Academic notes: Analysis

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I. METRIC SPACES

A. Basic notions

The field of real numbers $\mathbb{R}$ is a totally ordered field which also satisfies the completeness axiom, i.e. a non-empty bounded set $A \subseteq \mathbb{R}$ has a supremum and/or an infimum. The supremum of $A \subseteq \mathbb{R}$ is a real number s where $a \leq s$ for all $a \in A$. If m is also such that $a \leq m$ for $a \in A$, then $s \leq m$, denoted $\sup A$. The infimum of A is where the inequalities signs are swapped, denoted $\inf A$.

Lemma I.1 *Let $I_n = [a_n, b_n]$ be a sequence of closed intervals such that $a_n \leq a_{n+1} < b_{n+1} \leq b_n$ for all $n \geq 1$, then $\cap_{n=1}^{\infty} I_n$ is non-empty.*

Proof Let $a = \sup\{a_n\}$. Since $a_n \leq b_1$ for all n exists by completeness axiom, $a_n \leq b_k$ for any value of n and k, and so $a \leq b_k$. Hence $a_k \leq a \leq b_k$ for all k, and that $a \in \cap_{n=1}^{\infty} I_n$.

Let M be a set. A function $d : M \times M \to [0, \infty)$ is called a metric on M if

1. $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in M$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

The pair (M, d) is then called a metric space. It is easy to see any $N \subseteq M$ is also a metric space using the same d.

Example
1. On $\mathbb{R}$, $d(x, y) = |y - x|$ gives a metric.
2. On $\mathbb{R}^2$, $d_1(\boldsymbol{x}, yb) = |y_1 - x_1| + |y_2 - x_2|$ is also a metric, but notice that, for example, $d_1((1,1),(0,0)) = 2$ as opposed to the expected distance of $\sqrt{2}$.

The standard (Euclidean) metric in $\mathbb{R}^2$ is given by

$$d(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}.$$

Let V be a real vector space. An inner product on V is a function $(\cdot, \cdot) : V \times V \to \mathbb{R}$ that, for all $\boldsymbol{x}, \boldsymbol{y} \in V$, satisfies the following:

- linearity in the first factor;
- $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{y}, \boldsymbol{x})$;
- $(\boldsymbol{x}, \boldsymbol{x}) \geq 0$ and is zero iff $\boldsymbol{x} = 0$.

Example
1. For $V = \mathbb{R}^n$, the standard inner product is given by $(\boldsymbol{x}, \boldsymbol{y}) = x_i y_i$ (where Einstein notation is understood). If $\mathbf{A}$ is a symmetric matrix, then $(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^T \mathbf{A} \boldsymbol{y}$ is an inner product if all eigenvalues of $\mathbf{A}$ are positive.
2. For $V = C[a, b]$, $(f, g) = \int_a^b f(x) g(x)\, \mathrm{d}x$ is an inner product since V is a vector space of continuous functions, and the only function that is everywhere zero and continuous is $f(x) = 0$ for all $x \in [a, b]$.

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Theorem I.2 (Cauchy–Schwartz inequality) *Let V be a real vector space, and $(\cdot,\cdot)$ an inner product on V. Then*

$$|(\boldsymbol{x},\boldsymbol{y})| \le \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|,$$

where $\|\cdot\|$ is the standard Euclidean norm of the vector, and there is equality iff $\boldsymbol{x} = \lambda\boldsymbol{y}$ for some $\lambda \in \mathbb{R}$.

Proof Note that $(\boldsymbol{x},\boldsymbol{0}) = (\boldsymbol{x},\boldsymbol{x}-\boldsymbol{x}) = (\boldsymbol{x},\boldsymbol{x}) - (\boldsymbol{x},\boldsymbol{x}) = \boldsymbol{0}$, so we may assume that $\boldsymbol{y} \neq \boldsymbol{0}$. Then, with $\lambda = -(\boldsymbol{x},\boldsymbol{y})/\|\boldsymbol{y}\|^2$,

$$\begin{aligned} 0 \le (\boldsymbol{x}+\lambda\boldsymbol{y},\boldsymbol{x}+\lambda\boldsymbol{y}) &= \|\boldsymbol{x}\|^2 + 2\lambda(\boldsymbol{x},\boldsymbol{y}) + \lambda^2\|\boldsymbol{y}\|^2 \\ &= \|\boldsymbol{x}\|^2 - \frac{(\boldsymbol{x},\boldsymbol{y})^2}{\|\boldsymbol{y}\|^2}. \end{aligned}$$

So $(\boldsymbol{x},\boldsymbol{y})^2 \le \|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2$ and the result follows.

