Academic notes: 1B Analysis

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I. SEQUENCES AND LIMITS

A. Inequalities

Here we deal mainly with real analysis. Recall that we have the following number sets:

- natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$;
- integers $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$;
- rationals $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z},\ q \neq 0\}$;
- reals, $\mathbb{R}$.

$\mathbb{R}$ contains numbers that are not algebraic, i.e., numbers that are not roots of $\sum_i^n a_i x^i = 0$ (e.g., π, e); these numbers are called transcendental. We note that $\mathbb{R}$ is ordered and complete (roughly speaking, there is a real number between any two chosen real numbers).

Given distinct $x, y \in \mathbb{R}$, we have the following properties for 'less than' (and analogous ones for other inequalities):

- if $x < y$, $y < z$, then $x < z$ (transitivity);
- if $x < y$, $a < b$, then $a + x < b + y$;
- if $x < y$, $c > 0$, then $cx < cy$;
- if, on the other hand, $x < y$ but $c < 0$, then $cx > cy$;
- if $x < y$ and $x, y > 0$, then $1/x > 1/y$.

Example Some examples with inequalities:

1. Define for all $x \in \mathbb{R}$ such that $-3(4 - x) \leq 12$. This gives
$$-12 + 3x \leq 12 \qquad \Leftrightarrow \qquad 3x \leq 24 \qquad \Leftrightarrow \qquad x \leq 8.$$
2. Solve $(x + 2)/3 < (5 - 2x)/4$.
$$4x + 8 < 15 - 6x \qquad \Leftrightarrow \qquad 10x < 7 \qquad \Leftrightarrow \qquad x < 7/10.$$
3. Solve $x^2 - 4x + 3 > 0$.
$$(x - 3)(x - 1) > 0 \qquad \Leftrightarrow \qquad x > 3 \text{ or } x < 1,$$
after taking into account of the same of the quadratic.
4. Solve $3/(x - 2) \leq x$.
Since $(x - 2)$ could be less than zero, instead of multiplying across, we note that
$$\frac{3}{x-2} - x \leq 0 \qquad \Leftrightarrow \qquad \frac{x^2 - 2x - 3}{x - 2} \geq 0 \qquad \Leftrightarrow \qquad \frac{(x-3)(x+1)}{x-2} \geq 0.$$
Assuming $x \neq 2$, the inequality is not valie for $x < -1$ and $2 < x < 3$, so the solutions are $-1 \leq x < 2$ or $x \geq 3$.

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B. Absolute values

The absolute value of x is defined to be

$$|x| \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

We then have the following identities:

- the triangle inequality, $|x+y| \leq |x| + |y|$;
- $||x| - |y|| \leq |x - y|$, which is a variation of the above;
- $|x| < c$ iff $-c < x < c$;
- $x^2 < c$ iff $0 \leq |x| < c$.

Example Solve $|x+2| \leq |2x-1|$.

$$(x+2)^2 \leq (2x-1)^2 \qquad \Leftrightarrow \qquad 0 \leq 3x^2 - 8x - 3 = (3x+1)(x-3),$$

so $x \geq 3$ or $x \leq -1/3$.

C. Sequence

A sequence is a function which maps $\mathbb{N}$ to $\mathbb{R}$. Sequences are notatied as

$$\{x_n\}_{n=1}^{\infty} = \{x_1, x_2, \cdots x_n\},$$

where the list is ordered.

Example We have the following sequences:

- $x_n = 17$ for all n gives $\{17, 17, \cdots\}$;
- $x_n = n$ gives $\{1, 2, 3 \cdots\}$;
- $x_n = 1/n$ gives $\{1, 1/2, 1/3, \cdots\}$;
- $x_n = (-1)^{n+1}$ gives $\{1, -1, 1, \cdots\}$.

Let $\{x_n\}$ be a sequence. We say that the sequence tends to the limit L, written $\lim_{n\to\infty} x_n = L$ (or $x_n \to L$ as $n \to \infty$) if, given any $\epsilon > 0$, there is some N such that

$$|x_n - L| < \epsilon \quad \text{for all } n \geq N,$$

or, in words, the sequence becomes arbitrarily close to L at some point.

Example To show that $x_n = 1/n \to 0$ as $n \to \infty$, we observe that

$$|x_n - L| = \frac{1}{n} - 0 = \frac{1}{n} < \epsilon \qquad \Leftrightarrow \qquad \frac{1}{\epsilon} < n.$$

So, given some ϵ, we take $N > 1/\epsilon$ as required.

Suppose $\{x_n\}$ and $\{y_n\}$ are sequences, with $x_n \to L$ and $y_n \to K$ as $n \to \infty$, then, for constants A and B, we have:

- $Ax_n + By_n \to AL + BK$ as $n \to \infty$;
- $x_n y_n \to KL$ as $n \to \infty$;
- $x_n / y_n \to L/K$ as $n \to \infty$ for $K \neq 0$.

