

## Academic notes: Topological Fluid Dynamics

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(Dated: May 29, 2018)

Notes on the MAGIC “Topological Fluid Dynamics” course.

### I. INTRODUCTORY CONCEPTS

**Definition** The Lie bracket is an anti-symmetric bilinear form  $[\cdot, \cdot]$  satisfying the Jacobi condition

$$[[\mathbf{u}, \mathbf{v}], \mathbf{w}] + [[\mathbf{v}, \mathbf{w}], \mathbf{u}] + [[\mathbf{w}, \mathbf{u}], \mathbf{v}] = 0. \quad (1)$$

The Lie bracket that we will be interested in for fluid dynamics will usually be

$$[\mathbf{u}, \mathbf{v}] = \mathbf{u} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{u}.$$

We distinguish between vectors and co-vectors (or 1-forms): if  $\phi$  is materially conserved under a divergence-free flow, i.e.,

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi = 0,$$

then for a gradient of the scalar  $\mathbf{g} = \nabla \phi$ , we have (after integration by parts assuming appropriate boundary conditions)

$$\frac{D\mathbf{g}}{Dt} = -(\nabla \mathbf{u}) \cdot \mathbf{g}, \quad (2)$$

which is different to the transport for vectors, where

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}, \quad (3)$$

and  $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$  is the vorticity, and the wedge  $\wedge$  can be thought of as the cross product.

**Lemma I.1**  $D(\boldsymbol{\omega} \cdot \mathbf{g})/Dt = 0$  if  $\boldsymbol{\omega}$  is a vector and  $\mathbf{g}$  is a 1-form.

**Proof**

$$\frac{D}{Dt}(\boldsymbol{\omega} \cdot \mathbf{g}) = \boldsymbol{\omega} \cdot (\nabla \mathbf{u}) \cdot \mathbf{g} = 0.$$

If we consider three vectors  $\mathbf{b}, \mathbf{c}, \mathbf{d}$  that are not multiples of each other, then this defines an infinitesimal volume element. Since the flow is divergence free, the volume is conserved, so

$$\frac{D}{Dt}(\mathbf{b} \wedge \mathbf{c} \cdot \mathbf{d}) = 0.$$

With the above lemma,  $\mathbf{b} \wedge \mathbf{c}$  may be thought of as a 1-form, representing a surface element  $d\mathbf{S} = \mathbf{u} \, dS$ . Accordingly,

$$\frac{D}{Dt}d\mathbf{r} = d\mathbf{r} \cdot \nabla \mathbf{u}, \quad \frac{D}{Dt}d\mathbf{S} = -(\nabla \mathbf{u}) \cdot d\mathbf{S}, \quad \frac{D}{Dt}dV = 0,$$

so a line element ( $d\mathbf{r}$ ) transforms like a vector, a surface element ( $d\mathbf{S} = \mathbf{b} \wedge \mathbf{c}$ ) transforms like a 1-form, and a volume element ( $dV = \mathbf{b} \wedge \mathbf{c} \cdot \mathbf{d}$ ) transform like a scalar.

**Definition** The Lie derivative evaluates the change of a vector field along the flow of another vector field.

In our case,  $D/Dt$  acts as the Lie derivative, and we use this in the following example:

**Kelvin’s circulation theorem** Consider a material curve  $C(t)$  and that

$$\Gamma_C = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{r} = \int_{S(t)} \boldsymbol{\omega} \cdot d\mathbf{S}, \quad (4)$$

so

$$\frac{d}{dt}\Gamma_C = \int_{S(t)} \frac{D}{Dt}(\boldsymbol{\omega} \cdot d\mathbf{S}) = 0 \quad (5)$$

since  $\boldsymbol{\omega} \cdot d\mathbf{S}$  is a passive scalar. Thus circulation  $\Gamma_C$  is materially conserved for ideal Euler flows.

## II. INTEGRAL QUANTITIES

**Definition** The kinetic and magnetic energy is defined as

$$E_K = \frac{1}{2} \int |\mathbf{u}|^2 dV, \quad E_M = \frac{1}{2} \int |\mathbf{B}|^2 dV. \quad (6)$$

**Definition** The kinetic helicity is defined as

$$H_K = \int \mathbf{u} \cdot \boldsymbol{\omega} dV. \quad (7)$$

**Lemma II.1**  $H_K$  is conserved in an ideal flow when  $V$  is taken to be a material volume, with the boundary conditions being  $\boldsymbol{\omega} \cdot \mathbf{n} = 0$ , where  $\mathbf{n}$  is the outward normal of the material volume.

**Proof**

$$\begin{aligned} \frac{d}{dt} H_K(V) &= \int \left[ \frac{D\mathbf{u}}{Dt} \cdot \boldsymbol{\omega} + \mathbf{u} \cdot \frac{D\boldsymbol{\omega}}{Dt} \right] dV \\ &= \int [-\nabla p \cdot \boldsymbol{\omega} + \mathbf{u} \cdot (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + (\mathbf{u} \wedge \boldsymbol{\omega}) \cdot \boldsymbol{\omega}] dV \\ &= \int \nabla \cdot \left( -p\boldsymbol{\omega} + \frac{1}{2} |\mathbf{u}|^2 \boldsymbol{\omega} \right) dV \\ &= \int_S \left( -p + \frac{|\mathbf{u}|^2}{2} \right) \boldsymbol{\omega} \cdot \mathbf{n} dS \\ &= 0. \end{aligned}$$

If  $V(0)$  is bounded by vortex lines, then it will do so for  $V(t)$  as vorticity is conserved.

**Definition** The magnetic flux through a surface  $S$  is given by

$$\Phi_S = \int_S \mathbf{B} \cdot d\mathbf{S}. \quad (8)$$

**Theorem II.2 (Alfvén's theorem)** Magnetic flux  $\Phi_S$  is conserved for ideal MHD where  $S(t)$  is a material surface.

**Proof** Proof as for Kelvin's circulation theorem but with the boundary condition  $\mathbf{B} \cdot \mathbf{n} = 0$ .

**Definition** With  $\nabla \cdot \mathbf{B} = 0$ , we define the vector potential  $\mathbf{A}$  of  $\mathbf{B}$  by  $\mathbf{B} = \nabla \wedge \mathbf{A}$ .

