Calculus 1H

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- *Last compiled: May 2021*
- Adapted from notes of A. Taormina and I. MacPhee, Durham
- This was part of the Durham Core A module given in the first year. Re-arranged here slightly as some of the triple integral stuff was actually given in the probability section (for time constraint reasons I guess). Basic concepts of calculus and analysis, going up to multi-dimensional integrals.

- **TODO!** diagrams

1 Functions

A value which depends on another value implies something is a **function** or another. For example, we may have day d and temperature T, assuming that day affects the temperature. Then $t = f(d)$ and has a unique value of temperature associated with it. Finding a function which represents a given situation is called **mathematical modelling**.

The **domain** is the set of values the independent variable can take. The **range** or **image** is the set of values the dependent variable can take. More formally, assuming we are dealing with functions of a real variable x,

$$\begin{aligned} \text{dom } f &= \{x \in \mathbb{R} \mid f(x) \text{ exists}\}, \\ \text{im } f &= \{y \in \mathbb{R} \mid y = f(x),\ x \in \text{dom } f\}. \end{aligned}$$

Therefore we have

$$f : \text{dom } f \to \text{im} f, \quad x \mapsto f(x) = y.$$

Example For $f(x) = |x|$, dom $f = \mathbb{R} = (-\infty, +\infty)$, whilst im $f = [0, \infty)$, that is, including 0. For $f(x) = \sqrt{1 - x^2}$, dom $f = [-1, 1]$ and im $f = [0, 1]$.

The **graph** of f is defined to be the set $\{(x, y) \mid x \in \text{dom } f,\ y = f(x)\}$. We say a function is not **well-defined** if it is multi-valued for a given element in the domain; for example, if the branch is not defined, then the square root function is ill-defined.

If $f(x) = f(-x)$ for all $x \in$ dom f, then we say the function is **even**. If $f(x) = -f(-x)$ for all $x \in$ dom f, then we say the function is **odd**. For example, x^2 is even whilst x^3 is odd. Even functions are symmetrical about the y-axis, whilst odd functions are symmetrical about the origin.

Example It may be shown that

$$f(x) = \begin{cases} x^3, & x > 0, \\ 0, & x = 0, \\ -x^3, & x < 0 \end{cases}$$

is an even function.

Note that some functions are neither even nor odd. Also, in some calculations, it pays to spot the parity of the functions (e.g., when integrating over a symmetric domain).

any function $f(x)$ may be written as the sum of an even and odd function, as

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

We notice that the product of two odd/even functions are even, whilst the product of an odd and even function is odd.

For $f(x)$ a function with $a, b, \in \mathbb{R}$ and $\alpha \in \mathbb{R}_0^+$, some basic manipulations are as follows:

- $f(x - a)$ shifts the graph by $(a, 0)$, i.e., a translation in x;
- $f(x) + b$ shifts the graph by $(0, b)$, i.e., a translation in y;
- $f(-x)$ reflects the graph in the y-axis;
- $-f(-x)$ reflects the graph about the origin;
- $|f(x)|$ reflects any part of the graph with $f(x) < 0$ about the x-axis;
- $af(x)$ stretches the graph in the vertical by α;
- $f(\alpha x)$ stretches the graph in the horizontal by $1/\alpha$.

1.1 *Combination of functions*

A function f is given by (i) a set dom f, (ii) a set im f, (iii) by a **rule** which assigns, to each element of dom f, at most one element $f(x)$ in im f. Then we have the following:

- the **sum** of f and g is defined to be

$$(f+g)(x) = f(x) + g(x), \qquad \operatorname{dom}(f+g)(x) = \operatorname{dom} f \cap \operatorname{dom} g;$$

- the **product** of f and g is

$$(f \cdot g)(x) = f(x) \cdot g(x), \qquad \operatorname{dom}(f \cdot g)(x) = \operatorname{dom} f \cap \operatorname{dom} g;$$

- the **quotient** is

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \qquad g(x) \neq 0,$$
$$\operatorname{dom}\left(\frac{f}{g}\right)(x) = \operatorname{dom} f \cap \operatorname{dom} g.$$

- **scalar multiplication** by $\alpha \in \mathbb{R} - \{0\}$ gives

$$(\alpha f)(x) = \alpha f(x), \qquad \text{dom}(\alpha f) = \text{dom } f;$$

- **linear combinations** then give

$$\begin{aligned}(\alpha f + \beta g)(x) &= \alpha f(x) + \beta g(x), \\ \text{dom}(\alpha f + \beta g) &= \text{dom } f \cap \text{dom } g;\end{aligned}$$

- f **composed** with g is

$$(g \circ f)(x) = g(f(x)), \qquad f(x) \in \text{dom } g$$

  for the composition to exist. Note that $(g \circ f) \neq (f \circ g)$ necessarily.

Example Take $f(x) = x^2 - 1$, $g(x) = \sqrt{3-x}$, we have

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{3-x}) \\ &= (\sqrt{3-x})^2 - 1 \\ &= 2 - x, \qquad \text{dom}(f \circ g) = (-\infty, 3],\end{aligned}$$

while

$$\begin{aligned}(g \circ f)(x) &= g(x^2 - 1) \\ &= \sqrt{3 - (x^2 - 1)} \\ &= \sqrt{4 - x^2}, \qquad \text{dom}(g \circ f) = [-2, 2].\end{aligned}$$

1.2 *Inverses*

A function $f : A \to B$ is **injective** iff, for all $x_1, x_2 \in \text{dom } f$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. So if f is injective, then any element of B is the image of at most one element in A.

Example Consider $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2 + x - 6$. By drawing the graph of $f(x)$, we realise that g is not injective because for $y = 0$, $x = 2, -3$.

A function $f : A \to B$ is **surjective** iff for $y \in B$, there exists at least one element in A with $y = f(x)$. A function is **bijective** if f is both injective and surjective.

Theorem 1.2.1 *If f is an injective function, then there exists a function f^{-1} such that dom f^{-1} = im f, and $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$, with $x \in$ im f.* □

f^{-1} is called the **inverse** of f in the sense of function composition.

Example Suppose $f(x) = \sin x$, then $f(x) : [-\pi/2, \pi/2] \to [-1, 1]$. Then $f^{-1}(x) = \arcsin x$, with $f^{-1}(x) : [-1, 1] \to [-\pi/2, \pi/2]$.

1.3 *Periodic and hyperbolic functions*

A function is **periodic** if it satisfies $f(x+p) = f(x)$ for $p \in \mathbb{R} - \{0\}$. The smallest value of such a p is known as the **period** of the function. For example, $\sin x$ and $\tan x$ are periodic with period 2π and π respectively.

The **hyperbolic sine** and **hyperbolic cosine** function are

$$\sinh x = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}, \qquad \cosh x = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2},$$

and notice they are odd and even functions respectively. We also have **hyperbolic tangent** function to be $\tanh x = \sinh x / \cosh x$. Several properties to note:

- $\sinh 0 = 0$ and $\cosh 0 = 1$;
- $\cosh^2 x - \sinh^2 x = 1$;
- the two functions traces out the right-hand portion of a hyperbola;
- $0 < \sinh x < \cosh x$ for $x > 0$;
- $\mathrm{e}^{x} = \cosh x + \sinh x$ while $\mathrm{e}^{-x} = \cosh x - \sinh x$.

2 Limits, continuity and differentiation

2.1 Limits

A function may have a **limit** even if the the limit is not in the image of the function, or that the input value associated with the limit is not in the domain of the function.

Example For $f(x) = (x^2 - 9)/(x - 3)$, dom $f = \mathbb{R} - \{3\}$. However, we observe that $f(x) = x + 3$, so $\lim_{x\to 3} f(x) = 6$.

In $\mathbb{R}$, a limit may be approach from below or above, and we may have the case that

$$\lim_{x\nearrow c} f(x) = a, \qquad \lim_{x\searrow c} = b,$$

but a does not have to be equal to b necessarily (e.g., at a discontinuity). A well-defined limit only exists if $a = b$.

Example So the following do not have well-defined limits:

1. $f(x) = 1/(x - 2)$ for $x \to 2$. We have that, for $x > 2$, $f(x) > 0$, while for $x < 2$, $f(x) < 0$, so that

   $$\lim_{x\searrow 2} = +\infty, \qquad \lim_{x\nearrow 2} = -\infty,$$

   and limit does not exist.

2. $f(x) = \sin \pi/x$ for $x \searrow 0$. We may approach x as $\{1/n\}$, in which case $f(x) \to 0$, whilst $\{1/(4n - 1)\}$ gives $f(x) \to 1$, and $\{2/(4n - 1)\}$ gives $f(x) \to -1$, so limit does not exist as $x \to 0$.

   In summary:

- $\lim_{x\nearrow c} \neq \lim_{x\searrow c}$;
- $\lim f(x) = \pm\infty$;
- $f(x)$ oscillates as $\lim_{x\nearrow c}$ or $\lim_{x\searrow c}$.

2.2 Calculus of limits

There are several ways to calculates $\lim_{x\to c} f(x)$, the easiest of which is just just evaluate $f(c)$ if it exists. For $l, m, k \in \mathbb{R}$ and $\lim_{x\to b} f(x) = l$, $\lim_{x\to b} = m$, we have

- $\lim_{x\to b} f(x) \pm g(x) = l \pm m$ (**sum/difference**);
- $\lim_{x\to b} f(x) \cdot g(x) = lm$ (**product**);
- $\lim_{x\to b} f(x)/g(x) = l/m$ for $m \neq 0$ (**quotient**);
- $\lim_{x\to b} [f(x)]^{r/s} = l^{r/s}$ for $s \neq 0$ (**power**).

  Otherwise, there are several things we could try.

