Academic notes: 1H DAMS

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There are techniques that are used to construct mathematical models of physical problems, which can come from many areas, such as engineering, finance, biology, and so on.

I. LEAST SQUARES APPROXIMATION

Suppose we have a set of n pieces of data (x_i, y_i), $i \in [1, n]$ which we believe should lie on a straight line. Since the straight line has the general form $y = mx + c$, we have the error at each point to be

$$e_i = y_i - mx_i - c.$$

We wish to have a line so that $\sum e_i$ is minimised. One way to do this is to square the error and sum, i.e.

$$S = \sum_{i=1}^{n} (y_i - mx_i - c)^2.$$

By taking a derivative and let it equal to zero, we find a minimum where both partial derivatives of S are zero:

$$\frac{\partial S}{\partial c} = \sum_{i=1}^{n} 2(y_i - mx_i - c)(-1) = 0,$$
$$\frac{\partial S}{\partial x} = \sum_{i=1}^{n} 2(y_i - mx_i - c)(-x_i) = 0.$$

This may be rewritten as

$$\begin{pmatrix} \sum_{i=1}^n 1 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{pmatrix} \begin{pmatrix} c \\ m \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n y_i x_i \end{pmatrix}.$$

Example Suppose we have the set of data

i	1	2	3	4	5	6
x_i	0	1	2	3	4	5
y_i	0.0	9.9	29.6	59.1	98.4	147.5

It can be shown that the resulting matrix equation is

$$\begin{pmatrix} 6 & 15 \\ 15 & 55 \end{pmatrix} \begin{pmatrix} c \\ m \end{pmatrix} = \begin{pmatrix} 344.5 \\ 1377.5 \end{pmatrix}.$$

Solving this gives $m = 29.5$ and $c = -16.33$. In this case a linear model performs poorly, because the data clearly exhibits a more quadratic behaviour.

Extending a linear model to the quadratic case, we see that the sum of errors becomes

$$S = \sum_{i=1}^{n} (y - a_0 - a_1 x - a_2 x)^2,$$

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and the appropriate manipulations give

$$\begin{pmatrix} \sum_{i=1}^n 1 & \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i^3 \\ \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i^3 & \sum_{i=1}^n x_i^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n y_i x_i \\ \sum_{i=1}^n y_i x_i^2 \end{pmatrix}.$$

Applying this to the previous example yields $y = 5x + 4.9x^2$, which fits much better to the data. This procedure may be extended to arbitrary degree, with weighting to certain data points as required. The disadvantages are that, (i) matrix equations are time consuming to solve, (ii) computers are prone to rounding errors, and (iii) there is generally no guide for the degree of polynomial to fit to.

If a system instead behaves like $y = ax^b$ or $y = ae^{bx}$, we can apply logarithms to obtain

$$\log y = \log a + b \log x, \qquad \log y = \log a + bx$$

but we now require $\sum \log y_i$ and so forth and differentiate with respect to $\log a$ and $\log b$, then invert appropriately to get the coefficients.

II. DISCRETE MODELS

These are models of situations where it takes discrete steps. For example, $y_{n+1} = ay_n$ is discrete and is a linear different equation, whose general solution is $y_n = a^n y_0$.

A. Malthus's population model

Suppose the rate of growth is proportional to the current population, and assume that we are in a closed system where no deaths occur. Then we can model the population as

$$p_{n+1} = p_n + ap_n, \qquad a > 0.$$

The closed form solution is $p_n = (1+a)^n p_0$, and we have unbounded growth as expected. This model is of course unrealistic, so if we instead consider

$$p_{n+1} = p_n + a\left(1 - \frac{p_n}{L}\right) p_n,$$

then we see that as $p_n \to L$, the rate of growth tends to zero, which is more realistic if resources and other factors limits growth. This model however is nonlinear which makes it difficult to solve.

