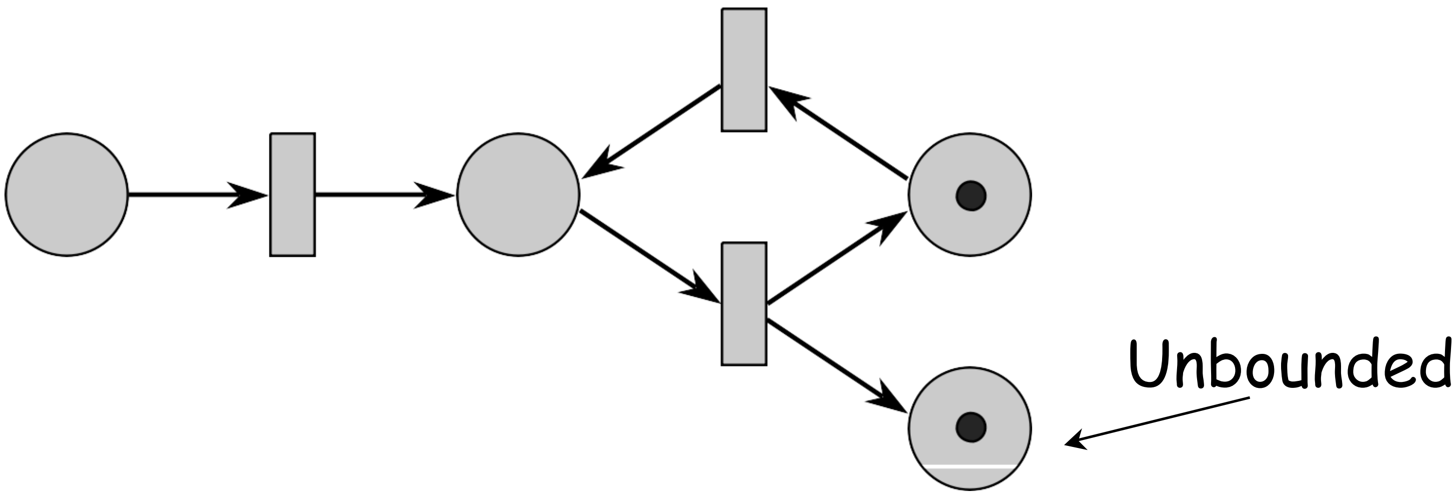
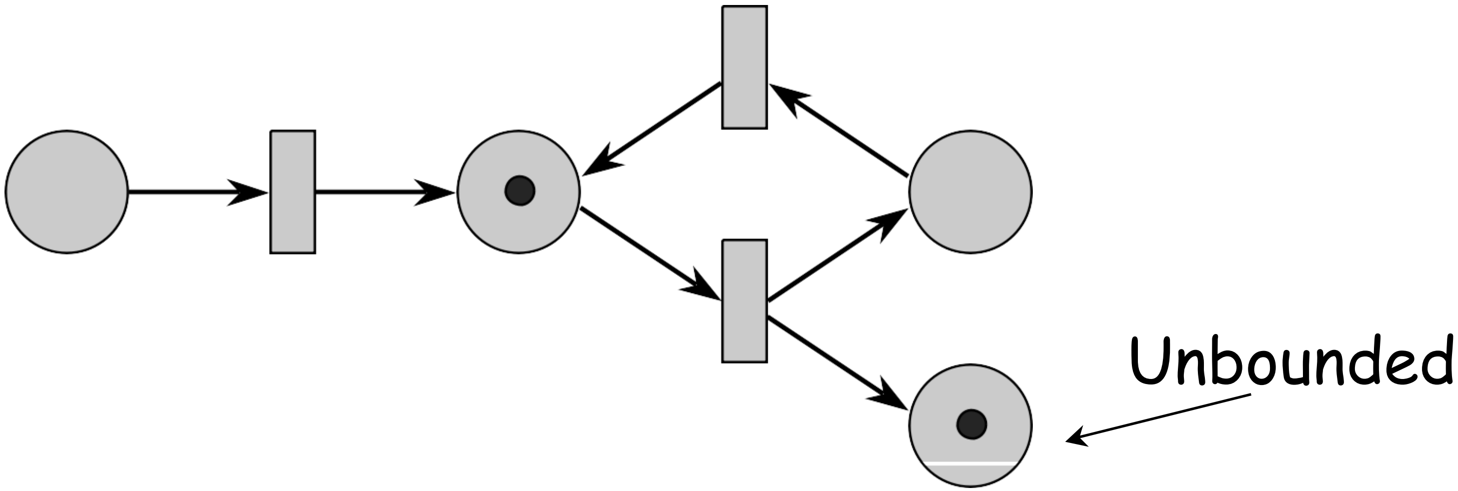


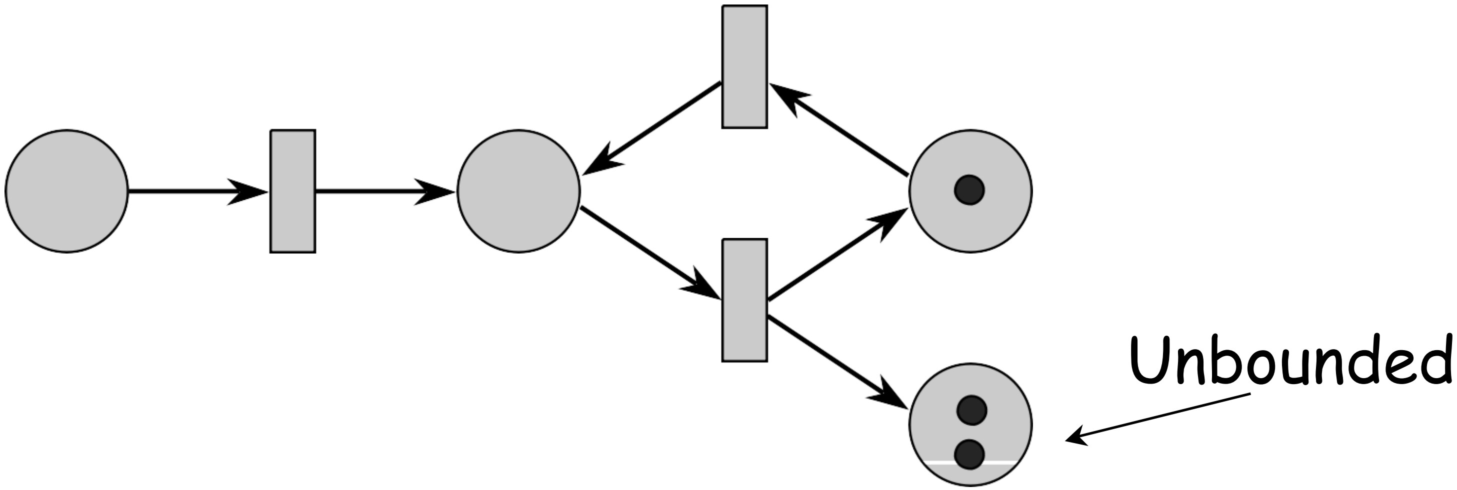
Boundedness (cont.)

Boundedness: the number of tokens in any place cannot grow indefinitely



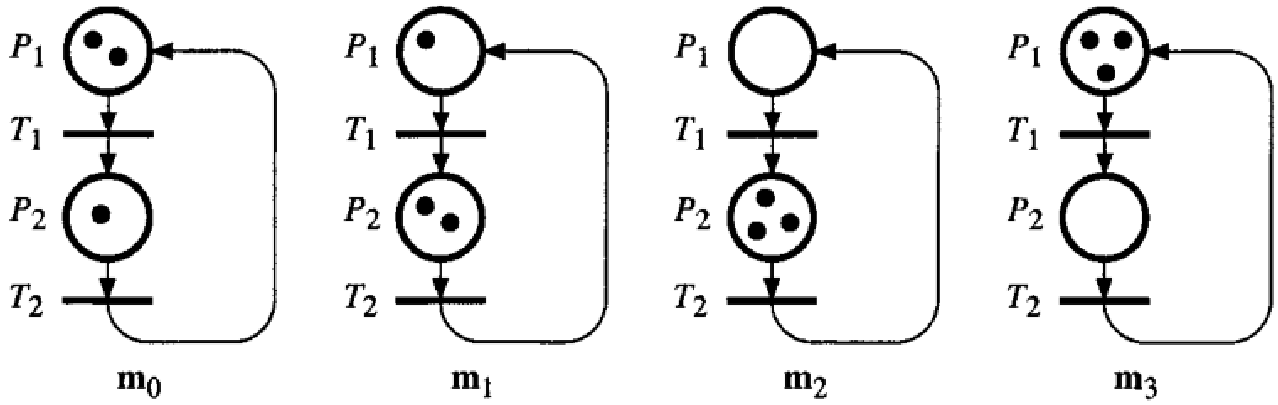
Boundedness (cont.)

Boundedness: the number of tokens in any place cannot grow indefinitely

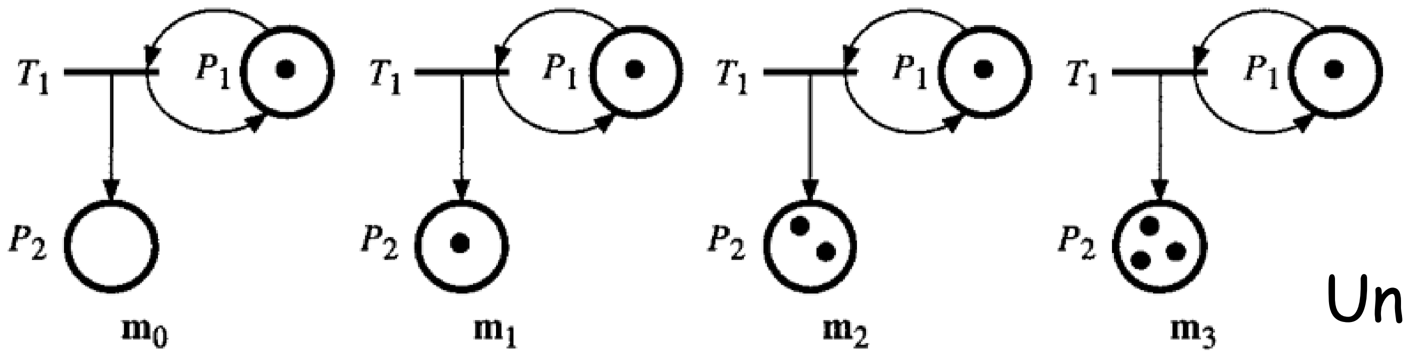


Safeness

- A Petri net is said to be safe for an initial marking M_0 if for all reachable markings, each place contains **zero** or **one** token.
- A **safe Petri net** is a particular case of bounded Petri net for which all the places are **1-bounded**.

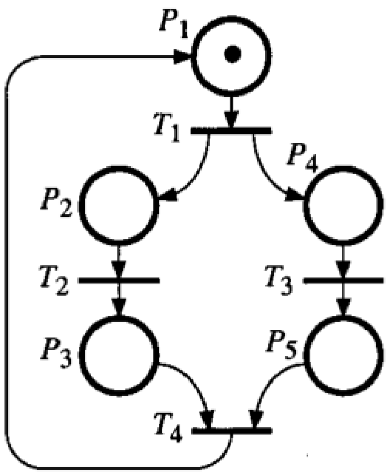


Bounded Petri net

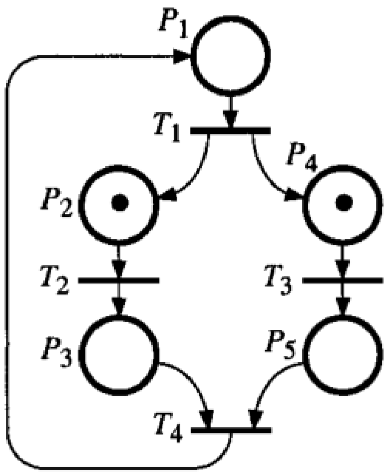


Unbounded Petri net

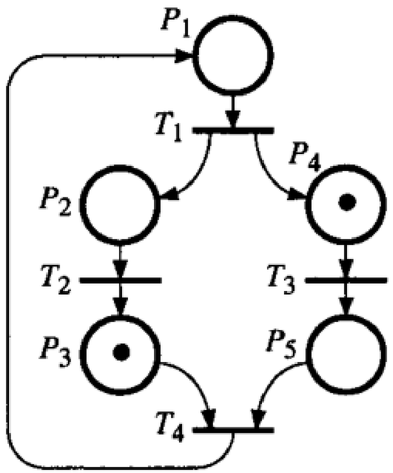
Safeness (cont.)



$\mathbf{m}_0$

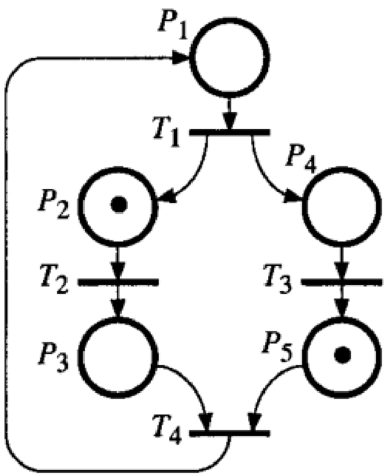


$\mathbf{m}_1$

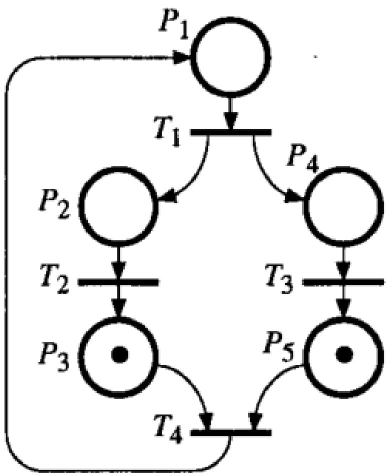


$\mathbf{m}_2$

Safe Petri net



$\mathbf{m}_3$



$\mathbf{m}_4$

Petri Net Analysis

Behavioural properties

1. Reachability
2. Liveness (Deadlock-free)
3. Boundedness
4. Safeness

1. Reachability
2. Deadlock
3. Boundedness

- Why do we need to analyse Petri net properties (e.g., reachability, deadlock-free, boundedness)?
- What do these properties mean to a system?

Conservation

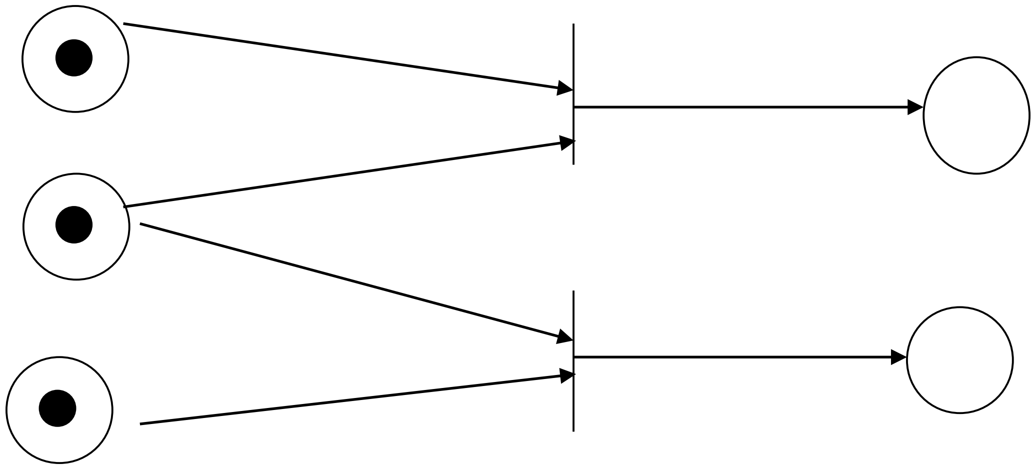
- Petri nets can be used to model *resource allocation systems*. For example, a Petri net can model the requests, allocations, and releases of input/output devices in a computer system. In these systems some tokens may represent the resources.

- For these systems, *conservation* is an important property. We would like to show that tokens which represent resources are created nor destroyed.

- *Conservative Petri nets* are that the *total number of tokens in the net remain constant*.