2.2.1 Eliminating zero from denominator

Sometimes it is possible to remove the apparently singularity. Such as

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2-x} = \lim_{x\to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x\to 1} \frac{x+2}{x} = 3,$$

or

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} &= \lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2} \frac{\sqrt{x^2+100}+10}{\sqrt{x^2+100}+10} \\ &= \lim_{x\to 0} \frac{1}{\sqrt{x^2+100}+10} = \frac{1}{20}.\end{aligned}$$

2.2.2 Squeezing theorem

Theorem 2.2.1 *If $g(x) \leq f(x) \leq k(x)$ for $x \in (a, c)$, where $b \in (a, c)$, then if $g(x) \to l$ and $k(x) \to l$ for $x \to b$, then $\lim_{x\to b} f(x) = l$.* □

Example Find the limit of:

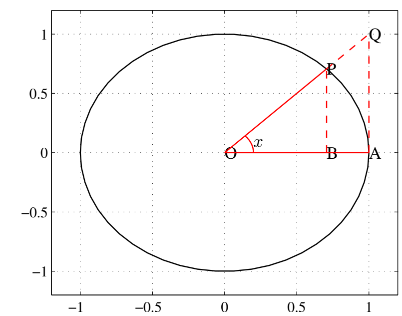
1. $(\sin x)/x$ for $x \to 0$.

   Consider the circle of radius 1 centred at the origin. For some angle $x \in [0, \pi/2]$, suppose we trace out the following.

   The area of the triangle OAP is $(1/2)(BP)(OA) = (1/2)\sin x$, while the area of the triangle OQA is $(1/2)(OA)(AQ) = (1/2)\tan x$. The sector OAP has area $x/2$, and since the sector area is bounded between the are of the two triangles, we have

   $$\frac{1}{2}\sin x < \frac{x}{2}, \qquad \frac{x}{2} < \frac{1}{2}\tan x \qquad \Rightarrow \qquad \frac{\sin x}{x} < 1, \qquad \cos x < \frac{\sin x}{x},$$

   so $\cos x < (\sin x)/x < 1$, and by squeezing theorem, $\lim_{x\to 0}(\sin x)/x = 1$.



2. $(1 - \cos x)/x$ for $x \to 0$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x} \frac{1 + \cos x}{1 + \cos x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \frac{\sin x}{1 + \cos x} \right) = 1$$

by the product rule.

2.2.3 *Rational functions when $x \to \pm\infty$*

Example Find the limits of the following:

1. a rational function where the degree of the numerator is smaller than the denominator,
$$\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^3 + 1} = \lim_{x \to \infty} \frac{5 + 8/x - 3/x^2}{3x + 1/x^2} = 0.$$
2. a rational function where the degree of the numerator is equal to the denominator,
$$\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 1} = \lim_{x \to \infty} \frac{5 + 8/x - 3/x^2}{3 + 1/x^2} = \frac{5}{3}.$$
3. a rational function where the degree of the numerator is less than the denominator,
$$\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x + 1} = \lim_{x \to \infty} \frac{5 + 8/x - 3/x^2}{3/x + 1/x^2} = \infty.$$

2.3 *Continuity*

A function f is **continuous** at a point c if (i), $f(c)$ is defined, (ii) the limit of $f(x)$ as $x \to c$ exists, and (iii), if the limit is there is $f(c)$. Trigonometric functions and polynomials are considered to be continuous. On an interval, if f is continuous at on (a, b), then f is continuous at each point in (a, b). If f is defined on $[a, b]$, f is continuous on $[a.b]$ if (i) f is continuous at (a, b), and (ii) if the right and left limits exists.

Discontinuities may be (i) **removable** (discontinuity at a point), (ii) a **jump** (discontinuity at an interval), (iii) **infinite** (vertical asymptotes).

Theorem 2.3.1 (Intermediate value theorem) *If f is continuous at (a, b) and k is any number between $f(a)$ and $f(b)$, then there exists at least one value $c \in (a, b)$ such that $f(c) = k$.* □

Example Show that $f(x) = (\cos x\pi/2) - x^2$ for $x \in [0, 1]$ has at least one root in $(0, 1)$.

Trigonometric and polynomials are continuous, so $f(x)$ is continuous here. Since $f(0) = 1$ and $f(1) = -1$, there must be at least one $c \in (0, 1)$ such that $f(c) = 0$ by the intermediate value theorem.

2.4 *Differentiation*

We define the **derivative** of $f(x)$ to be

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

The derivative exists provided the limit exists. Note that h can be of either sign, and that dom $f' \subset$ dom$f$ since

$$\text{dom } f' = \left\{ x \in \text{dom } f \mid \text{there exists } \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \right\}.$$

Example By definition, with $f(x) = \sqrt{x}$,

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Proposition 2.4.1 *If f is differentiable at x, then f is continuous at x.*

Proof We have

$$\lim_{h\to 0} [f(x+h) - f(x)] = \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \right] \cdot \lim_{h\to 0} h.$$

Since f is differentiable, the first limit on the right hand side is finite, and thus $\lim_{h\to 0}[f(x+h) - f(x)] = 0$. So $\lim_{h\to 0} f(x+h) = f(x)$, and f is continuous. ■

The converse is not true. for example, taking

$$f(x) = |x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0, \end{cases}$$

then at $x = 0$,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{|h|}{h},$$

which depends on the sign of h, so the derivative is not well-defined at $x = 0$ (although it is continuous there).

Example Is the function

$$f(x) = \begin{cases} x, & x \leq 1, \\ (x+1)/2, & x \geq 1, \end{cases}$$

differentiable on (i) $[0,2]$, (ii) $[1,2]$?

It may be checked that the two one-sided limits do not agree on $x = 1$, so it is not differentiable on $[0,2]$. However, only the one sided limits are required for $[1,2]$, so the function is differentiable on that domain.

Proposition 2.4.2 *Let f and g be differentiable at x, then the following are differentiable there also:*

- $f + g$;
- αf *where* $\alpha \in \mathbb{C}$;
- fg, *with* $(fg)' = f'g + fg'$ (**product rule**);
- f/g, *with* $(f/g)' = (fg' - fg')/g^2$ (**quotient rule**, *or write* $1/g = g^-1$ *and use chain rule)*;
- $(g \circ f)$, *with* $(g \circ f)' = g'(f(x)) \cdot f'$ (**chain rule**). $\square$

2.5 *Higher order derivatives and Taylor approximations*

We recall that

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Geometrically, this says that as $h \to 0$, the chord linking $f(x_0 + h)$ and $f(x_0)$ approaches the tangent to the curve at x_0. The line containing $(x_0, f(x_0))$ and whose slope is $f'(x_0)$ is the tangent line to the graph $f(x)$ at $(x_0, f(x_0))$. The tangent has the equation

$$y - f(x_0) = f'(x_0)(x - x_0),$$

and one can approximate $f(x)$ near x_0 by writing

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

In this case, $f(x)$ is approximated by a polynomial of degree 1 in $(x - x_0)$, called the **Taylor polynomial of degree** 1.

One can obtain better approximations to $f(x)$ near x_0. Denoting $f^{(n)}(x)$ to be the n^{th} derivative of f, the **Taylor polynomial of degree** n is given by

$$\begin{aligned} P_n(x - x_0) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \\ &= \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j. \end{aligned}$$

Example Find the Taylor polynomials of degree n for $f(x)$ near x_0:

1. $f(x) = \cos x$, $n = 4$, $x_0 = 0$.

   Note that the odd derivatives give a $\sin x$ which is zero at $x = 0$, and since $\cos 0 = 1$,

   $$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!},$$

   which is only valid near $x = 0$.

2. $f(x) = e^x$, $n = 3$, $x_0 = 1$.

$$P_3(x) = e + e(x-1) + \frac{e}{2!}(x-1)^2 + \frac{e}{3!}(x-1)^3.$$

Theorem 2.5.1 (Taylor's theorem) *If f has $n+1$ derivatives that are continuous on an open interval I, $x_0 \in I$ for all $x \in I$, then*

$$f(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + R_n(x),$$
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}, \qquad c \in (x_0, x),$$

where this is an equality rather than an approximation. □

$R_n(x)$ is the remainder and it is important to bound $R_n(x)$. One useful one is

$$|R_n(x)| \leq \frac{|x-x_0|^{n+1}|}{(n+1)!} \max_{t\in(x_0,x)} \left| f^{(n+1)}(t) \right|.$$

If $R_n(x) \to 0$ as $n \to \infty$, then

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j,$$

and the right hand side is a **Taylor series expansion** near x_0 the converges to $f(x)$.

We have the following useful expansions

1. if $f(x)$ is a polynomial then trivially the Taylor series is $f(x)$ for all $x \in \mathbb{R}$;

2. $$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + \frac{x^2}{2!} + \cdots, \qquad x \in \mathbb{R};$$

3. Remembering that $\cosh x + \sinh x = e^x$, we have that
$$\cosh x = \sum_{j=0}^{\infty} \frac{x^{2j}}{(2j)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$$
$$\sinh x = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

4. using $e^{i\theta} = \cos\theta + i\sin\theta$, we have that
$$\cos x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$$
$$\sin x = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

5.
$$\log(1+x) = \sum_{j=1}^{\infty}(-1)^{j-1}\frac{x}{j} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad |x| < 1$$

(note that $x = 1$ is actually defined, with $\log 2 = 1 - 1/2 + 1/3 + \cdots$);

6.
$$(1+c)^c = 1 + \sum_{n=1}^{\infty}\binom{c}{n}x^n, \qquad \binom{c}{n} = \frac{c(c-1)\cdots(c-n+1)}{n!}.$$

7.
$$\frac{1}{1-x} = 1 + \sum_{j=1}^{\infty}x^n, \qquad |x| < 1.$$

Example Calculate a remainder estimate for $f(x) = \cos x$ near $x = 0$.
Since the derivative of $\cos x$ is bounded above by 1, we have

$$|R_n(x)| \leq 1 \cdot \frac{|x|^{n+1}}{(n+1)!}.$$

Fixing x and choosing $k \in \mathbb{Z}$ with $k > |x|$ and $n > k+1$, we have

$$\frac{k^n}{k!} = \left(\frac{k^k}{k!}\right)\left(\frac{k}{k+1} + \frac{k}{k+2} + \cdots \frac{k}{n-1}\right)\left(\frac{k}{n}\right) < \frac{k^{k+1}}{k!}\frac{1}{n}.$$

Since $k > |x| \geq 0$, we have

$$0 \leq \frac{|x|^{n+1}}{(n+1)!} < \frac{k^n}{k!} < \frac{k^{k+1}}{k!}\frac{1}{n},$$

so $R_n(x) \to 0$ as $n \to \infty$ by squeezing.