B. Z-transform

Let

$$y = \{y_n\}_{n=0}^{\infty},$$

then define the <u>Z-transform</u> as

$$Z(y) = \sum_{n=0}^{\infty} y_n z^n.$$

The sequence of solutions become the coefficients of the infinite sum in z, which we can solve and turn into a neater way to obtain the solution.

Example With $u_n = \{1, 1, 1 \ldots\}$,

$$Z(u) = 1 + z + z^2 + \ldots = \frac{1}{1-z} \qquad (|z| < 1).$$

With $r_n = \{0, 1, 2 \ldots\}$,

$$Z(r) = z + 2z^2 + 3z^3 + \ldots = \frac{z}{(1-z)^2} \qquad (|z| < 1).$$

There are some properties of the Z transform which can help us solve equations:

1. Linearity,
$$Z(\{ax_n + by_n\}) = aZ(\{x_n\}) + bZ(\{y_n\}).$$
2. Backward shift, with $u_n = \{1, 1, 1 \ldots\}$,
$$Z(\{y_{n-m}u_n\}) = z^m Z(y).$$
3. Forward shift,
$$Z(\{y_{n+m}u_n\}) = z^{-m}Z(y) - (z^{-m}y_0 + z^{-m+1}y_1 + \cdots z^{-1}y_{m-1}).$$
4. Differentiation,
$$Z(\{y_n r_n\}) = z\frac{\mathrm{d}}{\mathrm{d}z}Z(y).$$
5. Convolution,
$$Z(\{x_n * y_n\}) = Z(x)Z(y).$$
6. Periodic,
$$Z(\{y_n\}) = \frac{y_0 + zy_1 + \cdots z^{p-1}y_{p-1}}{1 - z^p},$$
for $y_{n+p} = y_n$.

Example 1. For $Z(\{1, -2, 1, 1, -2, 1 \ldots\})$, the periodic property gives
$$Z(y_n) = \frac{1 - 2z + z^2}{1 - z^3} = \frac{1 - z}{1 + z + z^2}.$$
The are no real roots for the denominator so cannot use partial fractions. However, observing that
$$Z(\{\cos(\omega n)\}) = \frac{1 - z\cos\omega}{1 - 2z\cos\omega + z^2},$$
if $\cos\omega = -1/2$, $\omega = 2\pi/3$, then $\sin(2\pi/3) = \sqrt{3}/2$ gives
$$Z\left(\left\{\cos\frac{2\pi}{3}\right\}\right) = \frac{1 + z/2}{1 + z + z^2}, \qquad Z\left(\left\{\sin\frac{2\pi}{3}\right\}\right) = \frac{\sqrt{3}z/2}{1 + z + z^2},$$
so that
$$\frac{1 - z}{1 + z + z^2} = aZ\left(\left\{\cos\frac{2\pi}{3}\right\}\right) - bZ\left(\left\{\sin\frac{2\pi}{3}\right\}\right) = Z\left(\left\{\cos\frac{2\pi n}{3} - \sqrt{3}\sin\frac{2\pi n}{3}\right\}\right).$$

2. For
$$Z(\{x_n\}) = \frac{z^2}{(2 - z)(1 - z)},$$
we use the shift properties. With partial fractions, we have
$$\frac{1}{(2 - z)(1 - z)} = \frac{1}{1 - z} - \frac{1}{2 - z} = Z(\{u_n\}) - \frac{1}{2}\frac{1}{1 - z/2} = Z(\{u_n\}) - \frac{1}{2}Z\left(\left\{\frac{1}{2}\right\}\right)$$
using $Z(\{\alpha^n\}) = (1 - \alpha n)^{-1}$. So $x_n = 1 - (1/2)^{n+1}$ with a forward shift of 2, and so $x_n = \{0, 0, 1 - (1/2)^{n+1}, \ldots\}$.