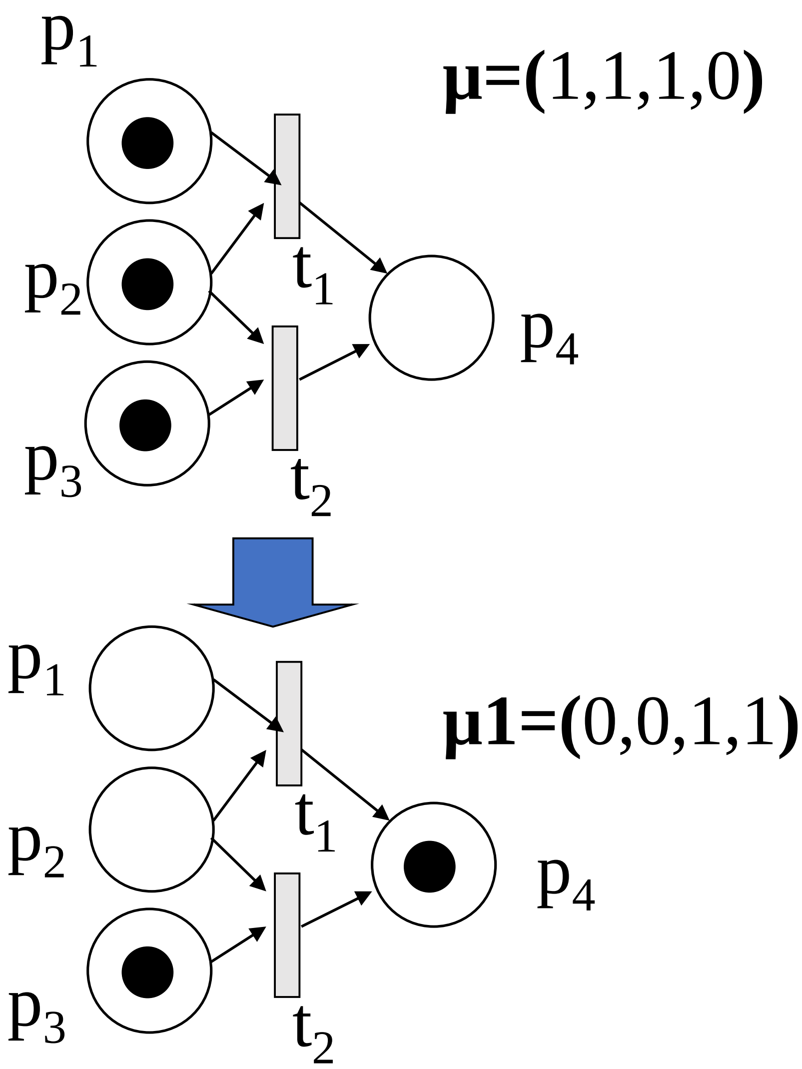
Conflict

In the following Petri net, the two enabled transitions are conflict. Only one transition can fire, since, in firing, it removes the token in the sharing input and disables the other transition.



Examples

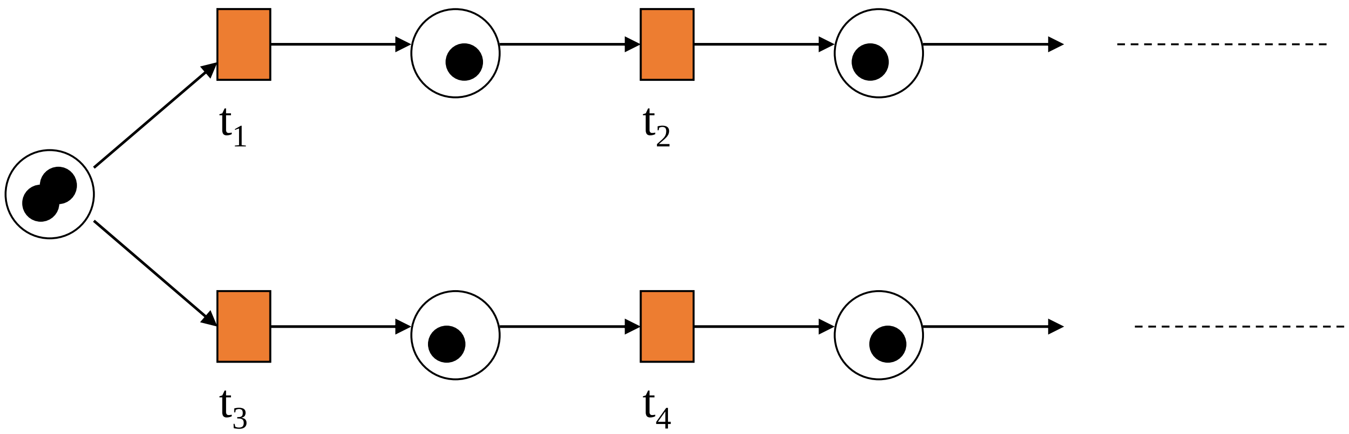
- Conflict
  t_1 and t_2 are both ready to fire but the firing of any leads to the disabling of the other transitions.



Examples (cont.)

- Conflict
  - the resulting conflict may be resolved in a purely non-deterministic way or in a probabilistic way, by assigning appropriate probabilities to the conflicting transitions.

there is a choice of either t_1 and t_2, or t_3 and t_4



Petri Net properties (further comments)

1. **Reachability**
   - Verify whether the system could reach some state from some initial state
   - Whether the system can be executed as planned
2. **Deadlock**
   - Resource-sharing environment
   - Improper resource distribution/allocation in the system
   - Resources are run out of
3. **Boundedness**
   - Checking overflow

Petri Nets: Structural Analysis

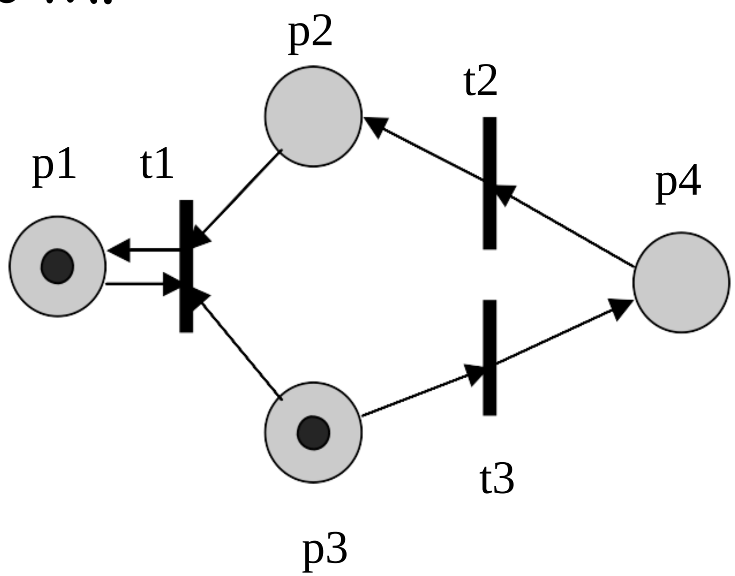
A class of techniques that can extract information about the behaviour of the system

- Describe and analyse the dynamic behaviour of concurrent systems modelled by Petri nets
- **Place invariants** and **Transition invariants**, i.e., through matrix equations

Place Invariant

Reachability Problem :

Marking M is **reachable** from marking M_0 if there exists a sequence of firings $\sigma = M_0\ t_1\ M_1\ t_2\ M_2 \ldots M$ that transforms M_0 to M.



$M_0 = (1,0,1,0)$

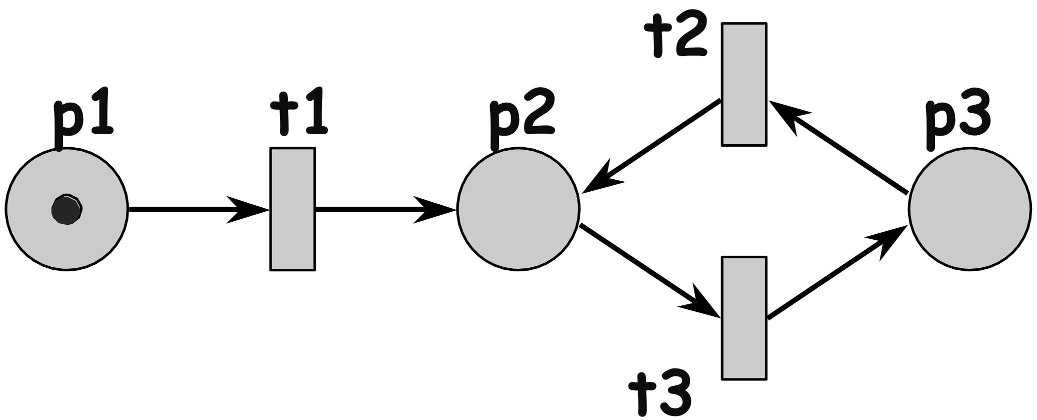
$M = (1,1,0,0)$

$$M_0 = (1,0,1,0) \xrightarrow{t3} M_1 = (1,0,0,1) \xrightarrow{t2} M = (1,1,0,0)$$

Place Invariant: Opening question

- How to solve the following question?

Verifying Reachability of $M = (0\ \ 0\ \ 1)^T$ from $M_0 = (1\ \ 0\ \ 0)^T$



Incidence Matrix

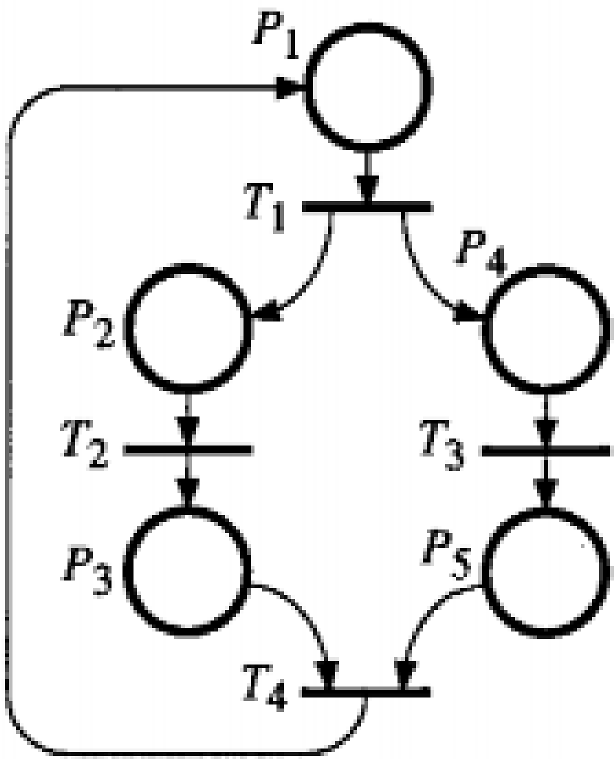
- Input incidence application: Pre: $P \times T \to \{0,1,2,\ldots\}$
  - $Pre(P_i, T_j)$ is the weight of the arc $P_i \to T_j$
- Output incidence application: Post: $P \times T \to \{0,1,2,\ldots\}$
  - $Post(P_i, T_j)$ is the weight of the arc $T_j \to P_i$
- Some relevant notations:
  - ${}^\circ T_j = \{P_i \in P | Pre(P_i, T_j) > 0\}$: set of input places of T_j;
  - $T_j{}^\circ = \{P_i \in P | Post(P_i, T_j) > 0\}$: set of output places of T_j;
  - ${}^\circ P_i = \{T_j \in T | Post(P_i, T_j) > 0\}$: set of input transitions of P_i;
  - $P_i{}^\circ = \{T_j \in T | Pre(P_i, T_j) > 0\}$: set of output transitions of P_i;

Incidence Matrix

- Transition T_j is enable for a marking M_k if $M_k(P_i) \geq Pre(P_i, T_j)$ for every $P_i \in {}^{\circ}T_j$.
- Input incidence matrix:
  - $W^- = [w_{ij}^-]$, where $w_{ij}^- = Pre(P_i, T_j)$
- Output incidence matrix:
  - $W^+ = [w_{ij}^+]$, where $w_{ij}^+ = Post(P_i, T_j)$
- Incidence matrix:
  - $W = W^+ - W^- = [w_{ij}]$

Exercise 3

Please calculate the **input incidence matrix**, **output incidence matrix** and the **incidence matrix** of the following Petri net.



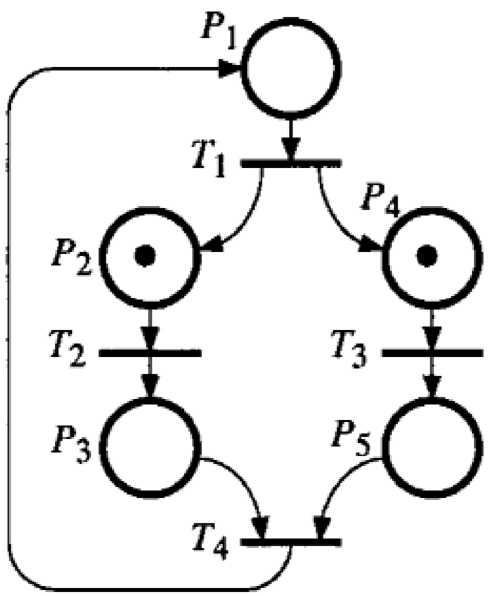
Fundamental equation

- The **characteristic vector** of sequence S, written as $\boldsymbol{s}$, is the m-component vector whose component number j corresponds to the number of firings of transition T_j in sequence S, e.g., $\boldsymbol{s_1} = (0, 1, 0, 0)$.
- If the firing sequence S is such that $M_i \xrightarrow{S} M_k$, then a fundamental equation is obtained:

$$M_k = M_i + W \cdot \boldsymbol{s}$$

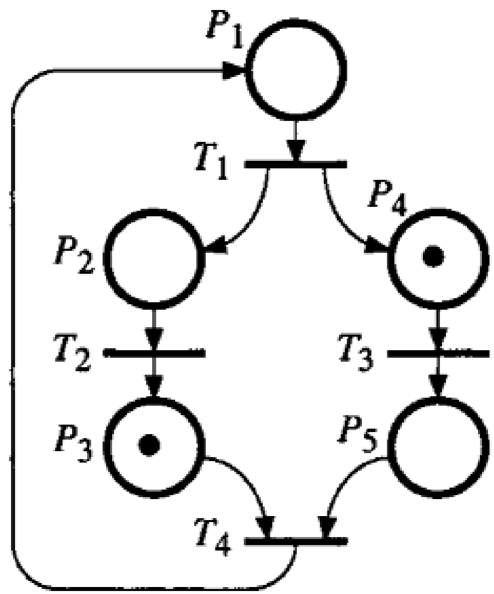
Fundamental equation (cont.)

example



M_1

$s_1 = (0, 1, 0, 0)$



M_2

$$\underbrace{\begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix}}_{M_1} + \underbrace{\begin{bmatrix}-1 & 0 & 0 & +1\\ +1 & -1 & 0 & 0\\ 0 & +1 & 0 & -1\\ +1 & 0 & -1 & 0\\ 0 & 0 & +1 & -1\end{bmatrix}}_{\mathbf{W}} \cdot \underbrace{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}}_{\mathbf{s}_1} = \begin{bmatrix}0\\1\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\-1\\+1\\0\\0\end{bmatrix} = \underbrace{\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}}_{M_2}.$$

Fundamental equation (cont.)

Another example:

- We have $M_2 \xrightarrow{T_3 T_4 T_1 T_3}$ and for $S_2 = T_3 T_4 T_1 T_3$, we have $\boldsymbol{s_2} = (1, 0, 2, 1)$
- The carrying out of this firing sequence gives the marking M_3 obtained by:

$$\underset{\substack{\mathbf{m}_2 \\ M_2}}{\begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix}} + \underset{\mathbf{W}}{\begin{bmatrix}-1 & 0 & 0 & +1\\+1 & -1 & 0 & 0\\0 & +1 & 0 & -1\\+1 & 0 & -1 & 0\\0 & 0 & +1 & -1\end{bmatrix}} \cdot \underset{\mathbf{s}_2}{\begin{bmatrix}1\\0\\2\\1\end{bmatrix}} = \begin{bmatrix}0\\0\\1\\1\\0\end{bmatrix} + \begin{bmatrix}0\\+1\\-1\\-1\\+1\end{bmatrix} = \underset{\substack{\mathbf{m}_3 \\ M_3}}{\begin{bmatrix}0\\1\\0\\0\\1\end{bmatrix}}.$$

Fundamental equation (cont.)

