Academic notes: Topological Fluid Dynamics

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Notes on the MAGIC "Topological Fluid Dynamics" course.

I. INTRODUCTORY CONCEPTS

Definition The Lie bracket is an anti-symmetric bilinear form $[\cdot,\cdot]$ satisfying the Jacobi condition

$$[[\boldsymbol{u},\boldsymbol{v}],\boldsymbol{w}]+[[\boldsymbol{v},\boldsymbol{w}],\boldsymbol{u}]+[[\boldsymbol{w},\boldsymbol{u}],\boldsymbol{v}]=0. \tag{1}$$

The Lie bracket that we will be interested in for fluid dynamics will usually be

$$[\boldsymbol{u},\boldsymbol{v}]=\boldsymbol{u}\cdot\nabla\boldsymbol{v}-\boldsymbol{v}\cdot\nabla\boldsymbol{u}.$$

We distinguish between vectors and co-vectors (or 1-forms): if ϕ is materially conserved under a divergence-free flow, i.e.,

$$\frac{\mathrm{D}\phi}{\mathrm{D}t}=\frac{\partial\phi}{\partial t}+\boldsymbol{u}\cdot\nabla\phi=0,$$

then for a gradient of the scalar $\boldsymbol{g}=\nabla\phi$, we have (after integration by parts assuming appropriate boundary conditions)

$$\frac{D\boldsymbol{g}}{Dt}=-(\nabla\boldsymbol{u})\cdot\boldsymbol{g}, \tag{2}$$

which is different to the transport for vectors, where

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t}=\frac{\partial\boldsymbol{\omega}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}, \tag{3}$$

and $\boldsymbol{\omega}=\nabla\wedge\boldsymbol{u}$ is the vorticity, and the wedge $\wedge$ can be thought of as the cross product.

Lemma I.1 $\mathrm{D}(\boldsymbol{\omega}\cdot\boldsymbol{g})/\mathrm{D}t=0$ *if* $\boldsymbol{\omega}$ *is a vector and* $\boldsymbol{g}$ *is a 1-form.*

Proof

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega}\cdot\boldsymbol{g})=\boldsymbol{\omega}\cdot(\nabla\boldsymbol{u})\cdot\boldsymbol{g}=0.$$

If we consider three vectors $\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}$ that are not multiples of each other, then this defines an infinitesimal volume element. Since the flow is divergence free, the volume is conserved, so

$$\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{b}\wedge\boldsymbol{c}\cdot\boldsymbol{d})=0.$$

With the above lemma, $\boldsymbol{b}\wedge\boldsymbol{c}$ may be thought of as a 1-form, representing a surface element $\mathrm{d}\boldsymbol{S}=\boldsymbol{u}\,\mathrm{d}S$. Accordingly,

$$\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\boldsymbol{r}=\mathrm{d}\boldsymbol{r}\cdot\nabla\boldsymbol{u},\qquad\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\boldsymbol{S}=-(\nabla\boldsymbol{u})\cdot\mathrm{d}\boldsymbol{S},\qquad\frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}V=0,$$

so a line element ($\mathrm{d}\boldsymbol{r}$) transforms like a vector, a surface element ($\mathrm{d}\boldsymbol{S}=\boldsymbol{b}\wedge\boldsymbol{c}$) transforms like a 1-form, and a volume element ($\mathrm{d}V=\boldsymbol{b}\wedge\boldsymbol{c}\cdot\boldsymbol{d}$) transport form like a scalar.

Definition The Lie derivative evaluates the change of a vector field along the flow of another vector field.

In our case, $\mathrm{D}/\mathrm{D}t$ acts as the Lie derivative, and we use this in the following example:

Kelvin's circulation theorem Consider a material curve $C(t)$ and that

$$\Gamma_C=\oint_{C(t)}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{r}=\int_{S(t)}\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{S}, \tag{4}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_C=\int_{S(t)}\frac{\mathrm{D}}{\mathrm{D}t}(\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{S})=0 \tag{5}$$

since $\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{S}$ is a passive scalar. Thus circulation Γ_C is materially conserved for ideal Euler flows.

II. INTEGRAL QUANTITIES

Definition The kinetic and magnetic energy is defined as

$$E_K = \frac{1}{2}\int |\boldsymbol{u}|^2 \,\mathrm{d}V, \qquad E_M = \frac{1}{2}\int |\boldsymbol{B}|^2 \,\mathrm{d}V. \tag{6}$$

Definition The kinetic helicity is defined as

$$H_K = \int \boldsymbol{u}\cdot\boldsymbol{\omega} \,\mathrm{d}V. \tag{7}$$

Lemma II.1 *H_K is conserved in an ideal flow when V is taken to be a material volume, with the boundary conditions being $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$, where $\boldsymbol{n}$ is the outward normal of the material volume.*

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}H_K(V) &= \int \left[\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot\boldsymbol{\omega} + \boldsymbol{u}\cdot\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t}\right] \mathrm{d}V \\
&= \int \left[-\nabla p\cdot\boldsymbol{\omega} + \boldsymbol{u}\cdot(\boldsymbol{\omega}\cdot\nabla)\boldsymbol{u} + (\boldsymbol{u}\wedge\boldsymbol{\omega})\cdot\boldsymbol{\omega}\right] \mathrm{d}V \\
&= \int \nabla\cdot\left(-p\boldsymbol{\omega} + \frac{1}{2}|\boldsymbol{u}|^2\boldsymbol{\omega}\right) \mathrm{d}V \\
&= \int_S \left(-p + \frac{|\boldsymbol{u}|^2}{2}\right)\omega\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{S} \\
&= 0.
\end{aligned}$$

If $V(0)$ is bounded by vortex lines, then it will do so for $V(t)$ as vorticity is conserved.

Definition The magnetic flux through a surface S is given by

$$\Phi_S = \int_S \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}. \tag{8}$$

Theorem II.2 (Alfvén's theorem) *Magnetic flux Φ_S is conserved for ideal MHD where $S(t)$ is a material surface.*

Proof Proof as for Kelvin's circulation theorem but with the boundary condition $\boldsymbol{B}\cdot\boldsymbol{n} = 0$.

Definition With $\nabla\cdot\boldsymbol{B} = 0$,we define the vector potential $\boldsymbol{A}$ of $\boldsymbol{B}$ by $\boldsymbol{B} = \nabla\wedge\boldsymbol{A}$.

This operation exists at least locally, from $\nabla\cdot\boldsymbol{B} = 0$. This $\boldsymbol{A}$ can be defined locally in infinite space, or in a bounded space provided that it is simply connected. Otherwise there may be problems uncurling $\boldsymbol{B}$.

Note that $\boldsymbol{B}$ is unchanged if we consider $\boldsymbol{A}\to\boldsymbol{A}+\nabla\phi$. Uncurling the ideal MHD equation gives

$$\frac{\partial\boldsymbol{A}}{\partial t} = \boldsymbol{u}\wedge\boldsymbol{B} + \nabla\phi = \boldsymbol{u}\wedge(\nabla\wedge\boldsymbol{A}) + \nabla\phi,$$

where $\nabla\phi$ corresponds to the choice of gauge. One choice is the Coulomb gauge $\nabla\cdot\boldsymbol{A} = 0$, which results in the Poisson equation

$$\nabla^2\phi = -\nabla\cdot(\boldsymbol{u}\wedge\boldsymbol{B}),$$

cf. the pressure equation in fluid systems. Note that with other gauge choices (e.g. Lorenz gauge), such an equation may change form or not arise.

Definition The magnetic helicity is defined as

$$H_M = \int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V. \tag{9}$$

Note that this definition is gauge invariant, since

$$\frac{\partial}{\partial t} H_M = \int \left[(\boldsymbol{u} \wedge \boldsymbol{B} + \nabla \phi) \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) \right] \, \mathrm{d}V.$$

The first integral vanishes because of the definition of the cross product and with $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ as the boundary condition. Additionall, noting that

$$\boldsymbol{A} \cdot \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) = \nabla \cdot \left[(\boldsymbol{u} \wedge \boldsymbol{B}) \wedge \boldsymbol{A} \right],$$

the second integral vanishes, thus there is no flow to transport helicity density $\boldsymbol{A} \cdot \boldsymbol{B}$ out of the domain.