Theorem I.1 *If $f(x)$ is continuous and $x_n \to L$ as $n \to \infty$, then $f(x_n) \to f(L)$ as $n \to \infty$.*

Theorem I.2 *Suppose* $0 \leq x_n \leq y_n$ *and* $y_n \to 0$ *as* $n \to \infty$*, then* $x_n \to 0$ *as* $n \to \infty$.

Proof Let $\epsilon > 0$ be given, then $|x_n - 0| = x_n \leq y_n = |y_n - 0|$. Since $y_n \to 0$, by definition, there is some N that $|y_n - 0| < \epsilon$ for all $n > N$, so $|x_n \to -0| \leq |y_n - 0| \leq \epsilon$ for all $n \geq N$, and indeed $x_n \to 0$ as $n \to \infty$.

In general, for $z_n \leq x_n \leq y_n$, if $z_n, y_n \to L$, then $x_n \to L$ as $n \to \infty$.

Example

1. Find the limit of $x_n = (x+3)/\sqrt{4n^2 - 2}$ as $n \to \infty$.

$$x_n \equiv \frac{1 + 3/n}{4 - 2/n^2} \to \frac{1}{\sqrt{4}} = \frac{1}{2}$$

since $n^{-p} \to 0$ if $p > 0$.

2. Find the limit of $x_n = (n^2 n!)/(n+2)!$ as $n \to \infty$.

$$x_n \equiv \frac{n^2}{(n+1)(n+2)} = \frac{1}{3/n + 2/n^2 + 1} \to 1.$$

3. Find the limit of $x_n = (n + \sin^2 n)/\sqrt{4n-1}$ as $n \to \infty$. Since $\sin^2 n$ fluctuates between 0 and 1, this is small compared to n at large n. However, at large n, the sequence goes like $\sqrt{n}$, so there is no limit.

4. Find the limit of $x_n = \sqrt{n}(\sqrt{n-1} - \sqrt{n})$ as $n \to \infty$.

$$x_n \equiv= \frac{\sqrt{n}(\sqrt{n-1} - \sqrt{n})^2}{(\sqrt{n-1} - \sqrt{n})} = \frac{1}{\sqrt{1 + 1/n} + 1} \to \frac{1}{2}.$$

5. Find the limit of $x_n = n^{-1} \log(3^n + n^3)$ as $n \to \infty$.

We use the observation that exponentials increase at a rate such that $e^n > n^p > \log n$, so

$$x_n \equiv \frac{1}{n} \log[3^n(1 + \frac{n^3}{3^n})] = \frac{1}{n}[n \log 3 + \log(1 + \frac{n^3}{3^n})] \to \log 3,$$

since the exponential kills the algebraic term, and the algebraic term kills the log term.

6. Find the limit of $x_n = t^{1/n}$ as $n \to \infty$.

Either define $y_n = \log x_n$ and find the limit of y_n accordingly, or simply observe that the exponent goes to 0, so $x_n \to 1$.

7. Find the limit of $x_n = (n^2 + n^3 \mathrm{e}^{-n})/(\log 2^n + \log n^8)^2$ as $n \to \infty$.

$$x_n = \frac{1 + n\mathrm{e}^{-n}}{(\log 2 - (8/n)\log n)^2} \to \frac{1}{(\log 2)^2}.$$

Theorem I.3 *If* $x \to L$ *as* $n \to \infty$*, and* $x_n < 0$ *for all* n*, then* $L \leq 0$.

Proof We proceed with a proof by contradiction. Assume that $L > 0$ and $x_n \to L$ as $n \to \infty$, and that we may choose and ϵ small enough. Since $x_n \to L$, we can find an integer N such that $|x_N - L| < \epsilon$ for all $n \geq N$. But $|x_N - L| < L$, so

$$-L < x_N - L < L \qquad \to \qquad x_N > 0,$$

which is the contradiction we need, and thus $L \leq 0$.

Theorem I.4 *If* $|t| < 1$*, then* $x_n = t^n \to 0$ *as* $n \to \infty$.

Proof Using the definition and that $u^{1/n} \to 0$ as $n \to \infty$, let $\epsilon > 0$ be given. We want N such that $|t|^n < \epsilon$ for all $n \geq N$. For $|t|^N < \epsilon$, $|t| < \epsilon^{1/N}$. Since $|t| < 1$, $|t|^{N+1} = |t|^N |t| < |t|^N$, and it is clear that if $|t| < \epsilon^{1/N}$, then $|t|^N < \epsilon$, so

$$0 \leq |t|^n \leq |t|^N < \epsilon$$

for all $n \geq N$. Since $\epsilon^{1/n} \to 1$ as $n \to \infty$, there is some n such that $\epsilon^{1/n} > |t|$ is possible, as required.

Corollary I.5 *If $t > 1$, t^n has no limit as $n \to \infty$.*

Proof Suppose $t > 1$ and we assume $t \to L$ as $n \to \infty$. Then

$$1 = 1^n = t^n \left(\frac{1}{t}\right)^n \to L \cdot 0 = 0,$$

which is a contradiction, so there is no limit L.

Example Find the limit of $x_n = [(2n+3)/(n-1/2)]^n$ as $n \to \infty$.
For large n, $x_n \sim (2n/n)^n = 2^n < x_n$, so by comparison, x_n does not have a limit.