3. Solve $y_{n+2} - 3y_{n+1} + 2y_n = 0$, $y_0 = 1$, $y_1 = 3$.
   Taking a Z transform of both sides and noting that
   $$Z(\{y_{n+2}\}) = z^{-2}Z(\{y_n\}) - z^{-2}y_0 - z^{-1}y_1, \qquad Z(\{y_{n+1}\}) = z^{-1}Z(\{y_n\}) - z^{-1}y_0,$$
   this gives, using the initial conditions,
   $$(z^{-2} - 3z^{-1} + 2)Z(\{y_n\}) - z^{-2} = 0.$$
   Rearranging, this gives
   $$Z(\{y_n\}) = \frac{z^{-2}}{z^{-2} - 3z^{-1} + 2} = \frac{1}{(1-2z)(1-z)} = \frac{2}{1-2z} - \frac{1}{1-z} = 2Z(\{2^n\}) - Z(\{u_n\}),$$
   so that $y_n = 2^{n+1} - 1$.

In theory we can solve any linear constant coefficient difference equation with the Z transform. Non-linear equations are more tricky, although we can still get some information from them. For example, for
$$p_{n+1} = p_n + a\left(1 - \frac{1}{L}p_n\right)p_n,$$
we can see that the fixed points of the model satisfy
$$p = p + a\left(1 - \frac{1}{L}p\right)p, \qquad \Rightarrow \qquad p = 0, L.$$
We can see that when p is close to zero, $a(1 - p_n/L)p_n > 0$, so it is an unstable point; the model predicts growth and so we never reach zero. By contrast, L is a stable point, as p_n tends towards L for from values of a.

We can make this approach more rigourous by linearising around critical points:

- $p = 0$. Assume p_n is close to zero, so that
  $$p_{n+1} - p_n = \left(1 - \frac{1}{L}p_n\right)p_n \approx ap_n,$$
  as the quadratic term is small. This has solution $p_n = (1+a)^n p_0$, and since $(1+a) > 1$, p_n grows and moves away from zero, and is thus an unstable solution.
- $p = L$. Assume that p_n is close to $p = L$, then
  $$(p_{n+1} - L) - (p_n - L) = (p_{n+1} - p_n) = a\left(1 - \frac{1}{L}p_n\right)p_n = -a(p_n - L) - \frac{a}{L}(p_n - L)^2 \approx -a(p_n - L).$$
  The linearised difference equation has solution $p_n = (1-a)^n p_0$, so L is stable if $|1-a| < 1$, and thus L is conditionally stable for $a \in (0, 2)$.

We can make this argument more abstract. Suppose that a system satisfies $p_{n+1} = f(p_n)$ with fixed point $p = f(p)$. Then
$$p_{n+1} - p = f(p_n) - f(p) = f'(\theta)(p_n - p), \qquad \theta \in (p_n, p)$$
by the mean value theorem. Thus by triangle inequality,
$$|p_{n+1} - p| \leq |f'(\theta)| \cdot |p_n - p|.$$
As $p_n \to p$, $\theta \to p$, so the fixed point p is stable if $|f'(p)| < 1$, and unstable if $|f'(p)| > 1$.

Example 1. From the population model, $f(x) = x + a(1 - x/L)x$, so
$$f'(x) = 1 + a\left(1 - \frac{2x}{L}\right).$$
$|f'(0)| = |1+a| > 1$ whilst $|f'(L)| = |1-a| < 1$, so they are unstable and stable respectively.

2. For $p_{n+1} = ap_n^3$, the critical points are $p = 0$ and $p = \pm 1/\sqrt{a}$. We have $f'(x) = 3ax^2$, so $f'(0) = 0$ whilst $f'(\pm 1/\sqrt{a}) = 3$, so $p = 0$ is the stable critical point.

3. For $p_{n+1} = ap_n(1 - p_n)$, the stable points are $p = 0$ and $p = 1 - 1/a$. $f'(x) = a(1-2x)$, and so $f'(0) = a$ and $f'(1 - 1/a) = 2 - a$. The former is stable if $|a| < 1$, whilst the latter is stable if $|2 - a| < 1$.