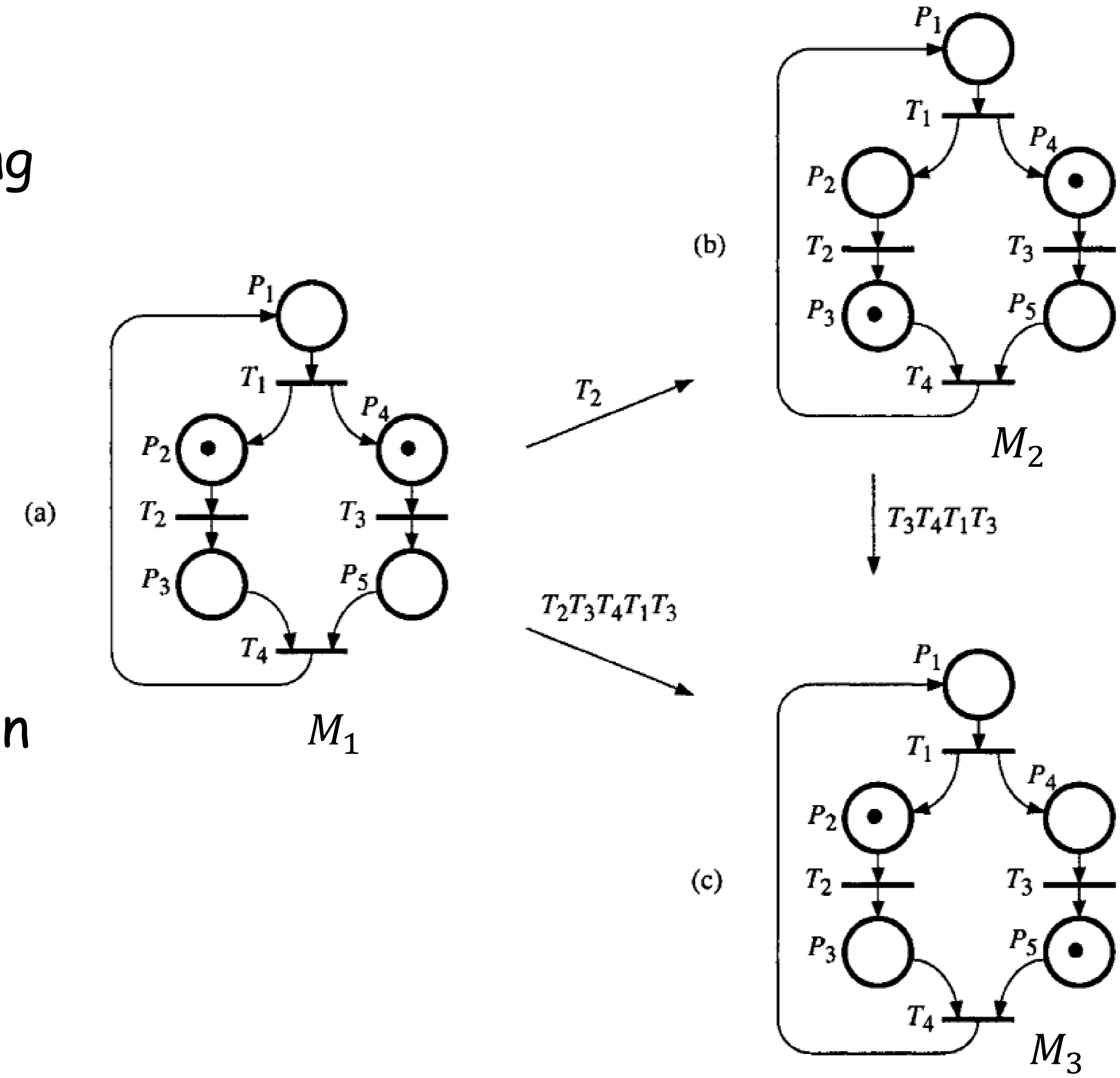
- If we consider the carrying out of the sequence $S_1 \cdot S_2$ from the marking M_1, we have

$$S_1 \cdot S_2 = T_2 T_3 T_4 T_1 T_3$$

and $$\boldsymbol{s_1} + \boldsymbol{s_2} = (1, 1, 2, 1).$$

Then, we have the equation

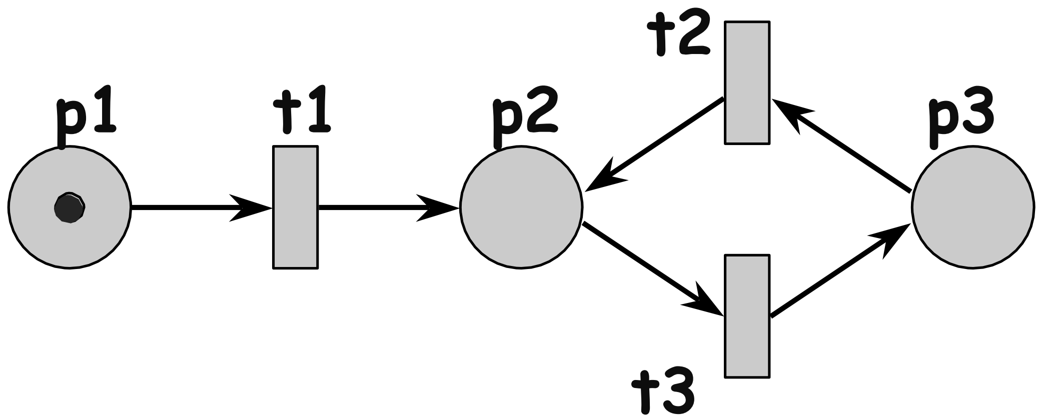
$$M_1 + W \cdot (\boldsymbol{s_1} + \boldsymbol{s_2}) = M_3$$



Incidence Matrix Example

Opening question:

reachability of $M = (0\ \ 0\ \ 1)^T$ from $M_0 = (1\ \ 0\ \ 0)^T$

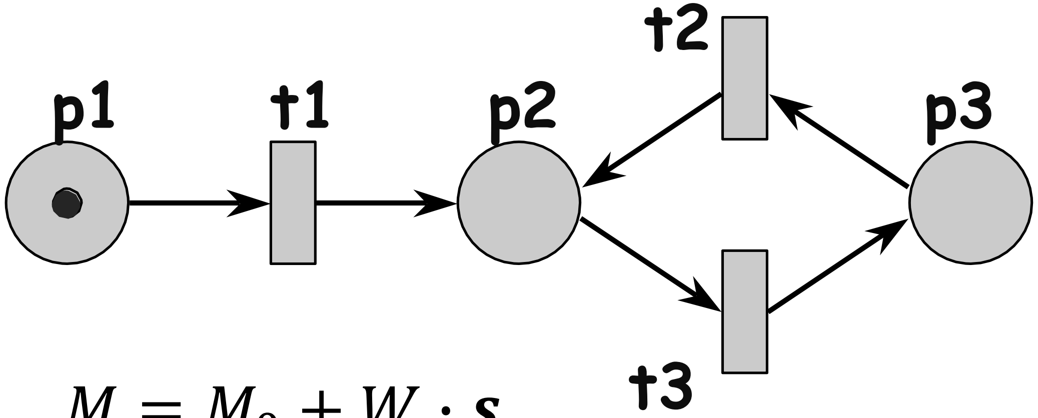


incidence matrix:

$$
\begin{array}{cc}
 & \begin{array}{ccc} t1 & t2 & t3 \end{array} \\
W = & \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\end{array}
\begin{array}{c} p1 \\ p2 \\ p3 \end{array}
$$

Incidence Matrix Example (cont.)

Reachability of $M = (0\ \ 0\ \ 1)^T$ from $M_0 = (1\ \ 0\ \ 0)^T$



$$W = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{cc} \begin{array}{ccc} \textbf{t1} & \textbf{t2} & \textbf{t3} \end{array} & \\ \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} & \begin{array}{c} \textbf{p1} \\ \textbf{p2} \\ \textbf{p3} \end{array} \end{array}$$

$$M = M_0 + W \cdot \boldsymbol{s}$$

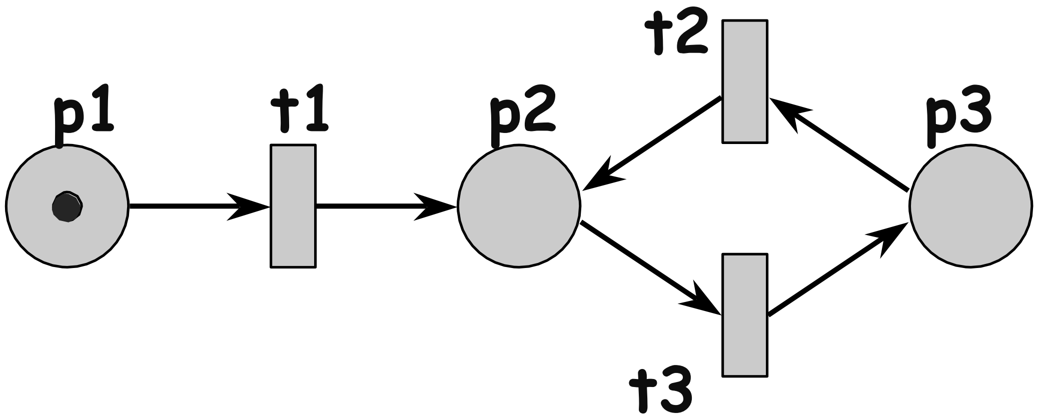
$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \boldsymbol{s}$$

$$\begin{array}{llll} & \textbf{t1} & \textbf{t2} & \textbf{t3} \\ \boldsymbol{s_1} = & [1 & 0 & 1]^T \\ \boldsymbol{s_2} = & [1 & 1 & 2]^T \\ & & \vdots & \\ \boldsymbol{s_k} = & [1 & k-1 & k]^T \end{array}$$

$$\boldsymbol{s_k} = [1 \quad k-1 \quad k]^T$$

Incidence Matrix Example (cont.)

Reachability of $M = (0\ \ 0\ \ 1)^T$ from $M_0 = (1\ \ 0\ \ 0)^T$



$$W = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccc} t1 & t2 & t3 & \\ \left[\begin{array}{c} -1 \\ 1 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 1 \\ -1 \end{array} & \left.\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right] & \begin{array}{c} p1 \\ p2 \\ p3 \end{array} \end{array}$$

$$M = M_0 + W \cdot \boldsymbol{s}$$

$$\boxed{\boldsymbol{s_k} = [1 \quad k-1 \quad k]^T}$$

P-invariants

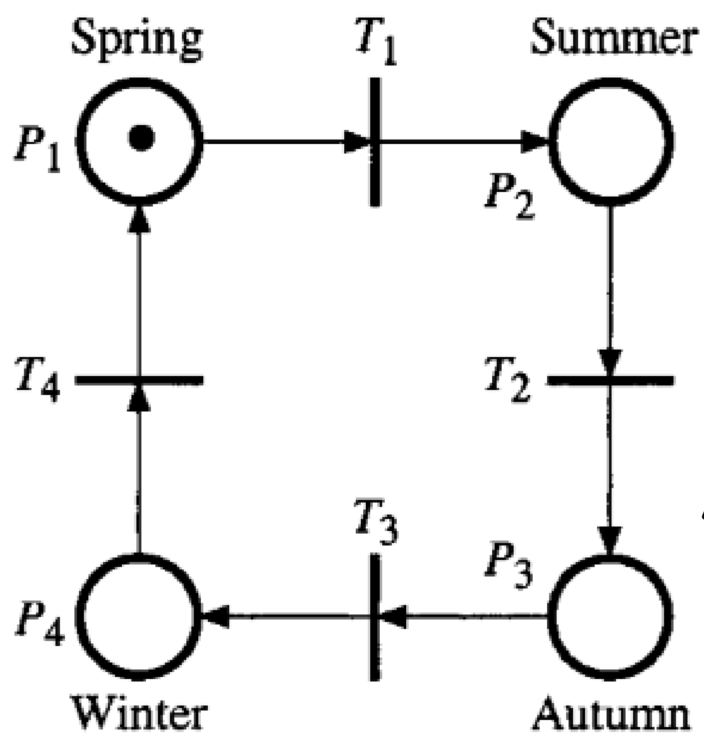
- Let N be a net and W be its incidence matrix.
  - A natural solution of the equation $\boldsymbol{x}^T \cdot W = 0$ such that $\boldsymbol{x} \geq \boldsymbol{0}$ is called a **place invariant** (or **P-invariant**) of N.
  - A P-invariant indicates that the number of tokens in all reachable markings satisfies some linear invariant.

- Let M be marking reachable with a transition sequence whose firing count is expressed by $\boldsymbol{s}$, i.e., $M = M_0 + W \cdot \boldsymbol{s}$.
  - Let $\boldsymbol{x}$ be a P-invariant. Then, the following holds:

$$\boldsymbol{x}^T \cdot M = \boldsymbol{x}^T \cdot (M_0 + W \cdot \boldsymbol{s}) = \boldsymbol{x}^T \cdot M_0 + \boldsymbol{x}^T \cdot W \cdot \boldsymbol{s} = \boldsymbol{x}^T \cdot M_0$$

P-invariants (cont.)

Example



- At all times, there will always be one and only one token for all places.

$$m_1 + m_2 + m_3 + m_4 = 1$$

At all times, we are at one and only one season.

How to prove?

P-invariants (cont.)

- Let $\boldsymbol{x}^T = (x_1, x_2, x_3, x_4)$

$$\boldsymbol{x}^T \cdot W = (x_1, x_2, x_3, x_4)\begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = 0 \Rightarrow \boldsymbol{x}^T = (1,1,1,1)$$

- With the initial marking $m_0 = (1,0,0,0)$,
- The constant of marking invariant is

$$\boldsymbol{x}^T \cdot M_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\Downarrow$$

$$\boldsymbol{x}^T \cdot M_k = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix} = \boldsymbol{x}^T \cdot m_0 = 1$$

The marking invariant is:

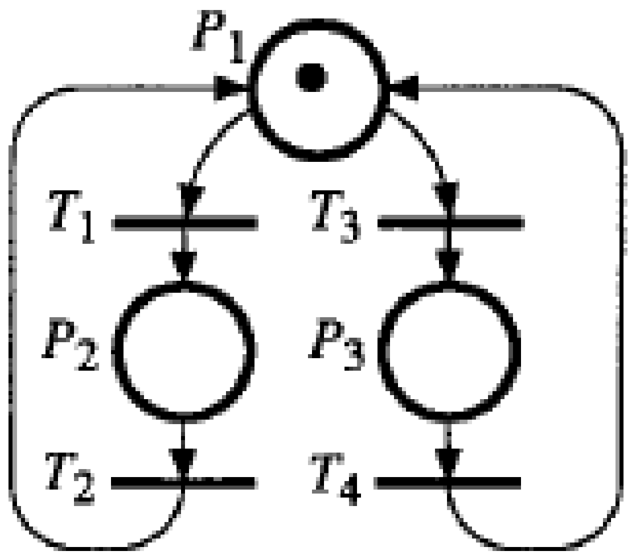
$$m_1 + m_2 + m_3 + m_4 = 1$$

T-invariants

- Let N be a net and W be its incidence matrix.
  - A natural solution of the equation $\boldsymbol{W} \cdot \boldsymbol{y} = \mathbf{0}$ $(\boldsymbol{y} \neq \mathbf{0})$ is known as a transition invariant (or **T-invariant**) of N.

Exercise 4

Please calculate the T-invariants of the following Petri net.



Siphons and Traps

Let $P' = \{P_1, \ldots P_r\}$ be a set of places of a PN.

- ${}^\circ P'$:the set of input transitions of the places of P' ,
- P'° : the set of output transitions of the places of P'.

That is to say,

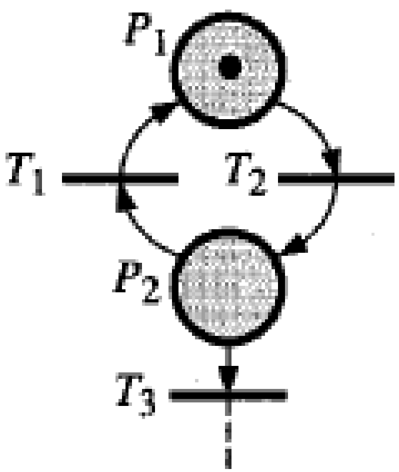
$${}^\circ P' = {}^\circ P_1 \cup \cdots \cup {}^\circ P_r \text{ and } P'^\circ = P_1^\circ \cup \cdots \cup P_r^\circ$$

- A **siphon** is a set of places P' such that the set of input transitions of P' is included in the set of output transitions of P', i.e., ${}^\circ P' \subseteq P'^\circ$.
- A **trap** is a set of places P' such that the set of output transitions of P' is included in the set of input transitions of P', i.e., ${}^\circ P' \supseteq P'^\circ$.

Siphons example

- *A* **siphon** is a set of places P' such that the set of input transitions of P' is included in the set of output transitions of P', i.e., ${}^{\circ}P' \subseteq P'^{\circ}$.