Is $\boldsymbol{A}$ a vector of a 1-form? At present it is *neither*: if one uses the Coulomb gauge then one has to solve a global Poisson equation for ϕ, and thus ϕ is generally not transported locally. However, if we write

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{u} \wedge (\nabla \wedge \boldsymbol{A}) + \nabla \phi = -\boldsymbol{u} \nabla \boldsymbol{A} + (\nabla \boldsymbol{A}) \cdot \boldsymbol{u} + \nabla \phi,$$

then the gauge choice $\phi = -\boldsymbol{u} \cdot \boldsymbol{A}$ results in

$$\frac{\mathrm{D}\boldsymbol{A}}{\mathrm{D}t} = -(\nabla \boldsymbol{u}) \cdot \boldsymbol{A}, \tag{10}$$

and $\boldsymbol{A}$ may then be identified as a 1-form. With this choice of gauge, there is the integral conservation of $H_M = \int h_M \, \mathrm{d}V$ with $h_M = \boldsymbol{A} \cdot \boldsymbol{B}$, as well as material conservation of the helicity density h_M by Lemma I.1. Although this is attractive, having a materially conserved quantity, it does depend on following the field $\boldsymbol{A}$ in this gauge. As it evolves, $\boldsymbol{A}$ may become very fine-scaled and develop substantial complexity.

Definition The cross helicity is defined as

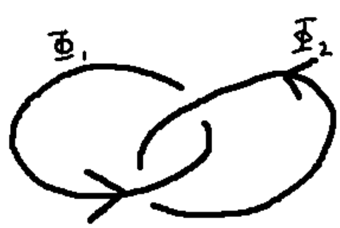
$$H_X = \int \boldsymbol{u} \cdot \boldsymbol{B} \, \mathrm{d}V. \tag{11}$$

Lemma II.3 *The cross helicity H_X is conserved for ideal MHD flow, but kinetic helicity H_K is not.*

Proof Just to the algebra, and note that in the calculation for H_K there is a $j \wedge \boldsymbol{B}$ term that appears, which is generically not zero.

A. Helicity as linking number

We either work with H_K or H_M (for Euler or ideal MHD equations). Consider two linked flux tubes, where $\boldsymbol{B} = 0$ except in the tubes, and each tube carries a flux $\Phi_j > 0$.



$$\Phi_j = \int_{\mathcal{A}_j} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}, \qquad \text{span}(\mathcal{A}_j) = \text{the tubes } T_j$$

We assume each individual tube is unknotted. Now, using Stokes' theorem,

$$H_j = \int_{T_j} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = \Phi_j \oint_{C_j} \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \Phi_j \oint_{C_j} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \Phi_j \Phi_i, \tag{12}$$

so that

$$H_M = H_1 + H_2 = \pm 2\Phi_1\Phi_2. \tag{13}$$

If the tubes are unlinked, then the helicity is zero. In general, we have

$$H_M = \Phi_i\Phi_j\alpha_{ij}, \tag{14}$$

where α_{ij} are integers that label the signed number of times that the centre lines C_j of the j^{th} tube crossing the surface S_i of the i^{th} tube. Here, the sign is "+" when the direction of C_j agrees with the outward normal of S_i, obtained from the right-hand rule from the direction of the centre line C_i. If a tube is knotted, then we could break it up accordingly to compute the helicity, thus defining α_{ii} in this slightly more complicated case.

Note that the topological nature of the invariant in terms of linkage of field lines or vortex lines explains why it is invariant under ideal flow, as linkages cannot be broken. Note also that we cannot define linkage easily if the lines do not close up (e.g., in most astrophysical situations). This is resolved later.

There is a connection between helicity and the topological invariant called the Gauss linking number. Consider a magnetic field $\boldsymbol{B}$ in unbounded space that occupies tubes T_j, with centrelines C_j and fluxes Φ_j. The corresponding $\boldsymbol{A}$ (in the Coulomb gauge) may be obtained at any time by a Biot–Savart integral (as can the velocity from the vorticity), namely, as

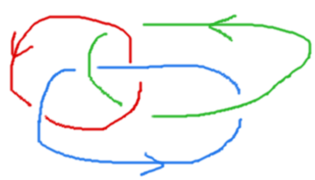
$$\boldsymbol{A}(\boldsymbol{x}) = -\frac{1}{4}\int_D \frac{(\boldsymbol{x}-\boldsymbol{x}')\wedge \boldsymbol{B}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\,\mathrm{d}V' = \sum_j \left(-\frac{\Phi_j}{4\pi}\right)\oint_{C_j} \frac{(\boldsymbol{x}-\boldsymbol{x}')\wedge \mathrm{d}\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}. \tag{15}$$

This then implies that the constants α_{ij} in equation (14) is given by

$$\alpha_{ij} = \frac{1}{4\pi}\oint_{C_i}\oint_{C_j} \frac{(\boldsymbol{x}-\boldsymbol{x}')\cdot \mathrm{d}\boldsymbol{x}\wedge \mathrm{d}\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|^3}. \tag{16}$$

This integral gives an integer, which is the Gauss linking number of curve i with curve j ($i \neq j$). Note that if $\alpha_{ij} \neq 0$, then the curves are linked.

Example An interesting example however is the Borromean rings, where $\alpha_{ij} = 0$ for all pairs i and j, so there is no helicity even though there is linkage.



We have assumed that there is no linkage associated with the field lines inside a given flux tube. This is not always correct and leads on to how one measures the twist and writhe of the magnetic field inside a flux tube.

B. Beltrami and ABC flows

We have focused on the magnetic field that is concentrated in discrete flux tubes. In the case of fluid flows however we have a continuous distribution, and very few field lines actually close up. The work of topological invariants have been generalised by Arnol'd to cover this case, and he refers to this as the asymptotic Hopf invariant. The idea is to take magnetic field lines starting at points $\boldsymbol{a}$ and integrate trajectories $\boldsymbol{x}(\boldsymbol{a},t)$ along them for some large time $T \gg 1$. One then uses a set of paths to link the end points $\boldsymbol{x}(\boldsymbol{a},T)$ to $\boldsymbol{a}$, with paths being short and straight. This gives a collection of closed curves from which the Gauss linking number may be computed. Taking $T \to \infty$ then yields the invariant (with technical details omitted here).

Now, consider an ideal fluid in a finite region D with $H_K \neq 0$. Then what is the minimum value of enstrophy

$$\Omega_K = \int_D |\boldsymbol{\omega}|^2\,\mathrm{d}V \tag{17}$$

for this given helicity? By the Cauchy–Schwartz inequality,

$$H_K^2 = \left(\int_D \boldsymbol{u} \cdot \boldsymbol{\omega} \, \mathrm{d}V \right)^2 \leq \left(\int_D |\boldsymbol{u}|^2; \mathrm{d}V \right) \left(\int_D |\boldsymbol{\omega}|^2; \mathrm{d}V \right) = 2E_K \Omega_K. \tag{18}$$

By Poincaré's inequality,

$$\Omega_K = \int_D |\boldsymbol{\omega}|^2 \, \mathrm{d}V \geq \frac{1}{L^2} \left(\int_D |\boldsymbol{u}|^2; \mathrm{d}V \right) = \frac{2E_K}{L^2} \tag{19}$$

for any $\boldsymbol{u}$ such that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, and L is given by the smallest eigenvalue of the curl operator, i.e.

$$\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u} = \frac{1}{L} \boldsymbol{u}. \tag{20}$$

Together, we have

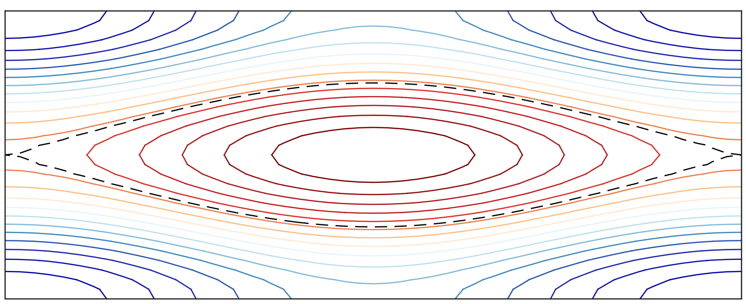
$$H_K^2 \leq 2E_K \Omega_K \leq L^2 \Omega_K \qquad \Rightarrow \qquad \Omega_K \geq \frac{1}{L} |H_K| \,. \tag{21}$$

We have equality if $\boldsymbol{\omega} = \lambda \boldsymbol{u}$ everywhere in D; such flows are called Beltrami flows, and satisfy the Euler equations since $\boldsymbol{u} \wedge \boldsymbol{\omega} = 0$.