2.6 *Taylor approximations and limits*

Let g by a single variable function defined on an interval containing h. Then $g(h)$ is said to be $o(h)$ if $\lim_{h\to 0} g(h)/h = 0$.

Example Find the limit of $\lim_{x\to 0}(\sin x)/x$.
$\sin x$ admits the Taylor series $\sin x = x - x^3/3! + o(x^3)$, so

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0} 1 - \frac{x^2}{3!} + o(x^2) = 1,$$

which we know already.

Example Let $f(x) = \sin(\mathrm{e}^{-x}/(x+1) - 1)$. Find the Taylor polynomial of order 3, and hence calculate the limit of $\lim_{x\to 0}(f(x) + 2x)/\sinh x^2$.

We note that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + o(x^3), \qquad (1+x)^{-1} = 1 - x + x^2 - x^3 + o(x^3).$$

Then

$$\frac{e^{-x}}{x+1} - 1 = -2x + \frac{5x^2}{2} - \frac{8x^3}{3} + o(x^3) = y.$$

Then since $\sin y = y - y^3/3! + o(y^3)$, we have

$$f(x) = -2x + \frac{5}{2}x^2 - \frac{4}{3}x^3 + o(x^3).$$

Now, $\sinh x^2 = x^2 + o(x^3)$, so

$$\begin{aligned} \lim_{x\to 0} \frac{f(x) - 2x}{\sinh x^2} &= \lim_{x\to 0} \frac{5x^2/2 - 4x^3/3 - 2x + 2x + o(x^3)}{x^2 + o(x^3)} \\ &= \lim_{x\to 0} \frac{5/2 - 4x^2/3 + o(x)}{1 + o(x)} = \frac{5}{2}. \end{aligned}$$

Example What is the maximum error when using $P_6(x)$ to approximate e^x, $x \in [0,1]$?

With $e^x = P_6(x) + R_6(x)$, we note that $\max_{t\in(0,1)} |f^{(7)}(t)|$ occurs when $t = 1$, i.e., e, so

$$|R_6(x)| \leq e\frac{|x|^7}{7!} < \frac{e}{7!},$$

and the error is no more than $e/7!$.

Example Give an estimate of $e^{0.2}$ correct to 3d.p., i.e., find n where $R_n(x) < 0.0005$.

We use $P_n(x)$ near 0. We have

$$P_n(x) + R_n(x) = 1 + x + \cdots \frac{x^n}{n!},$$

and we want $|R_n(0.2)| \leq 0.0005$. So

$$|R_n(0.2)| \leq e^{0.2}\frac{|0.2|^{n+1}}{(n+1)!} < e\frac{|0.2|^{n+1}}{(n+1)!} = \frac{e}{5^{n+1}(n+1)!} < 0.0005,$$

and we see $n = 3$ satisfies this.

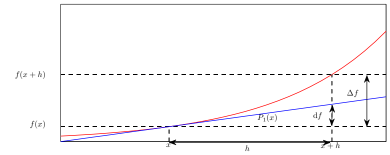
2.7 *Differentials*

By linearising to get

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0),$$

we see that if we take $x_0 = x$, $x = x + h$, we have

$$f(x+h) \approx f(x) + f'(x)h, \qquad \Delta f = f(x+h) - f(x) \approx f'(x)h.$$



Here, Δf is the **increment** of f from x to $x+h$, and $f'(x)h$ is the **differential** with increment h.

P_1 is valid near x, and here $\mathrm{d}f$ is an approximation to Δf. If

$$\frac{\Delta f - \mathrm{d}f}{h} \to 0$$

as $h \to 0$, then we can replace Δf by $\mathrm{d}f$. For example, with $f(x) = x^2$, $\Delta f = h^2 + 2xh$, while $\mathrm{d}f = 2xh$, so $(\Delta f - \mathrm{d}f)/\delta x = \delta x \to 0$.

Let $y = f(x)$ be differentiable. The differentiable $\mathrm{d}x$ is an independent variable (replacing h for $h \to 0$). The differential is then $\mathrm{d}y = f'(x)\mathrm{d}x$, so

$$f'(x) = \frac{\mathrm{d}y}{\mathrm{d}x}.$$

This is known as the **differential form** of the derivative.

2.8 *Implicit differentiation*

When an equation is not expressed as $y = f(x)$ with x as the sole variable, it is an **implicit function**. y depends on x, and we differentiate the whole equation with respect to x.

Example $x^2 + y^2 = 1$ is an implicit function. We have

$$\frac{\mathrm{d}(x^2)}{\mathrm{d}x} + \frac{\mathrm{d}(y^2)}{\mathrm{d}x} = \frac{\mathrm{d}1}{\mathrm{d}x} \qquad \Rightarrow \qquad 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0,$$

and so $\mathrm{d}y/\mathrm{d}x = -x/y$.

Another method to calculate a differential implicitly is to manufacture $F(x,y)$ such that $F(x,y) = 0$ encodes the content of the equation.

Example With the above, we have $F(x,y) = x^2 + y^2 - 1 = 0$. Since $F(x,y) = 0$, by chain rule,

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}F(x,y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x},$$

and we cover the previous result. Note also that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F/\partial x}{\partial F/\partial y}.$$

2.9 *Partial derivatives*

Suppose we have functions of n independent variables. Let D be a set of n-tuple $(x_1, \cdots x_n)$ real valued function on D. The function is a rule which assigns an unique valued $(\omega_1, \cdots \omega_n) = f(x_1, \cdots x_n)$ to each n-tuple on D. ω's here are the dependent variable, and x's are the independent variables. For $n = 2$, we usually use (x,y) or

(x, t) if we are interested in two-dimensional space or space plus time dimension.

The **partial derivative** with are defined analogously as

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}, \qquad \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h},$$

provided the limits exist.

Example Find the partial derivatives of $f(x, y) = 1 - x + y + 3x^2 y$.

From first principles,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \to 0} \frac{1 - (x+h) + y + 3(x+h)^2 y - (1 - x + y + 3x^2 y)}{h} \\ &= \lim_{h \to 0} \frac{-h + 6xyh + 3yh^2}{h} = 6xy - 1. \end{aligned}$$

Or, we treat y as a constant and differentiate with respect to x. Similarly, we have

$$\frac{\partial f}{\partial y} = 1 - 3x^2.$$

Geometrically, we can see that, like the case of one variables, the derivative generates a tangent line to the surface described by $z = f(x, y)$. Since there are two tangent lines for each partial derivative, they span a tangent plan to the surface at (x_0, y_0).

Implicit partial differential is as before.

Example Find $\partial z / \partial x$ of $yz - \log z = x + y$, where z is the independent variable.

We have

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1 \qquad \Rightarrow \qquad \frac{\partial z}{\partial x} = \frac{z}{zy - 1}.$$

Theorem 2.9.1 (Chain rule) *If $\omega = f(x, y)$ has continuous partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$, and if x and y are differentiable functions of t, then the composite function $\omega = f(x(t), y(t))$ is a differentiable function of t, with*

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\partial \omega}{\partial x} \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \omega}{\partial y} \frac{\mathrm{d}y}{\mathrm{d}t}.$$

□

Example $\omega = x \sin$, $x = \cos t$, $y = t^2$, then

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\sin y \sin t + 2xt \cos y = -\sin t^2 \sin t + 2t \cos t \cos t^2.$$

We get the same answer if we first substitute x and y to give $\omega = \cos t \sin t^2$.

The chain rule generalises to functions of more than two variables. In particular, if $\omega = f(x,y,z)$, and x,y,z are differentiable functions of t, we have
$$\frac{d\omega}{dt} = \frac{\partial\omega}{\partial x}\frac{dx}{dt} + \frac{\partial\omega}{\partial y}\frac{dy}{dt} + \frac{\partial\omega}{\partial z}\frac{dz}{dt}$$

Theorem 2.9.2 (Extended chain rule) *Suppose $\omega = f(x,y,z)$, and x,y,z are differentiable functions of u and t, then ω has partial derivatives given by*
$$\frac{\partial\omega}{\partial t} = \frac{\partial\omega}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\omega}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\omega}{\partial z}\frac{\partial z}{\partial t},$$
and similarly for $\partial\omega/\partial u$. □

Example For $\omega = x + 2y + z^2$ with $x = u/v$, $y = u^2 + \log v$, $z = 2u$,

$$\begin{aligned}
\frac{\partial\omega}{\partial u} &= 1\frac{1}{v} + 2(2u) + 2z(2) = 12u + \frac{1}{v},\\
\frac{\partial\omega}{\partial v} &= 1\left(-\frac{u}{v}\right) + 2\frac{1}{v} + 2z(0) = \frac{2v-u}{v^2}.
\end{aligned}$$

2.10 *Mixed partial derivatives*

A function $f(x,y)$ can have partial derivative with respect to x and y at a point without being continuous at that point. The situation is different from the case of single variable functions where existence of derivative implies the continuity of the function. The limiting procedure in taking partial derivatives is such that all independent variables but one are kept constant. The issue of continuity requires that one takes the limit of the type

$$(x_1, \cdots x_n) \to (x_{0,1}, \cdots x_{0,n}).$$

Example Investigate the continuity of the function

$$f(x,y) = \frac{xy + y^3}{x^2 + y^2}$$

at $(x,y) = (0,0)$.