Siphon



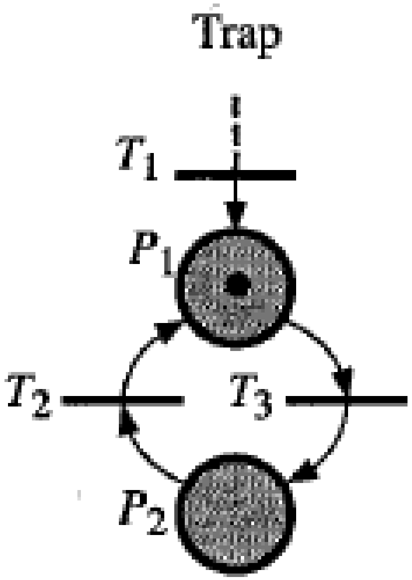
$P' = \{P_1, P_2\}$

${}^{\circ}P' = \{T_1, T_2\}$ and $P'^{\circ} = \{T_1, T_2, T_3\}$

We have ${}^{\circ}P' \subseteq P'^{\circ}$

Traps example

- A **trap** is a set of places P' such that the set of output transitions of P' is included in the set of input transitions of P', i.e., $^{\circ}P' \supseteq P'^{\circ}$.



$P' = \{P_1, P_2\}$

$^{\circ}P' = \{T_1, T_2, T_3\}$ and $P'^{\circ} = \{T_2, T_3\}$

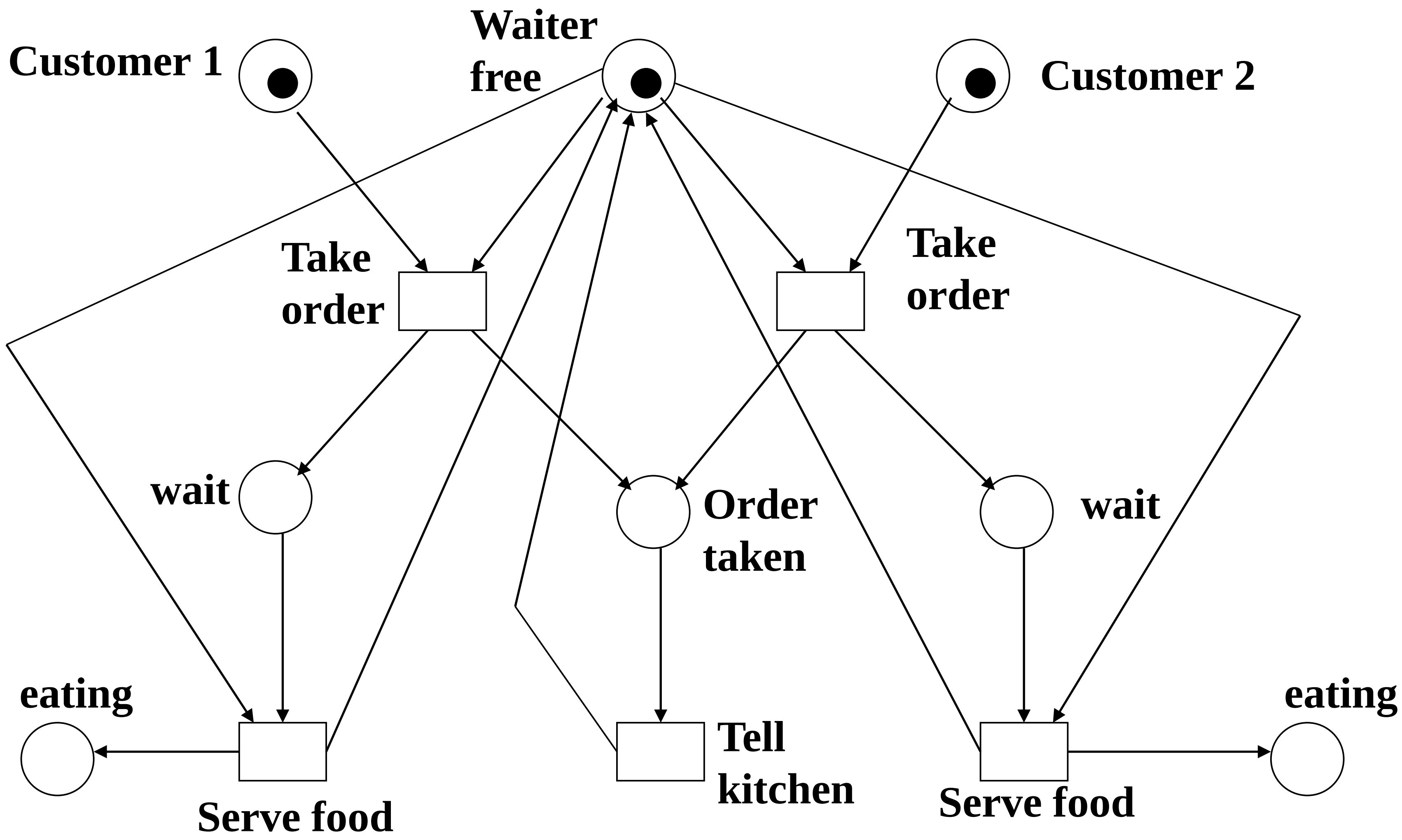
We have $^{\circ}P' \supseteq P'^{\circ}$

Modelling with Petri Nets

Petri nets were designed for and are used mainly for *modelling*. Many systems, especially those with independent components, can be modelled by a Petri net.

The simple Petri net view of a system concentrates on two primitive concepts: *events (transitions)* and *conditions (places)*. Events are actions which take place in the system. The occurrence of these events is controlled by the state of the system. The state of the system can be described as a set of conditions. A condition is a predicate or logical description of the state of the system.

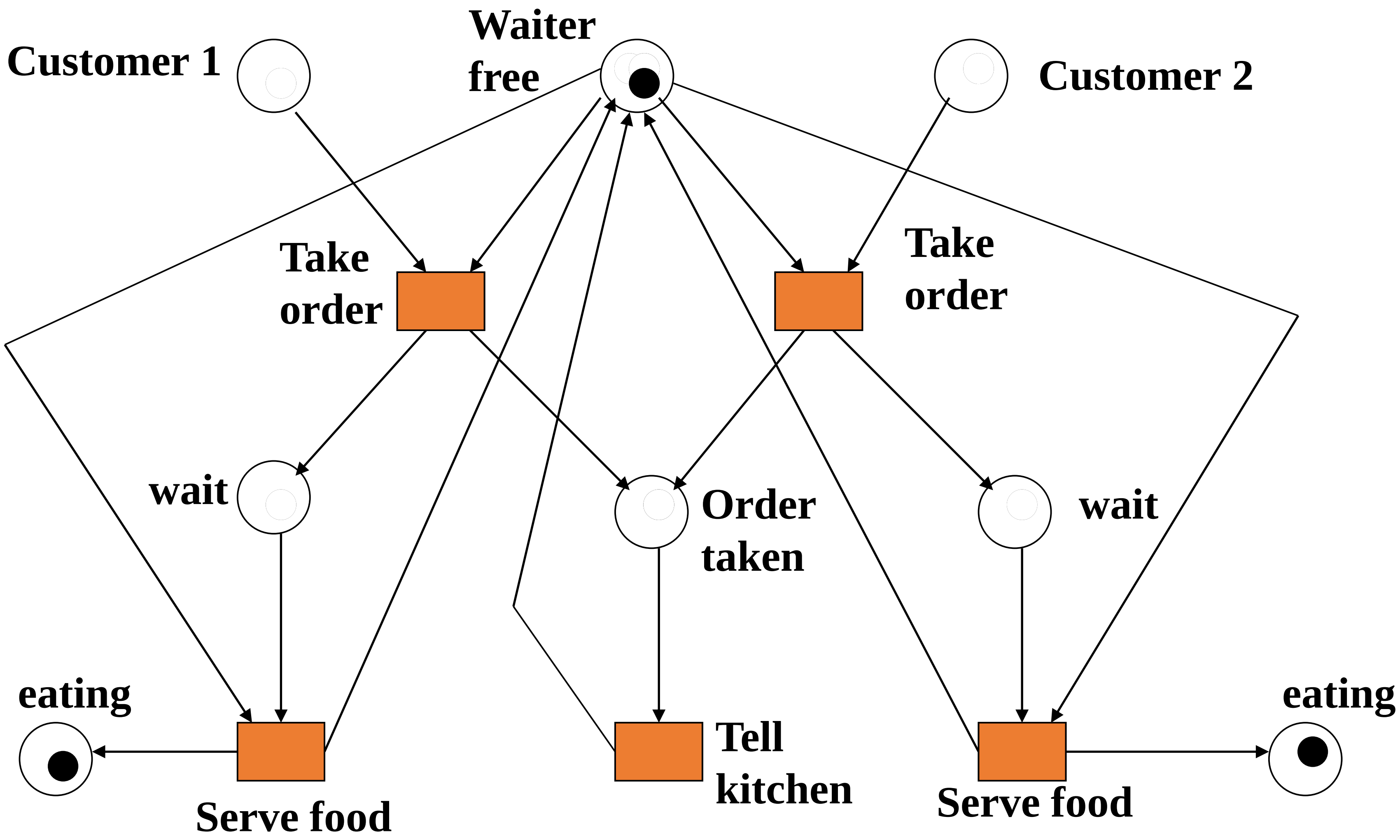
Example: In a Restaurant (A Petri Net)



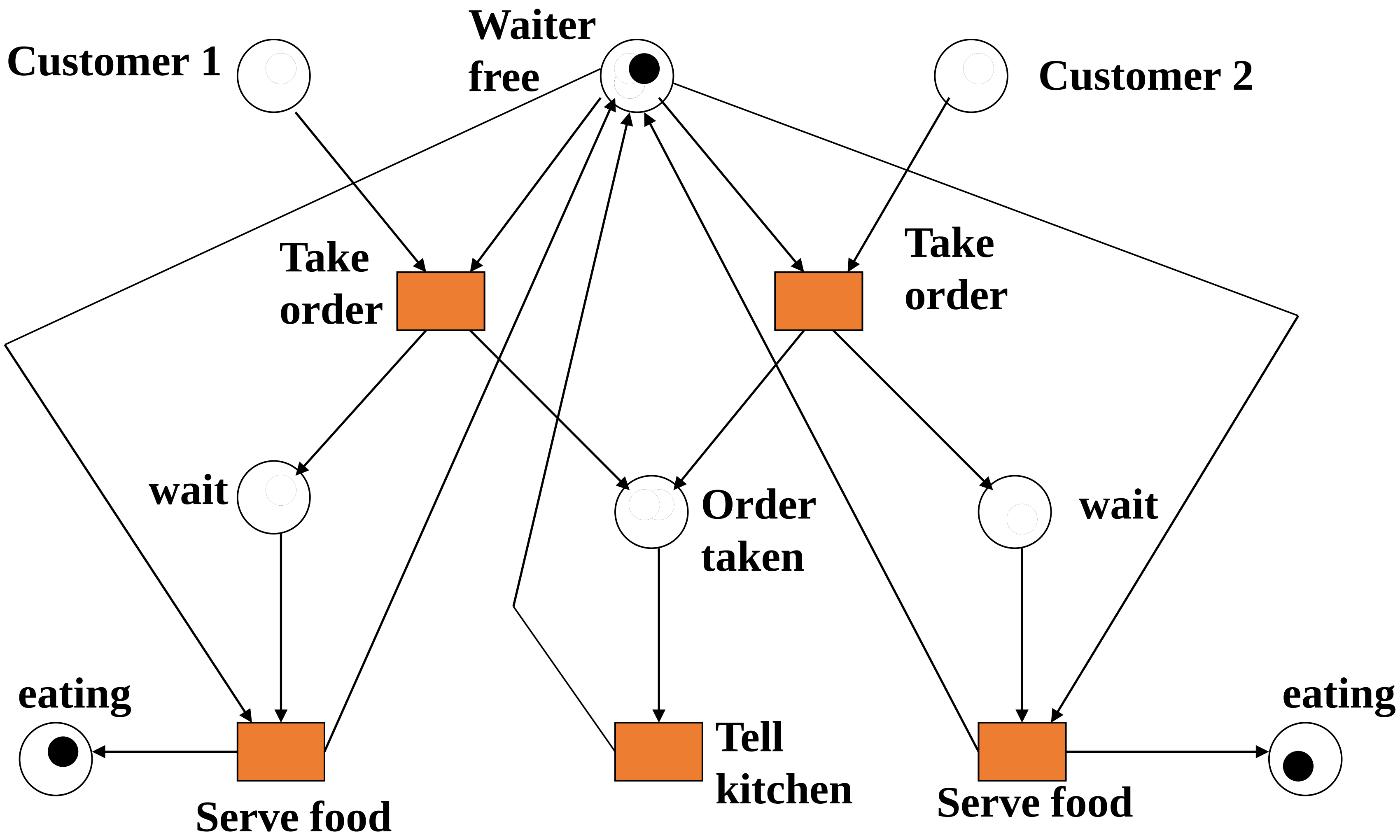
Example: In a Restaurant (Two Scenarios)

- Scenario 1:
  - Waiter takes order from customer 1; serves customer 1; takes order from customer 2; serves customer 2.
- Scenario 2:
  - Waiter takes order from customer 1; takes order from customer 2; serves customer 2; serves customer 1.

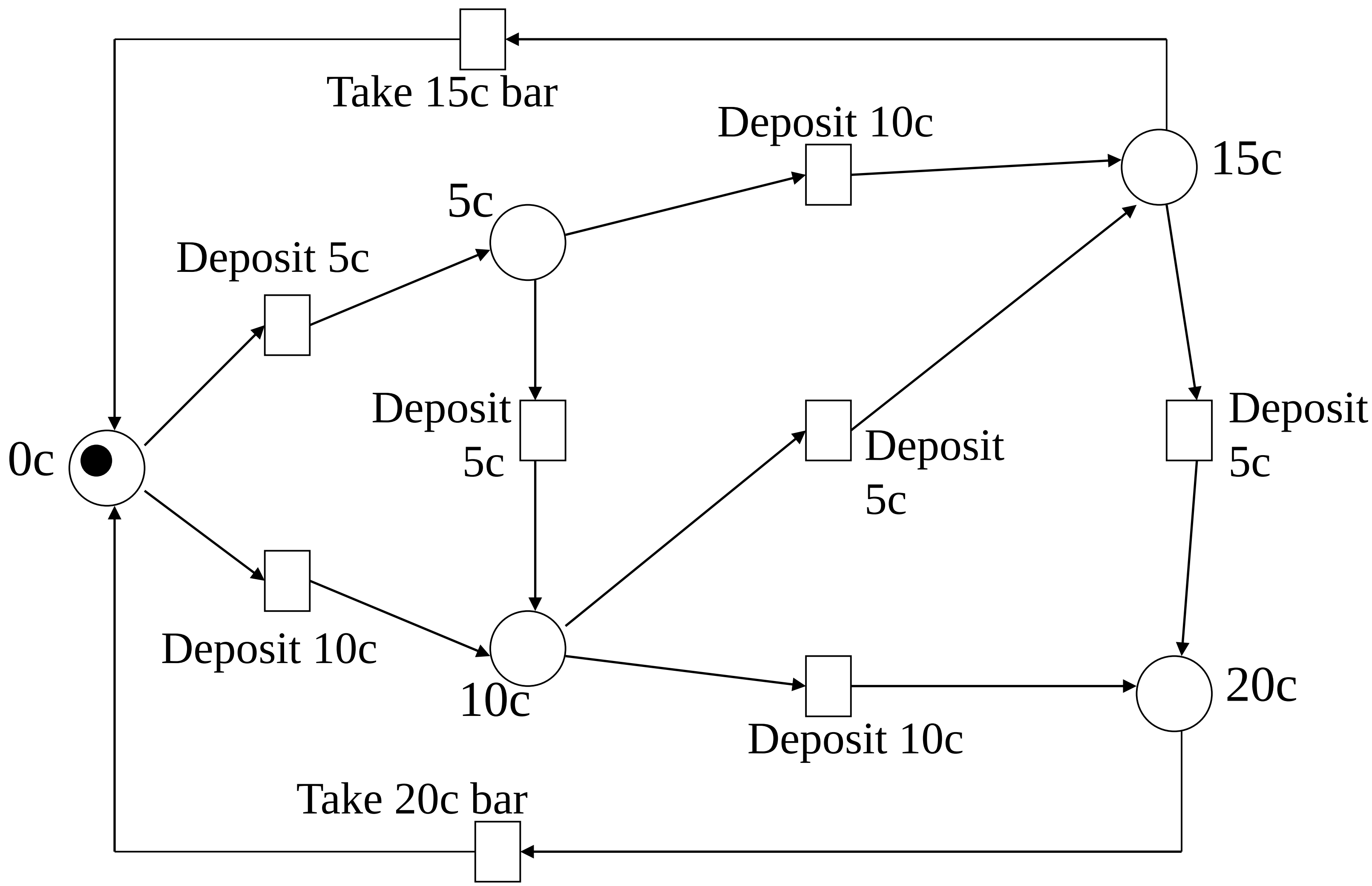
Example: In a Restaurant (Scenario 1)