This operation exists at least locally, from  $\nabla \cdot \mathbf{B} = 0$ . This  $\mathbf{A}$  can be defined locally in infinite space, or in a bounded space provided that it is simply connected. Otherwise there may be problems uncurling  $\mathbf{B}$ .

Note that  $\mathbf{B}$  is unchanged if we consider  $\mathbf{A} \rightarrow \mathbf{A} + \nabla\phi$ . Uncurling the ideal MHD equation gives

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \wedge \mathbf{B} + \nabla\phi = \mathbf{u} \wedge (\nabla \wedge \mathbf{A}) + \nabla\phi,$$

where  $\nabla\phi$  corresponds to the choice of gauge. One choice is the Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$ , which results in the Poisson equation

$$\nabla^2 \phi = -\nabla \cdot (\mathbf{u} \wedge \mathbf{B}),$$

cf. the pressure equation in fluid systems. Note that with other gauge choices (e.g. Lorenz gauge), such an equation may change form or not arise.

**Definition** The magnetic helicity is defined as

$$H_M = \int \mathbf{A} \cdot \mathbf{B} dV. \quad (9)$$

Note that this definition is gauge invariant, since

$$\frac{\partial}{\partial t} H_M = \int [(\mathbf{u} \wedge \mathbf{B} + \nabla \phi) \cdot \mathbf{B} + \mathbf{A} \cdot \nabla \wedge (\mathbf{u} \wedge \mathbf{B})] dV.$$

The first integral vanishes because of the definition of the cross product and with  $\mathbf{B} \cdot \mathbf{n} = 0$  as the boundary condition. Additionall, noting that

$$\mathbf{A} \cdot \nabla \wedge (\mathbf{u} \wedge \mathbf{B}) = \nabla \cdot [(\mathbf{u} \wedge \mathbf{B}) \wedge \mathbf{A}],$$

the second integral vanishes, thus there is no flow to transport helicity density  $\mathbf{A} \cdot \mathbf{B}$  out of the domain.

Is  $\mathbf{A}$  a vector of a 1-form? At present it is *neither*: if one uses the Coulomb gauge then one has to solve a global Poisson equation for  $\phi$ , and thus  $\phi$  is generally not transported locally. However, if we write

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \wedge (\nabla \wedge \mathbf{A}) + \nabla \phi = -\mathbf{u} \nabla \mathbf{A} + (\nabla \mathbf{A}) \cdot \mathbf{u} + \nabla \phi,$$

then the gauge choice  $\phi = -\mathbf{u} \cdot \mathbf{A}$  results in

$$\frac{D\mathbf{A}}{Dt} = -(\nabla \mathbf{u}) \cdot \mathbf{A}, \quad (10)$$

and  $\mathbf{A}$  may then be identified as a 1-form. With this choice of gauge, there is the integral conservation of  $H_M = \int h_M dV$  with  $h_M = \mathbf{A} \cdot \mathbf{B}$ , as well as material conservation of the helicity density  $h_M$  by Lemma I.1. Although this is attractive, having a materially conserved quantity, it does depend on following the field  $\mathbf{A}$  in this gauge. As it evolves,  $\mathbf{A}$  may become very fine-scaled and develop substantial complexity.

**Definition** The cross helicity is defined as

$$H_X = \int \mathbf{u} \cdot \mathbf{B} dV. \quad (11)$$

**Lemma II.3** *The cross helicity  $H_X$  is conserved for ideal MHD flow, but kinetic helicity  $H_K$  is not.*

**Proof** Just to the algebra, and note that in the calculation for  $H_K$  there is a  $\mathbf{j} \wedge \mathbf{B}$  term that appears, which is generically not zero.

#### A. Helicity as linking number

We either work with  $H_K$  or  $H_M$  (for Euler or ideal MHD equations). Consider two linked flux tubes, where  $\mathbf{B} = 0$  except in the tubes, and each tube carries a flux  $\Phi_j > 0$ .



$$\Phi_j = \int_{\mathcal{A}_j} \mathbf{B} \cdot d\mathbf{S}, \quad \text{span}(\mathcal{A}_j) = \text{the tubes } T_j$$

We assume each individual tube is unknotted. Now, using Stokes' theorem,

$$H_j = \int_{T_j} \mathbf{A} \cdot \mathbf{B} dV = \Phi_j \oint_{C_j} \mathbf{A} \cdot d\mathbf{l} = \Phi_j \oint_{C_j} \mathbf{B} \cdot d\mathbf{S} = \Phi_j \Phi_i, \quad (12)$$

so that

$$H_M = H_1 + H_2 = \pm 2\Phi_1\Phi_2. \quad (13)$$

If the tubes are unlinked, then the helicity is zero. In general, we have

$$H_M = \Phi_i\Phi_j\alpha_{ij}, \quad (14)$$

where  $\alpha_{ij}$  are integers that label the signed number of times that the centre lines  $C_j$  of the  $j^{\text{th}}$  tube crossing the surface  $S_i$  of the  $i^{\text{th}}$  tube. Here, the sign is “+” when the direction of  $C_j$  agrees with the outward normal of  $S_i$ , obtained from the right-hand rule from the direction of the centre line  $C_i$ . If a tube is knotted, then we could break it up accordingly to compute the helicity, thus defining  $\alpha_{ii}$  in this slightly more complicated case.

Note that the topological nature of the invariant in terms of linkage of field lines or vortex lines explains why it is invariant under ideal flow, as linkages cannot be broken. Note also that we cannot define linkage easily if the lines do not close up (e.g., in most astrophysical situations). This is resolved later.

There is a connection between helicity and the topological invariant called the Gauss linking number. Consider a magnetic field  $\mathbf{B}$  in unbounded space that occupies tubes  $T_j$ , with centrelines  $C_j$  and fluxes  $\Phi_j$ . The corresponding  $\mathbf{A}$  (in the Coulomb gauge) may be obtained at any time by a Biot–Savart integral (as can the velocity from the vorticity), namely, as

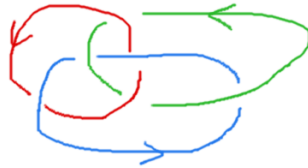
$$\mathbf{A}(\mathbf{x}) = -\frac{1}{4} \int_D \frac{(\mathbf{x} - \mathbf{x}') \wedge \mathbf{B}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} dV' = \sum_j \left( -\frac{\Phi_j}{4\pi} \right) \oint_{C_j} \frac{(\mathbf{x} - \mathbf{x}') \wedge d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (15)$$

This then implies that the constants  $\alpha_{ij}$  in equation (14) is given by

$$\alpha_{ij} = \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\mathbf{x} - \mathbf{x}') \cdot d\mathbf{x} \wedge d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}. \quad (16)$$

This integral gives an integer, which is the Gauss linking number of curve  $i$  with curve  $j$  ( $i \neq j$ ). Note that if  $\alpha_{ij} \neq 0$ , then the curves are linked.