C. System of equations

Consider the predator-prey equations. Let the population of rabbits and foxes be R_n and F_n respectively. Then

$$\begin{aligned} R_{n+1} &= R_n + aR_n - bR_nF_n, \\ F_{n+1} &= F_n - cF_n + dR_nF_n, \end{aligned}$$

where a is the growth rate of rabbits, b the death rate of rabbits due to predation, c the death rate of foxes due to absence of rabbit,s and d the growth rate of foxes due to predation, all coefficients being positive. In general, a system would look like

$$\begin{aligned} p_{n+1} &= f(p_n, q_n), \\ q_{n+1} &= g(p_n, q_n), \end{aligned}$$

with fixed points $p = f(p,q)$ and $q = g(p,q)$. By a Taylor expansion we get

$$\begin{aligned} p_{n+1} - p = f(p_n, q_n) - f(p,q) &\approx \left.\frac{\partial f}{\partial p}\right|_{(p,q)} (p_n - p) + \left.\frac{\partial f}{\partial q}\right|_{(p,q)} (q_n - q), \\ q_{n+1} - q = g(p_n, q_n) - g(p,q) &\approx \left.\frac{\partial g}{\partial p}\right|_{(p,q)} (p_n - p) + \left.\frac{\partial g}{\partial q}\right|_{(p,q)} (q_n - q). \end{aligned}$$

The stability then depends on the Jacobian matrix

$$\mathsf{J} = \begin{pmatrix} \partial f/\partial p & \partial f/\partial q \\ \partial g/\partial p & \partial g/\partial q \end{pmatrix}.$$

Example 1. For the system

$$p_{n+1} = \frac{1}{2}p_n + ap_nq_n, q_{n+1} = ap_nq_n + \frac{1}{2}q_n,$$

the fixed points satisfy $p - 2apq = 0$ and $q - 2apq = 0$, so the fixed points are $(0,0)$ and $(1/(2a), 1/(2a))$. For the former,

$$\mathsf{J}(0,0) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Letting $X_n = p_n - p$ and $Y_n = q_n - q$, we have

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}.$$

This gives $X_n = (1/2)^n X_0$ and $Y_n = (1/2)^n Y_0$, so the differences tend to zero as $n \to \infty$, so this critical point is stable. A similar manipulation for the other fixed point gives

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}.$$

The second one gives $Y_{n+1} = (1/2)X_n + Y_n$ whilst the first one gives $(1/2)Y_n = X_{n+1} - X_n$, thus we may write

$$\begin{aligned} X_{n+2} &= X_{n+1} + \frac{1}{2}Y_{n+1} \\ &= X_{n+1} + \frac{1}{4}X_n + \frac{1}{2}Y_n \\ &= X_{n+1} + \frac{1}{4}X_n + X_{n+1} - X_n, \end{aligned}$$

so we have $X_{n+2} - 2X_{n+1} + (3/4)X_n = 0$. By a Z transform we get

$$X_n = -A\left(\frac{1}{2}\right)^n + B\left(\frac{3}{2}\right)^n \to \infty$$

as $n \to \infty$. Doing the same procedure for Y_n shows that this critical point is unstable.

2. The system

$$p_{n+1} = \frac{3}{2}p_n - bp_nq_n, q_{n+1} = dp_nq_n + \frac{1}{2}q_n,$$

has fixed points $(0, 0)$ and $(1/(2b), 1/(2d))$. For the first critical point,

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} 3/2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}.$$

so it is unstable. A similar manipulation to the previous example gives for the second critical point

$$Y_{n+2} - 2Y_{n+1} + \frac{5}{4}Y_n = 0,$$

so a Z transform gives

$$Y_n = A\left(\frac{\sqrt{5}}{2}\right)^n \cos(\omega n) + B\left(\frac{\sqrt{5}}{2}\right)^n \sin(\omega n)$$

which does not tend to zero, so the critical point is also unstable.