A special case of a Beltrami flow is the ABC flow (Arnold–Beltrami–Childress)

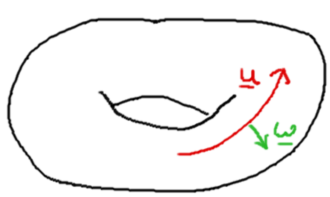
$$\boldsymbol{u} = A(0, \sin x, \cos x) + B(\cos y, 0, \sin y) + C(\sin z, \cos z, 0) \tag{22}$$

for constants A, B, C, and is a flow in $\mathbb{T}^3$ with no mean component. If one of the constants is zero, we obtain the Cat's eyes pattern with regular streamlines:

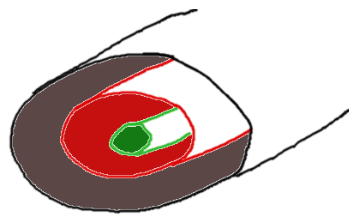


If all three are non-zero, then typically we see a complex mixture of regular vortices, with stream-surfaces that take the form of tori, interlaced with regions of chaos. To make life difficult, the flow is highly three dimensional. These flows have been studied by means of Poincaré sections, in which trajectories $\boldsymbol{x}(\boldsymbol{a}, t)$ are followed and points plotted whenever they intersect one of a family of planes (e.g., $x = \text{const}$).

Just how complicated can steady Euler flows be? Are there any constraints on their structure? An answer to this (and pretty much all that is known) is provided in a rigourous result by Arnol'd: in a finite domain D, a steady Euler flow obeys $\boldsymbol{u} \wedge \boldsymbol{\omega} = \nabla P$, so that $\boldsymbol{u} \cdot \nabla P = \boldsymbol{\omega} \cdot \cdot \cdot \nabla P = 0$. Then, provided ∇P does not vanish identically in a 3D region of space,, there will exist a family of 2D surfaces $S(P_0)$ on which $P = P_0 = \text{const}$. This says that the streamlines and vortex lines will be tangent to such surfaces. For example:



To understand what form the surfaces can take, we need to be careful about points where $\nabla P = 0$; we assume there are only finitely many such points in D under consideration. Choosing a value $P_0 = \text{const}$ gives a surface $S(P_0)$ which does not intersect any points with $\nabla P = 0$. We assume this surface does not intersect ∂D. Now, $\boldsymbol{u}$ and $\boldsymbol{\omega}$ are tangent to $S(P_0)$, so does not vanish on $S(P_0)$. It can be shown that the only such surface satisfying such conditions is topologically a torus, and that $\boldsymbol{u}$ and $\boldsymbol{\omega}$ wraps around it. As we vary the value of P_0, we obtain *nested* tori $S(P_0)$, provided we do not encounter a point where $\nabla P = 0$; at such a point, surfaces may intersect.



The region in which we must have tori is linked to the non-vanishing flow field $\boldsymbol{u}$ (or $\boldsymbol{\omega}$) on the surfaces. It cannot be a sphere (cf. hairy ball theorem). For more and for links with the Euler characteristic, see the book "Topological Fluid Dynamics" by Arnoĺd & Khesin.

We assumed P is such that $\nabla P = 0$ at only finitely many points, so nested 2D surfaces $P_0 = \text{const}$ may be defined. If $P = \text{const}$ over some region $V \subset D$, then $\boldsymbol{u} \wedge \boldsymbol{\omega} = 0$ on V, and $\boldsymbol{\omega} = \alpha(\boldsymbol{x})\boldsymbol{u}$, so we must have a Beltrami flow on V. On the other hand, the fields are divergence-free, so $\boldsymbol{u} \cdot \nabla\alpha = \omega \cdot \nabla\alpha = 0$. The above argument then extends over to this case: $\alpha(\boldsymbol{x})$ defines 2D tori $S(\alpha_0)$, with $\nabla\alpha = 0$ assumed to only happen on a finitely many number of points.

The final possibility then is that $\alpha = \lambda = \text{const}$, over a 3D region (Beltrami flows). Here,$\boldsymbol{u}$ and $\boldsymbol{\omega}$ lines lie on nested tori except in the special case of constant Bernoulli function $P = p + |\boldsymbol{u}|^2/2$ and $\boldsymbol{\omega} = \lambda\boldsymbol{u}$.

(Note that Arnol'd assumes $\boldsymbol{u}$ to be complex analytic, which means critical points $\nabla P = 0$ or $\nabla\alpha = 0$ cannot accumulate, and if P or α are constant in any 3D volume then it is constant everywhere. With this, no messy mixtures are allowed. The argument needs to be extended in more general situations in which we may only require $\boldsymbol{u} \in C^\infty$.)

C. Magnetic relaxation

Consider the case where there is viscous but no magnetic diffusion, so that the governing equations are

$$\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{u} \wedge \boldsymbol{\omega} + \boldsymbol{j} \wedge \boldsymbol{B} - \nabla P + \nu\nabla^2\boldsymbol{u}, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}), \tag{23}$$

with $\nabla \cdot \boldsymbol{u} = 0 = \nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{j} = \nabla \wedge \boldsymbol{B}$. We work in a domain D which is finite, with $\boldsymbol{u} = 0$ and $\boldsymbol{B} \cdot \boldsymbol{n} = 0$ on ∂D. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_k + E_M) = -\nu \int_D |\nabla\boldsymbol{u}|^2 \,\mathrm{d}V.$$

Can this energy have a zero limit? The Lorentz force will tend to drive a flow, but field lines are frozen in, so any helicity integrals must be conserved, which constraints how small E_M can actually become.

Recall that the enstrophy satisfies the bound $\Omega_K \geq H_K/L$, and by a similar argument,

$$2E_M = \int |\boldsymbol{B}|^2 \,\mathrm{d}V \geq \frac{1}{L}\left|\int \boldsymbol{A} \cdot \boldsymbol{B} \,\mathrm{d}V\right| = \frac{H_M}{L}. \tag{24}$$

So a non-zero magnetic helicity gives a non-trivial lower bound on the magnetic energy (note however L depends on the boundary conditions on $\boldsymbol{A}$). Hence a state with an initially non-zero magnetic helicity must lead to a non-trivial final state in this relaxation process. It may in fact be be shown that if there is a non-trivial linkage (e.g. Borommean rings has no magnetic helicity), this conclusion is still valid (see Freedman, 1988). As $t \to \infty$, we expect $E_K \to 0$, so we should get a magnetic equilibrium satisfying

$$\boldsymbol{j} \wedge \boldsymbol{B} = \nabla P. \tag{25}$$

Note this is analogous to the steady Euler equations

$$\boldsymbol{u}_E \wedge \boldsymbol{\omega}_E = \nabla P_E, \tag{26}$$

so if we find a steady Euler flow from the relaxation process. For example, we can start by having a set of nested surfaces $S(\gamma)$ in which helicity will be conserved. If we then relax it, we obtain a corresponding Euler flow where helicities are preserved. This helicity unction gives the signature of the vortex.

Despite this appealing picture, there are many (or unknown) unknowns. Does $\boldsymbol{u}$ tend to zero pointwise? Can singularities occur in the fields as they relax in the limit $t \to \infty$? Or, rather, how ban can singularities be? How can a sensible limit be defined? For example, consider the following (Bajer, 2005):

The strained flow results in a current sheet being formed, i.e. a discontinuity in $\boldsymbol{B}$ (numerically numerical diffusion would keep this the computation stable). Suppose a sensible limit could be taken, then how does the resulting flow $\boldsymbol{u}_E$ fit into Arnol'd's classification? Presumably the requirement of analyticity will no longer hold and will allow all sorts of complex mixtures of surfaces and chaotic regions, with discontinuities and may be devil's staircases arising in functions such as P (continuous but not absolutely continuous; see Cantor function). These questions are heard to investigate numerically in the difficult limit of zero magnetic diffusion and $t \to \infty$.

A variational formulation could be attempted. Going back to dealing with $\boldsymbol{u}$ and $\boldsymbol{\omega}$, we have

$$E_K = \frac{1}{2}\int_D |\boldsymbol{u}|^2 \,\mathrm{d}V.$$

Suppose some fictitious, divergence-less, infinitesimal flow field $\boldsymbol{\eta}$ acts and moves the vortex lines, so that

$$\boldsymbol{\omega} \to \boldsymbol{\omega} + \delta\boldsymbol{\omega}, \qquad \boldsymbol{u} \to \boldsymbol{u} + \delta\boldsymbol{u},$$

so that

$$\delta\boldsymbol{\omega} = \nabla \wedge (\boldsymbol{\eta} \wedge \boldsymbol{\omega}), \qquad \delta\boldsymbol{u} = \boldsymbol{\eta} \wedge \boldsymbol{\omega} + \nabla\phi, \tag{27}$$

where ϕ is chosen so that $\nabla \cdot \delta\boldsymbol{u} = 0$. The displacement $\boldsymbol{\eta}$ is called an isovortical displacement. Then

$$\delta E_K = \int_D \boldsymbol{u} \cdot \delta\boldsymbol{u} \,\mathrm{d}V = \int_D \boldsymbol{u} \cdot (\boldsymbol{\eta} \wedge \boldsymbol{\omega} + \nabla\phi) \,\mathrm{d}V. \tag{28}$$

Now, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on the boundary means that $\int_D \boldsymbol{u} \cdot \nabla\phi \,\mathrm{d}V = 0$. Now if $\boldsymbol{\omega} \wedge \boldsymbol{u}$ is the gradient of some scalar field, then the divergence theorem also makes its contribution zero. Conversely, it can be shown that if δE_K vanishes for all divergence-less $\boldsymbol{\eta}$, then

$$\boldsymbol{u} \wedge \boldsymbol{\omega} = \nabla P \tag{29}$$

for some scalar function P. Thus we obtain the results that *"stead Euler flows are stationary points of E_K with respect to general isovortical displacements"*. Further study in the second variation of E_K leads to a powerful stability theorem in 2D (Arnol'd, 1965, 1966).