Theorem I.6 *If $c \in \mathbb{R}$, then $(1 + c/n)^n \to \mathrm{e}^c$ as $n \to \infty$.*

Proof Notice that

$$\log y = \int_1^y \frac{1}{x}\,\mathrm{d}x, \qquad \log\left(1 + \frac{c}{n}\right)^n = n \log\left(1 + \frac{c}{n}\right) = n \int_1^{1+c/n} \frac{1}{x}\,\mathrm{d}x,$$

so

$$\frac{c}{n}\frac{1}{1+c/n} < n \int_1^{1+c/n} \frac{1}{x}\,\mathrm{d}x < 1 \cdot \frac{c}{n} \qquad \to \qquad \frac{cn}{n+c} \log\left(1 + \frac{c}{n}\right)^n < c.$$

By squeezing, $\log(1 + c/n)^n \to c$, so $(1 + c/n)^n \to \mathrm{e}^c$ as $n \to \infty$.

Example

1. We have

$$x_n = \left(\frac{n-2}{n+1}\right)^{3n} = \left(1 - \frac{3}{n+1}\right)^{3n} = \left[\left(1 - \frac{3}{n+1}\right)^n\right]^3 = \left[\frac{(1 - 3/(n+1))^{n+1}}{(1 - 3/(n+1))}\right]^3 \to \left[\frac{\mathrm{e}^{-3}}{1-0}\right]^3 = \mathrm{e}^{-9}$$

as $n \to \infty$.

2. To find the limit of $x_n = (3^n + 2^n)^{1/n}$, we observe that

$$3^n < 3^n + 2^n < 3^n + 3^n = 2 \cdot 3^n \qquad \Leftrightarrow \qquad (3^n)^{1/n} < x_n < 2^{1/n} \cdot (3^n)^{1/n},$$

so, by squeezing, since $2^{1/n} \to 1$, $x_n \to 3$. Alternatively, since $3^n > 2^n$, we have, for large n,

$$\log x_n = \frac{1}{n} \log(3^n + 2^n) \to \frac{1}{n} \log 3^n = \log 3,$$

so $x_n \to 3$.

Proposition I.7 $(\log n)/\sqrt{n} \to 0$ *as $n \to \infty$.*

Proof We observe that

$$0 \le \frac{\log n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \int_1^n \frac{1}{x}\,\mathrm{d}x \le \frac{1}{\sqrt{n}} \int_1^n \frac{1}{x^{3/4}}\,\mathrm{d}x = \frac{1}{\sqrt{n}} \left[4x^{1/4}\right]_1^n \to 0,$$

so $(\log n)/\sqrt{n} \to 0$ as $n \to \infty$ by squeezing.

This result generalises to $(\log n)/n^p \to 0$ as $n \to \infty$ for any $p > 0$.

Proposition I.8 $n^p/\mathrm{e}^n \to 0$ *as $n \to \infty$ for $p \in \mathbb{R}$.*

Proof We observe that

$$\frac{p}{n} \log n \to 0 \quad \Leftrightarrow n^{p/n} \to \mathrm{e}^0 = 1,$$

and thus

$$\frac{n^{p/n}}{\mathrm{e}} \to \frac{1}{e} < \frac{1}{2}.$$

Then there exists an n such that $0 < n^{p/n}/\mathrm{e} < 1/2$, and raising this to the n-th power gives $0 < n^p/\mathrm{e}^n < 1/2^n \to 0$ as $n \to \infty$, as required.

D. Sup and inf

A set $X \in \mathbb{R}$ has a maximum $k = \max(X)$ if $k \in X$ and $x \leq k$ fro all $x \in X$. The minimum $\min(X)$ is similarly defined. For example, for $X = \{n^{-1} \mid n \in \mathbb{N}\}$, $\max(X) = 1$ but $\min(X)$ is not defined. A set X is bounded above if there exists k such that $x \leq k$ for $x \in X$, and similar for X to be bounded below. With the above example, the set is bounded below and above by 0 and 1 respectively. Although there are infinitely many bounds for X, there is the largest of the lower bound and the smallest of the upper bound.

Let X be bounded above, then the supremum $\sup X$ exists if, (i), $\sup X$ is an upper bound of X, and (ii), for any other upper bound of X denoted K, $\sup X \leq K$. We note that the second condition is equilvalent to $\sup \in X$, or there exists $x_n \in X$ where $x_n \to \sup X$ as $n \to \infty$. The definition is similar for the infimum, denoted $\inf X$.

Example For the following sets, find the supremum and infimum.

1. $X = (0, 3)$. The guess is that the supremum and infimum are respectively 3 and 0. To show this for the supremum, we note that clearly $x \in X$ has $x \leq 3$. Then defining the sequence $x_n = 3 - 1/n$, we have $x_n \in X$, and clearly $x_n \to 3$ as $n \to \infty$, as required (similarly for the infimum).
2. For $X = \{n/(1 + n^2) \mid n \in \mathbb{N}\}$, we guess that $\sup X = 1/2$ whilst $\inf X = 0$.