We must explore the various ways of limiting in which $(x,y) \to (0,0)$. Existence of one cases where the limit is inconsistent proves discontinuity.

1. $x = 0$, $y \to 0$ gives $\lim_{y\to 0} f(0,y) = \lim_{y\to 0} y^3/y^2 = 0$;
2. $x \to 0$, $y = 0$ gives $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} 0/x^2 = 0$;
3. $y = 2x$ gives

$$\lim_{x\to 0} f(x,2x) = \lim_{x\to 0} \frac{2x^2 + 8x^3}{5x^2} = \frac{2}{5}.$$

We have an inconsistency, so $f(x,y)$ is not continuous at $(0,0)$.

For $f(x,y)$, we have the following order two derivatives:

$$\frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial y^2}, \qquad \frac{\partial^2 f}{\partial x \partial y}, \qquad \frac{\partial^2 f}{\partial y \partial x},$$

and so on up to order n.

Theorem 2.10.1 *If $f(x,y)$ and all its derivatives are defined throughout an open region containing (a,b) and care continuous at a and b, then*

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$

□

2.11 Differentiability and the gradient

Recall that differentiability for functions of one independent variable requires

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

to exist. To generalise it to n variables, let $f(x_1, \cdots x_n) = f(\boldsymbol{x})$ and $(h_1, \cdots h_n) = \boldsymbol{h}$. Since division of vectors does not exist, we use the alternative form

$$\Delta f = (\mathrm{d}f) \cdot \boldsymbol{h} + o(\boldsymbol{h})$$

to say that f is differentiable if there exists $\boldsymbol{y}$ such that

$$f(\boldsymbol{x} + \boldsymbol{h}) - f(\boldsymbol{x}) = \boldsymbol{y} \cdot \boldsymbol{h} + o(\boldsymbol{h}).$$

$\boldsymbol{y}$ is unique is it exists, and is called the **gradient** of f evaluated at $\boldsymbol{x}$, denoted

$$\boldsymbol{y} = \nabla f(\boldsymbol{x}).$$

Note that $\boldsymbol{y}$ is a vector.

Example Find the gradient of $f(x,y,z) = 2xy - 3z^2$.

$$\begin{aligned} f(\boldsymbol{x} + \boldsymbol{h}) &= 2(x + h_1)(y + h_2) + 3(zh_3)^2 \\ &= 2yh_1 + 2xh_2 - 6zh_3 + 2h_1h_2 - 3h_3^2 \\ &= \boldsymbol{h} \cdots (2y, 2x, -6z) + o(\boldsymbol{h}). \end{aligned}$$

So

$$\boldsymbol{y} = \begin{pmatrix} 2y \\ 2x \\ -6z \end{pmatrix} = \nabla f(\boldsymbol{x}) = \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{pmatrix}.$$

Theorem 2.11.1 *If f has continuous first partial derivatives in region of $\boldsymbol{x}$, then f is differentiable, and $\nabla f(\boldsymbol{x}) = (\partial f / \partial x_1, \cdots \partial f / \partial x_n)$ for n variables.*

□

With this,

$$\Delta f = \mathrm{d}f + o(\boldsymbol{h}) \equiv f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) = \nabla f \cdot \boldsymbol{h} + o(\boldsymbol{h}).$$

In two variables it is common to write $\boldsymbol{h} = (\mathrm{d}x, \mathrm{d}y)$, then

$$\mathrm{d}f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (\mathrm{d}x, \mathrm{d}y) = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y,$$

the differential of f at $\boldsymbol{x}$. This may be extended to n variables.

2.12 *Mean value theorem*

Theorem 2.12.1 (Mean value theorem) *If f is a differentiable function on (a, b), and if it is continuous on $[a, b]$, then there exists $c \in [a, b]$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

i.e., $f'(c)$ is parallel to the chord between $f(b)$ and $f(a)$ for at least one value of c. □

Corollary 2.12.2 (Rolle's theorem) *Suppose $f(a) = f(b) = 0$, and f satisfies the m ean value theorem, then there exists $c \in [a, b]$ such that $f'(c) = 0$.* □

This is useful in proving how many roots a polynomial has.

Example Find the roots for $f(x) = x^3 - 4x$ on $[0, 3]$.
A polynomial is continuous. We observe that

$$f'(c) = \frac{(3^3 - 12) - 0}{3 - 0} = 5, \qquad 5 = 3c^3 - 4 \qquad \Rightarrow \qquad c^2 = \pm\sqrt{3},$$

and since $c \in [0, 3]$, $c = \sqrt{3}$

Example Show $p(x) = 2x^3 + 5x - 1$ has exactly one root.
Since $p(0) = -1$ and $p(1) = 6$, by the intermediate value theorem, there exists at least one root. Suppose there are two roots with $p(a) = p(b) = 0$, then by Rolle's theorem, there exist $c \in (a, b)$ such that $p'(c) = 0$, but $p'(c) = 6c^2 + 5$ with no real c, so c is unique and $p(x)$ only has one root.

Corollary 2.12.3 (fundamental limits) *We have, for $\alpha \in \mathbb{R}^+$,*

$$\lim_{x\to\infty} \frac{1}{\log x} = 0, \qquad \lim_{x\to\infty} \frac{x^\alpha}{\mathrm{e}^x} = 0, \qquad \lim_{x\to\infty} \frac{\log x}{x^\alpha} = 0.$$

Proof Observe that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log x = \frac{1}{x},$$

and since $\log x$ is defined on $(0, \infty)$. By the mean value theorem, $\log x$ is an increasing function. For $x > 1$, let n be the largest integer such that $x > 2^n$. Then

$$0 > \log x > \log 2^n \qquad \Leftrightarrow \qquad 0 < \frac{1}{\log x} < \frac{1}{n \log 2},$$

so $1/\log x \to 0$ as $n \to \infty$ by squeezing theorem.

Now, for $f(x) = x/\mathrm{e}^x$, we have $f'(x) = \mathrm{e}^{-x}(1-x)$; by the mean value theorem, $f(x)$ is decreasing on $[1, \infty)$. Let n be the largest integer such that $n < x$, and let $b = (\mathrm{e} - 1) > 0$. We obtain

$$0 < f(x) < \frac{n}{\mathrm{e}^n} = \frac{n}{(b+1)^n} < \frac{n}{1 + nb + n(n-1)b^2/2} < \frac{2}{(n-1)b^2},$$

so $f(x) \to 0$ as $n \to \infty$ since b is a constant. So then we have

1. $y = \alpha \log x$, giving $x^\alpha = \mathrm{e}^y$, and so

$$\lim_{x \to \infty} \frac{y/\alpha}{\mathrm{e}^y} = \frac{1}{\alpha} \lim_{y \to \infty} \frac{y}{\mathrm{e}^y} = 0,$$

   and so we have $\log x / x^\alpha \to 0$.

2. let $z = \mathrm{e}^x$, then $x = \log z$, and

$$\lim_{z \to \infty} \frac{(\log z)^\alpha}{z} = \lim_{z \to \infty} \left(\frac{\log z}{z^{1/\alpha}} \right) = 0,$$

   so $x^\alpha / \mathrm{e}^x \to 0$. ■

Corollary 2.12.4 (L'Hopital's rule) *If $f(x)$ and $g(x)$ are differentiable at a, with $g'(a) \neq 0$ and $f(a) = g(a) = 0$, then*

$$\lim_{x \to a} = \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof We have

$$\begin{aligned} \lim_{x \to a} \frac{f(x)}{g(x)} &= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} \\ &= \lim_{x \to a} (f(x) - f(a))/(x-a)(g(x) - g(a))/(x-a) \\ &= \frac{f'(a)}{g'(a)}. \end{aligned}$$

■

Example

$$\lim_{x \to 1} \frac{2\log(2x-1)}{x^2 - 1} = \frac{2(2/(2-1))}{2} = 2$$

by L'Hopital's rule.

This may be re-iterated as appropriate to find the limit.

Corollary 2.12.5 (Increasing/decreasing functions) *Let f be differentiable on an open interval I, and $x_1 < x_2$, $x_1,\ x_2 \in I$. If*

- $f'(x) > 0$ *for all* $x \in I$, *then* f *is increasing on* I;
- $f'(x) < 0$ *for all* $x \in I$, *then* f *is decreasing on* I;
- $f'(x) = 0$ *for all* $x \in I$, *then* f *is constant on* I.

Proof Let $x_1, x_2 \in I$, $x_1 < x_2$, and f is differentiable. Applying the mean valute theorem on $[x_1, x_2] \subseteq I$, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Then it is easy to see that the above statements hold. ■

Example Show $x^3 - 2x^2 - 2x - 3 = 0$ has exactly one root on $[2, 5]$.

We have $f(2) = -7$ and $f(5) = 2$, so by the intermediate value theorem, there is at least one root. Now, $f'(x) = 3x^2 - 4x - 2$, and $f'(x) = 0$ when $x = (2 \pm \sqrt{10})/3$, with the positive root less than 2. Thus for all $x \geq 2$, $f'(x) > 0$, so it is a strictly increasing function and thus only has one root in the interval.

2.13 *Extreme values of continuous functions*

The (local) extrema of functions occurs where $f'(x) = 0$; when $f''(x) > 0$, we have a minima, whilst for $f''(x) > 0$, we have a local maxima.

Example Find and characterise the extrema of $f(x) = (x^3 - 3x^2/3 - 6x + 2)/4$ on $[-2, \infty)$.

We have

$$f'(x) = \frac{3x^2 - 3x - 6}{4}, \qquad f''(x) = \frac{6x - 3}{4},$$

so we have a maxima at $x = 1$ and minima at $x = 2$.