III. MORE ON MAGNETIC HELICITY

A. Decomposition of vector fields

We consider finite 3D volumes, and ask ourselves some questions about the volume and vector field. We will consider the following topologies:

1. Simply-connected. Any closed curve within the volume may be shrunk to a point (closed but not knotted).

2. Multiply-connected. Some closed curves cannot be shrunk to a point.

3. 2-connected. If simple surfaces may be shrunk to a point.

For example, a 2-sphere is 2-connected (and simply connected), torus is a multiple connected, and a spherical shell is simply connected but not 2-connected.

Some conditions/properties a vector field may have is that they could be:

1. divergence free ($\nabla \cdot \boldsymbol{B} = 0$);
2. curl free ($\nabla \wedge \boldsymbol{B} = 0$);
3. potential (there is some scalar function ψ where $\boldsymbol{B} = \nabla\psi$);
4. closed ($\boldsymbol{B} \cdot \boldsymbol{n} = 0$).

Given some volume with a particular topology, what sort of vector fields can we have inside? Also, given a vector field, could we decompose it so we obtain pieces with special properties?

Theorem III.1 (Hodge decomposition theorem) *Any differential form may be decomposed as*

$$\omega = \mathrm{d}\alpha + \delta\beta + \gamma,$$

where d*is the exterior derivative,* δ *the co-differential, and* γ *is harmonic (vanishes under* d *and* δ*).*

In our case, it means we have the possibility for decomposing the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ into

1. Grounded gradients (electrostatic fields, grounded on boundary)

$$\nabla \cdot \boldsymbol{E} \neq 0, \qquad \boldsymbol{E} = \nabla\psi, \qquad \psi|_{\partial D} = 0.$$

2. Harmonic gradients (electric field with perfectly conducting boundaries)

$$\nabla \cdot \boldsymbol{E} = 0, \qquad \boldsymbol{E} = \nabla\psi, \qquad \psi|_{\partial D} = \text{const}, \qquad \nabla^2\psi = 0.$$

3. Fluxless knots (closed magnetic fields)

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{B} \cdot \boldsymbol{n} = 0, \qquad \Psi = 0 \text{ in interior}$$

   where Φ is the magnetic flux.

4. Curly gradients (potential magnetic field)

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \boldsymbol{B} = \nabla\psi, \qquad \boldsymbol{B} \cdot \boldsymbol{n} \neq 0,$$

   but the net flux through ∂D is zero.

5. Harmonic knots (only exists in multiply-connected volumes)

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \wedge \boldsymbol{B} = 0, \qquad \boldsymbol{B} \cdot \boldsymbol{n} = 0, \qquad \nabla^2\boldsymbol{B} = 0.$$

Since magnetic monopoles are not known to exist in nature, 1) and 2) should not be relevant in magnetic problems. Also, 1) is only relevant in volumes that are not 2-connected.

Note that a $\boldsymbol{B}$ field in a simply connected volume will divide uniquely into a 3) and 4) field. (Note also that periodic boxes/volumes are not connected in the way we have mentioned, and we will avoid these here.)

B. Helicity

Consider two vector fields $\boldsymbol{V}$ and $\boldsymbol{W}$. We define the helicity as

$$H(\boldsymbol{V}, \boldsymbol{W}) = \int_V \boldsymbol{A_V} \cdot \boldsymbol{W} \, \mathrm{d}V, \qquad \nabla \wedge \boldsymbol{A_V} = \boldsymbol{V}. \tag{30}$$

We could write this ina more symmetric way by using the Biot-Savart integral for the potential $\boldsymbol{A_V}$:

$$\boldsymbol{A_V}(\boldsymbol{x}) = \frac{1}{4\pi} \int_V \frac{(\boldsymbol{y} - \boldsymbol{x})}{r^3} \wedge \boldsymbol{V}(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}, \qquad \boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}, \tag{31}$$

so that

$$H(\boldsymbol{V}, \boldsymbol{W}) = \frac{1}{4\pi} \iint_V \boldsymbol{V}(\boldsymbol{x}) \cdot \left(\frac{\boldsymbol{r}}{r^3} \wedge \boldsymbol{W}(\boldsymbol{y}) \right) \, \mathrm{d}\boldsymbol{y} \, \mathrm{d}\boldsymbol{x}. \tag{32}$$

We note that, because of the property of the scalar triple product in (32), we have that

$$H(\boldsymbol{V}, \boldsymbol{W}) = H(\boldsymbol{W}, \boldsymbol{V}), \qquad H(\alpha \boldsymbol{V} + \beta \boldsymbol{X}, \boldsymbol{W}) = \alpha H(\boldsymbol{V}, \boldsymbol{W}) + \beta H(\boldsymbol{X}, \boldsymbol{W}), \tag{33}$$

so the helicity is a symmetric bilinear form. We will adopt this more general definition.

C. Toroidal and poloidal fields

It is often useful to decompose $\boldsymbol{B}$ into toroidal and poloidal components. Let $\mathcal{L}$ be an operator (similar to the angular momentum operatory in quantum theory), defined as

$$\mathcal{L} = \begin{cases} -\boldsymbol{e}_z \wedge \nabla, & \text{Cartesian,} \\ -\boldsymbol{r} \wedge \nabla, & \text{spherical.} \end{cases} \tag{34}$$

Then we could write

$$\boldsymbol{B} = \mathcal{L}T + \nabla \wedge \mathcal{L}P, \tag{35}$$

where T is the toroidal function, and P is the poloidal function. We examine the Cartesian case. We note that

$$\boldsymbol{e}_z \cdot \mathcal{L} = 0, \qquad \nabla \cdot \mathcal{L} = 0. \tag{36}$$

The vector potentials for $\boldsymbol{B}_T = \mathcal{L}T$ and $\boldsymbol{B}_P = \nabla \wedge \mathcal{L}P$ are

$$\boldsymbol{A}_T = \mathrm{curl}^{-1} \boldsymbol{B}_T = T\boldsymbol{e}_z + \nabla \Psi_T, \qquad \boldsymbol{A}_P = \mathrm{curl}^{-1} \boldsymbol{B}_P = \mathcal{L}P + \nabla \Psi_P, \tag{37}$$

where Ψ are gauge functions. P and T can be obtained from $\boldsymbol{B}$ by solving the Poisson equations

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = B_z, \qquad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} = (\nabla \wedge \boldsymbol{B})_z. \tag{38}$$

Proposition III.2 *Toroidal fields do not link with each other, nor do two poloidal fields.*

Proof Since helicity integrals are gauge invariant, we may ignore Ψ. Now, since $\boldsymbol{e}_z \cdot \mathcal{L} = 0$,

$$H(\boldsymbol{B}_{T_1} + \boldsymbol{B}_{T_2}) = \int T_1 \boldsymbol{e}_z \cdots \mathcal{L} T_2 \lceil z = 0.$$

Also, note that

$$
\begin{aligned}
H(\boldsymbol{B}_{P_1}+\boldsymbol{B}_{P_2}) &= \int \mathcal{L}P_1 \cdot \boldsymbol{B}_{P_2}\ \boldsymbol{x} \\
&= \int \mathcal{L}P_1 \cdot \left(\nabla \frac{\partial P_2}{\partial z} - \boldsymbol{e}_z \nabla^2 P_2\right)\ \mathrm{d}\boldsymbol{x} \\
&= \int \mathcal{L}P_1 \cdot \left(\nabla \frac{\partial P_2}{\partial z}\right)\ \mathrm{d}\boldsymbol{x} \\
&= \int \nabla \wedge P_1 \boldsymbol{e}_z \cdot \left(\nabla \frac{\partial P_2}{\partial z}\right)\ \mathrm{d}\boldsymbol{x} \\
&= \int \nabla \cdot \left(P_1 \boldsymbol{e}_z \wedge \nabla \frac{\partial P_2}{\partial z}\right)\ \mathrm{d}\boldsymbol{x} \\
&= \oint \boldsymbol{e}_z \cdot \left(P_1 \boldsymbol{e}_z \wedge \nabla \frac{\partial P_2}{\partial z}\right)\ \mathrm{d}\ell \\
&= 0.
\end{aligned}
$$

Theorem III.3 *Consider $\boldsymbol{B} = \boldsymbol{B}_T + \boldsymbol{B}_P$ in V. Assume that V is either all space, half space bounded by a plane, layer bounded by two planes, interior or exterior of a sphere, or a spherical shell bounded by two concentric spheres. Then:*

1. $\boldsymbol{B}_P = 0 \quad \Rightarrow \quad H(\boldsymbol{B}, \boldsymbol{B}) = 0$;
2. $\boldsymbol{B}_T = 0 \quad \Rightarrow \quad H(\boldsymbol{B}, \boldsymbol{B}) = 0$;
3. $H(\boldsymbol{B}, \boldsymbol{B}) = 2\int \mathcal{L}T \cdot \mathcal{L}P\ \mathrm{d}\boldsymbol{x}$.