**Example** An interesting example however is the Borromean rings, where  $\alpha_{ij} = 0$  for all pairs  $i$  and  $j$ , so there is no helicity even though there is linkage.



We have assumed that there is no linkage associated with the field lines inside a given flux tube. This is not always correct and leads on to how one measures the twist and writhe of the magnetic field inside a flux tube.

## B. Beltrami and ABC flows

We have focused on the magnetic field that is concentrated in discrete flux tubes. In the case of fluid flows however we have a continuous distribution, and very few field lines actually close up. The work of topological invariants have been generalised by Arnol'd to cover this case, and he refers to this as the asymptotic Hopf invariant. The idea is to take magnetic field lines starting at points  $\mathbf{a}$  and integrate trajectories  $\mathbf{x}(\mathbf{a}, t)$  along them for some large time  $T \gg 1$ . One then uses a set of paths to link the end points  $\mathbf{x}(\mathbf{a}, T)$  to  $\mathbf{a}$ , with paths being short and straight. This gives a collection of closed curves from which the Gauss linking number may be computed. Taking  $T \rightarrow \infty$  then yields the invariant (with technical details omitted here).

Now, consider an ideal fluid in a finite region  $D$  with  $H_K \neq 0$ . Then what is the minimum value of enstrophy

$$\Omega_K = \int_D |\boldsymbol{\omega}|^2 dV \quad (17)$$

for this given helicity? By the Cauchy–Schwartz inequality,

$$H_K^2 = \left( \int_D \mathbf{u} \cdot \boldsymbol{\omega} \, dV \right)^2 \leq \left( \int_D |\mathbf{u}|^2 \, dV \right) \left( \int_D |\boldsymbol{\omega}|^2 \, dV \right) = 2E_K \Omega_K. \quad (18)$$

By Poincaré’s inequality,

$$\Omega_K = \int_D |\boldsymbol{\omega}|^2 \, dV \geq \frac{1}{L^2} \left( \int_D |\mathbf{u}|^2 \, dV \right) = \frac{2E_K}{L^2} \quad (19)$$

for any  $\mathbf{u}$  such that  $\mathbf{u} \cdot \mathbf{n} = 0$ , and  $L$  is given by the smallest eigenvalue of the curl operator, i.e.

$$\boldsymbol{\omega} = \nabla \wedge \mathbf{u} = \frac{1}{L} \mathbf{u}. \quad (20)$$

Together, we have

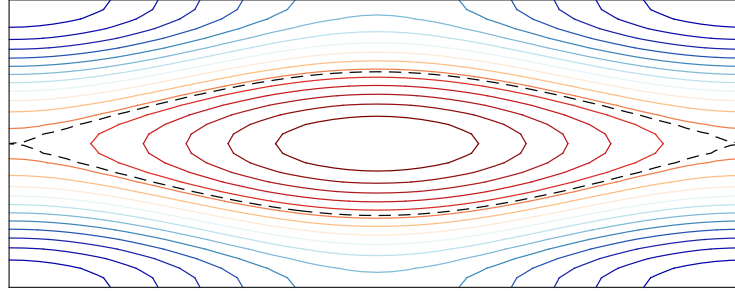
$$H_K^2 \leq 2E_K \Omega_K \leq L^2 \Omega_K \quad \Rightarrow \quad \Omega_K \geq \frac{1}{L} |H_K|. \quad (21)$$

We have equality if  $\boldsymbol{\omega} = \lambda \mathbf{u}$  everywhere in  $D$ ; such flows are called Beltrami flows, and satisfy the Euler equations since  $\mathbf{u} \wedge \boldsymbol{\omega} = 0$ .

A special case of a Beltrami flow is the ABC flow (Arnold–Beltrami–Childress)

$$\mathbf{u} = A(0, \sin x, \cos x) + B(\cos y, 0, \sin y) + C(\sin z, \cos z, 0) \quad (22)$$

for constants  $A, B, C$ , and is a flow in  $\mathbb{T}^3$  with no mean component. If one of the constants is zero, we obtain the Cat’s eyes pattern with regular streamlines:

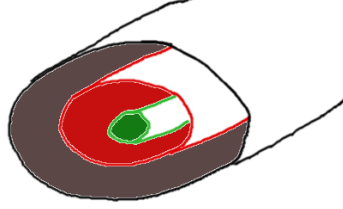


If all three are non-zero, then typically we see a complex mixture of regular vortices, with stream-surfaces that take the form of tori, interlaced with regions of chaos. To make life difficult, the flow is highly three dimensional. These flows have been studied by means of Poincaré sections, in which trajectories  $\mathbf{x}(\mathbf{a}, t)$  are followed and points plotted whenever they intersect one of a family of planes (e.g.,  $x = \text{const}$ ).

Just how complicated can steady Euler flows be? Are there any constraints on their structure? An answer to this (and pretty much all that is known) is provided in a rigorous result by Arnol’d: in a finite domain  $D$ , a steady Euler flow obeys  $\mathbf{u} \wedge \boldsymbol{\omega} = \nabla P$ , so that  $\mathbf{u} \cdot \nabla P = \boldsymbol{\omega} \cdot \nabla P = 0$ . Then, provided  $\nabla P$  does not vanish identically in a 3D region of space,, there will exist a family of 2D surfaces  $S(P_0)$  on which  $P = P_0 = \text{const}$ . This says that the streamlines and vortex lines will be tangent to such surfaces. For example:



To understand what form the surfaces can take, we need to be careful about points where  $\nabla P = 0$ ; we assume there are only finitely many such points in  $D$  under consideration. Choosing a value  $P_0 = \text{const}$  gives a surface  $S(P_0)$  which does not intersect any points with  $\nabla P = 0$ . We assume this surface does not intersect  $\partial D$ . Now,  $\mathbf{u}$  and  $\boldsymbol{\omega}$  are tangent to  $S(P_0)$ , so does not vanish on  $S(P_0)$ . It can be shown that the only such surface satisfying such conditions is topologically a torus, and that  $\mathbf{u}$  and  $\boldsymbol{\omega}$  wraps around it. As we vary the value of  $P_0$ , we obtain *nested* tori  $S(P_0)$ , provided we do not encounter a point where  $\nabla P = 0$ ; at such a point, surfaces may intersect.