   For the infimum, it is easy to see that $x \in X$ satisfies $x \geq 0$. Also, $n/(1 + n^2) \to 0$ as $n \to \infty$, as required.

   for the supremum, we observe that $n/(1 + n^2)$ is a decreasing function bounded by $x_1 = 1/2$, and since $1/2 \in X$, we have $\sup X = 1/2$.
3. For $X = \{mn/(1 + m^2 + n^2) \mid m, n \in \mathbb{N}\}$, we have that $x \in X$ is positive, and either m or $n \to \infty$ yields $mn/(1 + m^2 + n^2) \to 0$, so $\inf X = 0$. Taking $m = n$, we have $n^2/(1 + 2n^2) \to 1/2$, so we make a guess that $\sup X = 1/2$. Now,

   $$\frac{mn}{1 + m^2 + n^2} \leq \frac{1}{2} \qquad \Leftrightarrow \qquad 0 \leq (m - n)^2 + 1,$$

   which is obviously true, and since $n^2/(1 + 2n^2) \to 1/2$, $\sup X = 1/2$.
4. For $X = \{(n^2 - 4n + 4)/(2n^2 + 1) \mid n \in \ltimes\} = \{1/3, 0, 1/19, \cdots (1/2)\}$, where the last term is in brackets because it is the limit of $n \to \infty$. Then $\inf X = 0$ because $(n - 2)^2/(2n^2 + 1) \geq 0$, and $0 \in X$. To show that $\sup X = 1/2$, we observe that

   $$\frac{(n-2)^2}{2n^2 + 1} \leq \frac{1}{2} \qquad \Leftrightarrow \qquad 2n^2 - 8n + 8 \leq 2n^2 + 1 \qquad \Leftrightarrow \qquad n \geq \frac{7}{8},$$

   which is true since $n \in \mathbb{N}$. Furthermore, the limit of the sequence tends to $1/2$, as required.
5. Find the supremum and infimum of $S = \{n/(4 + n^2) \mid n \in \mathbb{N}\}$.

   Note that $S = \{1/5, 1/4, 3/15 \cdots (0)\}$, so we guess that $\inf S = 0$ and $\sup S = 1/4$. Since $n/(4 + n^2) > 0$ as $n > 0$ and $n/(4 + n^2) \to 0$ as $n \to \infty$, we have the former. Observing that

   $$\frac{n}{4 + n^2} \leq \frac{1}{4} \qquad \Leftrightarrow \qquad (n - 2)^2 \geq 0$$

   and $1/4 \in S$, we have the latter.

Remark $\mathbb{R}$ is constructed such that, for $X \subseteq \mathbb{R}$, there exists $\sup X$ and $\inf X$, i.e., $\mathbb{R}$ is continuous for the interval $(-\infty, +\infty)$, a property known as completeness.

Let X be a set, and $f : X \to \mathbb{R}$ a function, and we denote the image as $f(X)$. We say that f is bounded above if $f(X)$ is bounded above, and then $\sup f = \sup f(X)$, and similarly for the infimum.

Example For the following f and X, find the infimum and supremum if they exist.

1. For $f(x) = x^2$ and $x \in \mathbb{R}$, $f(X) = [0, \infty)$. $f(X)$ is not bounded above so the supremum does not exist, but it is bounded below, and $\inf f = 0$.

2. Let $f(x) = (x^2 \cos x)/(1+x^2)$, $x > 0$. We note that

$$\leq 0 \frac{x^2}{1+x^2} < 1, \qquad -1 \leq \cos x \leq 1,$$

so $-1 < f(x) < 1$, so the guess is that $\sup f = 1$ and $\inf f = -1$.

For the supremum, we noted already that $f(x) < 1$. Letting $x = 2\pi n$, then $x \cos x = 1$, and $f(2\pi n) = (2\pi n)^2/(1 + (2\pi n)^2) \to 1$, as required.

For the infimum, we also noted that $f(x) > -1$, so taking $x = 2\pi n + \pi$, so that $\cos x = -1$, we have $(-1)(2\pi n + \pi)^2/(1 + (2\pi n + \pi)^2) \to -1$, as required.

3. For $f(x) = (x+1)/(x+2)$ with $x > 0$, find the supremum and infimum.

We note that $f(0) = 1/2$ and that $f(x) \to 1$ as $x \to \infty$. We could differentiate to find the extrema, or just guess that $\inf f = 1/2$ and $\sup f = 1$. For the supremum, we see that

$$\frac{x+1}{x+2} \leq 1 \qquad \Leftrightarrow \qquad 1 \leq 2$$

since $x + 2 > 0$, and we have already showed $f(x) \to 1$, thus $\sup f = 1$. For the infimum, we have

$$\frac{x+1}{x+2} \geq \frac{1}{2} \qquad \Leftrightarrow \qquad x \geq 0,$$

and that $f(x) \to 1/2$ as $x \to 0$, so $\inf f = 0$.