3 Integration

3.1 Definite integrals as limit to Riemann sums

Let f be a continuous function on $[a, b]$, and $\mathcal{P}_{ab}$ be a **partition** of $[a, b]$. $\mathcal{P}_{ab} = \{x_0, x_1, \cdots x_n\}$ is a finite subset of $[a, b]$ which contains a and b, and it breaks $[a, b]$ into n subintervals $\Delta x_1, \cdots \Delta x_n$. The definite integral of f from a to b can be viewed as the limit of **Riemann sums**

$$S^*(\mathcal{P}_{ab}) = \sum_{i=1}^{n} f(x_1')\Delta x_1.$$

$S^*(\mathcal{P}_{ab})$ may be negative as $f(x_j)$ is unsigned. Defining the **norm** of a partition as

$$\|\mathcal{P}_{ab}\| = \max\{\Delta x_i,\ i = 1, \cdots n\}\|,$$

then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\|\mathcal{P}_{ab}\|\to 0} \sum_{i=1}^{n} f(x_i')\Delta x_i$$

provided the limit exists.

A Riemann sum represents the signed area of the rectangles and differences between the signed area bounded by $f(x)$ and the x-axis. The integral is defined is the actual error we make may be made arbitrarily small with shrinking interval lengths.

3.2 Functions defined by an integral

Theorem 3.2.1 *Let*

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,$$

f continuous on $[a, b]$. If $F(x)$ is defined and continuous on $[a, b]$, and differentiable on (a, b), then, for all $x \in (a, b)$,

$$F'(x) = \frac{\mathrm{d}F}{\mathrm{d}x} = f(x).$$

□

Example Find the derivatives of the following $F(x)$.

1.
$$F(x) = \int_0^x \sin \pi t \, \mathrm{d}t \qquad \Rightarrow \qquad F'(x) = \sin x\pi$$

2. Using $u = x^3$,
$$F(x) = \int_0^x t \cos t \, \mathrm{d}t \qquad \rightarrow \qquad F'(x) = \frac{\mathrm{d}F}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = (u \cos u) 3x^2 = 3x^5 \cos x^3$$

3. For $x > 0$,
$$F(x) = \int_x^{x^2} t \, \mathrm{d}t = \int_0^{x^2} t \, \mathrm{d}t - \int_0^x t \, \mathrm{d}t \qquad \Rightarrow \qquad F'(x) = 2x^3 - x$$

4. Let
$$F(x) = \int_0^x f(t) \, \mathrm{d}t, \qquad f(x) = \begin{cases} 2 - x, & 0 \le x \le 1, \\ 2 + x, & 1 < x \le 3. \end{cases}$$

   For $x \in (0, 1)$,
$$F(x) = \int_0^x (2 - t) \, \mathrm{d}t \qquad \Rightarrow \qquad F'(x) = 2 - x,$$

   while for $x \in (1, 3)$,
$$F(x) = \int_0^1 (2 - t) \, \mathrm{d}t + \int_1^x (2 + t) \, \mathrm{d}t \qquad \Rightarrow \qquad F'(x) = 2 + x,$$

   so
$$\begin{cases} 2 - x, & 0 \le x \le 1, \\ 2 + x, & 1 < x \le 3. \end{cases}$$

3.3 *Fundamental theorem of calculus*

Let f be a continuous function on $[a, b]$, then g is called the **anti-derivative** of f on $[a, b]$ if g is continuous and $g'(x) = f(x)$ for all $x \in (a, b)$.

Theorem 3.3.1 (Fundamental theorem of calculus) *Let f be continuous on $[a, b]$. If g is the anti-derivative of f, then*
$$\int_a^b f(t) \, \mathrm{d}t = g(b) - g(a) = [g(t)]_a^b.$$

□

Some observations:

- If $f(t) > 0$ for all $t \in [a, b]$, then the area bounded by $f(t)$ and the x-axis is $\in_a^b f(t) \, \mathrm{d}t$.

- If $f(t), g(t) > 0$ for all $t \in [a, b]$, $f(t) \geq g(t)$, then the area bounded between $f(t)$ and $g(t)$ is

$$\int_a^b f(t)\,\mathrm{d}t - \int_a^b g(t)\,\mathrm{d}t = \int_a^b [f(t) - g(t)]\,\mathrm{d}t.$$

- Suppose f is continuous on $[-a, a]$, $a > 0$, then if f is an odd or even function, we have respectively

$$\int_{-a}^{a} f(t)\,\mathrm{d}t = 0, \qquad \int_{-a}^{a} f(t)\,\mathrm{d}t = 2\int_0^a f(t)\,\mathrm{d}t.$$

The **Gaussian integral** is deinfed as

$$\int_{-\infty}^{\infty} \mathrm{e}^{-ax^2}\,\mathrm{d}x = \sqrt{\frac{\pi}{a}}$$

for $a > 0$. This intergral shows up quite often in probability theory.

4 Ordinary Differential Equations (ODEs)

Let x be the independent variable, $y(x)$ be dependent on x. An **ordinary differential equation** (ODE) is an equation which relates an unknown function like $y(x)$ to one or more of its derivatives. The general form of a first **order** (order here refers to the highest derivative) ODE is

$$Q(x,y)y' + P(x,y) = 0 \qquad \text{or} \qquad Q(x,y)\,\mathrm{d}y + P(x,y)\,\mathrm{d}x = 0.$$

4.1 Forms of first order ODEs

4.1.1 Separable ODEs

These have the form

$$Q(y)y' + P(x) = 0 \qquad \text{or} \qquad Q(y)\,\mathrm{d}y + P(x)\,\mathrm{d}x = 0.$$

These may be solved by integration.

Example Solve $(y+1)y' = x^2 y - y$.

$$\int \left(1 + \frac{1}{y}\right)\,\mathrm{d}y = \int (x^2 - 1)\,\mathrm{d}x \qquad \Rightarrow \qquad y + \log|y| = \frac{x^3}{3} - x + c.$$

We have one free constant in this case because the ODE is of first order.

4.1.2 Exact ODEs

If $\partial P/\partial y = \partial Q/\partial x$, then the ODE is exact, and there exists $F(x,y)$ such that

$$Q(x,y) = \frac{\partial F}{\partial y}, \qquad P(x,y) = \frac{\partial F}{\partial x}.$$

Then, by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x,y) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = P(x,y) + Q(x,y)y',$$

so $F(x,y) = $ constant is a solution of the ODE.

Example Solve $(xy - x^3) + (x^3 - y)y' = 0$.

Note that $\partial P/\partial x = \partial Q/\partial x = 2xy$, so ODE is exact. To find $F(x,y)$, we have

$$xy^2 - x^3 = \frac{\partial F}{\partial x} \qquad \Rightarrow \qquad F = \frac{x^2y^2}{2} + \frac{x^4}{4} + \phi(y),$$

while

$$xy^2 - y = \frac{\partial F}{\partial y} \qquad \Rightarrow \qquad F = \frac{x^2y^2}{2} - \frac{y^2}{2} + \psi(x),$$

so

$$F = \frac{x^2y^2}{2} + \frac{x^4}{4} - \frac{y^2}{2} + k.$$

The solution is then

$$\frac{x^2y^2}{2} + \frac{x^4}{4} - \frac{y^2}{2} = c.$$

4.1.3 *Linear ODEs*

If $Q(x,y) = Q(x)$ and $P(x,y) = R(x)y + S(x)$, we introduce the **integrating factor**

$$\mu(x) = \mathrm{e}^{\int R(x)/Q(x)\,\mathrm{d}x} \qquad \left(\mu'(x) = \frac{R(x)}{Q(x)}\mu(x)\right).$$

The ODE could be written as

$$y' + \frac{R(x)}{Q(x)}y = -\frac{S(x)}{Q(x)} \qquad \Rightarrow \qquad \mu y' + \mu\frac{R(x)}{Q(x)}y = -\mu\frac{S(x)}{Q(x)}.$$

We notice that

$$\mu y' + \frac{R(x)}{Q(x)}\mu y = (\mu(x)y)',$$

so using this, integrating and rearranging gives

$$y(x) = -\frac{1}{\mu}\int \mu\frac{S(x)}{Q(x)}\,\mathrm{d}x.$$

Example Solve $xy' + 2y = (\cos x)/x$.

Rearranging to $y' + 2y/x = (\cos x)/x^2$, we notice the integrating factor is $\exp(\int^x 2/s\,\mathrm{d}s) = x^2$, so

$$\begin{aligned} x^2y' + 2yx = \cos x \qquad &\Rightarrow \qquad x^2y = \int \cos x\,\mathrm{d}x = \sin x + c \\ &\Rightarrow \qquad y = \frac{\sin x}{x^2} + \frac{c}{x^2}. \end{aligned}$$

4.1.4 *Homogeneous ODEs*

If $Q(\lambda x, \lambda y) = \lambda^n Q(x,y)$ for $n \in \mathbb{R}$ and similarly for P, then the ODE is homogeneous, with

$$\lambda^n Q(x,y)y' + \lambda^n P(x,y) = 0.$$

To solve this, we let $y = v(x)x$, so that $y' = v'x + v$, and the resulting PDE becomes separable in v and x.

Example Solve $y' = y/x + \sin(y/x)$.

The ODE is homogeneous with $n = 0$, so the above substitution yields

$$\begin{aligned} xv' + v = v + \sin v \qquad &\Rightarrow \qquad \int \frac{\mathrm{d}v}{\sin v} = \frac{\mathrm{d}x}{x} \\ &\Rightarrow \qquad \frac{1 - \cos(y/x)}{\sin(y/x)} = Ax \end{aligned}$$

after reverting for y and x.