The region in which we must have tori is linked to the non-vanishing flow field  $\mathbf{u}$  (or  $\boldsymbol{\omega}$ ) on the surfaces. It cannot be a sphere (cf. hairy ball theorem). For more and for links with the Euler characteristic, see the book “Topological Fluid Dynamics” by Arnold & Khesin.

We assumed  $P$  is such that  $\nabla P = 0$  at only finitely many points, so nested 2D surfaces  $P_0 = \text{const}$  may be defined. If  $P = \text{const}$  over some region  $V \subset D$ , then  $\mathbf{u} \wedge \boldsymbol{\omega} = 0$  on  $V$ , and  $\boldsymbol{\omega} = \alpha(\mathbf{x})\mathbf{u}$ , so we must have a Beltrami flow on  $V$ . On the other hand, the fields are divergence-free, so  $\mathbf{u} \cdot \nabla \alpha = \boldsymbol{\omega} \cdot \nabla \alpha = 0$ . The above argument then extends over to this case:  $\alpha(\mathbf{x})$  defines 2D tori  $S(\alpha_0)$ , with  $\nabla \alpha = 0$  assumed to only happen on a finitely many number of points.

The final possibility then is that  $\alpha = \lambda = \text{const}$ , over a 3D region (Beltrami flows). Here,  $\mathbf{u}$  and  $\boldsymbol{\omega}$  lines lie on nested tori except in the special case of constant Bernoulli function  $P = p + |\mathbf{u}|^2/2$  and  $\boldsymbol{\omega} = \lambda \mathbf{u}$ .

(Note that Arnold'd assumes  $\mathbf{u}$  to be complex analytic, which means critical points  $\nabla P = 0$  or  $\nabla \alpha = 0$  cannot accumulate, and if  $P$  or  $\alpha$  are constant in any 3D volume then it is constant everywhere. With this, no messy mixtures are allowed. The argument needs to be extended in more general situations in which we may only require  $\mathbf{u} \in C^\infty$ .)

### C. Magnetic relaxation

Consider the case where there is viscous but no magnetic diffusion, so that the governing equations are

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \wedge \boldsymbol{\omega} + \mathbf{j} \wedge \mathbf{B} - \nabla P + \nu \nabla^2 \mathbf{u}, \quad \frac{\partial \mathbf{B}}{\partial t} = \nabla \wedge (\mathbf{u} \wedge \mathbf{B}), \quad (23)$$

with  $\nabla \cdot \mathbf{u} = 0 = \nabla \cdot \mathbf{B}$  and  $\mathbf{j} = \nabla \wedge \mathbf{B}$ . We work in a domain  $D$  which is finite, with  $\mathbf{u} = 0$  and  $\mathbf{B} \cdot \mathbf{n} = 0$  on  $\partial D$ . Then we have

$$\frac{d}{dt}(E_k + E_M) = -\nu \int_D |\nabla \mathbf{u}|^2 dV.$$

Can this energy have a zero limit? The Lorentz force will tend to drive a flow, but field lines are frozen in, so any helicity integrals must be conserved, which constraints how small  $E_M$  can actually become.

Recall that the enstrophy satisfies the bound  $\Omega_K \geq H_K/L$ , and by a similar argument,

$$2E_M = \int |\mathbf{B}|^2 dV \geq \frac{1}{L} \left| \int \mathbf{A} \cdot \mathbf{B} dV \right| = \frac{H_M}{L}. \quad (24)$$

So a non-zero magnetic helicity gives a non-trivial lower bound on the magnetic energy (note however  $L$  depends on the boundary conditions on  $\mathbf{A}$ ). Hence a state with an initially non-zero magnetic helicity must lead to a non-trivial final state in this relaxation process. It may in fact be shown that if there is a non-trivial linkage (e.g. Borromean rings has no magnetic helicity), this conclusion is still valid (see Freedman, 1988). As  $t \rightarrow \infty$ , we expect  $E_K \rightarrow 0$ , so we should get a magnetic equilibrium satisfying

$$\mathbf{j} \wedge \mathbf{B} = \nabla P. \quad (25)$$

Note this is analogous to the steady Euler equations

$$\mathbf{u}_E \wedge \boldsymbol{\omega}_E = \nabla P_E, \quad (26)$$

so if we find a steady Euler flow from the relaxation process. For example, we can start by having a set of nested surfaces  $S(\gamma)$  in which helicity will be conserved. If we then relax it, we obtain a corresponding Euler flow where helicities are preserved. This helicity uncton gives the signature of the vortex.

Despite this appealing picture, there are many (or unknown) unknowns. Does  $\mathbf{u}$  tend to zero pointwise? Can singularities occur in the fields as they relax in the limit  $t \rightarrow \infty$ ? Or, rather, how ban can singularities be? How can a sensible limit be defined? For example, consider the following (Bajer, 2005):



The strained flow results in a current sheet being formed, i.e. a discontinuity in  $\mathbf{B}$  (numerically numerical diffusion would keep this the computation stable). Suppose a sensible limit could be taken, then how does the resulting flow  $\mathbf{u}_E$  fit into Arnol'd's classification? Presumably the requirement of analyticity will no longer hold and will allow all sorts of complex mixtures of surfaces and chaotic regions, with discontinuities and may be devil's staircases arising in functions such as  $P$  (continuous but not absolutely continuous; see Cantor function). These questions are heard to investigate numerically in the difficult limit of zero magnetic diffusion and  $t \rightarrow \infty$ .