Proof The first two were shown above. For the last one, since the helicity is a bilinear form,

$$
H(\boldsymbol{B}, \boldsymbol{B}) = H(\boldsymbol{B}_P + \boldsymbol{B}_T, \boldsymbol{B}_P + \boldsymbol{B}_T) = 2H(\boldsymbol{B}_P, \boldsymbol{B}_T) = 2\int_V \mathcal{L}P \cdot \mathcal{L}T\ \mathrm{d}\boldsymbol{x}.
$$

D. Helicity in open volumes

So far we have considered helicity in closed volumes, where field lines never cross the boundary. In most physical and mathematical problems this is far too restrictive. For example, we take the photosphere as the outer or inner boundary depending on which portion we want to study. We need a form of helicity integral which works just for the interior or exterior. Furthermore, the sum of interior and exterior helicities should sensible relate to the helicity integral of all space.

Here, we consider $H(\boldsymbol{B}) = H(\boldsymbol{B}, \boldsymbol{B})$ in some arbitrary D. We give a definition which remains topologically meaningful and is gauge invariant. Let the space be divided into D and D', with magnetic fields $\boldsymbol{B}$ and $\boldsymbol{B}'$. At the boundary S,

$$
\boldsymbol{B} \cdot \boldsymbol{n} = \boldsymbol{B}' \cdot \boldsymbol{n},
$$

We define the magnetic field defined in all space to be

$$
\{\boldsymbol{B}, \boldsymbol{B}'\}(\boldsymbol{x}) = \begin{cases} \boldsymbol{B}, & \boldsymbol{x} \in D, \\ \boldsymbol{B}', & \boldsymbol{x} \in D'. \end{cases} \tag{39}
$$

Now, note that: a), $H(\{\boldsymbol{B}, \boldsymbol{B}'\})$ includes information about all the magnetic structure in $\boldsymbol{B}'$ when we just want $\boldsymbol{B}$, and we need to subtract this extra information out; b) simply integrating (32) will not do, if D contains curly gradients or harmonic knots. Instead, we measure the helicity relative to a minimal base state. Thus we look for some simple vector field $\boldsymbol{P}$ inside D for which we can calculate the reference helicity $H(\{\boldsymbol{P}, \boldsymbol{B}'\})$. One we subtract this reference helicity, the dependence on the external field will vanish.

The boundary information $\boldsymbol{B} \cdot \boldsymbol{n}$ tells us the distribution of flux crossing S. It also determines a unique vector field, the vaccum field $\boldsymbol{P}$ satisfying

$$
\boldsymbol{P} \cdot \boldsymbol{n}|_S = \boldsymbol{B} \cdot \boldsymbol{n}, \qquad \nabla \wedge \boldsymbol{P}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in D. \tag{40}
$$

If D is multiply-connected, the net flux of $\boldsymbol{P}$ through any closed curve on S should also be the same for $\boldsymbol{B}$ and $\boldsymbol{P}$. In other words, if $\boldsymbol{B}$ with D is decomposed (uniquely) into fluxless knots, harmonic knots and curly gradients, then

$$\boldsymbol{P} = \text{Harmonic knots} + \text{Curly gradients}.$$

A simple variational calculation will show that $\boldsymbol{P}$ is the minimum energy state consistent with the boundary data $\boldsymbol{B}\cdot\boldsymbol{n}$. Magnetic field lines spiral about current lines. As the vaccum field has zero current, it has minimum helical structure. It also requires a minimum of information for its specification. So the magnetic helicity inside D is

$$H_D = H(\{\boldsymbol{B}, \boldsymbol{B}'\}) - H(\{\boldsymbol{P}, \boldsymbol{B}'\}). \tag{41}$$

Theorem III.4 *H_D satisfies the following:*

1. *H_D is gauge invariant;*
2. *$H_D = \int_V (\boldsymbol{A} + \boldsymbol{A}_P)\cdot(\boldsymbol{B} - \boldsymbol{P})\,\mathrm{d}\boldsymbol{x}$, where $\nabla\wedge\boldsymbol{A}_P = \boldsymbol{P}$;*
3. *H_D is independent of $\boldsymbol{B}'$ outside of D.*

Proof 1. Both terms in H_D are integrated over all space, so they are gauge invariant, or we could check directly.

2. Let $\{\boldsymbol{A}, \boldsymbol{A}'\}$ be the potential of $\{\boldsymbol{B}, \boldsymbol{B}'\}$, the true magnetic field. The vector potential for the reference field $\{\boldsymbol{B}, \boldsymbol{B}'\}$ is $\boldsymbol{A}_P$ in D, but is $\boldsymbol{A}' + \nabla\psi$ in D', since $\nabla\psi$ is required as a boundary condition at S to match it smoothly with $\boldsymbol{A}_P$. So the reference potential is $\{\boldsymbol{A}_P, \boldsymbol{A}' + \nabla\psi\}$. Let the normal be $\boldsymbol{n}$, pointing from D into D'. We need continuity of the parallel components (parallel to S), so

$$\boldsymbol{n}\wedge\boldsymbol{A} = \boldsymbol{n}\wedge\boldsymbol{A}', \qquad \boldsymbol{n}\wedge\boldsymbol{A}_P = \boldsymbol{n}\wedge(\boldsymbol{A}' + \nabla\psi),$$

so that

$$H(\{\boldsymbol{B}, \boldsymbol{B}'\}) = \left(\int_D \boldsymbol{A}\cdot\boldsymbol{B} + \int_{D'} \boldsymbol{A}'\cdot\boldsymbol{B}'\right)\,\mathrm{d}\boldsymbol{x}, \qquad H(\{\boldsymbol{P}, \boldsymbol{B}'\}) = \left(\int_D \boldsymbol{A}_P\cdot\boldsymbol{P} + \int_{D'} (\boldsymbol{A}' + \nabla\psi)\cdot\boldsymbol{B}'\right)\,\mathrm{d}\boldsymbol{x},$$

which implies that

$$\begin{aligned} H_D &= \int_D (\boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{A}_P\cdot\boldsymbol{P})\,\mathrm{d}\boldsymbol{x} - \int_{D'} \nabla\psi\cdot\boldsymbol{B}'\,\mathrm{d}\boldsymbol{x} \\ &= \int_D (\cdots) + \oint_S \psi\boldsymbol{B}\cdot\boldsymbol{n} + 0, \end{aligned}$$

since $\nabla\cdot\boldsymbol{B} = 0$. Now, note that

$$\begin{aligned} \int_V (\boldsymbol{A}_P\cdot\boldsymbol{B} - \boldsymbol{A}\cdot\boldsymbol{P})\,\mathrm{d}\boldsymbol{x} &= \int_V \left[\boldsymbol{A}_P\cdot(\nabla\wedge\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\wedge\boldsymbol{A}_P)\right]\,\mathrm{d}\boldsymbol{x} \\ &= \int_V \nabla\cdot(\boldsymbol{A}\wedge\boldsymbol{A}_P)\,\mathrm{d}\boldsymbol{x} \\ &= \oint_S (\boldsymbol{A}\wedge\boldsymbol{A}_P)\cdot\boldsymbol{n} \\ &= \oint_S \boldsymbol{A}\wedge(\boldsymbol{A}_P - \boldsymbol{A})\cdot\boldsymbol{n}\,\mathrm{d}S \\ &= \oint_S (\boldsymbol{A}\wedge\nabla\psi)\cdot\boldsymbol{n}\,\mathrm{d}S \\ &= \oint_S (\psi\nabla\wedge\boldsymbol{A} - \nabla\wedge(\psi\boldsymbol{A}))\cdot\boldsymbol{n}\,\mathrm{d}S \\ &= \oint_S \psi\boldsymbol{B}\cdot\boldsymbol{n}\,\mathrm{d}S, \end{aligned}$$

where the line integral of $\nabla\wedge(\psi\boldsymbol{A})$ vanishes by Stokes' theorem. With this,

$$H_D = \int_D (\boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{A}_P\cdot\boldsymbol{P})\,\mathrm{d}\boldsymbol{x} + \int_D (\boldsymbol{A}_P\cdot\boldsymbol{B} - \boldsymbol{A}\cdot\boldsymbol{P})\,\mathrm{d}\boldsymbol{x} = \int_V (\boldsymbol{A} + \boldsymbol{A}_P)\cdot(\boldsymbol{B} - \boldsymbol{P})\,\mathrm{d}\boldsymbol{x}.$$