Theorem I.9 *Let $f, g : X \to \mathbb{R}$ be two functions, both bounded above. Then $f + g$ is bounded, and*

$$\sup f + \inf g \leq \sup(f+g) \leq \sup f + \sup g.$$

Proof By definition, $f(x) \leq \sup f$, $g(x) \leq \sup g$ for all $x \in X$, so $f(x) + g(x) \leq \sup f + \sup g$. Since $\sup f + \sup g$ is an upper bound, but the $\sup(f+g)$ is the least upper bound, so we have

$$\sup(f+g) \leq \sup f + \sup g.$$

With $f(x) + g(x) \leq \sup(f+g)$, we have $f(x) \leq \sup(f+g) - g(x)$, and since $g(x) \geq \inf g$, we have $-g(x) \leq -\inf g$, so $f(x) \leq \sup(f+g) - \inf g$. This is an upper bound, but $\sup f$ is the least upper bound, so

$$\sup f + \inf g \leq \sup(f+g).$$

E. Sequences revisited

A sequence $\{x_n\}_{n=1}^{\infty}$ is increasing if $x_{n+1} \geq x_n$ for all n, and similarly it is decreasing if $x_{n+1} \leq x_n$.

Theorem I.10 *If $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded, then $x_n \to \sup\{x_n\}$ as $n \to \infty$.*

Proof Since $\{x_n\}$ is bounded, $\sup\{x_n\}$ exists by completeness. Given an $\epsilon > 0$, we have that $x_n \leq L$ and there exists an N such that $x_n > \sup\{x_n\} - \epsilon$ (because otherwise $\sup\{x_n\} - \epsilon$ will be a smaller upper bound). Then, for $n \geq N$, $x_n \geq x_N > \sup\{x_n\} - \epsilon$ since it is increasing, so

$$\sup\{x_n\} + \epsilon > \sup\{x_n\} \geq x_n > L - \epsilon \qquad \Rightarrow \qquad \epsilon > x_n - \sup\{x_n\} > -\epsilon \qquad \Leftrightarrow \qquad |x_n - \sup\{x_n\}| < \epsilon,$$

thus $x_n \to \sup\{x_n\}$ as required.

If $\{x_n\}_{n=1}^{\infty}$ is a sequence, then a subsequence is $\{x_{n_i}\}_{i=1}^{\infty}$ with $n_1 < n_2 < \cdots$. For example, $\{x_{2n}\} = \{x_1, x_3 \cdots\}$ is a subsequence of $\{x_n\}$.

Theorem I.11 (Bolzano–Weierstrass) *Every bounded sequence contains a subsequence which has a a limit.*

Remark This actually applies to any complete field, not just for $\mathbb{R}$ (e.g., $\mathbb{R}^n$).

Example $\{(-1)^{n+1}\}_{n=1}^{\infty}$ is bounded but has no limit. However, the subsequences $\{x_{2n-1}\}_{n=1}^{\infty} \to 1$ and $\{x_{2n}\}_{n=1}^{\infty} \to -1$.

II. SERIES AND CONVERGENCE

Given a sequence $\{x_n\}_{n=1}^{\infty}$, what is $\sum_{n=1}^{\infty} x_n$? This series could converge or diverge depending on whether the infinite sum is defined. Usually a test is done to test the convergence of a series.

Given $\{x_n\}_{n=1}^{\infty}$, the partial sum $S_k = \sum_{n=1}^{k} x_n$. $\{S_k\}_{n=1}^{\infty}$ is a sequence, and if $S_k \to S$ as $k \to \infty$, then $\sum_{n=1}^{\infty} x_n$ converges to S. If S_k has no limit, then the series $\{x_n\}_{n=1}^{\infty}$ diverges.

Example Find the partial sums and determine whether the series associated with the following sequences converge:

1. $x_n = 1$. This gives $S_k = k$, and there is no limit as $k \to \infty$, so the sum diverges.

2. $x_n = 1/[n(n+1)]$. By partial fractions, we have
$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$
so
$$S_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{k+1},$$
so $S_k \to 1$ as $k \to \infty$, and hence the series converges.

3. Fixing $t \in \mathbb{R}$, and take $x_n = t^n$, we have
$$\sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + \cdots,$$
which is known as a geometric series. We observe that
$$S_k = 1 + t + \cdots + t^k, \qquad tS_k = t + t^2 + \cdots + t^{k+1},$$
so that
$$S_k - tS_k = 1 - t^{k+1} \qquad \Rightarrow \qquad S_k = \frac{1 - t^{k+1}}{1 - t},$$
assuming $t \neq 1$. Observe that $1 - t^{k+1} < 1 - t$ iff $|t| < 1$, so the series converges to 1 as $k \to \infty$ iff $|t| < 1$, otherwise it diverges.

4. $x_n = 1/n$. The sum $\sum_{n=1}^{\infty} = 1 + 1/2 + 1/3 + \cdots$ is known as the harmonic series. Observer that
$$S_1 = 1 = \frac{2}{2}, \qquad S_2 = 3/2, \qquad S_4 = S_2 + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = \frac{4}{2}, \qquad S_8 = S_4 + \cdots > 2 + 4 \cdot \frac{1}{8} = \frac{5}{2},$$
and it may be shown by induction that $S_{2^p} \geq (p+2)/2$, and thus S_k has no limit, and the series diverges.