A variational formulation could be attempted. Going back to dealing with  $\mathbf{u}$  and  $\boldsymbol{\omega}$ , we have

$$E_K = \frac{1}{2} \int_D |\mathbf{u}|^2 dV.$$

Suppose some fictitious, divergence-less, infinitesimal flow field  $\boldsymbol{\eta}$  acts and moves the vortex lines, so that

$$\boldsymbol{\omega} \rightarrow \boldsymbol{\omega} + \delta\boldsymbol{\omega}, \quad \mathbf{u} \rightarrow \mathbf{u} + \delta\mathbf{u},$$

so that

$$\delta\boldsymbol{\omega} = \nabla \wedge (\boldsymbol{\eta} \wedge \boldsymbol{\omega}), \quad \delta\mathbf{u} = \boldsymbol{\eta} \wedge \boldsymbol{\omega} + \nabla\phi, \quad (27)$$

where  $\phi$  is chosen so that  $\nabla \cdot \delta\mathbf{u} = 0$ . The displacement  $\boldsymbol{\eta}$  is called an isovortical displacement. Then

$$\delta E_K = \int_D \mathbf{u} \cdot \delta\mathbf{u} dV = \int_D \mathbf{u} \cdot (\boldsymbol{\eta} \wedge \boldsymbol{\omega} + \nabla\phi) dV. \quad (28)$$

Now,  $\mathbf{u} \cdot \mathbf{n} = 0$  on the boundary means that  $\int_D \mathbf{u} \cdot \nabla\phi dV = 0$ . Now if  $\boldsymbol{\omega} \wedge \mathbf{u}$  is the gradient of some scalar field, then the divergence theorem also makes its contribution zero. Conversely, it can be shown that if  $\delta E_K$  vanishes for all divergence-less  $\boldsymbol{\eta}$ , then

$$\mathbf{u} \wedge \boldsymbol{\omega} = \nabla P \quad (29)$$

for some scalar function  $P$ . Thus we obtain the results that “*stead Euler flows are stationary points of  $E_K$  with respect to general isovortical displacements*”. Further study in the second variation of  $E_K$  leads to a powerful stability theorem in 2D (Arnol'd, 1965, 1966).

### III. MORE ON MAGNETIC HELICITY

#### A. Decomposition of vector fields

We consider finite 3D volumes, and ask ourselves some questions about the volume and vector field. We will consider the following topologies:

1. Simply-connected. Any closed curve within the volume may be shrunk to a point (closed but not knotted).
2. Multiply-connected. Some closed curves cannot be shrunk to a point.

3. 2-connected. If simple surfaces may be shrunk to a point.

For example, a 2-sphere is 2-connected (and simply connected), torus is a multiple connected, and a spherical shell is simply connected but not 2-connected.

Some conditions/properties a vector field may have is that they could be:

1. divergence free ( $\nabla \cdot \mathbf{B} = 0$ );
2. curl free ( $\nabla \wedge \mathbf{B} = 0$ );
3. potential (there is some scalar function  $\psi$  where  $\mathbf{B} = \nabla\psi$ );
4. closed ( $\mathbf{B} \cdot \mathbf{n} = 0$ ).

Given some volume with a particular topology, what sort of vector fields can we have inside? Also, given a vector field, could we decompose it so we obtain pieces with special properties?

**Theorem III.1 (Hodge decomposition theorem)** *Any differential form may be decomposed as*

$$\omega = d\alpha + \delta\beta + \gamma,$$

where  $d$  is the exterior derivative,  $\delta$  the co-differential, and  $\gamma$  is harmonic (vanishes under  $d$  and  $\delta$ ).

In our case, it means we have the possibility for decomposing the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$  into

1. Grounded gradients (electrostatic fields, grounded on boundary)

$$\nabla \cdot \mathbf{E} \neq 0, \quad \mathbf{E} = \nabla\psi, \quad \psi|_{\partial D} = 0.$$

2. Harmonic gradients (electric field with perfectly conducting boundaries)

$$\nabla \cdot \mathbf{E} = 0, \quad \mathbf{E} = \nabla\psi, \quad \psi|_{\partial D} = \text{const}, \quad \nabla^2\psi = 0.$$

3. Fluxless knots (closed magnetic fields)

$$\nabla \cdot \mathbf{B} = 0, \quad \mathbf{B} \cdot \mathbf{n} = 0, \quad \Psi = 0 \text{ in interior}$$

where  $\Phi$  is the magnetic flux.

4. Curly gradients (potential magnetic field)

$$\nabla \cdot \mathbf{B} = 0, \quad \mathbf{B} = \nabla\psi, \quad \mathbf{B} \cdot \mathbf{n} \neq 0,$$

but the net flux through  $\partial D$  is zero.

5. Harmonic knots (only exists in multiply-connected volumes)

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \wedge \mathbf{B} = 0, \quad \mathbf{B} \cdot \mathbf{n} = 0, \quad \nabla^2 \mathbf{B} = 0.$$

Since magnetic monopoles are not known to exist in nature, 1) and 2) should not be relevant in magnetic problems. Also, 1) is only relevant in volumes that are not 2-connected.

Note that a  $\mathbf{B}$  field in a simply connected volume will divide uniquely into a 3) and 4) field. (Note also that periodic boxes/volumes are not connected in the way we have mentioned, and we will avoid these here.)



### B. Helicity

Consider two vector fields  $\mathbf{V}$  and  $\mathbf{W}$ . We define the helicity as

$$H(\mathbf{V}, \mathbf{W}) = \int_V \mathbf{A}_V \cdot \mathbf{W} \, dV, \quad \nabla \wedge \mathbf{A}_V = \mathbf{V}. \quad (30)$$

We could write this in a more symmetric way by using the Biot-Savart integral for the potential  $\mathbf{A}_V$ :

$$\mathbf{A}_V(\mathbf{x}) = \frac{1}{4\pi} \int_V \frac{(\mathbf{y} - \mathbf{x})}{r^3} \wedge \mathbf{V}(\mathbf{y}) \, d\mathbf{y}, \quad \mathbf{r} = \mathbf{y} - \mathbf{x}, \quad (31)$$

so that

$$H(\mathbf{V}, \mathbf{W}) = \frac{1}{4\pi} \iint_V \mathbf{V}(\mathbf{x}) \cdot \left( \frac{\mathbf{r}}{r^3} \wedge \mathbf{W}(\mathbf{y}) \right) \, d\mathbf{y} \, d\mathbf{x}. \quad (32)$$

We note that, because of the property of the scalar triple product in (32), we have that

$$H(\mathbf{V}, \mathbf{W}) = H(\mathbf{W}, \mathbf{V}), \quad H(\alpha\mathbf{V} + \beta\mathbf{X}, \mathbf{W}) = \alpha H(\mathbf{V}, \mathbf{W}) + \beta H(\mathbf{X}, \mathbf{W}), \quad (33)$$

so the helicity is a symmetric bilinear form. We will adopt this more general definition.

