Academic notes: Discrete maths

J. Mak (December 18, 2015) [From notes of S. K. Darwin, Durham]*

I. NOTATION

The notation

$$S = \{n \in \mathbb{Z} \mid n/3 \in \mathbb{Z}\}$$

is to be read as $n \in \mathbb{Z}$ such that $n/3$ is also in $\mathbb{Z}$; in this case, $S = \{0, \pm 3, \pm 6, \ldots\}$. A collection/family of sets can be indexed as A_1, A_2 and so on. Then

$$\bigcup_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for some } 1 \le i \le n\}, \qquad \bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for all } 1 \le i \le n\},$$

where $\bigcup$ is the union over all sets, and $\bigcap$ is the intersection over all sets.

If X and Y are sets, then the Cartesian product $X \times Y$ is

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Note this is an ordered set.

The size (the number of elements) of the set X is denotes $|X|$.

Proposition I.1 $|X \times Y| = |X| \times |Y|$, *and* $|A^n| = |A|^n$.

The power set of A is the set of all subsets of A; it may be shown that the power set has size $2^{|A|}$ (an element can either be included or not, hence the 2).

II. COMBINATORICS

A. Lists and permutations

A list is a sequence of objects where order is essential. Formally, a list is written with square brackes, e.g. $[1, 2, -1, 2]$. The length of a list is the number of objects within the list.

Example How many lists of length k can we make from elements of a set S with $|S| = n$?

The solution is n^k from using the previous proposition. Another way of thinking of this is that there are k slots available, and n choices of each slot.

Example The number of four letter works that can be made from the English alphabet is 26^4, as there are 26 possibilities each time. Suppose no letter is to be repeated, then we have $26 \cdot 25 \cdot 24 \cdot 23 = 358000$ ways; alternatively, it is $26!/22!$.

This leads on to the multiplcation principle: if for $1 \le i \le k$, $i \in \mathbb{Z}$, there are n_i choices for the i^{th} object, then the number of lists of length k is given by $\prod_{i=1}^{k} n_i$.

Example: Canadian postcode How many length six lists eixst where:

- 1st, 3rd and 5th are distinct letters;
- 2nd, 4th and 6th are digits;

* julian.c.l.mak@googlemail.com

- 2^{nd} and 4^{th} are distinct?

The first, third and fifth slot has $26 \cdot 25 \cdot 24$ possibilities. The second and sixth slot each has 10 possibilities, while the second slot only has 9 possibilities, so there are $(26!/22!) \cdot 10^2 \cdot 9$ ways.

A permutation is a reshuffling of the list where ordering matters.

Proposition II.1 *If $|S| = n$, then there are $n!$ permutations of S. A k permutation on S is a list of k distinct elements taken from S. If $|S| = n$, then there are $P(n, k) = n!/(n-k)!$ permutations.*

Example Using the English alphabet, how many four letters words are there:

1. using 'E' at least once;
2. using 'E' exactly once;
3. never three consecutive letters the same;
4. use 'E' and 'T' at least once?

1. There are a total of 26^4 permutations and $(26-1)^4$ permutations that does not include 'E', so $26^4 - 25^4$.
2. 'E' can go into one of four slots and there are $25^{(}4-1)$ remaining choices, so $4 \cdot 25^3$.
3. The excluded options are $xxxy$ ($26 \cdot 25$ ways), $yxxx$ ($25 \cdot 26$ ways) and $xxxx$ (26 ways), so it is $26^4 - (26 \cdot 51)$ ways.
4. So here we want the to take the total (26^4 ways) subtracted by the following cases: (i) neither 'E' nor 'T' ($(26-2)^4$ ways); (ii) 'E' but no 'T'; (iii) 'T' but no 'E'. For the latter two cases, we want no 'T' (25^4 ways) minus 'neither 'E' nor 'T' (24^4 from case i). The same is true for the other case, so we have $26^4 - 2(25^4 - 24^4) - 24^4$ ways.

Example How many three letters words are there in strictly alphabetical order (i.e., no repeats)?
The number of distinct three letter words is $26 \cdot 25 \cdot 24$. Out of the possible $3! = 6$ permutations, only one of them will be in strict alphabetical order, so it is $26 \cdot 25 \cdot 24/6$ ways.

This example follows the addition principle: If X is a set of results which divides $X = \bigcup_{i=1}^{n} X_r$, where $X_i \cap X_j = \emptyset$ for $i \neq j$, then $|X| = \sum_{i=1}^{n} |X_i|$.

B. Combinations

A combination is a collection of objects that are un-ordered. For example, STOAT and TOAST are different permutations but same combination. Often, combinations do not allow repetitions.

Example From the set $\{P, Q, R, S\}$, how many combinations are there of

1. two distinct letters;
2. three distinct letters?

1. PQ is to be regarded as the same as QR, so we have PQ, PR, PS, QR, QS, RS, so six ways.
2. PQR, PQS, PRS, QRS, so four ways.

Generally speaking, from a set S with $|S| = n$, the number of combinations of size k is $C(n, k) = P(n, k)/k! = n!/[(n-k)!k!]$; there are $P(n, k)$ lists of length k, for which there are $k!$ permutations of those lists that are to be regarded as the same.

Example How many combinations are there to select (i) 10, (ii) 16 from the English alphabet?
We have

$$C(26, 10) = \frac{26!}{16!10!} \qquad \text{and} \qquad C(16, 10) = \frac{26!}{10!16!},$$

i.e., the same number of choices (cf. "choose 10 from 26" and "out of 26, do not choose 10").

III. INDUCTION, PIGEOHOLE PRINCIPLE AND INCLUSION-EXCLUSION PRINCIPLE

IV. RECURRENCE RELATIONS AND GENERATION FUNCTIONS

V. FINITE STATE MACHINES

VI. GRAPH THEORY