Theorem II.1 *If $\sum_{n=1}^{\infty} x_n$ converges, then $x_n \to 0$ as $n \to \infty$.*

Proof With the definition of the partial sum $S_k = \sum_{n=1}^{k} x_n$, we have $S_k - S_{k-1} = x_k$. Since $\sum x_n$ converges, then $S_k, S_{k-1} \to S$ as $k \to \infty$, so $S_k - S_{k-1} = x_k \to 0$.

Note the converse is not true. The harmonic series does not converge even though the sequence goes to zero.

Theorem II.2 *If $\sum_{n=1}^{\infty} x_n$ converges to S, and $\sum_{n=1}^{\infty} y_n$ converges to T, and $A, B \in \mathbb{R} - \{0\}$, then $\sum_{n=1}^{\infty}(Ax_n + By_n)$ converges to $AS + BT$.*

Proof Apply the corresponding limits of S_k of x_n and y_n.

A. Comparison test

Theorem II.3 (Comparison test) *Suppose $0 \leq x_n \leq y_n$ for all n, and $\sum_{n=1}^{\infty} y_n$ converges to T, then $\sum_{n=1}^{\infty} x_n$ converges to S with $0 \leq S \leq T$.*

Proof Let $S_k = \sum^k x_n$ and $T_k = \sum^k y_n$. We note that since $x_n \geq 0$, both S_k and T_k as a sequence is increasing, with $T_k \to T$ as $k \to \infty$. Then $S_k \leq T_k \leq T$, so S_k is bounded and $\sup\{S_k\} = S$ exists, with $0 \leq S \leq T$.

Example Test the convergence of the following series associated with the sequences:

1. $x_n = 1/n^2$. We observe that

$$z_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \qquad \sum_{n=1}^{\infty} = 1$$

from a previous example. We also note that $0 < 1/(n+1)^2 < 1/[n(n+1)]$ for all n. By comparison, $\sum_{n=1}^{\infty}(n+1)^-2$ converges to some L with $0 \leq 0 \leq 1$. Then

$$\sum_{n=1}^{\infty} = \frac{1}{4} + \frac{1}{9} + \cdots \leq 1, \qquad \Rightarrow \qquad \sum_{n=1}^{\infty} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots \leq 2,$$

so the sum converges (and in fact converges to $\pi^2/6$, a result which may be obtained for example by the consideration of a Fourier series problem).

2. Assuming that $\sum_{n=1}^{\infty} 1/n^p$ converges iff $p > 1$, test the convergence of the series with $x_n = n/\sqrt{n^8+2}$.

The rough argument is that $x_n \sim n/\sqrt{n^8} = 1/n^3$ for large n, which converges, so we set up the comparison to try and proof convergence. One such that works is $0 \leq x_n \leq 1/n^3$ for all n, so the series converges.

3. $x_n = (n+3)/\sqrt{2n^3-1}$.

a similar argument gives $x_n \sim n/\sqrt{2n^3} = 1/\sqrt{2n}$, so we expect a divergence, so we set up the comparison to proof a divergence. Noting that $n+3 > 3$ and $\sqrt{2n^3-1} < \sqrt{2n^3}$, we have $0 < \frac{1}{\sqrt{2n}} < x_n$, and since $\sum 1/\sqrt{2n}$ diverges, the series diverges.

4. $x_n = n^2/\mathrm{e}^n$.

We know already that $n^8/\mathrm{e}^n \to 0$, so we expect convergence. Since $n^10/\mathrm{e}^n > n^8/\mathrm{e}^n$ and $n^10/\mathrm{e}^n \to 0$ as $n \to \infty$, n^10/e^n is bounded above by some $K < \infty$ for all n, thus $0 \leq n^8/\mathrm{e}^n \leq K/n^2$, so by comparison, the series converges.

5. $x_n = 1/(2+\sqrt{n})$.

We expect this to diverge since $x_n \sim 1/\sqrt{n}$ for large n. One way is to note that since $2 \leq 2\sqrt{n}$, we have $2+\sqrt{n} \leq 3\sqrt{n}$ and so $x_n \geq 1/(3\sqrt{n})$, which shows divergence. Another is to note that

$$x_n\sqrt{n} = \frac{\sqrt{n}}{2+\sqrt{n}} \geq \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \frac{1}{2}$$

since we are increasing the denominator. Thus $x_n \geq 1/(2\sqrt{n})$, and so since $\sum 1/\sqrt{n}$ diverges, the $\sum x_n$ diverges.

6. $x_n = (\log n^2)/n^2$.

We assume that $\sum 1/n^p$ converges iff $p > 1$. Since $(\log n^2)/n^{1/2} \to 0$ as $n \to \infty$, the sequence is bounded and $(\log n^2)/n^{1/2} \leq k$ for all n. Then

$$0 \leq \frac{\log n^2}{\sqrt{n}} \frac{1}{n^{3/2}} = x_n \leq \frac{K}{n^{3/2}},$$

and so $\sum x_n$ converges by comparison.