3. The above is only integrated over D.

E. Addition of helicities

Let $H_T(\{\boldsymbol{B},\boldsymbol{B}'\})$ be the helicity of all space, $\boldsymbol{P}$ be the potential field in V, and $\boldsymbol{P}'$ the potential field in V' (given the boundary condition $\boldsymbol{B}\cdot\boldsymbol{n}$ on S, and the magnitude of any interior fluxes if V is multiply connected and contains a harmonic knot field). Then

$$H_T(\{\boldsymbol{B},\boldsymbol{B}'\}) = H_V(\boldsymbol{B}) + H_{V'}(\boldsymbol{B}') + H(\{\boldsymbol{P},\boldsymbol{P}'\}), \tag{42}$$

since

$$H_V(\boldsymbol{B}) = H_T(\{\boldsymbol{B},\boldsymbol{B}'\}) - H_T(\{\boldsymbol{P},\boldsymbol{B}'\}), \qquad H_{V'}(\boldsymbol{B}') = H_T(\{\boldsymbol{P},\boldsymbol{B}'\}) - H_T(\{\boldsymbol{P},\boldsymbol{P}'\}).$$

Lemma III.5 *If S is a plane or sphere, then $H_T(\{\boldsymbol{P},\boldsymbol{P}'\}) = 0$.*

Proof For the planar case, in the usual Biot–Savart form,

$$\boldsymbol{A}_V(\boldsymbol{x}) = -\frac{1}{4\pi}\int_V \frac{\nabla_{\boldsymbol{y}}\wedge \boldsymbol{V}(\boldsymbol{y})}{r}\,\mathrm{d}\boldsymbol{y}$$

where $\boldsymbol{r} = \boldsymbol{y} - \boldsymbol{x}$, and so

$$\boldsymbol{A}_P(\boldsymbol{x}) = -\frac{1}{4\pi}\int_V \frac{\nabla_{\boldsymbol{y}}\wedge \boldsymbol{P}(\boldsymbol{y})}{r}\,\mathrm{d}\boldsymbol{y}.$$

However, a potential field has zero curl. The only place the global field $\{\boldsymbol{P},\boldsymbol{P}'\}$ has non-zero curl is on S. For a planar boundary, $\nabla\wedge\boldsymbol{P}$ is parallel to S, but this means that $\boldsymbol{A}_P\cdot\boldsymbol{n} = 0$. Now, $\boldsymbol{P} = \nabla\phi$ and $\boldsymbol{P}' = \nabla\phi'$, so

$$\begin{aligned} H_T(\{\boldsymbol{P},\boldsymbol{P}'\}) &= \int_V \boldsymbol{A}_P\cdot\boldsymbol{P}\,\mathrm{d}\boldsymbol{x} + \int_V \boldsymbol{A}'_P\cdot\boldsymbol{P}'\,\mathrm{d}\boldsymbol{x} \\ &= \int_V \boldsymbol{A}_P\cdot\nabla\phi\,\mathrm{d}\boldsymbol{x} + \int_V \boldsymbol{A}'_P\cdot\nabla\phi'\,\mathrm{d}\boldsymbol{x} \\ &= 0 + 0 \end{aligned}$$

by the divergence theorem and boundary conditions.

Being able to calculate helicity of sub-volumes of space has special relevant in the Solar atmosphere (the boundary is then the photosphere).

F. Time derivative of helicity

From the definition of H_V, we can see that

$$\frac{\mathrm{d}}{\mathrm{d}t}H_V = -2\int_V \boldsymbol{E}\cdot\boldsymbol{B}\,\mathrm{d}\boldsymbol{x} - 2\oint_S \boldsymbol{A}_P\wedge\boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x}. \tag{43}$$

Here, $\boldsymbol{A}_P$ is uniquely defined by

$$\nabla\wedge\boldsymbol{A}_P = \boldsymbol{P}, \quad \boldsymbol{n}\cdot\nabla\wedge\boldsymbol{A}_P = B_n, \quad \nabla\cdot\boldsymbol{A}_P = 0, \quad \boldsymbol{A}_P\cdot\boldsymbol{n} = 0. \tag{44}$$

Suppose we have $\boldsymbol{E} = \eta\boldsymbol{j}$ with η being the Ohmic dissipation coefficient. The first term in (43) measures Ohmic dissipation of helicity. Now, let μ_0 be the vacuum magnetic permeability, then we have

$$E_M = \frac{1}{2\mu_0}\int_V |\boldsymbol{B}|^2\,\mathrm{d}\boldsymbol{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}H_V = 2\int \eta\boldsymbol{j}\cdot\boldsymbol{B}\,\mathrm{d}\boldsymbol{x}, \tag{45}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_M = -\int_V \eta|\boldsymbol{j}|^2\,\mathrm{d}\boldsymbol{x}$$

thus, by the Cauchy–Schwartz inequality,

$$\left|\frac{\mathrm{d}H_V}{\mathrm{d}t}\right| \leq \sqrt{8\eta\mu_0 E_M \left|\frac{\mathrm{d}E_M}{\mathrm{d}t}\right|}. \tag{46}$$

For $\eta \ll 1$ as in most astrophysical problems, $|\mathrm{d}H_V/\mathrm{d}t|$ is small, so helicity is fairly well conserved. In problems such as liquid metal experiments, this is not the case as η is comparatively large. Note that we only have exact conservation in ideal MHD where $\eta = 0$.

The boundary term in (43) gives helicity flow from one space to another. For ideal MHD where the first term in (43) is zero, $\boldsymbol{E} = \boldsymbol{B} \wedge \boldsymbol{u}$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t} H_V = -2 \oint_S [(\boldsymbol{A}_P \cdot \boldsymbol{u})\boldsymbol{B} - (\boldsymbol{A}_P \cdot \boldsymbol{B})\boldsymbol{u}] \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{x}. \tag{47}$$

This may then be seen as a helicity flux.

Example Consider the Solar cycle. The equator of the sun rotates faster than the poles, so magnetic features move with respect to polar features. This leads to twisting up of field lines (cf. Ω-effect), injecting negative helicity into the Southern hemisphere. At the same time, positive helicity streams out of the Northern hemisphere into the Northern Solar wind and into the Northern interplanetary space, and similarly for the Southern hemisphere. The diagnosed quantities from observations follows the budget (43) well.

IV. SOME GEOMETRY AND TOPOLOGY

A. The Frenet frame

Given a curve $C(t)$ and the arc-length $s(t)$, the <u>tangent vector</u> T, <u>velocity vector</u> V, <u>curvature</u> κ, <u>normal vector</u> N, <u>bi-normal vector</u> B and <u>torsion</u> τ are defined by the following:

$$T = \frac{\mathrm{d}C}{\mathrm{d}s}, \quad V = \frac{\mathrm{d}s}{\mathrm{d}t}, \quad \kappa = \left|\frac{\mathrm{d}T}{\mathrm{d}s}\right|, \quad N = \frac{1}{\kappa}\frac{\mathrm{d}T}{\mathrm{d}s}, \quad B = T \wedge N, \quad \tau = \frac{\mathrm{d}N}{\mathrm{d}t} \cdot B. \tag{48}$$

Note that the velocity changes if C is re-parameterised.

We have the following properties:

1. $|T| = 1$
2. $T \cdot N = 0$
3. $T \cdot \mathrm{d}N/\mathrm{d}s = -\kappa$
4. $\mathrm{d}N/\mathrm{d}s = -\kappa T + \tau B$
5. $\mathrm{d}B/\mathrm{d}s = -\tau N$

Proof 1. From definition,

$$\begin{aligned}
\left|\frac{\mathrm{d}C}{\mathrm{d}s}\right| &= \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2\right] \\
&= \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \left(\frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{\mathrm{d}t^2}\right) \\
&= \left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 \left(\frac{\mathrm{d}s^2}{\mathrm{d}t^2}\right)^2 \\
&= 1
\end{aligned}$$

2. From definition again, taking a derivative on both sides leads to

$$0 = 2T\frac{\mathrm{d}T}{\mathrm{d}s} = 2T \cdot \kappa N$$

3. Since $T \cdot N = 0$, taking a derivative leads to

$$0 = \frac{\mathrm{d}T}{\mathrm{d}s} \cdot N + T \cdot \frac{\mathrm{d}N}{\mathrm{d}s} = \kappa N \cdot N + T \cdot \frac{\mathrm{d}N}{\mathrm{d}s}$$

which leads to the desired result.