Lemma I.3 *Let V be a real vector space with inner product $(\cdot,\cdot)$. Then $d : V \times V \to [0,\infty)$ with $d(\boldsymbol{x},\boldsymbol{y}) = \|\boldsymbol{x}-\boldsymbol{y}\|$ gives a metric on V.*

Proof Clearly $d(\boldsymbol{x},\boldsymbol{x}) = 0$ and is symmetric, so we just need to check the triangle inequality. By Cauchy–Schwartz,

$$\|\boldsymbol{a}+\boldsymbol{b}\| = \sqrt{\|\boldsymbol{a}\|^2 + 2(\boldsymbol{a},\boldsymbol{b}) + \|\boldsymbol{b}\|^2} \le \sqrt{\|\boldsymbol{a}\|^2 + 2\|\boldsymbol{a}\|\|\boldsymbol{b}\| + \|\boldsymbol{b}\|^2} \le= \|\boldsymbol{a}\| + \|\boldsymbol{b}\|,$$

as required.

Let $f : M \to N$ be a function metric metric spaces (M, d_M) and (N, d_N). For $a \in M$, f is <u>continuous at a</u> if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $d_N(f(a), f(x)) < \epsilon$ for all $x \in M$ when $d_M(a,x) < \delta$.

B. Sequences and Cauchy sequences

Let M be a metric space. A <u>sequence</u> (a_n) in M consists of elements $a_n \in M$ for all $n \in \mathbb{N}$. Let $a \in M$, and (a_n) <u>converges to a</u> if, for all $\epsilon > 0$, $d(a_n, a) < \epsilon$ for some all $n \ge n_0$. We write $\lim_{n\to\infty} a_n = a$. The sequence (a_n) is called <u>convergent</u> if there exists $a \in M$ where $a_n \to a$.

Lemma I.4 *Let $f : M \to N$ be a function between metric spaces and $a \in M$. The function f is continuous at $a \in M$ iff $f(a_n) \to f(a)$ for $(a_n) \in M$ with $a_n \to a$. (Note that $f(a_n)$ is a sequence in N.)*

Proof Assume that f is continuous at $a \in M$, and let (a_n) be a sequence with $a_n \to a$. By continuity, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for $d(a,y) < \delta$, $d(f(a), f(y)) < \epsilon$ for arbitrary $y \in M$. Choose $n_0 \ge 0$ such that $d(a_n, a) < \delta$ for all $n \ge n_0$, then this implies $d(f(a_n), f(a)) < \epsilon$, and thus $f(a_n) \to f(a)$ as required.

On the other hand, assume $f(a_n) \to f(a)$ for all sequences such that $a_n \to a$. Given $\epsilon > 0$, assume that instead there is no $\delta > 0$ such that, for $d(a,y) < \delta$, $d(f(a), f(y)) < \epsilon$ for arbitrary $y \in M$. Then we can find $a_n \in M$ with $d(a, a_n) < 1/n$. However, this means $d(f(a), f(a_n)) \ge \epsilon$, which contradicts the assumption that $f(a_n) \to f(a)$ even though $a_n \to a$. So such δ exists and we have continuity.

Lemma I.5 *The limit of a sequence is unique.*

Proof Assume there are two limits a and b for the sequence a_n. Then $d(a,b) \le d(a,a_n) + d(a_n,b)$. As $n \to \infty$, the RHS tends to zero so $a = b$.

A Cauchy sequence (a_n) in the metric space M is a sequence such that, for all $\epsilon > 0$, there exists $n_0 \ge 0$ such that $d(a_p, a_q) < \epsilon$ for all $p, q \ge n_0$.