### C. Toroidal and poloidal fields

It is often useful to decompose  $\mathbf{B}$  into toroidal and poloidal components. Let  $\mathcal{L}$  be an operator (similar to the angular momentum operator in quantum theory), defined as

$$\mathcal{L} = \begin{cases} -\mathbf{e}_z \wedge \nabla, & \text{Cartesian,} \\ -\mathbf{r} \wedge \nabla, & \text{spherical.} \end{cases} \quad (34)$$

Then we could write

$$\mathbf{B} = \mathcal{L}T + \nabla \wedge \mathcal{L}P, \quad (35)$$

where  $T$  is the toroidal function, and  $P$  is the poloidal function. We examine the Cartesian case. We note that

$$\mathbf{e}_z \cdot \mathcal{L} = 0, \quad \nabla \cdot \mathcal{L} = 0. \quad (36)$$

The vector potentials for  $\mathbf{B}_T = \mathcal{L}T$  and  $\mathbf{B}_P = \nabla \wedge \mathcal{L}P$  are

$$\mathbf{A}_T = \text{curl}^{-1} \mathbf{B}_T = T\mathbf{e}_z + \nabla\Psi_T, \quad \mathbf{A}_P = \text{curl}^{-1} \mathbf{B}_P = \mathcal{L}P + \nabla\Psi_P, \quad (37)$$

where  $\Psi$  are gauge functions.  $P$  and  $T$  can be obtained from  $\mathbf{B}$  by solving the Poisson equations

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = B_z, \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = (\nabla \wedge \mathbf{B})_z. \quad (38)$$

**Proposition III.2** *Toroidal fields do not link with each other, nor do two poloidal fields.*

**Proof** Since helicity integrals are gauge invariant, we may ignore  $\Psi$ . Now, since  $\mathbf{e}_z \cdot \mathcal{L} = 0$ ,

$$H(\mathbf{B}_{T_1} + \mathbf{B}_{T_2}) = \int T_1 \mathbf{e}_z \cdots \mathcal{L}T_2 \, [z = 0].$$

Also, note that

$$\begin{aligned}
H(\mathbf{B}_{P_1} + \mathbf{B}_{P_2}) &= \int \mathcal{L}P_1 \cdot \mathbf{B}_{P_2} \, d\mathbf{x} \\
&= \int \mathcal{L}P_1 \cdot \left( \nabla \frac{\partial P_2}{\partial z} - \mathbf{e}_z \nabla^2 P_2 \right) d\mathbf{x} \\
&= \int \mathcal{L}P_1 \cdot \left( \nabla \frac{\partial P_2}{\partial z} \right) d\mathbf{x} \\
&= \int \nabla \wedge P_1 \mathbf{e}_z \cdot \left( \nabla \frac{\partial P_2}{\partial z} \right) d\mathbf{x} \\
&= \int \nabla \cdot \left( P_1 \mathbf{e}_z \wedge \nabla \frac{\partial P_2}{\partial z} \right) d\mathbf{x} \\
&= \oint \mathbf{e}_z \cdot \left( P_1 \mathbf{e}_z \wedge \nabla \frac{\partial P_2}{\partial z} \right) d\ell \\
&= 0.
\end{aligned}$$

**Theorem III.3** Consider  $\mathbf{B} = \mathbf{B}_T + \mathbf{B}_P$  in  $V$ . Assume that  $V$  is either all space, half space bounded by a plane, layer bounded by two planes, interior or exterior of a sphere, or a spherical shell bounded by two concentric spheres. Then:

1.  $\mathbf{B}_P = 0 \Rightarrow H(\mathbf{B}, \mathbf{B}) = 0$ ;
2.  $\mathbf{B}_T = 0 \Rightarrow H(\mathbf{B}, \mathbf{B}) = 0$ ;
3.  $H(\mathbf{B}, \mathbf{B}) = 2 \int \mathcal{L}T \cdot \mathcal{L}P \, d\mathbf{x}$ .

**Proof** The first two were shown above. For the last one, since the helicity is a bilinear form,

$$H(\mathbf{B}, \mathbf{B}) = H(\mathbf{B}_P + \mathbf{B}_T, \mathbf{B}_P + \mathbf{B}_T) = 2H(\mathbf{B}_P, \mathbf{B}_T) = 2 \int_V \mathcal{L}P \cdot \mathcal{L}T \, d\mathbf{x}.$$

#### D. Helicity in open volumes

So far we have considered helicity in closed volumes, where field lines never cross the boundary. In most physical and mathematical problems this is far too restrictive. For example, we take the photosphere as the outer or inner boundary depending on which portion we want to study. We need a form of helicity integral which works just for the interior or exterior. Furthermore, the sum of interior and exterior helicities should sensible relate to the helicity integral of all space.

Here, we consider  $H(\mathbf{B}) = H(\mathbf{B}, \mathbf{B})$  in some arbitrary  $D$ . We give a definition which remains topologically meaningful and is gauge invariant. Let the space be divided into  $D$  and  $D'$ , with magnetic fields  $\mathbf{B}$  and  $\mathbf{B}'$ . At the boundary  $S$ ,

$$\mathbf{B} \cdot \mathbf{n} = \mathbf{B}' \cdot \mathbf{n},$$

We define the magnetic field defined in all space to be

$$\{\mathbf{B}, \mathbf{B}'\}(\mathbf{x}) = \begin{cases} \mathbf{B}, & \mathbf{x} \in D, \\ \mathbf{B}', & \mathbf{x} \in D'. \end{cases} \quad (39)$$

Now, note that: a),  $H(\{\mathbf{B}, \mathbf{B}'\})$  includes information about all the magnetic structure in  $\mathbf{B}'$  when we just want  $\mathbf{B}$ , and we need to subtract this extra information out; b) simply integrating (32) will not do, if  $D$  contains curly gradients or harmonic knots. Instead, we measure the helicity relative to a minimal base state. Thus we look for some simple vector field  $\mathbf{P}$  inside  $D$  for which we can calculate the reference helicity  $H(\{\mathbf{P}, \mathbf{B}'\})$ . One we subtract this reference helicity, the dependence on the external field will vanish.