4.1.5 *ODEs that may be made exact*

For $Q(x, y)y' + P(x, y) = 0$, if $\partial Q/\partial x \neq \partial P/\partial y$, we might be able to find a factor μ so that $\mu Q(x, y)y' + P(x, y)\mu = 0$ with $\partial(\mu P)/\partial y = \partial(\mu Q)/\partial x$. This is potentially complicated because $\mu = \mu(x, y)$. We may impose either of the following requirements on μ, that μ is a function of x or y alone. Without loss of generality, let $\mu = \mu(x)$, then the equality of the partial derivatives yields

$$\mu \frac{\partial P}{\partial y} = \frac{\mathrm{d}\mu}{\mathrm{d}x} Q + \frac{\partial Q}{\partial x} \mu,$$

and separating this gives

$$\int \frac{\mathrm{d}\mu}{\mu} = \int \frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \mathrm{d}x,$$

and so the integrating factor in this case is $\mu = \exp(\int^x r(s)\, \mathrm{d}x)$. (If $r(x)$ is not a function of x alone, then try $r(y)$.)

Example Solve $(2y^2 + 3x + 2/x^2) + (2xy - y/x)y' = 0$.

We have that

$$\frac{\partial Q}{\partial x} = 2y + \frac{y}{x^2}, \qquad \frac{\partial P}{\partial y} = 4y,$$

so ODE is non-exact. However, we notice that

$$r(x) = \frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{1}{x},$$

so with the integrating factor $\mu = x$, we have the ODE $(2xy^2 + 3x^2 + 2/x) + (2x^2 y - y)y' = 0$, and that $\partial \tilde{P}/\partial y = \partial \tilde{P}/\partial x = 4xy$. Thus

$$\begin{aligned} F &= \int^x \tilde{P}\, \mathrm{d}s = x^2 y^2 + x^3 + 2 \log|x| + \phi(y), \\ F &= \int^y \tilde{Q}\, \mathrm{d}s = x^2 y^2 - \frac{y^2}{2} + \psi(x), \end{aligned}$$

and the solution is $x^2 y^2 + x^3 + \log x^2 - y^2/2 = c$.

4.1.6 Bernouilli ODEs: ODEs that may be made linear

When $Q(x,y) = 1$ and $P(x,y) = R(x)y - S(x)y^n$, we have $y' + R(x)y = S(x)y^n$, which is nonlinear in y. However, setting $z = y^{1-n}$, and so $z' = (1-n)y^{-n}y'$, the ODE is transformed into

$$z' + (1-n)R(x)z = (1-n)S(x).$$

Example Solve $y' + 4y = 3e^{2x}y^2$.

Dividing by y^2 gives $y'y^{-2} + 4y^{-1} = 3x^{2x}$. Letting v^{-1}, we notice that $v' = -y^{-2}y'$, so the resulting ODE becomes $v' - 4v = -3x^2$, which is linear. Using an integrating factor of $\mu = e^{-4x}$, we have

$$e^{-4x}v = -\int 3e^{-2x}\,dx \qquad \Rightarrow \qquad \frac{1}{y} = \frac{3}{2}e^{2x} + ce^{4x}.$$

4.2 Second order linear ODEs with constant coefficients

These are of the form

$$f_2y'' + f_1y' + f_0y = f(x).$$

A general solution of the ODE consists of the **complementary function** and the **particular integral**. The first of these is obtained by solving the homogeneous equation

$$f_2y'' + f_1y' + f_0y = 0.$$

Since this is linear, several complementary functions may be added together to form another solution (the **principle of superposition**). We solve for the complementary function by forming the **auxiliary equation** by trying solutions of the form $y(x) \sim e^{\lambda x}$, giving

$$f_2\lambda^2 + f_1\lambda + f_0 = 0.$$

If the determinant of the resulting quadratic equation is non-zero, we have exactly two solutions for $\lambda \in \mathbb{C}$, and the complementary solution takes the form

$$y_c(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$

Otherwise, there is a degeneracy, but we may restore the degree of freedom by letting

$$y_c(x) = Ae^{\lambda x} + Bxe^{\lambda x}.$$

For the particular integral, we have to make an intelligent guess:

- $f(x) = k =$ constant, then try $y_p(x) =$ constant;
- $f(x) = kx$, try $y_p(x) = bx + c$;

- $f(x) = Ae^{bx}$, try $y_p(x) = \alpha e^{bx}$, with α not necessarily equal to A;
- $f(x) = m \cos bx$ or $m \sin bx$, try $y_p(x) = m \cos bx + n \sin bx$.

Example Solve for the following ODEs:

1. $y'' + 2y' + 5y = 10e^{-2x}$.

   The auxiliary equation has roots $\lambda = -1 \pm 2i$, and so

   $$y_c(x) = Ae^{(-1+2i)x} + Be^{(-1-2i)x} = e^{-x}(C \cos 2x + Di \sin 2x),$$

   where $C = A + B$, $D = A - B$. For the particular integral, we try $y_p = me^{-2x}$, which gives $4m - 4m + 5m = 10$, so $m = 2$, and the general solution is

   $$y = e^{-x}(A \cos 2x + Bi \sin 2x) + 2e^{-2x},$$

   where the constants may be determined given sufficient number of conditions.

2. $y'' - 4y = 12x$, given that $y(0) = 2$ and $y'(0) = 0$.

   The auxiliary equation yields $\lambda = \pm 2$, so $y_c(x) = Ae^{2x} + Be^{-2x}$. Trying $y_p(x) = ax + b$, this gives $a = -3$ and $b = 0$, so the general solution is $y = Ae^{2x} + Be^{-2x} - 3x$. With the initial conditions, this implies $A = B = 1$, so the solution is

   $$y = e^{2x} + e^{-2x} - 3x.$$

5 Fourier series

Suppose $f(x)$ is given. We consider expanding the function as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx) + b_n \sin nx),$$

and the coefficients a_n and b_n depends on the function $f(x)$.

Example Some easy examples:

- $f(x) = 1$, $a_0 = 2$, otherwise zero.
- $f(x) = \cos 2x$, $a_2 = 1$, otherwise zero.
- $f(x) = \sin^2 x = (1 - \cos 2x)/2$, $a_0 = 1$, $a_2 = -1/2$, otherwise zero.

To work out the coefficients in general, we first state this orthogonality result.

Proposition 5.0.1 *Over the symmetric domain $(-\pi, \pi)$, we have the following orthogonality relations:*

$$\int_{-\pi}^{\pi} \cos nx \sin mx = 0,$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m = 0 \neq 0, , \\ 2\pi, & n = m = 0, \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases}.$$

Proof Expand using compound angle formulae, occasionally use the fact that $\cos(x)$ and $\sin(x)$ are even and odd on this interval, and that $\sin p\pi = 0$ for $p \in \mathbb{Z}$. ■

The general procedure is then as follows:

1. Multiply through by $\cos mx$, $m \in \mathbb{Z}^+$.
2. Integrate over $(-\pi, \pi)$, which results in

$$\int_{-\pi}^{\pi} f(x) \cos mx \,\mathrm{d}x = a_m \int_{-\pi}^{\pi} \cos^2 mx \,\mathrm{d}x = a_m \pi$$

by orthogonality, so

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,\mathrm{d}x, \qquad m = 0, 1, 2, \cdots$$

3. A similar procedure for the b_m terms gives

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,\mathrm{d}x, \qquad m = 1, 2, 3, \cdots$$

The Fourier series may not converge on a general domain, so instead we define the Fourier series on (π, π) and see if series converges, and if so, how the sum resembles $f(x)$ itself on this interval.

Example Find the Fourier transform $\mathcal{F}(f(x) = x)$ for $x \in (-\pi, \pi)$.

We notice that $f(x)$ is an odd function, so the integrals associated with the a_n coefficients are identically zero. On the other hand,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,\mathrm{d}x = \frac{2}{n}(-1)^{n+1},$$

after noticing the integrand is an even function, doing an integration by parts and that $\cos n\pi = (-1)^n$ for $n \in \mathbb{Z}$. So

$$\mathcal{F}(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx, \qquad x \in (-\pi, \pi).$$

(Using MAPLE or otherwise, keeping up to about the twelfth term of the partial sum gives a close approximation to the original function, and the sum appears to converge; this may be expected from the **Riemann–Lebesgue lemma**.)

Remark We have equality at $x = \pi/2$, and so since $f(\pi/2) = \pi/2 = \mathcal{F}(f(x = \pi/2))$, we have

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\cdots.$$

Outside of the interval $(-\pi, \pi)$, the Fourier transform gives a **periodic extension** of $f(x)$ from $(-\pi, \pi)$. At the edge of the intervals however the Fourier transform may have jumps (as in this case, since $f(x)$ is not periodic).

5.1 *Convergence of Fourier series*

Theorem 5.1.1 (Dirichlet's theorem) *For $f(x)$ defined on $(-\pi, pi)$, let $f_E(x) = f(x)$ on $(-\pi, pi)$ and $f_E(x) = f_E(x + 2n\pi)$ be the periodic extension over the real line. Then $\mathcal{F}(f(x))$ converges to $f_E(x)$ whenever $f_E(x)$ is continuous, and to the average of its left and right limits at any finite jumps, provided that*

1. *all jumps are finite, and elsewhere $f'_E(x)$ is finite,*
2. *in each 2π interval, $f_E(x)$ has finite number of discontinuities and extrema.* □

Example For $\mathcal{F}(f(x) = |x|)$ in $(-\pi, pi)$, $f(x)$ is an even function, so all the b_n coefficients associated with $\sin(nx)$ are zero. We have that

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi x\,\mathrm{d}x = \pi, \\
a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi x\left(\frac{\sin nx}{n}\right)\,\mathrm{d}x \\
&= \frac{2}{\pi}\frac{(-1)^n - 1}{n^2},
\end{aligned}$$

so

$$\mathcal{F}(|x|) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}.$$

We note that we have pointwise convergence here even though $f_E(x)$ is not differentiable at $x = 2n\pi$, $n \in \mathbb{Z}$.