Lemma I.6 *A convergent sequence is a Cauchy sequence (the converse is not true).*

Proof Suppose $a_n \to a$. Then, for all $\epsilon > 0$, there is some $n_0 \ge 0$ such that $d(a_n, a) < \epsilon/2$ for $n \ge n_0$. Let $p, q \ge n_0$, then $d(a_n, a_q) \le d(a_p, a) + d(a_q, a) < \epsilon$, so the sequence is Cauchy.

A metric space M is <u>complete</u> if all Cauchy sequences in M converges.

Theorem I.7 (Completeness of $\mathbb{R}$) *The real line $\mathbb{R}$ is complete.*

Proof Let (a_n) be a Cauchy sequence in $\mathbb{R}$. Define the sequence of integers (n_k) where $n_0 = 1$, and n_{k+1} is the smallest integer bigger than n_k where $|a_p - a_q| < 2^{-(k+2)}$ for $p, q \geq n_{k+1}$. Define the intervals $I_k = [a_{n_k} - 2^{-k}, a_{n_k} + 2^{-k}]$ and let $x \in I_{k+1}$. Now, since $x \in I_{k+1}$, this implies that $|x - a_{n_{k+1}}| < 2^{-(k+1)}$. By definition of the integer sequence, $|a_{n_k} - a_{n_{k+1}}| < 2^{-(k+1)}$, so then, by triangle inequality,

$$|a_{n_k} - x| \leq |x - a_{n_{k+1}}| + |a_{n_{k+1}} - a_{n_k}| < 2 \cdot 2^{-(k+1)} = 2^{-k},$$

so $x \in I_k$. However, $x \in I_{k+1}$, so $I_{k+1} \subset I_k$. By Lemma I.1, $\cap_{k=1}^{\infty} I_k \neq \emptyset$, so assume $a \in \cap_{k=1}^{\infty} I_k$. For $m \geq n_k$,

$$|a - a_m| \leq |a - a_{n_k}| + |a_{n_k} - a_m| \leq 2^{-k} + 2^{-(k+1)} \to 0$$

as $m \geq n_k \to \infty$. Thus $a_m \to a$ and this arbitrary Cauchy sequence converges in $\mathbb{R}$ and thus $\mathbb{R}$ is complete.

Proposition I.8 *For $X \neq \emptyset$, let $\mathcal{B}(X)$ be the set of functions $f : X \to \mathbb{R}$ such that f is bounded. For $f, g, \in \mathcal{B}(X)$, let $d(f,g) = \sup_{x \in X} |f(x) - g(x)$. Then $(\mathcal{B}(X), d(f,g))$ defines a complete metric space.*

Proof d is clearly a metric. For completeness, let (f_n) be a Cauchy sequence in $\mathcal{B}(X)$. For $x \in X$, $(f_n(x))$ is a Cauchy sequence of real numbers because, by definition of $d(f,g)$, $|f_q(x) - f_p(x)| \leq d(f_p - f_q)$, and since $\mathcal{R}$ is complete, the sequence $(f_n(x))$ converges.

Defining $f : X \to \mathbb{R}$ such that $f(x) = \lim_{n \to \infty} f_n(x)$, we need to show that $f \in \mathcal{B}(X)$, and that indeed $f_n(x) \to f(x)$ regardless of $x \in X$. Be definition of a Cauchy sequence, for $\epsilon > 0$, there exists $n_0 \geq 0$ such that $d(f_p, f_q) < \epsilon/2$ for $p, q \geq n_0$. Note also that, for all $x \in X$, there exists $n_1(x) \geq n_0$ such that $|f_{n_1(x)} - f| < \epsilon/2$. Then, let $x_\in X$ and $n \geq n_0$, we have

$$|f_n(x) - f(x)| \leq |f_n - f_{n_1(x)}| + |f_{n_1(x)} - f| < \epsilon.$$

Additionally note, $|f(x)| \leq |f(x) - f_{n_0(x)}| + |f_{n_0(x)}| \leq \epsilon + c_{f_{n_0}}$ since $f_{n_0(x)}$ is bounded, so $f \in \mathcal{B}(X)$. Further, $d(f_n - f) = \sup |f_n - f| = \delta < \epsilon$, so f_n converges to $f \in \mathcal{B}(x)$. Thus every Cauchy sequence converges and thus the space is complete and equipped with a metric.