7. $x_n = \sqrt{(2n+1)/(3n^2-1)}$.

Roughly, $x_n \sim 1/n$, so we expect divergence. We have

$$\sqrt{\frac{2n+1}{3n^2-1}} > \sqrt{\frac{2n}{3n^2-1}} > \sqrt{\frac{2n}{3n^2}},$$

so the series diverges by comparison.

Theorem II.4 *If x_n is <u>absolute convergent</u>, i.e., if $\sum_{n=1}^{\infty}|x_n|$ is convergence, then this implies that $\sum_{n=1}^{\infty}x_n$ is convergent (but the converse is not true).*

Proof Assuming absolute convergence, then since $-|x_n| \leq x_n \leq |x_n|$, we have $0 \leq x_n + |x_n| \leq 2|x_n|$, and so $\sum x_n$ converges by comparison.

Example For $x_n = \cos n^2/n^2$, we have $0 \leq |\cos n^2/n^2| \leq 1/n^2$, so the series converges.

B. Ratio test

Theorem II.5 (Ratio test) *Suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence of non-zero numbers, and $|x_{n+1}/x_n| \to L$ as $n \to \infty$. Then the associated series is divergent and convergent if $L > 1$ and $L < 1$ respectively. If $L = 1$, the series could be either (e.g., $x_n = 1$ and $x_n = 1/n^2$ is divergent and convergent respectively).*

Proof For $L > 1$, there exists N such that $|x_{n+1}/x_n| > 1$ for $n \geq N$, and so $0 < |x_N| < |x_{N+1}| < |x_{N+2}| < \cdots$, thus $\{x_n\} \not\to 0$, and the series diverges.

For $L < 1$, we consider a comparison test with the geometric series $\sum_{n=1}^{\infty} t^n$. Choosing $t \in (L, 1)$, we have, for some $n \geq N$, $|x_{n+1}/x_n| \to L < t$, so $|x_{N+1}| < t|x_N|$, $|x_{N+2}| < t|x_{N+1}| < t^2|x_N|$ etc., and $|x_{N+i}| < t^i|x_N|$. Then $\sum_{i=1}^{\infty} t^i|x_N| = |x_N|\sum_{i=1}^{\infty} t^i$, so $\sum_{i=1}^{\infty}|x_{n+i}|$ converges by comparison test, and thus $\sum_{i=1}^{\infty} x_{N+i}$ converges absolutely. Since we may add arbitrary finite values to convergent sums without violating convergence, $\sum_{n=1}^{\infty} x_n$ converges.

Example Test the convergence of the series associated with the following sequences:

1. $x_n = c^n/n!$, for $c \in \mathbb{R} - \{0\}$.
$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{c^{n+1}}{(n+1)!}\frac{n!}{c^n}\right| = \left|\frac{c}{n+1}\right| \to 0$$
as $n \to \infty$, so the series converges by ratio test (in fact converges to $\mathrm{e}^c - 1$, by considering a Taylor series for example).

2. $x_n = n!(2/n)^n$.
$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}}\frac{n^n}{n!2^n} = 2\left[1 + \frac{1}{n}^n\right]^{-1} \to \frac{2}{\mathrm{e}} < 1$$
as $n \to \infty$, so the series converges by the ratio test.

C. Integral test

Theorem II.6 (Integral test) *Let $f(x)$ by a positive decreasing function for $x \geq 1$. Let*
$$F(m) = \int_1^m f(x)\,\mathrm{d}x, \qquad x_n = F(n),$$
then $\sum_{n=1}^{\infty} x_n$ converges iff $F(m) \to L < \infty$ as $m \to \infty$.

Proof By definition,
$$F(m) = \int_1^m f(x)\,\mathrm{d}x = \left(\int_1^2 + \int_2^3 \cdots + \int_{m-1}^m\right) f(x)\,\mathrm{d}x = \sum_{k=1}^{m-1}\int_k^{k+1} f(x)\,\mathrm{d}x = \sum_{k=1}^{m-1} I_k,$$
so $F(m)$ is the partial sum of $\sum_{k=1}^{\infty} I_k$, has $F(m)$ has a limit iff $\sum I_k$ converges. Now, by assumption, $f(x)$ is a decreasing function, so, for $x \in (k, k+1)$,
$$I_k = \int_k^{k+1} f(x)\,\mathrm{d}x \leq \int_k^{k+1} f(k)\,\mathrm{d}x = f(k)[1]_k^{k+1} = f(k) = x_k,$$
and we can show that $x_{k+1} \leq I_k \leq x_k$. If $\sum x_k$ converges, then the partial sums converge and $F(m) \to L < \infty$ as $m \to \infty$. Conversely, if the partial sums converge, then $\sum x_{k+1}$ converges. Adding finite values does not affect convergence, thus $\sum_{n=1}^{\infty}$ converges.

Example Test the convergence of the series associated with the following sequences:

1. $x_n = 1/n^p$.