Example: In a Restaurant (Scenario 2)



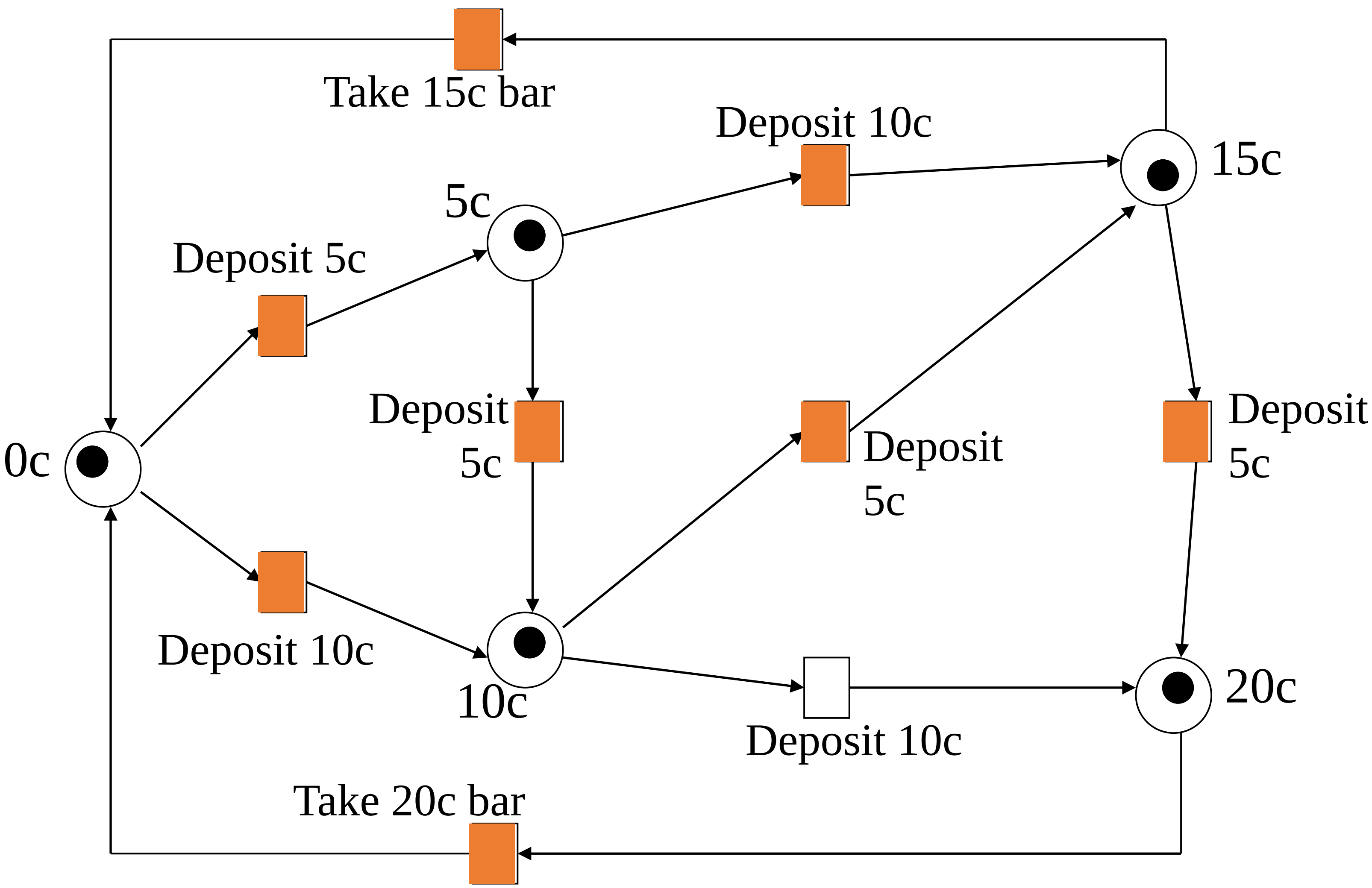
Example: Vending Machine (A Petri net)



Example: Vending Machine (3 Scenarios)

- Scenario 1:
  - Deposit 5c, deposit 5c, deposit 5c, deposit 5c, take 20c snack bar.
- Scenario 2:
  - Deposit 10c, deposit 5c, take 15c snack bar.
- Scenario 3:
  - Deposit 5c, deposit 10c, deposit 5c, take 20c snack bar.

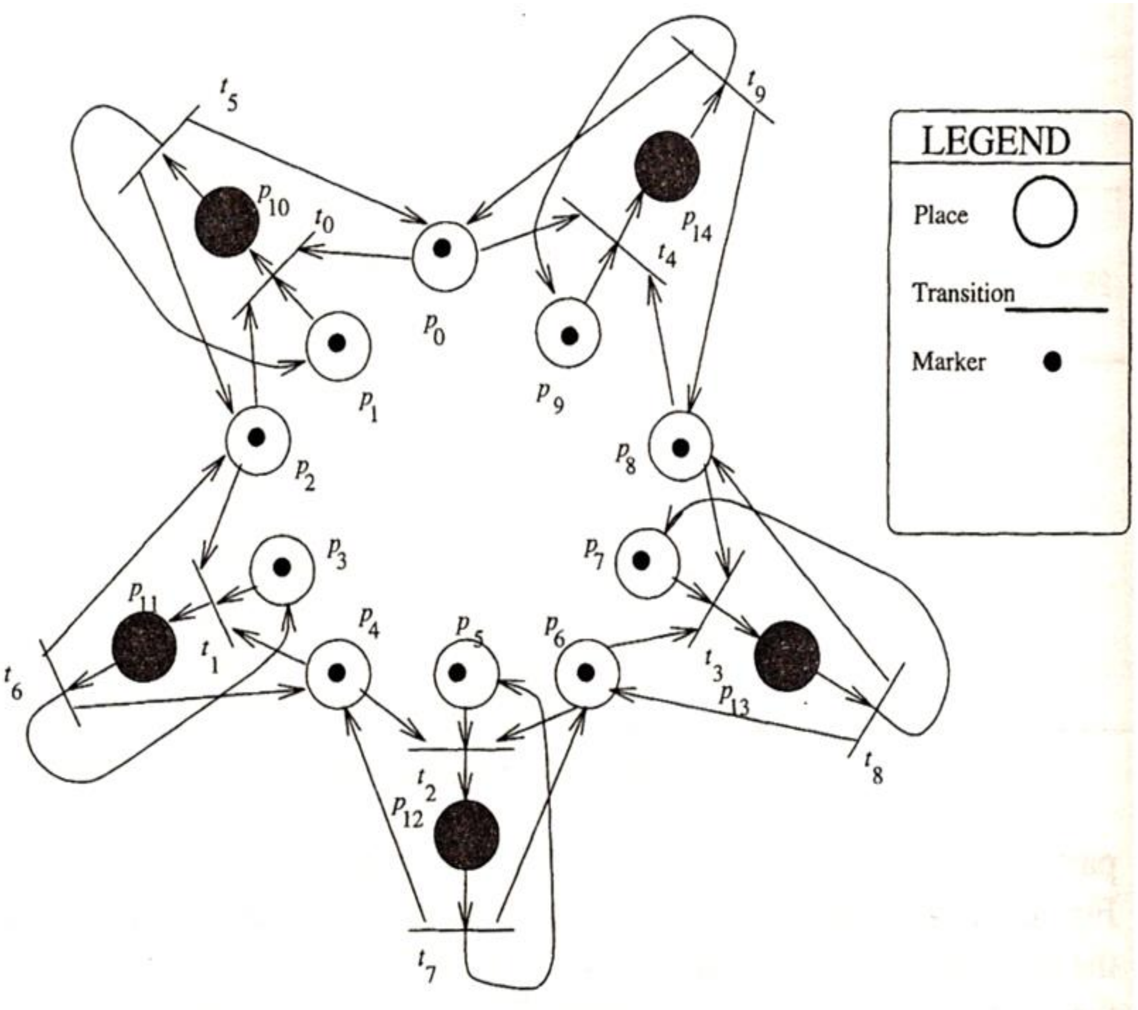
Example: Vending Machine (Token Games)



Concurrent modelling example
Dining Philosophers Problem

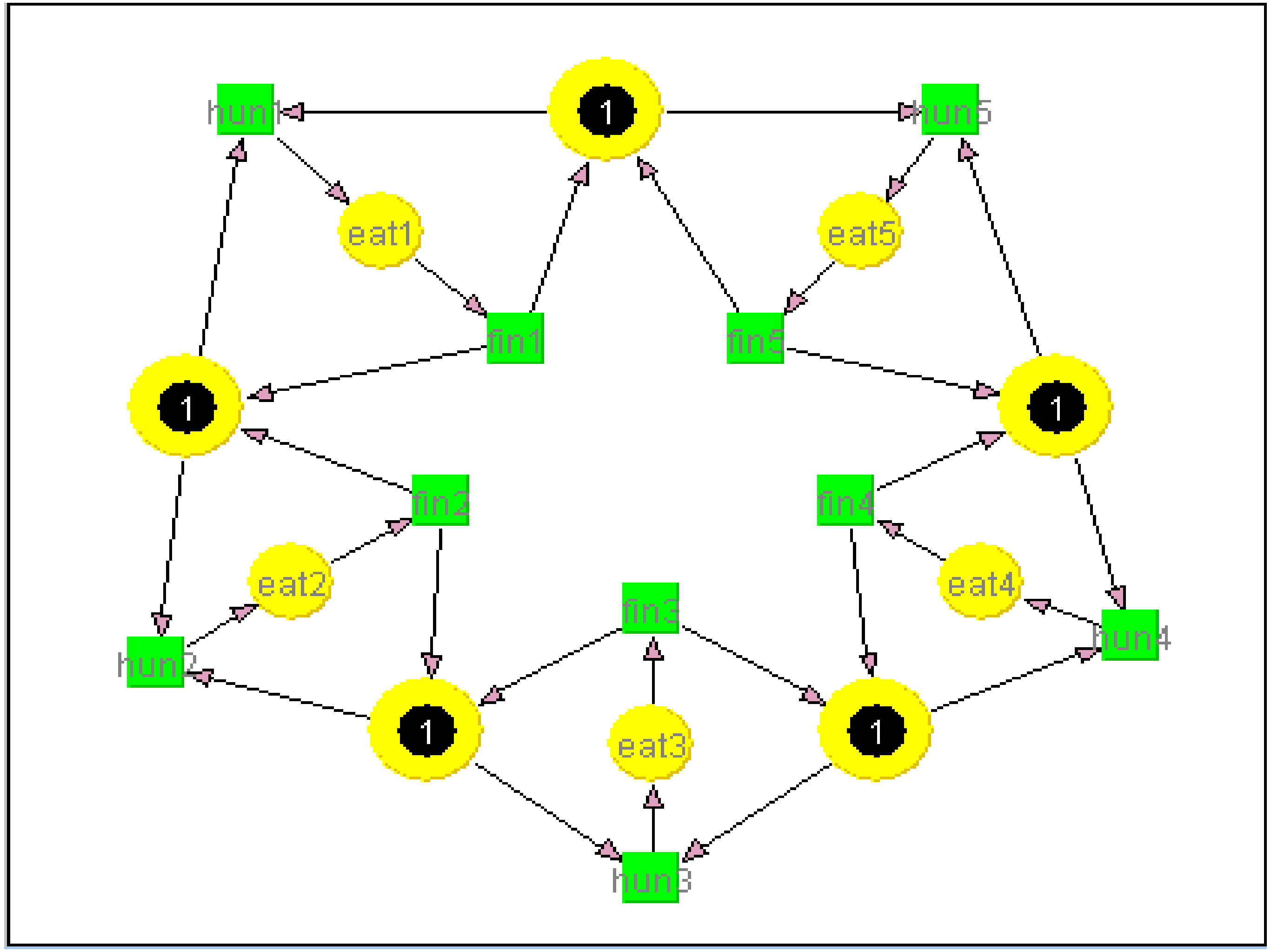
- The dining philosophers problem is summarized as five philosophers sitting at a table doing one of two things: *eating or thinking*. *While eating, they are not thinking, and while thinking, they are not eating*. The five philosophers sit at a circular table with a large bowl of noodle in the centre. A fork is placed in between each pair of adjacent philosophers, and as such, each philosopher has one fork to his left and one fork to his right. As noodle is difficult to serve and eat with a single fork, it is assumed that *a philosopher must eat with two forks*. Each philosopher can only use the forks on his immediate left and immediate right.

Petri Net examples (Dining Philosophers)

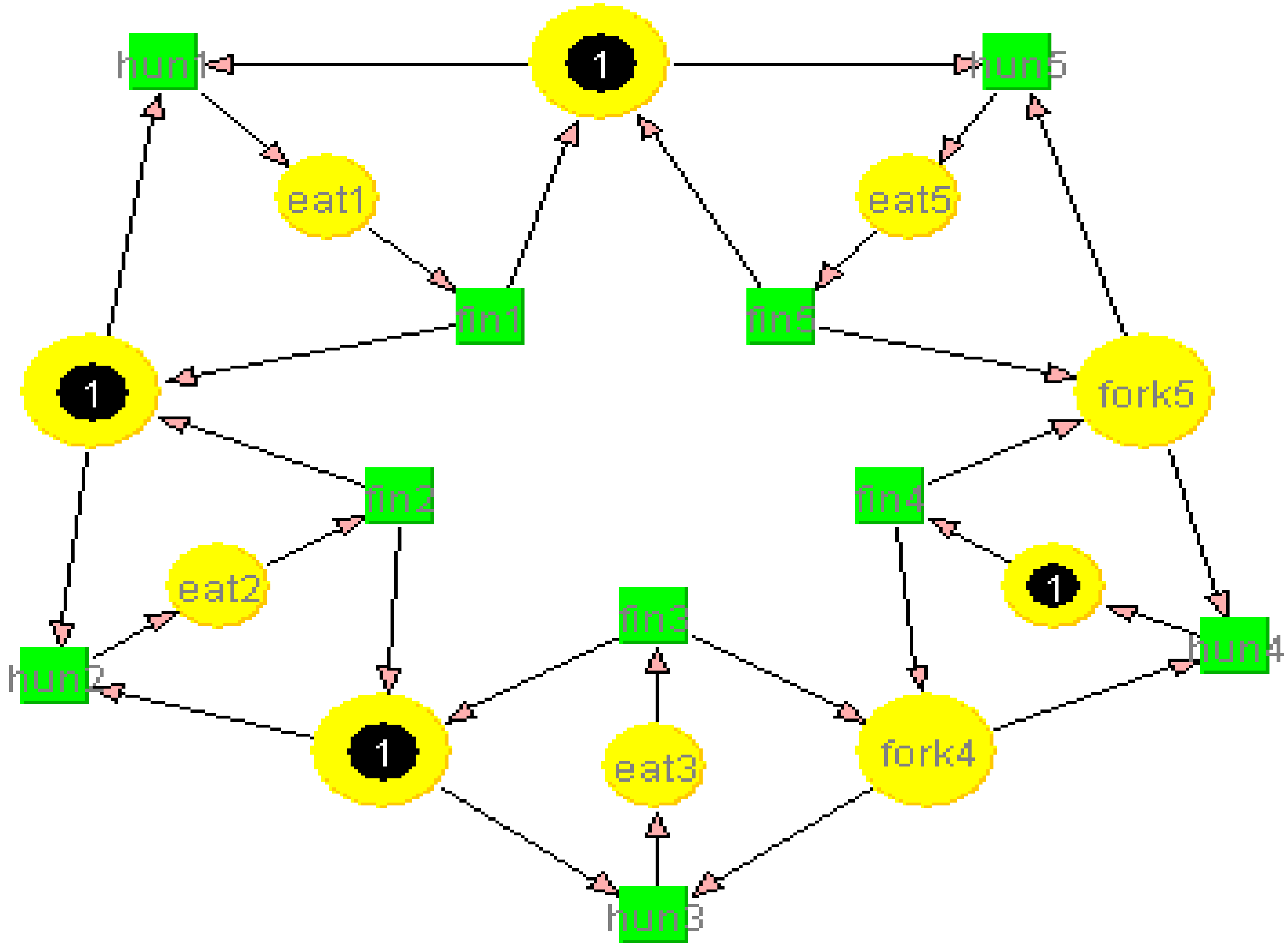


- Five philosophers alternatively think and eating
- forks:
  p_0, p_2, p_4, p_6, p_8
- Philosophers eating:
  p_{10}, p_{11}, p_{12}, p_{13}, p_{14}
- Philosophers thinking/meditating:
  p_1, p_3, p_5, p_7, p_9