C. Topology of metric spaces

Let (M, d) be a metric space with $\boldsymbol{x} \in M$ and $r > 0$. Define the open ball around $\boldsymbol{x}$ of radius r to be

$$B(\boldsymbol{x}; r) = \{\boldsymbol{y} \in M \mid d(\boldsymbol{x}, \boldsymbol{y}) < r\}.$$

The analogous closed ball $D(\boldsymbol{x}; r)$ is defined with the less than or equal to sign. A set $A \subset M$ is bounded if it can be contained in some $D(\boldsymbol{x}; r)$ for some $\boldsymbol{x} \in M$, $r > 0$. A set $U \subset M$ is open if, for all $\boldsymbol{x} \in U$, there exists $r_{\boldsymbol{x}} > 0$ such that $B(\boldsymbol{x}; r_{\boldsymbol{x}}) \subset U$. A set $A \subset M$ is closed if $M \setminus A$ is open.

Lemma I.9 *Let (M, d) be a metric space, then:*

1. *M and $\emptyset$ are open;*
2. *$\bigcup_i A_i$ is open if all $A_i \subset M$ are open;*
3. *$\bigcap_i^n$ is open if all $A_i \subset M$ are open and $n < \infty$;*
4. *$B(\boldsymbol{x}; r)$ is open for some $r > 0$.*

Proof The first two are obvious. For 3), suppose the open sets U_i indexed by i are open and $\boldsymbol{x} \in \bigcap_{i=1}^n U_i$. Then $\boldsymbol{x} in U_i$ for all i, so there is some $B(\boldsymbol{x}; r_i) \subset U_i$. Taking the minimum of such $r_i > 0$ means $B(\boldsymbol{x}; r_i) \subset \bigcap_{i=1}^n U_i$, and thus the collective finite union is open.

For 4), let $\boldsymbol{y} \in B(\boldsymbol{x}; r)$, $r_y = r - d(\boldsymbol{x}, \boldsymbol{y}) > 0$ and $\boldsymbol{z} \in B(\boldsymbol{y}; r_y)$. Then $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) < d(\boldsymbol{x}, \boldsymbol{y}) + r - d(\boldsymbol{x}, \boldsymbol{y}) = r$, so $B(\boldsymbol{y}; r_y) \subseteq B(\boldsymbol{x}; r)$.

Corollary I.10 *The following may be shown by considering the appropriate complements:*

1. *M and $\emptyset$ are closed;*
2. *$\bigcap_i A_i$ is closed if $A_i \subset M$ for all i;*

3. *$\bigcup_i A_i$ is closed if $A_i \subset M$ for all i and $n < \infty$;*

4. *$D(\boldsymbol{x}; r)$ is closed.*

Example Open intervals are open and closed intervals are closed.
(a, ∞) is open as it is a union of open bounded intervals.
$[a, \infty)$ is closed since $(-\infty, a)$ is open.
$\mathbb{Z}$ is closed as $\mathbb{R} \setminus \left(\cup_{n=-\infty}^{\infty}(n, n+1)\right)$ is closed.
$\mathbb{Q}$ and $[0, 1)$ are neither, while $\mathbb{R}$ is both.

Proposition I.11 *Suppose M is a metric space and $A \subseteq M$. A is closed iff every sequence converges to $a \in A$.*

Proof Assume A is closed and $a_n \to a$. Assume the converse so that $a \in U = M \setminus A$ which is an open set. Then there is some $r > 0$ such that $B(a; r) \in U$, and since $a_n \to a$, there exists $n_0 \geq 0$ where $d(a_n, a) < r$ for $n \geq n_0$. This implies $a_n \in B(a; r)$ for all n, but this is a contradiction since $a_n \in A$, and thus $a \in A$.