5.2 *Even and odd functions*

If $f(x)$ is even, then we have

$$b_n = 0, a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,\mathrm{d}x,$$

whilst for $f(x)$ odd, we have

$$a_n = 0, b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,\mathrm{d}x.$$

Some functions are neither even or odd, but we can always write a function as

$$f(x) = f_+(x) + f_-(x), \qquad f_\pm = \frac{f(x) \pm f(-x)}{2},$$

which are even and odd functions respectively. The Fourier series of $f_\pm$ are the cosine and sine parts of $\mathcal{F}(f(x))$ respectively.

5.3 *Fourier series on $(-L, L)$*

For $f(x)$ defined on $(-L, L)$, we replace x by $\pi x/L$, and we have

$$\mathcal{F}(f(x)) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L} + b_n\sin\frac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{1}{L}\int_{-L}^{L} \cos\frac{n\pi}{L}\,\mathrm{d}x,$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}\,\mathrm{d}x,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}\,\mathrm{d}x.$$

Example Find $\mathcal{F}(x/2)$ for $x \in (0,2)$.

Since $0 < x < 2$, we rescale this to $-\pi < \pi(x-1) < \pi$, so letting $y = \pi(x-1)$, this gives $x = 1 + y/\pi$, and so we consider instead the Fourier transform of $g(y) = (1 + y/\pi)/2$, $y \in (-\pi, \pi)$. We have

$$\begin{aligned}
\mathcal{F}\left(\frac{1}{2} + \frac{y}{2\pi}\right) &= \frac{1}{2\pi}\mathcal{F}(y) + \frac{1}{2} \\
&= \frac{2}{2\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin ny}{n} + \frac{1}{2} \\
&= \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi(x-1)}{n} + \frac{1}{2}.
\end{aligned}$$

Using double angle formulae, we have

$$\mathcal{F}\left(\frac{x}{2}\right) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}, \qquad x \in (0,2).$$

5.4 *Half range series*

Suppose we want to extend the Fourier series from $x \in (0, L)$ to $x \in (-L, L)$ or something similar.

Example Using the above example, we might consider the **even extension**, with $f(x) = |x|/2$, $x \in (-2,2)$. Then

$$\mathcal{F}_c(f) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}.$$

We have

$$\begin{aligned}
a_0 &= 1, \\
a_n &= \frac{1}{2}\int_{-2}^{2}\frac{x}{2}\cos\frac{n\pi x}{2}\,\mathrm{d}x \\
&= \left[\frac{x}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2 - \frac{1}{n\pi}\int_0^2 \sin\frac{n\pi x}{2}\,\mathrm{d}x \\
&= \frac{2}{(n\pi)^2}((-1)^n - 1),
\end{aligned}$$

so $a_0 = 1$ and $a_n = -4/(n\pi)^2$ only when n is odd.

Example If instead we consider the **odd extension** with $f(x) = x/2$, $x \in (-2, 2)$, we should have

$$\mathcal{F}_s(f) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}.$$

So then

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^{2} \frac{x}{2} \sin \frac{n\pi x}{2} \, \mathrm{d}x \\ &= \left[-\frac{x}{n\pi} \cos \frac{n\pi x}{2} \right]_0^2 + \frac{1}{n\pi} \int_0^2 \cos \frac{n\pi x}{2} \, \mathrm{d}x \\ &= \frac{2}{n\pi} (-1)^{n+1} + 0. \end{aligned}$$

5.5 An application to PDEs

Suppose we have the diffusion equation on the interval $(0, L)$ subject to initial and boundary conditions

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \qquad u(), t) = u(L, t) = 0, \qquad u(x, 0) = f(x).$$

We assume we have solutions of the form $u(x, t) = X(x)T(t)$. With this, we have

$$\kappa X T' = \kappa T X'' \qquad \Leftrightarrow \qquad \kappa \frac{X''}{X} = \frac{T'}{T} = -\alpha^2,$$

where $-\alpha^2$ is the **constant of separation** due to x and t being independent of each other; α is chosen to be positive here so that the boundary conditions are satisfied. We now have two ODEs to solve, i.e.,

$$X'' + \alpha^2 X = 0, \qquad T' + \kappa\alpha^2 T = 0,$$

related by $-\alpha^2$ that is to be determined.

For the spatial equation, we have $X(x) = A \cos \alpha x + B \sin \alpha x$. By the boundary conditions, $A = 0$, and we require $\sin \alpha L = 0$, i.e., $\alpha = n\pi/L$, so

$$X = X_n(x) = B_n \sin \frac{n\pi x}{L}.$$

The idea here is that, since the equations are linear, solutions may be added accordingly to give the full solution (**principle of superposition**). For the time equation, we have $T(t) = C\mathrm{e}^{-\kappa\alpha^2 t}$, so

$$T = T_n(t) = C_n \mathrm{e}^{-\kappa(n\pi/L)^2 t},$$

and so the general form of the solution is

$$u(x, t) = D_n \sin \frac{n\pi x}{L} \mathrm{e}^{-\kappa(n\pi/L)^2 t}, \qquad n \in \mathbb{N}.$$

D_n is determined by the initial condition. since

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} D_n \sin \frac{n\pi x}{L},$$

we have

$$D_n = \frac{2}{\pi} \int_0^L f(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x.$$

Example Suppose a bar of length L has the middle half portion heated to 100°, i.e.,

$$f(x) = \begin{cases} 100, & L/4 < x < 3L/4, \\ 0, \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} D_n &= \frac{2}{\pi} \int_{L/4}^{3L/4} 100 \sin \frac{n\pi x}{L} \, \mathrm{d}x \\ &= -\frac{200L}{n\pi^2} \left(\cos \frac{3\pi}{4} - \cos \frac{n\pi}{4} \right) \end{aligned}$$

$$= \frac{100L}{n\pi^2} \sin \frac{n\pi}{2} \sin n\pi 4,$$

using

$$\cos A - \cos B = \frac{1}{2} \sin \frac{A+B}{2} \sin \frac{A-B}{2}.$$

6 Integration in higer dimensions

We extend definitions of the one dimensional integral in the obvious way, by dividing our domain of integration D into partition R_{ij} and take the sum of it, i.e., $\sum f_{ij} \times \text{area}(R_{ij})$.

Given a set of indices $A = \{1 \le i \le m,\ 1 \le j \le n\}$ and for each pair (i, j) there is a number a_{ij}, their sum is

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}.$$

Since addition is associative and commutative, we know that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right).$$

Addition and multiplication are distributive, i.e., $a(b+c) = ab + ac$, so that

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (\alpha a_{ij} + \beta b_{ij}) = \alpha \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} + \beta \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij}.$$

If $a_{ij} = b_i c_j$ for all (i, j), then

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} b_i c_j = b_i \sum_{j=1}^{n} c_j,$$

so

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{m} \left(b_i \sum_{j=1}^{n} c_j \right) = \left(\sum_{i=1}^{m} b_i \right) \left(\sum_{j=1}^{n} c_j \right).$$

Sometimes the summation depends on how we add the terms together (e.g., summing the sequence $(-1)^n n^{-1}$). We may re-arrange sums if, however, $a_i \ge 0$ for all i, or that the sequence $\{a_i\}$ is absolutely convergent.

6.1 Integration in $\mathbb{R}^2$

Consider a rectangle $R = [a_0, a_1] \times [b_0, b_1]$. If we partition in the x and y direction as

$$\mathcal{P}_x = [x_0, \cdots x_m], \qquad \mathcal{P}_y = [y_0, \cdots y_n],$$

then
$$R_{ij} = \{(x, y) \in \mathbb{R}^2 \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}.$$
Suppose f is continuous on $\mathbb{R}^2$, we let

$$m_{ij} = \min_{R_{ij}} f(x, y), \qquad M_{ij} = \max_{R_{ij}} f(x, y).$$

Forming the sums

$$L_f(\mathcal{P}) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \Delta x_i \Delta y_j, \qquad U_f(\mathcal{P}) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} \Delta x_i \Delta y_j,$$

with $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, we can prove that there is a unique I such that
$$L_f(\mathcal{P}) \leq I \leq U_f(\mathcal{P})$$
for every partition $\mathcal{P}$.

The integral of f covering a domain $D \in \mathbb{R}^2$ is defnoted by

$$\iint_D f \, \mathrm{d}A \qquad \text{or} \qquad \iint_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$

The second form is useful as it makes the independent variables explicit.

6.1.1 *Repeated integration*

Looking at $L_f(\mathcal{P})$, fixing i and summing over j gives

$$\sum_{j=1}^{n} (m_{ij} \Delta y_j) x_i.$$

At a specific x, these upper and lower j sums are close to

$$\int_{b_0}^{b_1} f(x, y) \, \mathrm{d}y.$$

Denoting this by $F(x)$, we have the double sum being equal to

$$\int_{a_0}^{a_1} F(x) \, \mathrm{d}x,$$

so that
$$\iint_R f \, \mathrm{d}A = \int_{a_0}^{a_1} \left(\int_{b_0}^{b_1} f(x, y) \, \mathrm{d}y \right) \mathrm{d}x.$$
It turns out that, usually, the ordering of integrals do not matter, so

$$\iint_R f \, \mathrm{d}A = \int_{b_0}^{b_1} \left(\int_{a_0}^{a_1} f(x, y) \, \mathrm{d}x \right) \mathrm{d}y.$$

Example Compute the integrals of

1. $f(x,y) = (1+x+y)^{-1}$ over $R = [0,1]^2$.

$$\begin{aligned} I &= \int_0^1 \left(\int_0^1 (1+x+y)^{-1} \, \mathrm{d}y \right) \mathrm{d}x \\ &= \int_0^1 (\log 2 + x - \log 1 + x) \, \mathrm{d}x \\ &= 3\log 3 - 4\log 2. \end{aligned}$$

2. $f(x,y) = xy^2$ over $R = [a,b] \times [0,c]$.

$$\begin{aligned} \int_a^b \left(\int_0^c xy^2 \, \mathrm{d}y \right) \mathrm{d}x &= \int_a^b \left(x \int_0^c y^2 \, \mathrm{d}y \right) \mathrm{d}x \\ &= \left(\int_0^c y^2 \, \mathrm{d}y \right) \left(\int_a^b x \, \mathrm{d}x \right) \\ &= \frac{c^3}{3} \left(\frac{b^2 - a^2}{2} \right). \end{aligned}$$

6.1.2 *More complex domains*

We can evaluate integrals over D of types where

- $D = \{a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}$, i.e., domain bounded by two curves varying in x,
- $D = \{\psi_1(y) \le x \le \psi_2(y), a \le y \le b\}$, i.e., domain bounded by two curves varying in y.