4. Left as an excercise.

5. We have

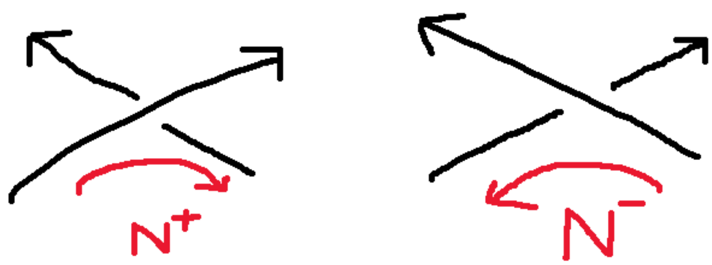
$$\frac{\mathrm{d}B}{\mathrm{d}s} = \frac{\mathrm{d}N}{\mathrm{d}s} \wedge N + T \wedge \frac{\mathrm{d}N}{\mathrm{d}s} = T \wedge (-\kappa T + \tau B) = \tau T \wedge B.$$

B. Links, twist and writhe

Recall that the Gauss integral for the linking number between a curve $X(\tau)$ and $Y(\sigma)$ is

$$\alpha = \frac{1}{4\pi} \oint_X \oint_Y \frac{\mathrm{d}X}{\mathrm{d}t} \cdot \frac{\boldsymbol{r}}{r^3} \wedge \frac{\mathrm{d}Y}{\mathrm{d}\sigma}\, \mathrm{d}\tau\, \mathrm{d}\sigma, \qquad \boldsymbol{r} = Y - X. \tag{49}$$

We can also describe the linking number in terms of the crossing seen when the curve is projected onto the plane. A crossing can be either positive or negative:



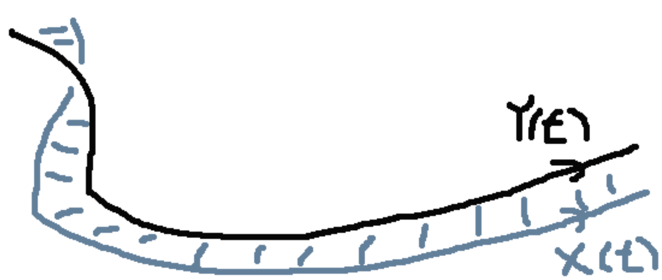
The linking number is then defined as

$$\alpha = \frac{1}{2}(N^+ - N^-), \tag{50}$$

where N^+ is the positive crossing number. This avoids having to do the integral altogether. However, care needs to be taken. If we are looking at a single closed curve (a knot), then $N^{+,-}$ depends on the viewing angle. This is undesirable, but we can get a topological quantity called the writhe, by averaging over all viewing angles.

What happens when two curves are nearly parallel? Let ℓ be the length of X, and that

$$Y(t) = X(t) + \epsilon \hat{V}(t), \qquad |\hat{V}| = 1, \epsilon \ll \ell.$$



As an example, consider the trefoil knot in a ribbon fashion. How much of α is due to twist, and how much is due to writhe? This is answered by Călugăreanu's theorem. Suppose we consider the Gauss linking number for a ribbon, so that

$$\alpha = \frac{1}{4\pi}\oint_X \oint_{X+\epsilon\hat{V}} \frac{\mathrm{d}X}{\mathrm{d}\tau}\cdot\frac{\boldsymbol{r}}{r^3}\wedge\frac{\mathrm{d}(X+\epsilon\hat{V}}{\mathrm{d}\sigma}\,\mathrm{d}\tau\,\mathrm{d}\sigma, \qquad \boldsymbol{r} = X(\tau) - (X(\sigma)+\epsilon\hat{V}(\sigma)).$$

What happens when $\epsilon \to 0$? We do not get the same thing as $\epsilon = 0$.

Theorem IV.1 (Călugăreanu's theorem) *In the limit $\epsilon \to 0$, $\alpha = \mathcal{T} + \mathcal{W}$, where*

$$\mathcal{W} = \frac{1}{4\pi}\oint_X \oint_X \frac{\mathrm{d}X}{\mathrm{d}\tau}\cdot\frac{\boldsymbol{r}}{r^3}\wedge\frac{\mathrm{d}X}{\mathrm{d}\sigma}\,\mathrm{d}\tau\,\mathrm{d}\sigma, \qquad \mathcal{T} = \frac{1}{2\pi}\oint_X \frac{\mathrm{d}X}{\mathrm{d}\tau}\cdot\hat{V}(\tau)\wedge\frac{\mathrm{d}\hat{V}(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{51}$$

Some properties of α are as follows:

1. α is invariant to deformations of the two curves, as long as the two curves are not allowed to cross through each other.
2. $\alpha \in \mathbb{Z}$.
3. $\alpha = (N^+ - N^-)/2$, as sene in any plane projection.
4. α is independent of the direction of axis curve, i.e., α is invariant if $s \to -s$.

(For two arbitrary curves, α changes sign if one of the two curves reverses its direction. For a ribbon, both must change sign together.)

The writhe is given by

$$\mathcal{W} = \frac{1}{4\pi}\oint_X \oint_X \hat{T}(s)\wedge\hat{T}(s')\frac{\boldsymbol{x}(s)-\boldsymbol{x}(s')}{|\boldsymbol{x}(s)-\boldsymbol{x}(s')|}\,\mathrm{d}s\,\mathrm{d}s' \tag{52}$$

The writhe only depends on the axis curve $\boldsymbol{x}$, and equals the signed number of crossings of the axis curve with itself, averages over all possible projection angles. Thus the writhe is independent of the direction of the axis curve.

The twist is given by

$$\mathcal{T} = \frac{1}{2\pi}\oint_X \hat{T}(s)\wedge\hat{V}(s')\wedge\frac{\mathrm{d}\hat{V}(s)}{\mathrm{d}s}\,\mathrm{d}s = \frac{1}{2\pi}\oint_X \frac{\hat{T}(s)}{|\boldsymbol{v}|^2(s)}\cdot\boldsymbol{v}(s)\wedge\frac{\mathrm{d}\hat{\boldsymbol{v}}(s)}{\mathrm{d}s}\,\mathrm{d}s, \qquad \boldsymbol{v} = \epsilon\hat{V}. \tag{53}$$

The twist has a local density along the curve, so that the quantity

$$\mathcal{T} = \frac{1}{2\pi}\oint \frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s}\,\mathrm{d}s \tag{54}$$

is well-defined. Here, $\mathrm{d}\mathcal{T}/\mathrm{d}s$ measures the rotation rate of the secondary curve about the axis curve. At each point on $\boldsymbol{x}$, consider the plane perpendicular to $\hat{T}(s)$. The offset vector $\hat{V}(s)$ lives in this plane, and rotates at a rate

$$\hat{T}(s)\cdot\hat{V}(s)\wedge\frac{\mathrm{d}\hat{V}(s)}{\mathrm{d}s} = 2\pi\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s}. \tag{55}$$

Consider a ribbon whose edges are two neighbouring magnetic field lines. In this case, the twist is related to the parallel electric current. Let $\boldsymbol{J} = \nabla\wedge\boldsymbol{B}/\mu_0$ by associated with $\boldsymbol{B}$, and $J_\parallel = \boldsymbol{J}\cdot\boldsymbol{B}/|\boldsymbol{B}|$, then

$$\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s} = \frac{\mu_0}{4\pi}\frac{J_\parallel}{|\boldsymbol{B}|}. \tag{56}$$

Similar, if we measure how much two neighbouring flow lines in a field twist about each other, then

$$\frac{\mathrm{d}\mathcal{T}}{\mathrm{d}s} = \frac{\mu_0}{4\pi}\frac{\omega_\parallel}{|\boldsymbol{u}|}, \tag{57}$$

where $\boldsymbol{u}$ is the fluid velocity and ω is the vorticity. Note that $\mathcal{T}$ is independent of the direction of the axis curve. For example, suppose the axis is a vertical line, and the secondary is a right helix (with positive twist). Turning the two upside down will still give a right helix of the same pitch.

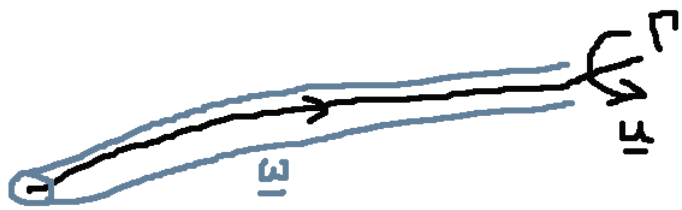
V. VORTEX DYNAMICS

We discuss motion of vortex filaments, building on the earlier work on helicity and transport of geometric properties of space curves. A vortex filament is an idealisation of a thin tube of vorticity whose internal structure we wish to ignore. It is assumed that it stays thing and moves in its own velocity field. If filament is straight, there is no motion of the filament: although obviously the flow around it is rotating, its average effect is zero. On the other hand, if the tube is curves, then there is average motion of the filament, proportional to its curvature, in the direction of its binormal B. Retaining just this component at leading order yields the local induction approximation (LIA). This has linkes to simulations of superfluid motions. We will discuss how the model breaks down and how this links to the million dollar question of regularity of the Navier–Stokes equation.