Assume $a_n \to a \in A$. Let $x \in M \setminus A$, $r > 0$, and assume there is no such $B(x; r) \subset M \setminus A$. Thus there is an intersection, i.e., $B(x; 1/n) \cap A \neq \emptyset$. This implies that there is some i where $a_i \in B(x; 1/n) \cap A$. However, (a_n) is a sequence in A and $d(a_m, x) < 1/n$ for $m \geq n + 1$, so $a_m \to a$, but this implies $x = a$ which is not possible since $x \in M \setminus A$. So $M \setminus A$ is open which means A is closed.

Theorem I.12 *Let M be a complete metric space and $A \subseteq M$ is closed. Then A is complete with the induced metric.*

Proof Let (a_n) be a Cauchy sequence in A. Since M is complete, (a_n) converges in M, but A is closed, so (a_n) converges in A by previous proposition, which implies A is complete.

Let M be a metric space. M is compact if every sequence $(a_n) \in M$ has a convergent subsequence (a_{n_k}).

Example

- $(a_n) = (-1)^n$ is non-convergent but has a convergent sequence.
- $M = (0, 1)$ is not compact since $a_n = 1/n$ and its subsequences do not converge in M.
- $\mathbb{R}$ is not compact as a_n has no subsequence converging in $\mathbb{R}$.
- $M = [0, 1]$ is compact. Let (a_n) be a subsequence in M. Let I_1 be either $[0, 1/2]$ or $[1/2, 1]$, and let (a_{n_k}) be the subsequences in I_1. Continuing this we have a sequence of intervals $I_{m+1} \subset I_m$ with I_m of length 2^{-m}. Denote the subsequences $(a^m_{m_k})$ to be those in I_m. Taking $b_m = a^m_{n_m} \in I_m$, we see that $b_{m+1} \in I_M$ since $I_{m+1} \subset I_m$, so that $d(b_m, b_q) \leq 2^{-m}$ for $q \geq m$. Thus (b_m) is a Cauchy sequence, which is a subsequence of (a_n). Since $M \subseteq \mathbb{R}$, M is complete, so $b_m \to b \in M$, and thus M is compact.

Proposition I.13 *By extension, closed n-gons in $\mathbb{R}^n$ are compact.*

Proposition I.14 *Let $f : M \to N$ be a continuous map between metric spaces. If M is compact, then $f(M) \subset N$ is compact.*

Proof Let (a_n) be a sequence in $f(M)$. Then $a_n = f(b_n)$ for some $b_n \in M$. The sequence (b_{n_k}) converges in M since M is compact, thus

$$\lim_{k \to \infty} a_{n_k} = \lim_{k \to \infty} f(b_{n_k}) = f\left(\lim_{k \to \infty} b_{n_k}\right) = f(b)$$

since f is continuous. So (a_{n_k}) is convergent, thus $f(M)$ is compact.

Proposition I.15 *A closed subset of a compact space is a compact set.*

Proof Let (a_n) be a sequence in $A \subset M$ where M is compact. Since $(a_n) \in M$, (a_{n_k}) is convergent, but A closed so $(a_{n_k}) \to a \in A$, thus A is compact.

Theorem I.16 (Heine–Borel) *A subset $A \subseteq \mathbb{R}^n$ is compact iff A is closed and bounded.*

Proof Suppose A is compact, so clearly A is closed. If A is unbounded, then there exists $(a_n) \in A$ where $d(a_n, 0) \geq n$, so (a_{n_k}) does not converge in $\mathbb{R}^n$. However A is compact, which is a contradiction, so A is bounded.

Suppose A is bounded, then $A \subseteq [a, b]^n$. If A is closed, then it is a closed subset of a compact set, so A is compact by previous proposition.

For example, if $f : M \to N$ with f is a scalar continuous function, then $f(M) \subset \mathbb{R}$ is closed and bounded since M is compact, and thus $f(M)$ compact implies $f(M)$ is closed and bounded.