The boundary information  $\mathbf{B} \cdot \mathbf{n}$  tells us the distribution of flux crossing  $S$ . It also determines a unique vector field, the vacuum field  $\mathbf{P}$  satisfying

$$\mathbf{P} \cdot \mathbf{n}|_S = \mathbf{B} \cdot \mathbf{n}, \quad \nabla \wedge \mathbf{P}(\mathbf{x}) = 0, \quad \mathbf{x} \in D. \quad (40)$$

If  $D$  is multiply-connected, the net flux of  $\mathbf{P}$  through any closed curve on  $S$  should also be the same for  $\mathbf{B}$  and  $\mathbf{P}$ . In other words, if  $\mathbf{B}$  with  $D$  is decomposed (uniquely) into fluxless knots, harmonic knots and curly gradients, then

$$\mathbf{P} = \text{Harmonic knots} + \text{Curly gradients}.$$

A simple variational calculation will show that  $\mathbf{P}$  is the minimum energy state consistent with the boundary data  $\mathbf{B} \cdot \mathbf{n}$ . Magnetic field lines spiral about current lines. As the vacuum field has zero current, it has minimum helical structure. It also requires a minimum of information for its specification. So the magnetic helicity inside  $D$  is

$$H_D = H(\{\mathbf{B}, \mathbf{B}'\}) - H(\{\mathbf{P}, \mathbf{B}'\}). \quad (41)$$

**Theorem III.4**  $H_D$  satisfies the following:

1.  $H_D$  is gauge invariant;
2.  $H_D = \int_V (\mathbf{A} + \mathbf{A}_P) \cdot (\mathbf{B} - \mathbf{P}) \, d\mathbf{x}$ , where  $\nabla \wedge \mathbf{A}_P = \mathbf{P}$ ;
3.  $H_D$  is independent of  $\mathbf{B}'$  outside of  $D$ .

**Proof** 1. Both terms in  $H_D$  are integrated over all space, so they are gauge invariant, or we could check directly.

2. Let  $\{\mathbf{A}, \mathbf{A}'\}$  be the potential of  $\{\mathbf{B}, \mathbf{B}'\}$ , the true magnetic field. The vector potential for the reference field  $\{\mathbf{B}, \mathbf{B}'\}$  is  $\mathbf{A}_P$  in  $D$ , but is  $\mathbf{A}' + \nabla\psi$  in  $D'$ , since  $\nabla\psi$  is required as a boundary condition at  $S$  to match it smoothly with  $\mathbf{A}_P$ . So the reference potential is  $\{\mathbf{A}_P, \mathbf{A}' + \nabla\psi\}$ . Let the normal be  $\mathbf{n}$ , pointing from  $D$  into  $D'$ . We need continuity of the parallel components (parallel to  $S$ ), so

$$\mathbf{n} \wedge \mathbf{A} = \mathbf{n} \wedge \mathbf{A}', \quad \mathbf{n} \wedge \mathbf{A}_P = \mathbf{n} \wedge (\mathbf{A}' + \nabla\psi),$$

so that

$$H(\{\mathbf{B}, \mathbf{B}'\}) = \left( \int_D \mathbf{A} \cdot \mathbf{B} + \int_{D'} \mathbf{A}' \cdot \mathbf{B}' \right) d\mathbf{x}, \quad H(\{\mathbf{P}, \mathbf{B}'\}) = \left( \int_D \mathbf{A}_P \cdot \mathbf{P} + \int_{D'} (\mathbf{A}' + \nabla\psi) \cdot \mathbf{B}' \right) d\mathbf{x},$$

which implies that

$$\begin{aligned} H_D &= \int_D (\mathbf{A} \cdot \mathbf{B} - \mathbf{A}_P \cdot \mathbf{P}) \, d\mathbf{x} - \int_{D'} \nabla\psi \cdot \mathbf{B}' \, d\mathbf{x} \\ &= \int_D (\dots) + \oint_S \psi \mathbf{B} \cdot \mathbf{n} + 0, \end{aligned}$$

since  $\nabla \cdot \mathbf{B} = 0$ . Now, note that

$$\begin{aligned} \int_V (\mathbf{A}_P \cdot \mathbf{B} - \mathbf{A} \cdot \mathbf{P}) \, d\mathbf{x} &= \int_V [\mathbf{A}_P \cdot (\nabla \wedge \mathbf{A}) - \mathbf{A} \cdot (\nabla \wedge \mathbf{A}_P)] \, d\mathbf{x} \\ &= \int_V \nabla \cdot (\mathbf{A} \wedge \mathbf{A}_P) \, d\mathbf{x} \\ &= \oint_S (\mathbf{A} \wedge \mathbf{A}_P) \cdot \mathbf{n} \\ &= \oint_S \mathbf{A} \wedge (\mathbf{A}_P - \mathbf{A}) \cdot \mathbf{n} \, dS \\ &= \oint_S (\mathbf{A} \wedge \nabla\psi) \cdot \mathbf{n} \, dS \\ &= \oint_S (\psi \nabla \wedge \mathbf{A} - \nabla \wedge (\psi \mathbf{A})) \cdot \mathbf{n} \, dS \\ &= \oint_S \psi \mathbf{B} \cdot \mathbf{n} \, dS, \end{aligned}$$

where the line integral of  $\nabla \wedge (\psi \mathbf{A})$  vanishes by Stokes' theorem. With this,

$$H_D = \int_D (\mathbf{A} \cdot \mathbf{B} - \mathbf{A}_P \cdot \mathbf{P}) \, d\mathbf{x} + \int_D (\mathbf{A}_P \cdot \mathbf{B} - \mathbf{A} \cdot \mathbf{P}) \, d\mathbf{x} = \int_V (\mathbf{A} + \mathbf{A}_P) \cdot (\mathbf{B} - \mathbf{P}) \, d\mathbf{x}.$$