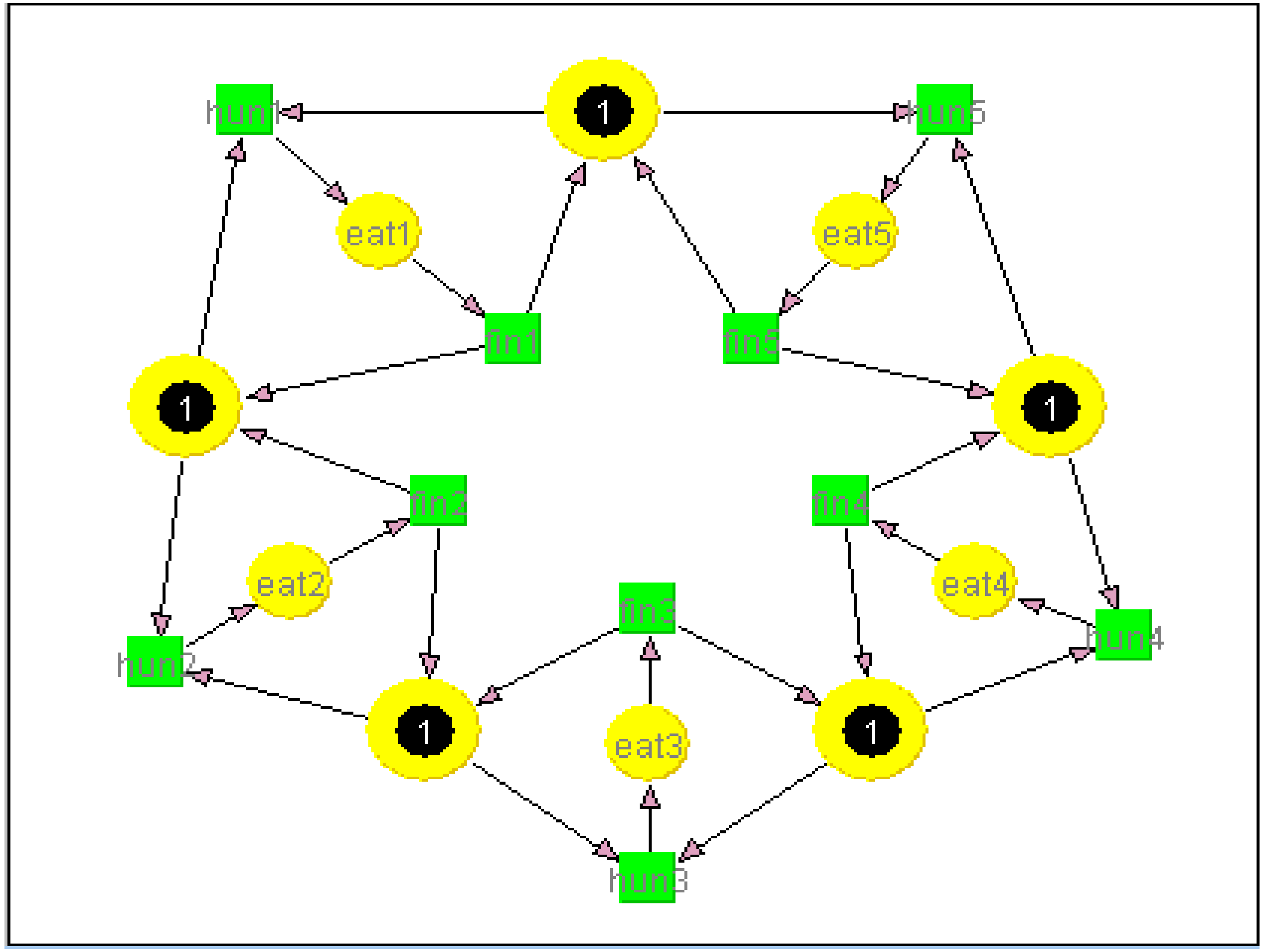
The Petri net structure



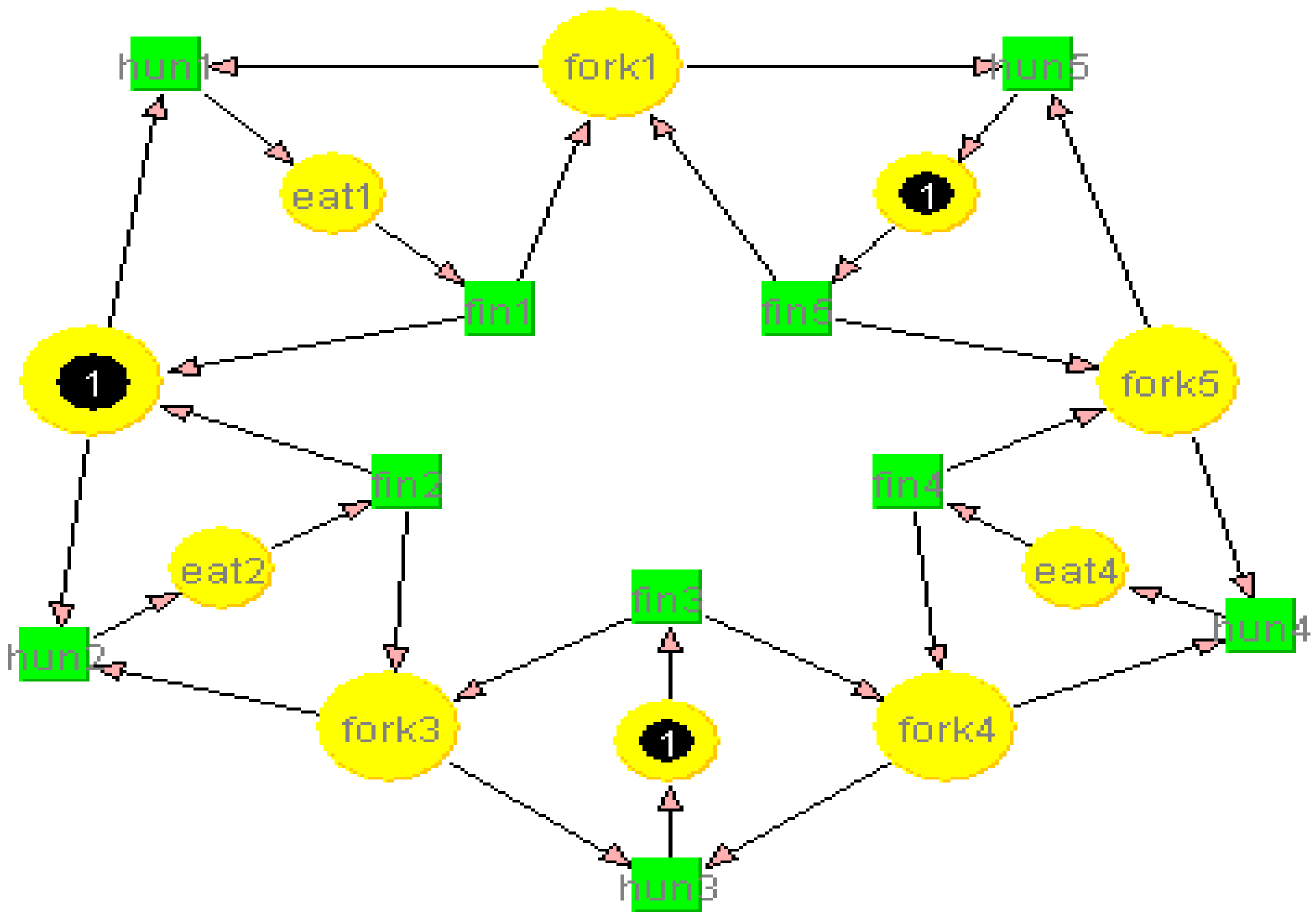
Suppose eater 4 picks up forks first so the transition hun4 fires. The state of Petri net as follows:



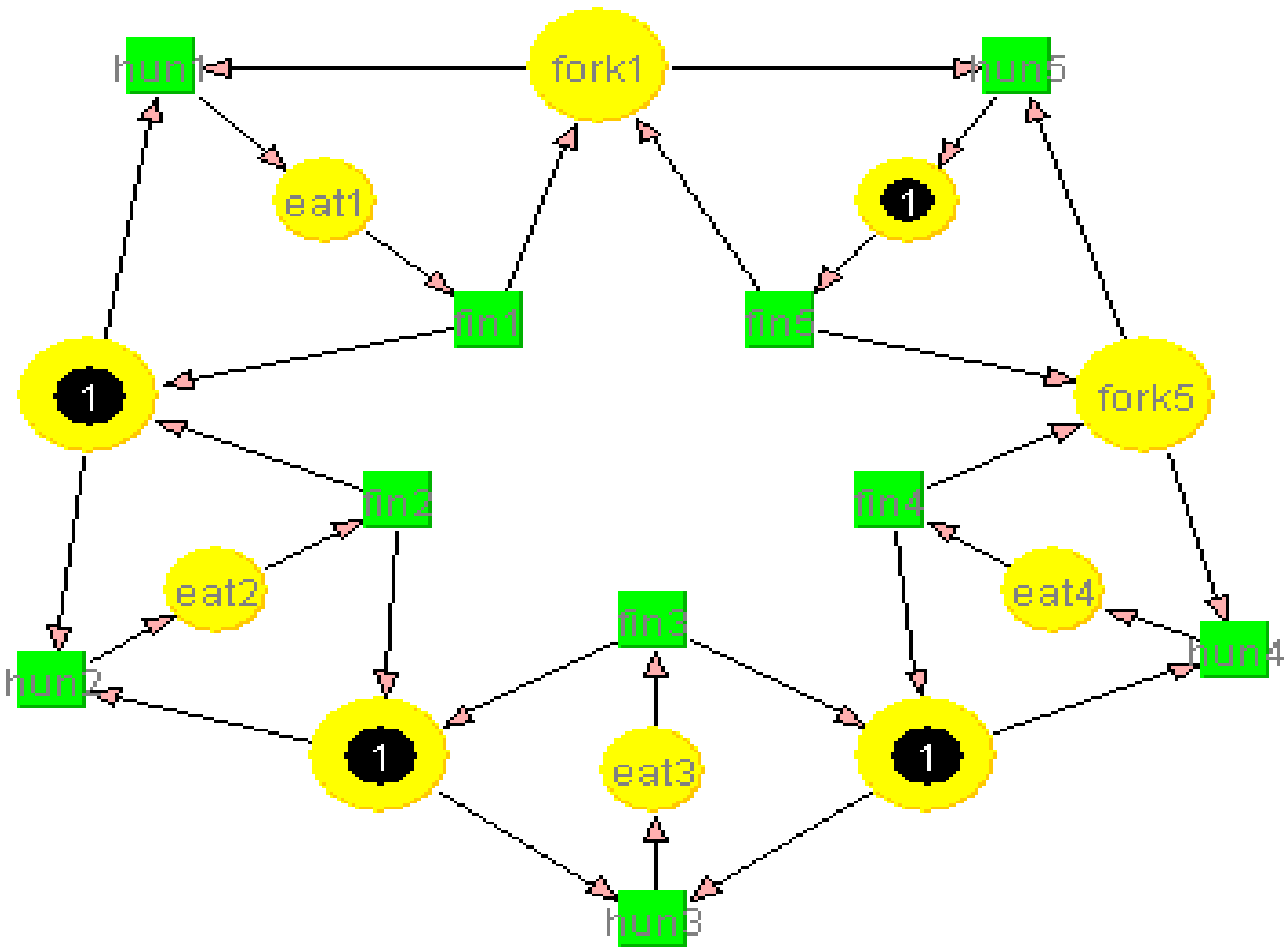
After finishing eating, transition fin4 fires.
The state of Petri net is:



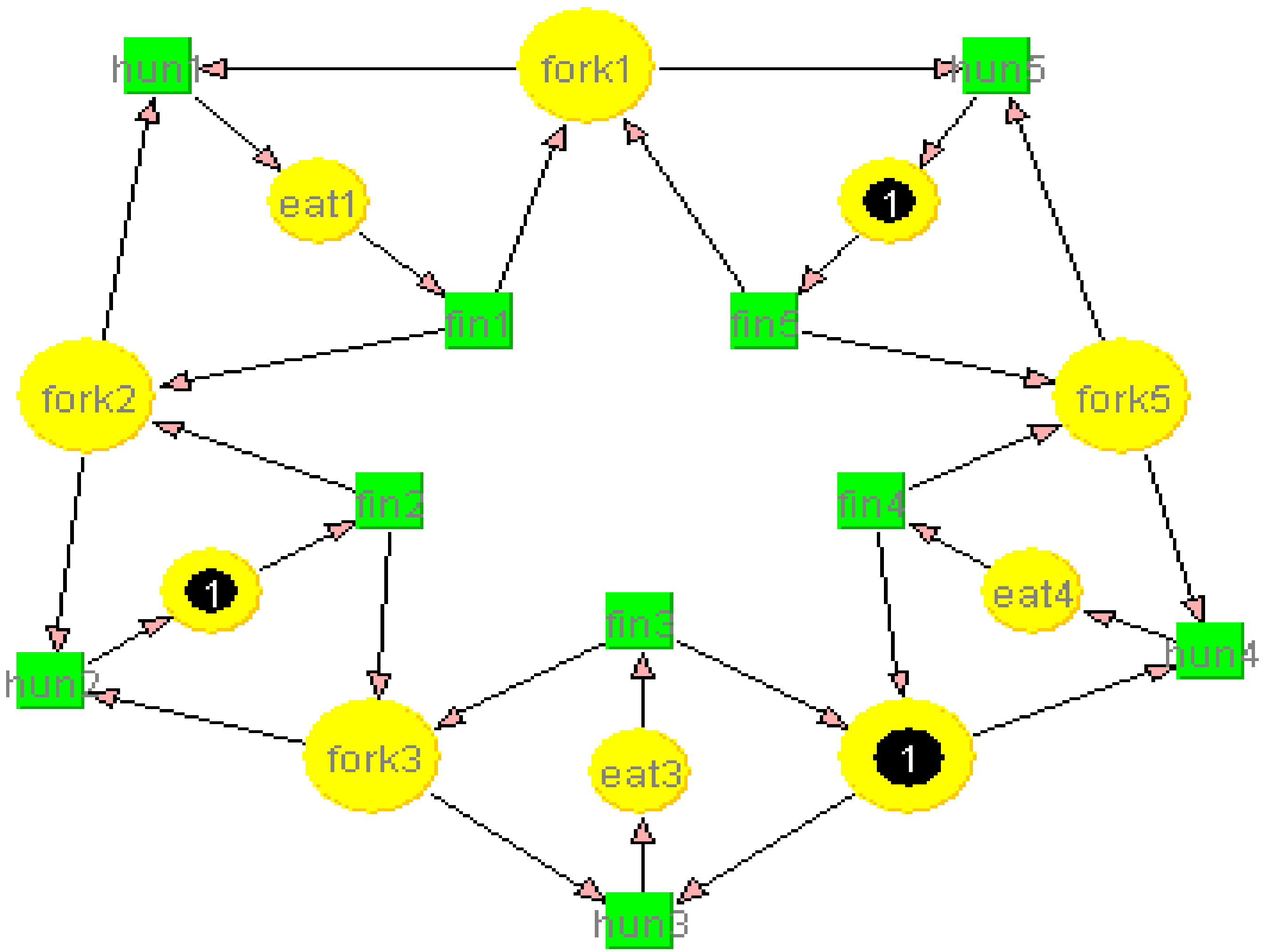
Suppose that eater 3 and eater 5 pick up the forks in the same time so the transition hun3 and hun5 fire in the same time. The state of Petri net became as:



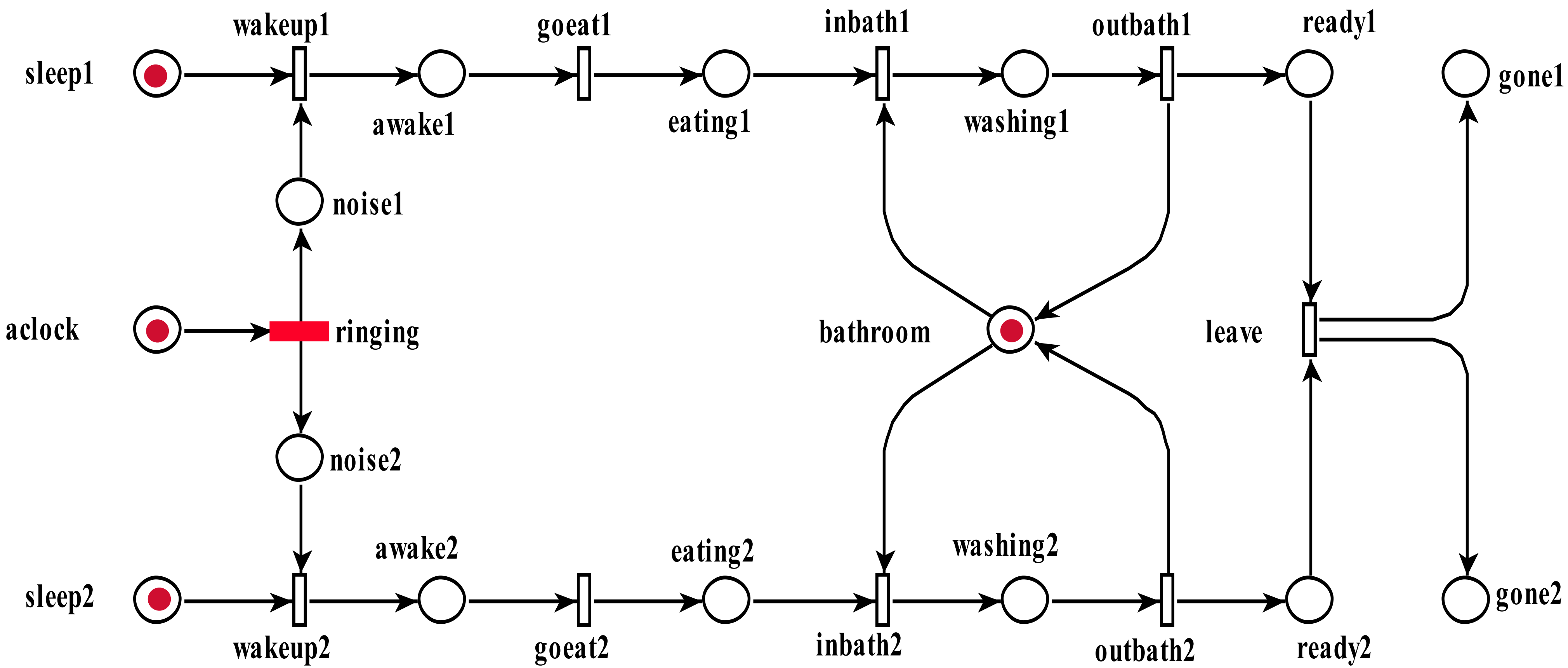
Suppose that eater 3 finished his eating first, so fin3 fires. The state of the Petri net becomes:



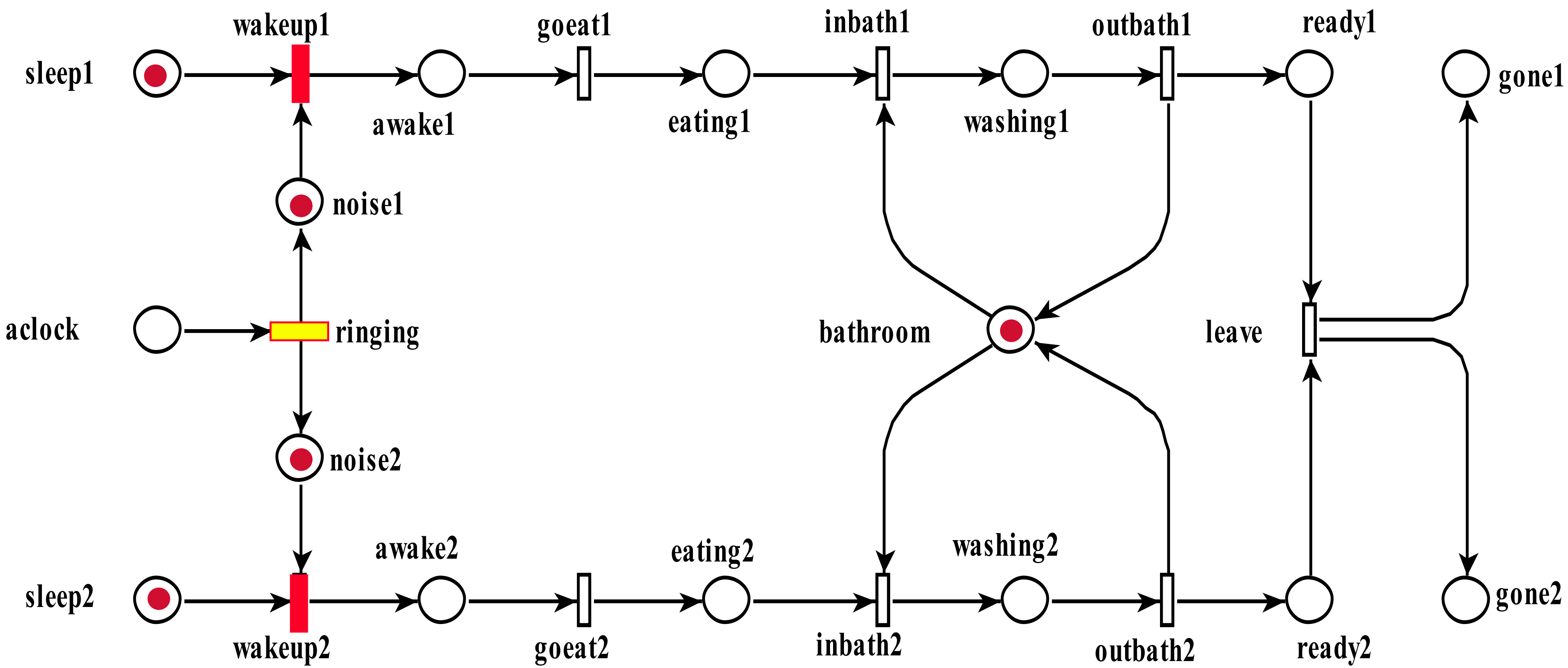
Eater 2 has two forks available. Transition hun2 fires. The state of the Petri net changed to the follows:



Two people waking up



Two people waking up



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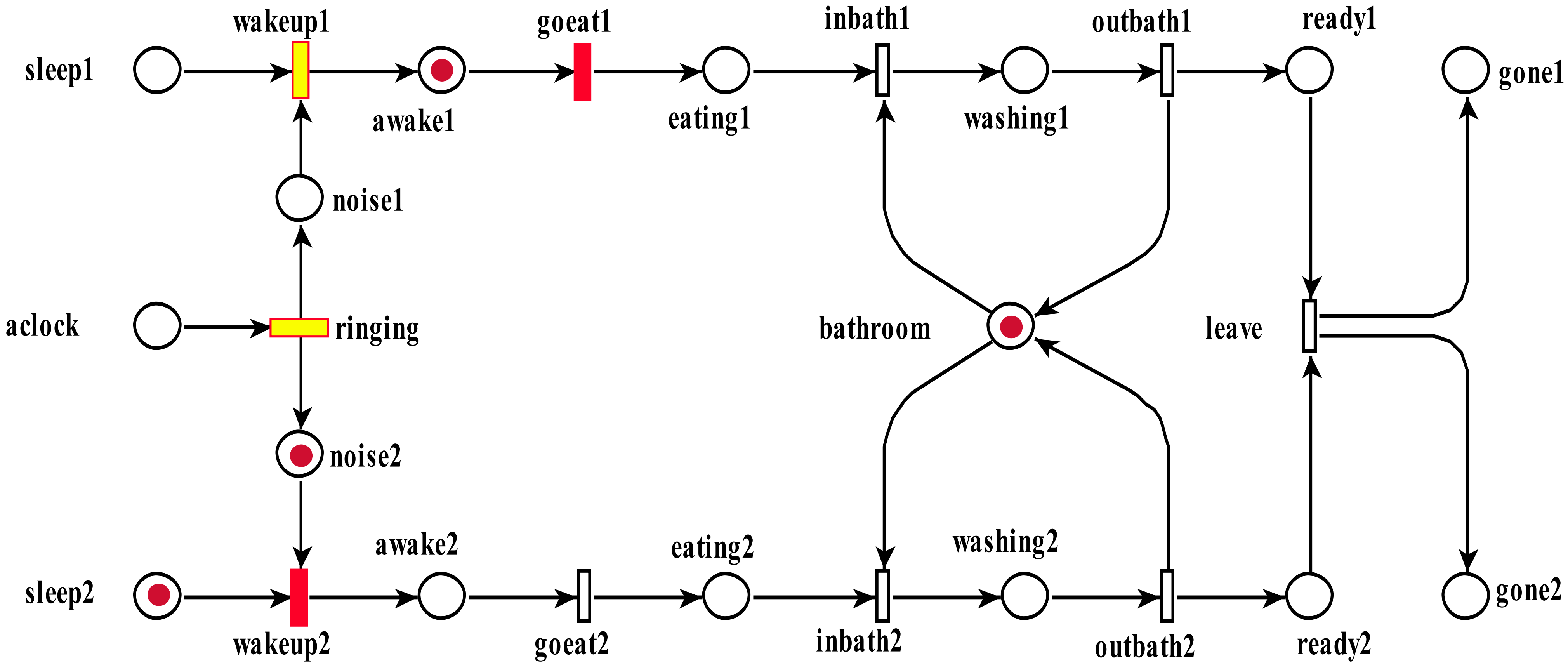
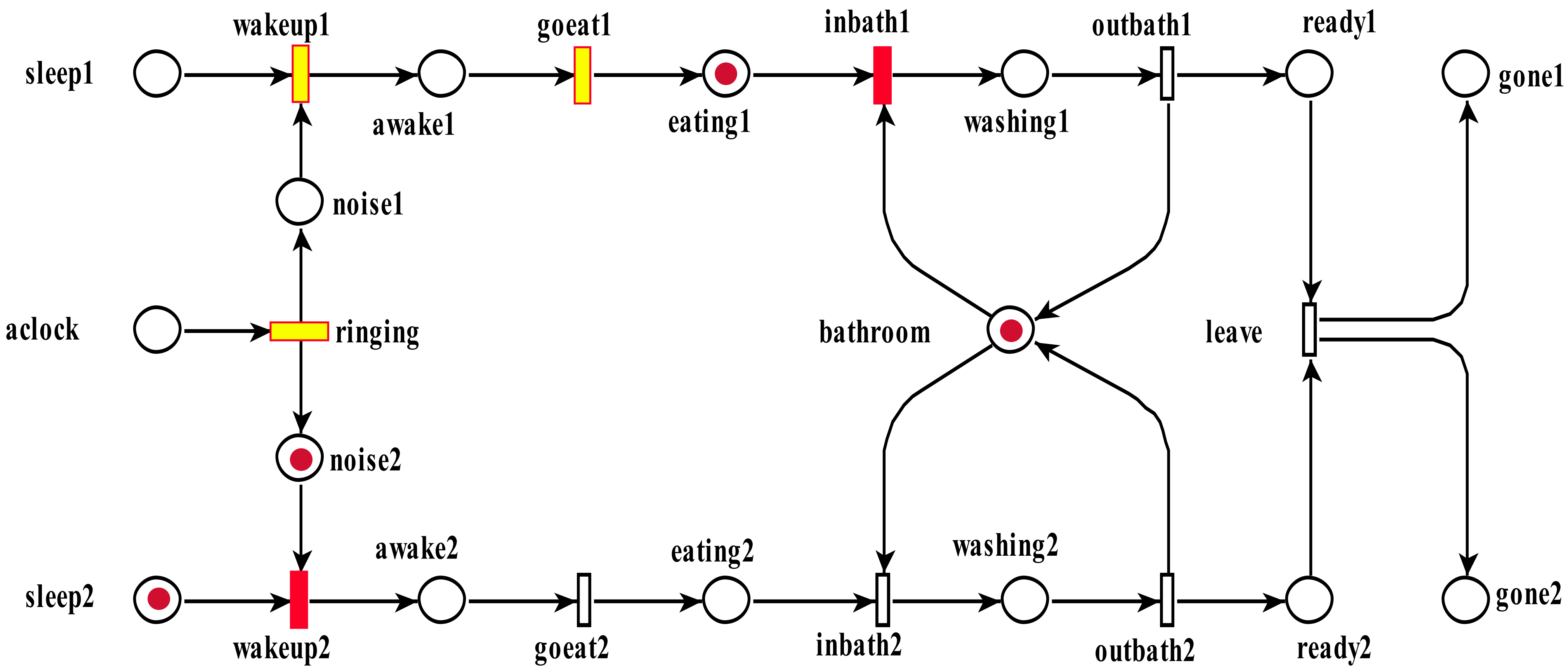
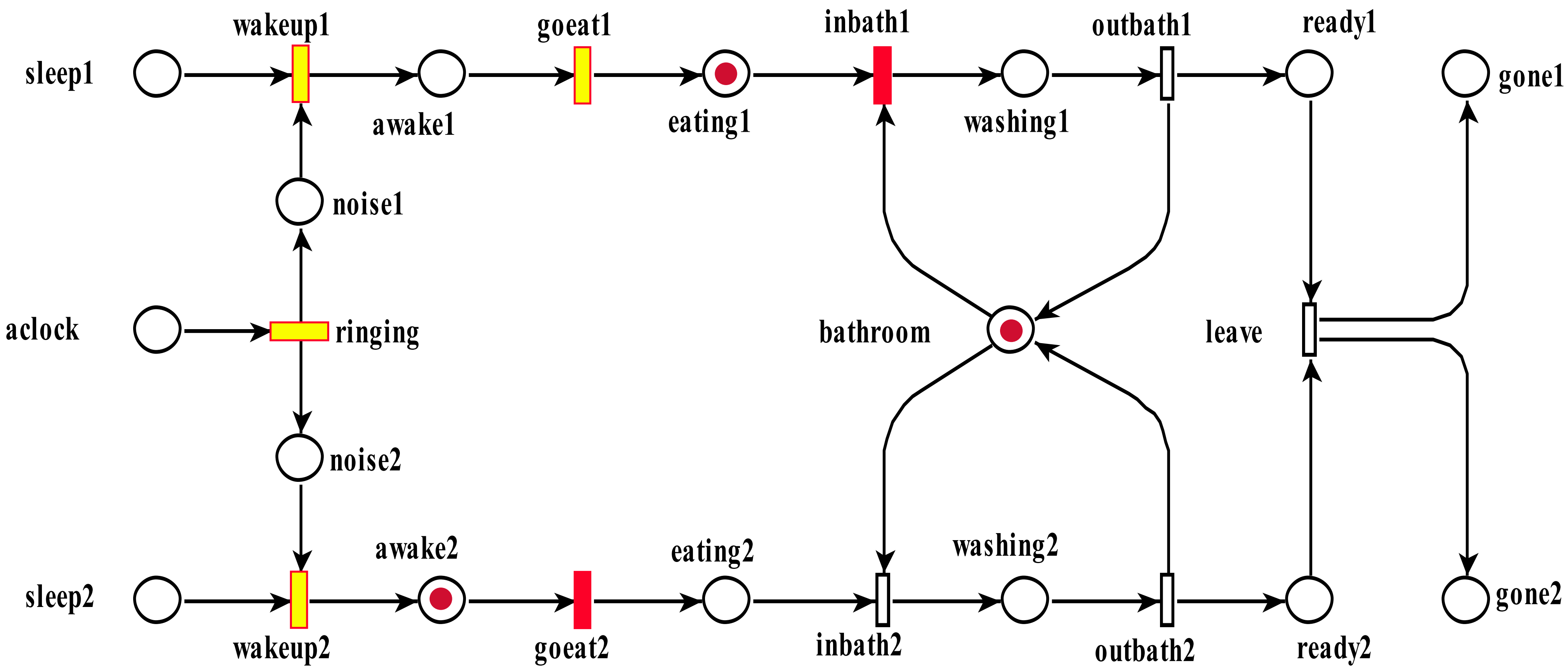