D. Banach and Hilbert spaces

Let V be a real vector space. The norm on V is a function $\|\cdot\| : V \to [0, \infty)$ where:

1. $\|\boldsymbol{x}\| = 0$ iff $\boldsymbol{x} = \boldsymbol{0}$;
2. $\|\lambda \boldsymbol{x}\| = |\lambda| \cdot \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in V$ and $\lambda \in \mathbb{R}$;
3. $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

The pair $(V, \|\cdot\|)$ gives a normed vector space.

Lemma I.17 *Let V be a normed vector space, then $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$ defines a metric on V.*

Proof Two of the properties follow from definition. To show the reflexive property, note that

$$d(\boldsymbol{y}, \boldsymbol{x}) = \|\boldsymbol{y} - \boldsymbol{x}\| = \|(-1)(\boldsymbol{x} - \boldsymbol{y})\| = \|\boldsymbol{x} - \boldsymbol{y}\| = d(\boldsymbol{x}, \boldsymbol{y}).$$

Example

1. It may be shown that the metrics

$$\sum_i |x_i|, \qquad \sum_i \sqrt{|x_i|^2}, \qquad \max\{|x_i| \in \mathbb{R}\}$$

   define norms on $\mathbb{R}^n$ (the ℓ^1, ℓ^2 and ℓ^∞ norms).

2. The supremum norm on $B(X)$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| \in \mathbb{R}\,;\; x \in X\}.$$

3. For X a metric space, $C_b(X) = \{f : x \to \mathbb{R} \mid f \text{ continuous and bounded}\}$ is also a normed vector space with the supremum norm.

If $C(X) = \{f : x \to \mathbb{R} \mid f \text{ continuous}\}$ then f does not have a supremum, however, we have the following:

Proposition I.18 *If X is compact, then $C(X) = C_b(X)$, so $C(X)$ is a normed vector space.*

Proof $C_b(X) \subseteq C(X)$ regardless of X. For the converse, assume $f \in C(X)$, so that $f(X)$ is compact. This implies $f(X)$ is bounded and closed by the Heine–Borel theorem, so $C(X) \subseteq C_b(X)$.

Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed vector spaces. A function $f : V \to W$ is continuous at $\boldsymbol{x} \in V$ if, for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|\boldsymbol{x} - \boldsymbol{y}\|_V < \delta$ implies that $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\|_W < \epsilon$.

Let V be a normed vector space. V is a Banach space if V with the metric induced by the norm is complete.

Theorem I.19 *Let X be a metric space, then $C_b(X)$ with the supremum norm is a Banach space.*

Proof Since $C_b(X) \subseteq B(X)$, if C_b is closed, then C_b is complete since $B(X)$ is complete. To show this, let $(f_n) \in C_b(X)$, and let $f_n \to f \in B(X)$. The convergene of f_n implies that there exists $n_0 \geq 0$ such that $\|f_n - f\| < \epsilon/3$ for any $\epsilon > 0$ with $n \geq n_0$. Also, $\|f_{n_0}(y) - f(y)\| < \epsilon/3$ for all $y \in X$. The functions are continuous, so there exists $\delta > 0$ where, if $d(x, y) < \delta$, $\|f_{n_0}(x) - f_{n_0}(y)\| < \epsilon/3$ for $x \in X$. Thus, for $d(x, y) < \delta$,

$$|f(x) - f(y)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)| < \epsilon,$$

so f is continuous, and $C_b(X)$ is closed and thus complete.

Corollary I.20 *For $a < b$, $C[a, b]$ with the supremum norm is a Banach space.*

Note that $C[a, b]$ is not a complete space with, for example, the L_2 norm

$$\|f\|_2 = \sqrt{\int_a^b (f(x))^2 \,\mathrm{d}x}.$$

For example, with $f_n = x^n$, $f_n \to 0$ but clearly $f_n(1) = 1$ for all n. The underlying reason is the sequence is not a Cauchy sequence with respect to the norm.

Convergence with respect to the supremum norm is called uniform convergence.

II. ORDINARY DIFFERENTIAL EQUATIONS

III. TANGENT SPACES AND VECTOR FIELDS

IV. DIFFERENTIAL FORMS ON $\mathbb{R}^n$

V. DIFFERENTIAL FORMS ON ORIENTED MANIFOLDS