III. CONTINUOUS MODELS

We can take discrete models and make them continuous. For example, taking the Malthus population model $p_{n+1} = p_n + ap_n$ gives

$$p(t + \delta t) = p(t) + (b\delta t)p(t) \qquad \Rightarrow \qquad \frac{p(t + \delta t) - p(t)}{\delta t} = bp(t)$$

as $\delta t \to 0$, giving

$$p'(t) = bp(t).$$

With $p(0) = p_0$, $p(t) = p_0 \mathrm{e}^{bt}$. We can use this model to work out half life and double time of population, or radioactive substances for example. If the half life is τ, then $p(\tau) = p(0)/2$ for $b < 0$, i.e., $\tau = -b \log 2$.

Example Suppose 100mg sample of radioactive material decays to 82.09mg in a week. Find the half life τ.
We have $p_0 = 100$, $p(\tau) = 82.09 = 100\mathrm{e}^{7b}$, which gives

$$b = \frac{1}{2} \log \frac{82.09}{100}.$$

This gives $\tau = 24.5$days.

A. Scalar autonomous equations

These are equations of the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f(p),$$

i.e., not explicitly dependent on t. Ideally we solve this by separation of variables:

$$\int \frac{\mathrm{d}p}{f(p)} = \int \mathrm{d}t.$$

However, the LHS may not exist or may be non-unique.

Example 1. $f(p) = bp$ gives $p(t) = p_0 \mathrm{e}^{bt}$.

2. $f(p) = p^2$ gives $p(t) = p_0/(1 - p_0 t)$ which blows up as $t \to 1/p_0$.
3. $f(p) = \sqrt{p}$ gives $p(t) = 0$ or $p(t) = t^2/4$.

It is sometimes possible to get information without solving the equation:

- Fixed points: for fixed point p^*, $f(p^*) = 0$ (which gives $\mathrm{d}p/\mathrm{d}t = 0$).
- Equilibria: We can analyse the stability of the fixed points p^* by linearising. For p close to p^* we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(p - p^*) = f(p) - f(p^*) \approx f'(p^*)(p - p^*).$$

  Let $x \approx p - p^*$, then the above equation becomes

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(p^*)x.$$

  By solving the linear ODE, we get

$$x(t) = x_0 \mathrm{e}^{f'(p^*)}.$$

  We see that a fixed point is stable if $f'(p^*) < 0$, and is unstable if $f'(p^*) > 0$.

Example 1. Malthus' model has $f(p) = bp$, with $p^* = 0$. We have $f'(p) = b$, $f'(p^*) = b$, so the fixed point is stable if $b < 0$ and unstable otherwise.

2. Phase separation of alloys may be modelled as $f(p) = p - p^3$. The fixed points are $p^* = 0, \pm 1$. Now, $f'(p) = 1 - 3p^2$, so that $f'(0) = 1$ and $f'(\pm 1) = -2$, i.e., $p = 0$ is unstable and $p = \pm 1$ is stable.
3. The logistic model is $f(p) = rp(1 - p/L)$ with $r, L > 0$. The fixed points are $f^* = 0, L$, and $f'(p) = r(1 - 2p/L)$, so that $f'(0) = r$ whilst $f'(L) = -r$. So $p = 0$ is unstable whilst $p = L$ is stable.

B. Systems of autonomous equations

Let a model be modelled by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f(p, q), \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = g(p, q).$$

Although it is possible to calculate the solution of the above, we can always derive useful information about them, such as fixed points and stability of fixed points. Additionally, we can compute the phase path by working out and solving

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{f(p, q)}{g(p, q)}.$$

Example 1. Let

$$\frac{\mathrm{d}p}{\mathrm{d}t} = q, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = p.$$

(This has exact solution $p(t) = A \sin t + B \cos t$, $q(t) = A \cos t - B \sin t$). The fixed points are $p = q = 0$. The phase paths in this case is obtained by solving $\mathrm{d}p/\mathrm{d}q = -q/p$. This is separable and gives $p^2 + q^2 =$ const, i.e., the phase paths are circular in the (p, q) plane.

2. Let

$$\frac{\mathrm{d}p}{\mathrm{d}t} = pq, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = -p^2.$$

The fixed points are ones where $p = 0$, i.e., a line. We see that the phase paths are the same ones as the above example, although the direction it travels around in phase space is different.