Two people waking up



Two people waking up



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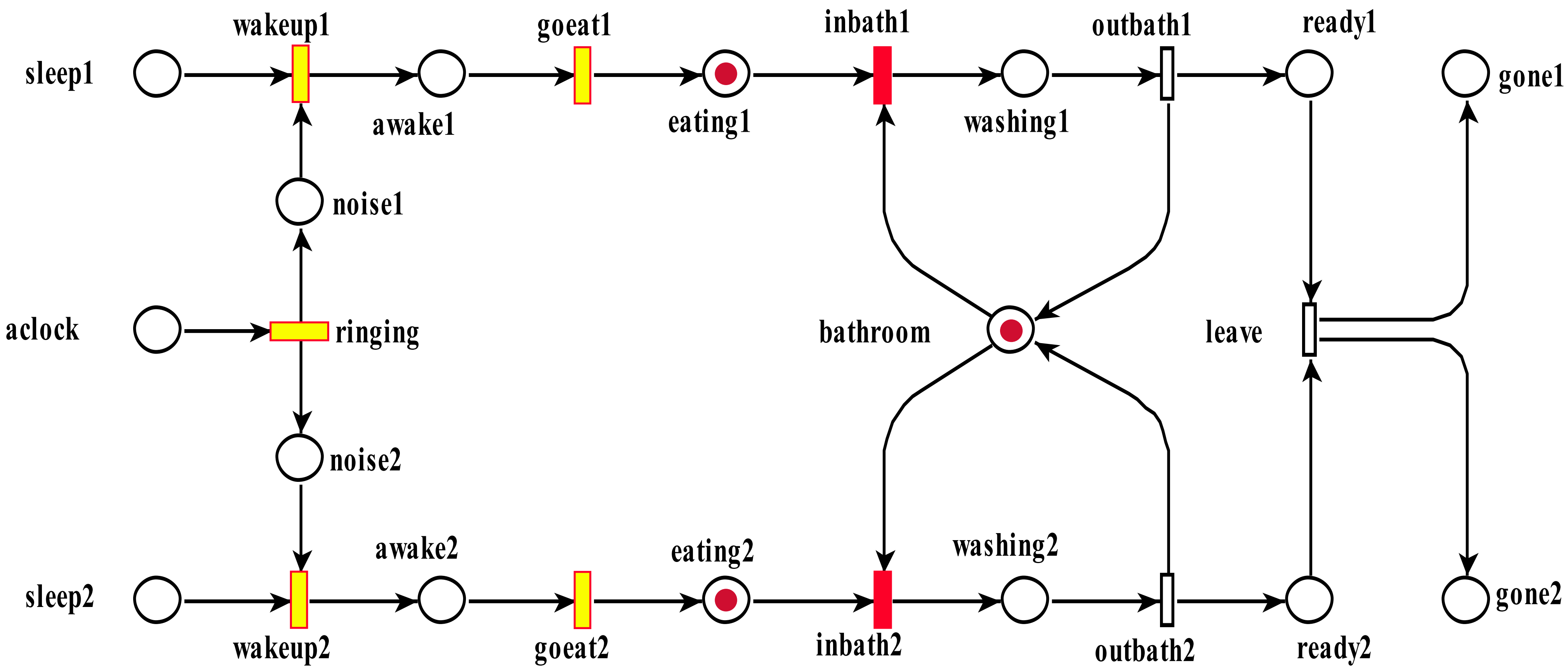

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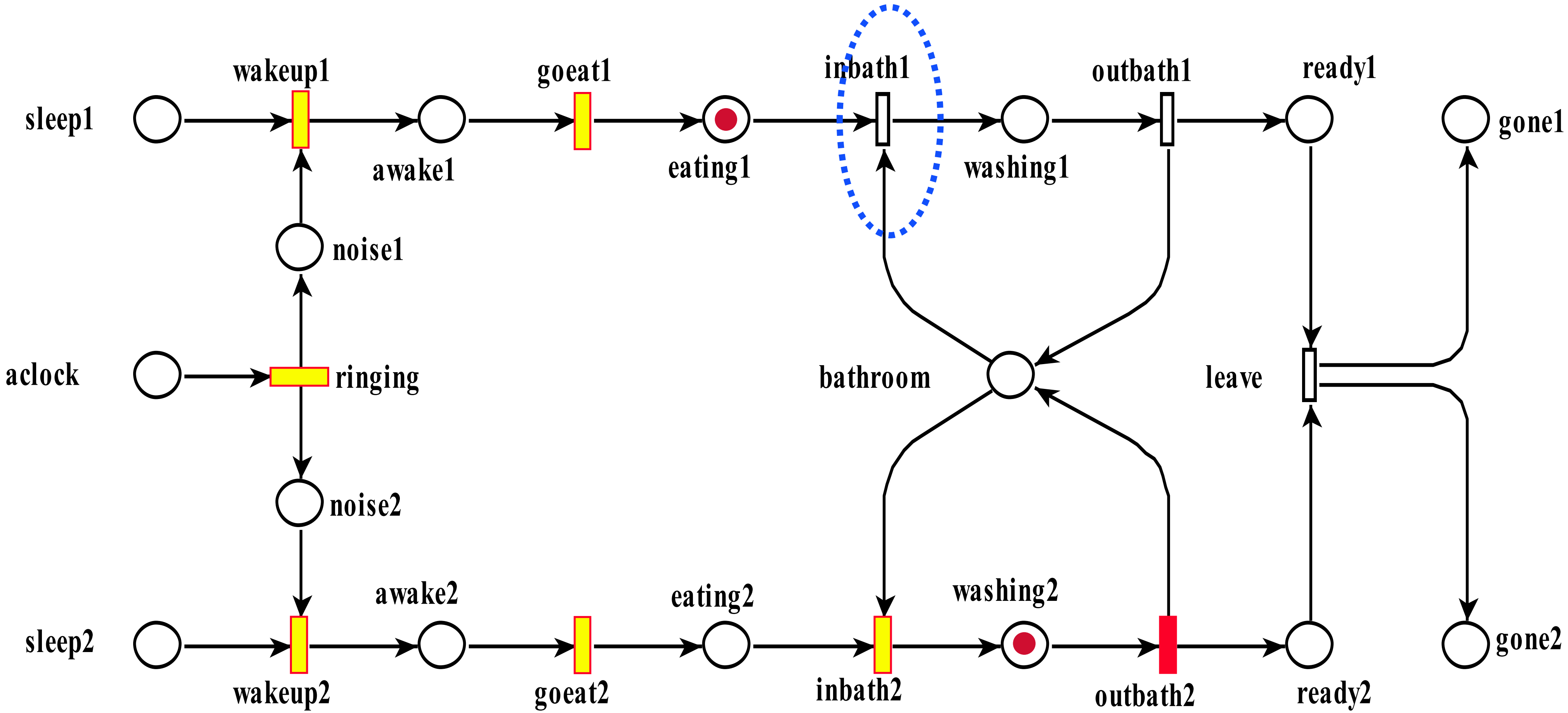

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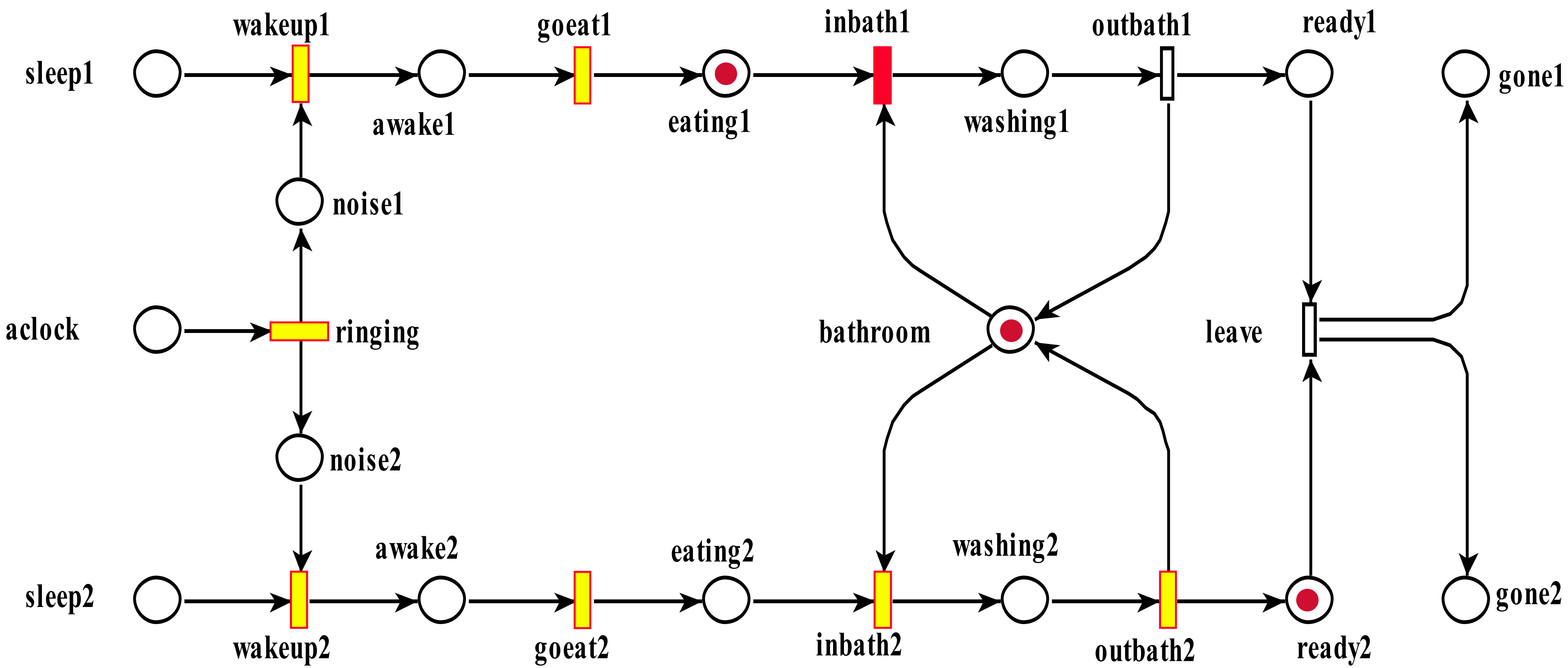

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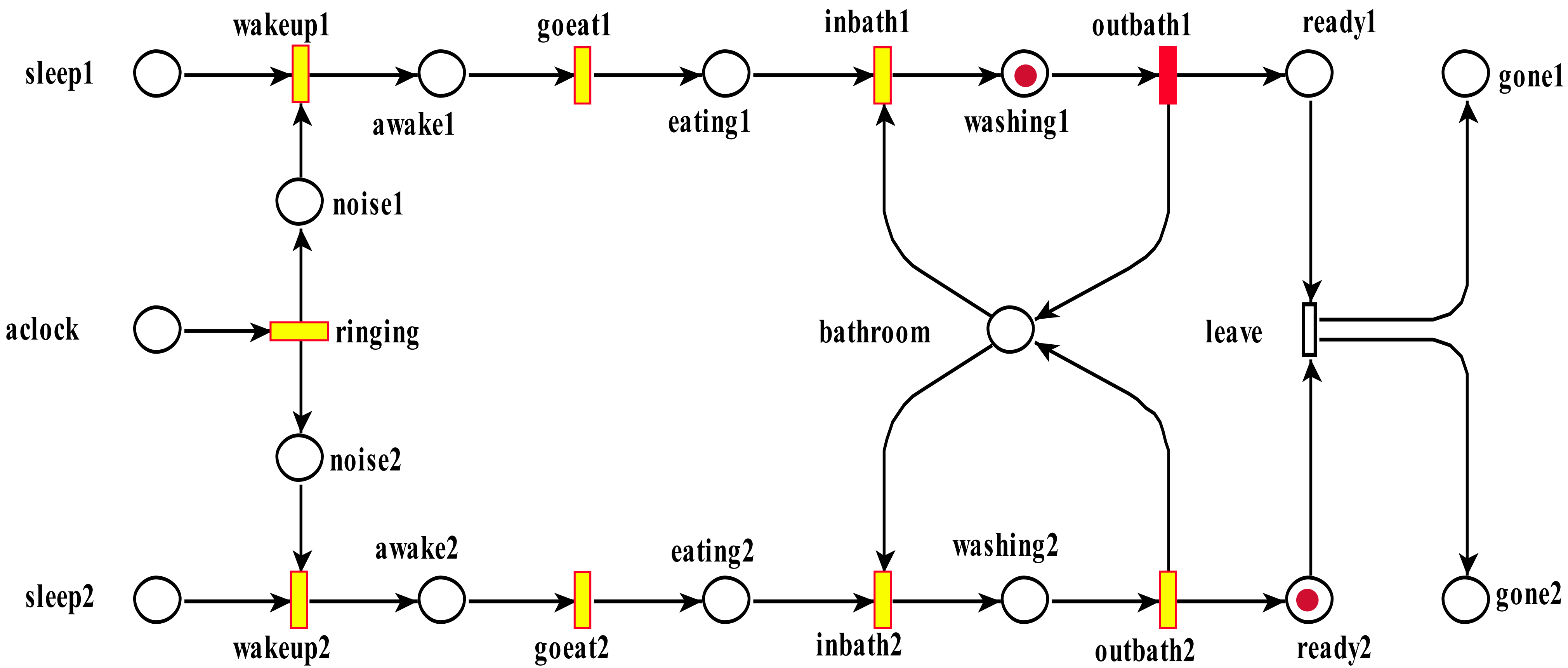

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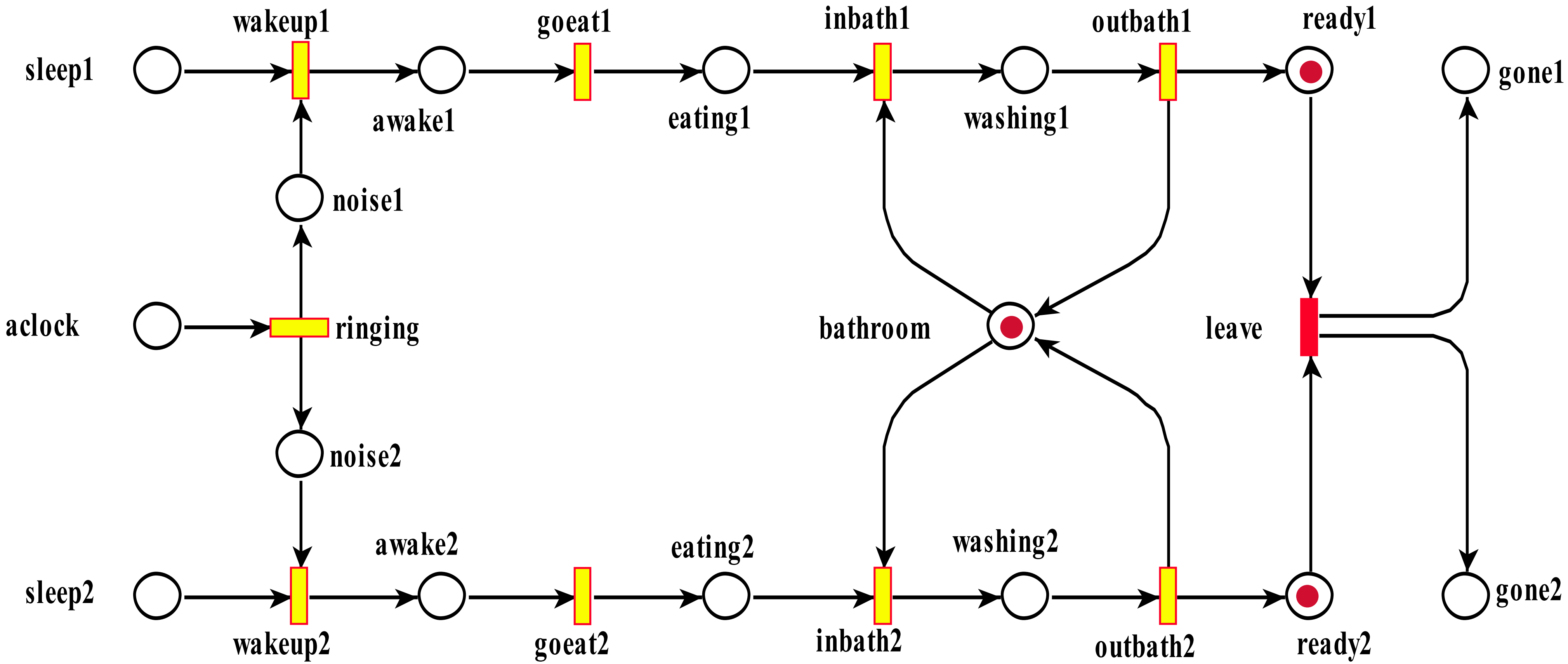

Two people waking up