We have

$$I = \int_a^b \left(\int_{\phi_2(x)}^{\phi_1(x)} f(x,y) \, \mathrm{d}y \right) \mathrm{d}x \qquad \text{or} \qquad I = \int_a^b \left(\int_{\psi_2(y)}^{\psi_1(y)} f(x,y) \, \mathrm{d}x \right) \mathrm{d}y.$$

We also have the following properties of integrals:

- **Linearity**: $\iint_D (\alpha f + \beta g) \, \mathrm{d}A = \alpha \iint_D f \, \mathrm{d}A + \beta \iint_D g \, \mathrm{d}A$.
- **Order**: For $f \ge 0$ on D, $f(x,y) \ge 0$ FOR $(x,y) \in D$ implies that $\iint_D f \, \mathrm{d}A \ge 0$, and if $f(x,y) \ge g(x,y)$ for $(x,y) \in D$, $\iint_D f \, \mathrm{d}A \ge \iint_D g \, \mathrm{d}A$.
- **Additivity**: If D is partition into disjoint regions D_i, then $\iint_D f \, \mathrm{d}A = \sum_{i=1}^n \iint_{D_i} f \, \mathrm{d}A$.
- **Mean value**: there exists some $(x_0, y_0) \in D$ such that $\iint_D f \, \mathrm{d}A = f(x_0, y_0) \times$ area of D.

Example Integrate the following over the appropriate domains:

1. $f(x,y) = (x\log(y+a))/(y-a)^2$ over D where D is the region between the circle of radius a centred on $(0,a)$ and the x-axis.

We integrate x first, and to do that, we work out $\psi_{1,2}(y)$. The circle has formula $x^2 + (y-a)^2 = a^2$, so

$$\psi_2(y) = a, \qquad \psi_1(y) = \sqrt{a^2 - (y-a)^2} = \sqrt{2ay - y^2}.$$

Thus

$$\begin{aligned} I = \iint_D f \, \mathrm{d}A &= \int_0^a \int_{\sqrt{2ay-y^2}}^a \frac{x \log(y+a)}{(y-a)^2} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^a \left[\frac{x^2}{2}\right]_{\sqrt{2ay-y^2}}^a \frac{\log(y+a)}{(y-a)^2} \, \mathrm{d}y \\ &= \frac{1}{2} \int_0^a \log(y+a) \, \mathrm{d}y. \end{aligned}$$

So this results in $I = (a/2)(\log 4a - 1)$.

2. $f(x,y) = xy$ over the triangle with vertices at $\{(0,0),(1,1),(2,0)\}$.

Doing this over y first, we have

$$\phi_1(x) = 0, \qquad \phi_2(x) = \begin{cases} x, & x \leq 1, \\ 2-x, & x > 1, \end{cases}.$$

By additivity, with have

$$\iint_D f \, \mathrm{d}A = \int_0^1 \int_0^x xy \, \mathrm{d}y \, \mathrm{d}x + \int_1^2 \int_0^{2-x} xy \, \mathrm{d}y \, \mathrm{d}x = \cdots = \frac{1}{3}.$$

6.2 *Change of variables*

One notable co-ordinates system is the polar co-ordinate system where

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

A change of variables from Cartesian to polar co-ordinates has a correction

$$\iint_D f \, \mathrm{d}A = \iint_D f \, r \, \mathrm{d}r \mathrm{d}\theta.$$

Example Compute the integral of $f(x,y)$ over the sector with vertices at $\{(0,0),(\sqrt{2},0),(1,1)\}$.

We see that our domain satisfies $\theta \in [0, \pi/4]$ and $r \in [0, \sec\theta]$. So the integral is

$$\begin{aligned} \iint xy \, \mathrm{d}A &= \int_0^{\pi/4} \int_0^{\sec\theta} r^2 \cos\theta \sin\theta \, r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_0^{\pi/4} \left[\frac{r^4}{4} \sin\theta\cos\theta\right]_0^{\sec\theta} \mathrm{d}\theta \\ &= \frac{1}{4} \int_0^{\pi/4} \frac{\sin\theta}{\cos^3\theta} \, \mathrm{d}\theta. \end{aligned}$$

A substitution with $y = \cos\theta$ results in $I = 1/8$.

6.2.1 The Jacobian

When we substitute for x, y using u, x we have $x = x(u, v)$ and $y = y(u, v)$. Consider a box $[u, u + \Delta u] \times [v, v + \Delta v]$ and a point $(u + \delta u, v + \delta v)$ in this box. Using Taylor's theorem,

$$\delta x = x(u + \delta u, v + \delta v) - x(u, v) = \frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v + \cdots,$$
$$\delta y = y(u + \delta u, v + \delta v) - y(u, v) = \frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v + \cdots.$$

At leading order, we have

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = \mathbf{J} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}.$$

In uv-space, our box has area $\Delta u \Delta v$, and its image in xy plane has area

$$\Delta u \Delta v \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = |\mathbf{J}| \Delta u \Delta v.$$

Here we replace the area element $\mathrm{d}x\,\mathrm{d}y$ with $|\mathbf{J}|\,\mathrm{d}u\,\mathrm{d}v$ when we substitute u, v for x, y. For example, for polar co-ordinates, we have

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Example Find the area between the curve $x^2 - 4xy + 4y^2 - (2x + y) = 1$ and the line $y = 2/5$.

Note that we have $x^2 - 4xy + 4y^2 - (2x + y) = (x - 2y)^2 - (2x + y) = 1$, and so letting $u = x - 2y$ and $v = 2x + y$, we the curve becomes $u^2 - v = 1$. On the other hand, $y = (x - 2u)/5 = 2/5$, so the line becomes $v = 2 + 2u$. We also have $|\mathbf{J}| = 1/5$. The two curves meet where $u^2 - 1 = 2u + 2$, so $u = -1, 3$, thus

$$A = \int_{-1}^{3} \int_{u^2-1}^{2u+2} \frac{1}{5}\,\mathrm{d}v\,\mathrm{d}u = \frac{1}{5} = \int_{-1}^{3} (3 + 2u - u^2)\,\mathrm{d}u = \frac{32}{15}.$$

6.3 Triple integrals

Considering a region I in R^3 which contains material of density $f(x, y, z)$ at $(x, y, z) \in D$, then $\iiint_D \mathrm{d}V$ is the volume of D, $M = \iiint_D f\,\mathrm{d}V$ is the mass of material in D, and $(\iint_x f(x, y, z)\,\mathrm{d}V)/M$ is the centre of mass x_0.

Example Consider a square base pyramid with base $[-1, 1]^2$ and apex at $(0, 0, 1)$. Find the volume and centre of mass assuming uniform density.

We integrate over z last. At height z, the (x, y) cross-section is the square $[z-1, 1-z]^2$, so, using the substitution $u = 1-z$,

$$V = \int_0 1^1 \int_{z-1}^{1-z} \int_{z-1}^{1-z} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_0^1 4(1-z)^2\,\mathrm{d}z = -4\int_1^0 u^2\,\mathrm{d}u = \frac{4}{3}.$$

Centre of mass is found by integrating in x, y and z respectively:

$$\int_0^1 \int_{-(1-z)}^{1-z} \int_{-(1-z)}^{1-z} x\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0$$

as x is an odd function, and similar for y. We know that since we have uniform density and the volume is $4/3$, the mass $M = 4/3$, so

$$\begin{aligned} z_0 &= \frac{3}{4}\int_0^1 z\left(\iint_{[z-1,1-z]^2} 1\,\mathrm{d}A\right)\,\mathrm{d}z \\ &= \frac{3}{4}\int_0^1 z4(1-x)^2\,\mathrm{d}z \\ &= -3\int_0^1 (1-u)u^2\,\mathrm{d}u \\ &= \frac{1}{4}, \end{aligned}$$

so the centre of mass is located at $(0, 0, 1/4)$.

Changing variables is done as before, and this time we have

$$\mathbf{J} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{pmatrix}.$$

Example In spherical polar co-ordinates, we have:

- r the distance from $(0, 0, 0)$ to (x, y, z);
- θ the angle in (x, y) plane from x-axis, $\theta ßn[0, 2\pi)$;
- ϕ the polar angle fro z, $\phi \in [0, \pi]$.

The change of co-ordinates is

$$x = r\sin\phi\cos\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\phi.$$

For the integral of $\iiint_D (x^2+y^2+z^2)^{-1}\,\mathrm{d}V$ over the region bounded by the sphere $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 9$, we have $|\mathbf{J}| = r^2\sin\phi$, so

$$\begin{aligned} I &= \int_1^3 \int_0^{2\pi} \int_0^{\pi} \frac{1}{r^2} r^2 \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\ &= -\int_1^3 \int_0^{2\pi} [\cos\phi]_0^{\pi}\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_1^3 4\pi\,\mathrm{d}r = 8\pi. \end{aligned}$$