3. The Lotka–Volterra predator–prey model is

$$\frac{\mathrm{d}p}{\mathrm{d}t} = ap - bpq, \qquad \frac{\mathrm{d}q}{\mathrm{d}t} = -cq + dpq,$$

with positive coefficients. The fixed points are $(0,0)$ and $(c/d, a/b)$. The phase path equation is

$$\frac{\mathrm{d}p}{\mathrm{d}q} = \frac{p(a - bq)}{q(-c + bq)},$$

which is separable and satisfies

$$a \log q + c \log p - bq - dp = \text{const.}$$

As before we can linearise around the fixed points to evaluate whether they are stable or not. We see that, defining $x \approx p - p^*$ and $y \approx q - q^*$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \partial f/\partial p & \partial f/\partial q \\ \partial g/\partial q & \partial g/\partial q \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Stability depends on the Jacobian matrix. In fact the solutions to the ODEs is a linear combination of terms of the form $\mathrm{e}^{\lambda_i t}$, where λ_i are the eigenvalues of the Jacobian matrix (formally, the solutions to these linear equations are exponentials). We thus only need to look at the real part of λ_i:

1. if all the eigenvalue have negative real part, the system is stable;
2. if some are positive then it is unstable;
3. if they are negative and/or zero the situation is more complicated.

Example We use the previous three examples.

1. In this case the matrix is

$$\mathsf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which results in eigenvalues $\lambda = \pm\mathrm{i}$. We know already that the phase path is circular, so really it is neither stable nor unstable (the fixed point is a focus).

2. At fixed point $(0, y)$, we have

$$\mathsf{A} = \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix},$$

so the eigenvalues are $\lambda = y, 0$. For $y < 0$ this has the possibility of being stable (the set of critical points form a stable and unstable node).

3. The Jacobian matrix is

$$\mathsf{A} = \begin{pmatrix} a - bq & -bp \\ dq & -c + bp \end{pmatrix}.$$

At $(0,0)$, $\lambda = a, -c$, so it is unstable. The point $(a/b, c/d)$ is complicated.

IV. STOCHASTIC MODELS

These are models for processes that evolve over time in a probabilistic manner. To predict long term behaviours we use the probabilistic approach and introduce the stochastic transition matrix. This is a square matrix where all entries are positive and all the row entries sum to 1. Rows represent the states and the columns represent the probabilities of changing to a particular state. The following is an example of a transition matrix:

$$\mathsf{A} = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0.5 & 0 & 0.5 \\ 0.5 & 0 & 0.5 & 0 \end{pmatrix}.$$

For example, this says that state 1 can remain at state 1 with probability 0.5 and move to state two with probability 0.5, and so on. These can be represented in a transition diagram. To calculate the probability of being at the state described by a row vector $\boldsymbol{p}(n+1)$ from $\boldsymbol{p}(n)$, we calculate $\boldsymbol{p}(n+1) = \boldsymbol{p}(n)\mathsf{A}$ (note the order of multiplication).

In the long term, we seek $\boldsymbol{p}$ such that $\boldsymbol{p} = \boldsymbol{p}\mathsf{A}$ (this may be regarded as a sort of eigenvalue problem). For the example above, the resulting constraints are

$$\begin{aligned} p_1 &= 0.5p_1 + 0.25p_2 + 0.25p_4, \\ p_2 &= 0.5p_1 + 0.25p_2 + 0.5p_3, \\ p_3 &= 0.25p_2 + 0.5p_4, \\ p_4 &= 0.25p_2 + 0.5p_3. \end{aligned}$$

Together with the fact they are probabilities, we have $p_1 + p_2 + p_3 + p_4 = 1$, and it may be shown that, in the long term, we expect to be in state i with probability

$$p_1 = \frac{1}{3}, \qquad p_2 = \frac{1}{3}, \qquad p_3 = \frac{1}{6}, \qquad p_4 = \frac{1}{6}.$$