Given any vorticity distribution $\boldsymbol{\omega}(\boldsymbol{r})$ (we assume time independence for now), the Biot-Savart law gives

$$\boldsymbol{u}(\boldsymbol{r}) = -\frac{1}{4\pi}\int_D \frac{(\boldsymbol{r}-\boldsymbol{r}')\wedge\boldsymbol{\omega}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}V'. \tag{58}$$

Analogous to what we did for $\boldsymbol{A}$ and $\boldsymbol{B}$, we suppose $\boldsymbol{\omega}$ is confined to a thin tube with centre C and that the total circulation is Γ.



We can then write

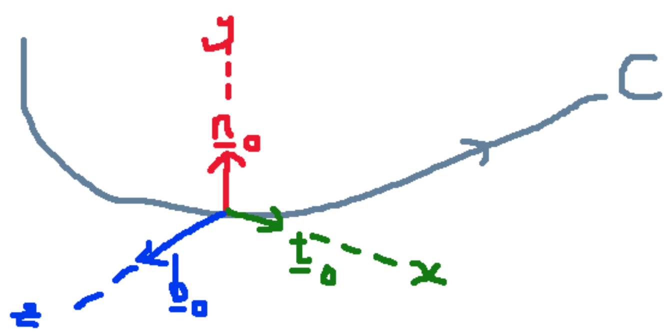
$$\boldsymbol{u}(\boldsymbol{r}) = -\frac{\Gamma}{4\pi}\int_D \frac{(\boldsymbol{r}-\boldsymbol{r}')\wedge\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}, \tag{59}$$

where $\boldsymbol{r}'$ is a dummy variable, transversing C. Now, suppose we use a Serret–Frenet basis along a curve. If s is the arc length and $\boldsymbol{r}(s)$ gives the position on C, we have the right-handed orthonormal basis $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ with

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \boldsymbol{t}, \qquad \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \kappa\boldsymbol{n}, \qquad \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}s} = -\kappa\boldsymbol{t}+\tau\boldsymbol{b}, \qquad \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}s} = -\tau\boldsymbol{n}. \tag{60}$$

We need to consider a vortex along C and work out the flow field. There will be circulatory motion around the vortex, which increases the close we are to the tube, but this will only rotate the vortex and cannot translate it. We will have subtract off this motion to find the mean velocity of the filament.

Consider a local piece of vortex filament, based at the origin without loss of generalisation, and use $\{x,y,z\}$ corresponding to $\{\boldsymbol{t}_0,\boldsymbol{n}_0,\boldsymbol{b}_0\}$ as the co-ordinates.



We will seek the velocity for a point $\boldsymbol{r} = y\boldsymbol{n}_0 + z\boldsymbol{b}_0$ in the plane perpendicular to the vortex at the origin. Along the curve, we have approximately, $\boldsymbol{r} = s\boldsymbol{t}_0 + (\kappa s^2)/2\boldsymbol{n}_0 + \ldots$, where s is the arc length. This is because

$$\boldsymbol{t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \boldsymbol{t}_0 + \kappa s\boldsymbol{n}_0 + \ldots, \qquad \boldsymbol{n} = \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = \kappa\boldsymbol{n}_0 + \ldots \tag{61}$$

Substituting this for the line vortex curve for $\boldsymbol{r}'$ with

$$\mathrm{d}\boldsymbol{r}' = (\boldsymbol{t}_0 + \kappa s \boldsymbol{n}_0 + \ldots)\mathrm{d}s,$$

so

$$\boldsymbol{r} - \boldsymbol{r}' = -s\boldsymbol{t}_0 + \left(y - \frac{1}{2}\kappa s^2\right)\boldsymbol{n} + z\boldsymbol{b}_0 + \ldots,$$

and so

$$(\boldsymbol{r}' - \boldsymbol{r}) \wedge \mathrm{d}\boldsymbol{r} = \left[-z\kappa s\boldsymbol{t}_0 + z\boldsymbol{n}_0 - \left(y + \frac{1}{2}\kappa s\right)\boldsymbol{b}_0 + \ldots\right]\mathrm{d}s, \tag{62}$$

as well as

$$|\boldsymbol{r} - \boldsymbol{r}'|^2 = y^2 + z^2 + s^2(1 - \kappa y) + \frac{1}{4}\kappa^2 s^4 + \ldots \tag{63}$$

We can now put together the integral to give the velocity at $(0, y, z)$ from the local portion $s \in [-L, L]$ of the curve, as

$$\boldsymbol{u} = -\frac{\Gamma}{4\pi}\int_{-L}^{L} \frac{-z\kappa s\boldsymbol{t}_0 + z\boldsymbol{n}_0 - (y + \kappa s^2/2)\boldsymbol{b}_0}{[y^2 + z^2 + s^2(1 - \kappa y) + \kappa^2 s^4/4]^{3/2}}\,\mathrm{d}s. \tag{64}$$

Now, $\boldsymbol{u}$ in $\boldsymbol{t}_0$ vanishes as it is an odd function, so setting $y = \sigma\cos\phi$ and $z = \sigma\sin\phi$ gives

$$\boldsymbol{u} = \frac{\Gamma}{4\pi}\int_{-L}^{L} \frac{\sigma(\boldsymbol{b}_0\cos\phi - \boldsymbol{n}_0\sin\phi) + \kappa s^2/2\boldsymbol{b}_0}{[\sigma^2 + s^2(1 - \kappa\sigma\cos\phi) + \kappa^2 s^4/4]^{3/2}}\,\mathrm{d}s. \tag{65}$$

Substituting $\zeta = s/\sigma$ gives

$$\boldsymbol{u} = \frac{\Gamma}{4\pi}\int_{-L/\sigma}^{L/\sigma} \frac{\sigma^{-1}(\boldsymbol{b}_0\cos\phi - \boldsymbol{n}_0\sin\phi) + \kappa\zeta^2/2\boldsymbol{b}_0}{[1 + \zeta^2(1 - \kappa\sigma\cos\phi) + \kappa^2\sigma^2\zeta^4/4]^{3/2}}\,\mathrm{d}\zeta. \tag{66}$$

We have in mind that $\sigma \to 0$ as we are interested $\boldsymbol{u}$ very close to the vortex filament. Approximating the denominator then gives

$$\boldsymbol{u} = \frac{\Gamma}{4\pi}\int_{-L/\sigma}^{L/\sigma} \frac{\sigma^{-1}(\boldsymbol{b}_0\cos\phi - \boldsymbol{n}_0\sin\phi) + \kappa\zeta^2/2\boldsymbol{b}_0}{(1 + \zeta^2)^{3/2}}\,\mathrm{d}\zeta. \tag{67}$$

The first integral scaling as $(1 + \zeta^2)^{-3/2}$ is easily done and converges as we take the limits L/σ to infinity. The second integral scaling as $\zeta^2(1 + \zeta^2)^{-3/2}$ has a logarithmic divergence at the upper and lower limit. The resulting integral is

$$\boldsymbol{u} = \frac{\Gamma}{2\pi\sigma}(\boldsymbol{b}_0\cos\phi - \boldsymbol{n}_0\sin\phi) + \frac{\Gamma\kappa}{4\pi}\boldsymbol{b}_0\log\frac{L}{\sigma}. \tag{68}$$

The first term is the local circulation around the vortex line, with a σ^{-1} increase as the line is approached, which will not give rise to the motion of the filament itself. Only the second term can do this, and we have a motion in the direction in $\boldsymbol{b}_0$, at a velocity proportional to the local curvature κ.

The second term is problematic as it depends on the <u>vortex scale</u> σ and the <u>cutoff scale</u> L. Essentially, the thinner the core,the faster the vortex will move. The notion of infinitely thin line vortices does not make sense in fluid mechanics, unless it is straight. Also, the cross sectional width of a vortex will not be uniform in general and could vary in time, especially if it is being stretched. This means the approximations we have introduced should be understand to have many limitations.

Nonetheless, one can proceed by taking $\log(L/\sigma) = \text{const}$. Being a log, it certainly varies slowly with L/σ. We then idealise the vortex as localised around $C(t)$ with positions $\boldsymbol{r}(s, t)$ and take the velocity of the vortex filament $\boldsymbol{v}$ to be proportional to $\kappa\boldsymbol{b}$,