3. The above is only integrated over  $D$ .

### E. Addition of helicities

Let  $H_T(\{\mathbf{B}, \mathbf{B}'\})$  be the helicity of all space,  $\mathbf{P}$  be the potential field in  $V$ , and  $\mathbf{P}'$  the potential field in  $V'$  (given the boundary condition  $\mathbf{B} \cdot \mathbf{n}$  on  $S$ , and the magnitude of any interior fluxes if  $V$  is multiply connected and contains a harmonic knot field). Then

$$H_T(\{\mathbf{B}, \mathbf{B}'\}) = H_V(\mathbf{B}) + H_{V'}(\mathbf{B}') + H(\{\mathbf{P}, \mathbf{P}'\}), \quad (42)$$

since

$$H_V(\mathbf{B}) = H_T(\{\mathbf{B}, \mathbf{B}'\}) - H_T(\{\mathbf{P}, \mathbf{B}'\}), \quad H_{V'}(\mathbf{B}') = H_T(\{\mathbf{P}, \mathbf{B}'\}) - H_T(\{\mathbf{P}, \mathbf{P}'\}).$$

**Lemma III.5** *If  $S$  is a plane or sphere, then  $H_T(\{\mathbf{P}, \mathbf{P}'\}) = 0$ .*

**Proof** For the planar case, in the usual Biot–Savart form,

$$\mathbf{A}_V(\mathbf{x}) = -\frac{1}{4\pi} \int_V \frac{\nabla_{\mathbf{y}} \wedge \mathbf{V}(\mathbf{y})}{r} d\mathbf{y}$$

where  $\mathbf{r} = \mathbf{y} - \mathbf{x}$ , and so

$$\mathbf{A}_P(\mathbf{x}) = -\frac{1}{4\pi} \int_V \frac{\nabla_{\mathbf{y}} \wedge \mathbf{P}(\mathbf{y})}{r} d\mathbf{y}.$$

However, a potential field has zero curl. The only place the global field  $\{\mathbf{P}, \mathbf{P}'\}$  has non-zero curl is on  $S$ . For a planar boundary,  $\nabla \wedge \mathbf{P}$  is parallel to  $S$ , but this means that  $\mathbf{A}_P \cdot \mathbf{n} = 0$ . Now,  $\mathbf{P} = \nabla\phi$  and  $\mathbf{P}' = \nabla\phi'$ , so

$$\begin{aligned} H_T(\{\mathbf{P}, \mathbf{P}'\}) &= \int_V \mathbf{A}_P \cdot \mathbf{P} d\mathbf{x} + \int_V \mathbf{A}'_P \cdot \mathbf{P}' d\mathbf{x} \\ &= \int_V \mathbf{A}_P \cdot \nabla\phi d\mathbf{x} + \int_V \mathbf{A}'_P \cdot \nabla\phi' d\mathbf{x} \\ &= 0 + 0 \end{aligned}$$

by the divergence theorem and boundary conditions.

Being able to calculate helicity of sub-volumes of space has special relevance in the Solar atmosphere (the boundary is then the photosphere).

### F. Time derivative of helicity

From the definition of  $H_V$ , we can see that

$$\frac{d}{dt} H_V = -2 \int_V \mathbf{E} \cdot \mathbf{B} d\mathbf{x} - 2 \oint_S \mathbf{A}_P \wedge \mathbf{E} \cdot \mathbf{n} d\mathbf{x}. \quad (43)$$

Here,  $\mathbf{A}_P$  is uniquely defined by

$$\nabla \wedge \mathbf{A}_P = \mathbf{P}, \quad \mathbf{n} \cdot \nabla \wedge \mathbf{A}_P = B_n, \quad \nabla \cdot \mathbf{A}_P = 0, \quad \mathbf{A}_P \cdot \mathbf{n} = 0. \quad (44)$$

Suppose we have  $\mathbf{E} = \eta \mathbf{j}$  with  $\eta$  being the Ohmic dissipation coefficient. The first term in (43) measures Ohmic dissipation of helicity. Now, let  $\mu_0$  be the vacuum magnetic permeability, then we have

$$E_M = \frac{1}{2\mu_0} \int_V |\mathbf{B}|^2 d\mathbf{x}, \quad \frac{d}{dt} H_V = 2 \int_V \eta \mathbf{j} \cdot \mathbf{B} d\mathbf{x}, \quad (45)$$

so that

$$\frac{d}{dt} E_M = - \int_V \eta |\mathbf{j}|^2 d\mathbf{x}$$

thus, by the Cauchy–Schwartz inequality,

$$\left| \frac{dH_V}{dt} \right| \leq \sqrt{8\eta\mu_0 E_M} \left| \frac{dE_M}{dt} \right|. \quad (46)$$

For  $\eta \ll 1$  as in most astrophysical problems,  $|dH_V/dt|$  is small, so helicity is fairly well conserved. In problems such as liquid metal experiments, this is not the case as  $\eta$  is comparatively large. Note that we only have exact conservation in ideal MHD where  $\eta = 0$ .

The boundary term in (43) gives helicity flow from one space to another. For ideal MHD where the first term in (43) is zero,  $\mathbf{E} = \mathbf{B} \wedge \mathbf{u}$ , so we have

$$\frac{d}{dt} H_V = -2 \oint_S [(\mathbf{A}_P \cdot \mathbf{u})\mathbf{B} - (\mathbf{A}_P \cdot \mathbf{B})\mathbf{u}] \cdot \mathbf{n} \, dx. \quad (47)$$

This may then be seen as a helicity flux.

**Example** Consider the Solar cycle. The equator of the sun rotates faster than the poles, so magnetic features move with respect to polar features. This leads to twisting up of field lines (cf.  $\Omega$ -effect), injecting negative helicity into the Southern hemisphere. At the same time, positive helicity streams out of the Northern hemisphere into the Northern Solar wind and into the Northern interplanetary space, and similarly for the Southern hemisphere. The diagnosed quantities from observations follows the budget (43) well.

## IV. SOME GEOMETRY AND TOPOLOGY

### A. The Frenet frame

Given a curve  $C(t)$  and the arc-length  $s(t)$ , the tangent vector  $T$ , velocity vector  $V$ , curvature  $\kappa$ , normal vector  $N$ , bi-normal vector  $B$  and torsion  $\tau$  are defined by the following:

$$T = \frac{dC}{ds}, \quad V = \frac{ds}{dt}, \quad \kappa = \left| \frac{dT}{ds} \right|, \quad N = \frac{1}{\kappa} \frac{dT}{ds}, \quad B = T \wedge N, \quad \tau = \frac{dN}{dt} \cdot B. \quad (48)$$

Note that the velocity changes if  $C$  is reparameterised.