   Defining $F(m) = \int_1^m 1/x^p \, \mathrm{d}x$, if $p = 1$, $F(m) = \log m$, otherwise we have $F(m) = (M^{1-p} - 1)/(1-p)$. Since $\log m \to \infty$ and $(M^{1-p} - 1)/(1-p) \to L < \infty$ iff $1 < p$, $\sum_{n=1}^{\infty} 1/n^p$ converges iff $p > 1$.

2. $x_n = 1/[4n(\log n)^2]$with $n \geq 2$.

   An application of the ratio or comparison test fails to yield any conclusion about this. If we define

   $$f(x) = \frac{1}{4x(\log x)^2},$$

   we observe that $f(x) > 0$ for $x > 0$, and $f(x$ is a decreasing function. With the substitution $u = \log x$, we have

   $$F(m) = \int_2^m \frac{\mathrm{d}x}{4x(\log n)^2} = \frac{1}{4}\int_{\log 2}^{\log m} \frac{\mathrm{d}u}{u^2} = \frac{1}{4\log 2} - \frac{1}{4\log m} \to \frac{1}{4\log 2}$$

   as $m \to \infty$, so the series converges by the integral test.

D. Alternating sign test

Note that all previous test proves absolute convergence. This proves conditional convergence.

Theorem II.7 (Alternating sign test) *Suppose $\{x_n\}_{n=1}^{\infty}$ is a positive decreasing sequence and $x_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ converges.*

Proof We look at $S_k = \sum_{n=1}^{k}(-1)^{n+1}x_n$. Suppose k is odd, then

$$S_{2p-1} = (x_1 - x_2) + (x_3 - x_4)\cdots + (x_{2p-3} - x_{2p-2}) + x_{2p-1}.$$

Since $\{x_n\}$ is a positive decreasing sequence, $(x_{2p-3} - x_{2p-2}) > 0$ for all admissible p values, so $\{S_{2p-1}\}$ is bounded below and has limit M.

Suppose now k is even, then a similar manipulation has

$$S_{2p} = y_1 - (x_2 - x_3)\cdots - (x_{2p-2} - x_{2p-1}) - x_{2p},$$

and, with this grouping, the brackets terms all all positive, so $S_{2p} \leq x_1$, thus it is bounded above with limit L. So then $S_{2p} = S_{2p-1} - x_{2p}$, and since $x_n \to 0$, the relation tends to $L = M - 0$, thus $L = M$ as $p \to \infty$, and since the partial sums tend to a limit, the series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ converges.

Example Test the convergence of the series associated with the following sequences:

1. $x_n = (-1)^{n+1}/n = 1 - 1/2 + 1/3 - 1/4 + \cdots$.

   Since $1/n \to 0$ and is a positive decreasing sequence, the sequence $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ converges by the alternative sign test(in fact converges to $\log 2$).

2. $x_n = \tan(\pi/n)\cos(\pi n)$ for $n \geq 3$.

   Note that $\cos(\pi n) = (-1)^n$, and since $\tan(\pi/n)$ is positive and decreasing, the series $\sum_{n=3}^{\infty}\tan(\pi/n)\cos(\pi n)$ converges by the alternative sign test.

E. Complex sequences and series

A complex sequence $\{z_n\}_{n=1}^{\infty}$ is like a sequence except it is for complex numbers $z_n \in \mathbb{C}$. By definition, since $\{|z_n - c|\}_{n=1}^{\infty} \in \mathbb{R}$, $z_n \to c \in \mathbb{C}$ as $n \to \infty$ if $|z - c| \to 0$ as $n \to \infty$ by analogous definitions.

Example Determine whether the following sequences converge, and if they do, find the limit:

1. $z_n = 1/(n+\mathrm{i})$.

   There are several ways to see this. One is to guess the limit. We expect the limit in this case to go to zero. To show this, observe that

   $$|z_n - 0| = |z_n| = \frac{1}{|n+\mathrm{i}|} = \frac{1}{\sqrt{n^2+1}} \to 0,$$

   so $z_n \to 0$ as $n \to \infty$.

   Another possible way to see this is to make use of the rules for calculus of limits. We have

   $$z_n = \frac{1/n}{1+1/n} \to \frac{0}{1+0} = 0$$

   as $n \to \infty$.

   We could also show determine how the real and imaginary parts behave as $n \to \infty$. We have

   $$z_n = \frac{n}{n^2+1} - \mathrm{i}\frac{1}{n^2+1} \to 0 + 0\mathrm{i} = 0,$$

   as required.

2. $z_n = [\sqrt{n^2+1}/(n+2\mathrm{i})] \cdot \exp[\mathrm{i}\pi n/(\sqrt{n^2+1}+\sqrt{n^2-1})]$.

   $$z_n = \frac{\sqrt{1+1/n^2}}{1+2\mathrm{i}/n} \exp\left(\frac{\mathrm{i}\pi}{\sqrt{1+1/n^2}+\sqrt{1-1/n^2}}\right) \to \mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}$$

   as $n \to \infty$.

3. $z_n = \sqrt{n^3+1}/(n^2+2\mathrm{i})\mathrm{e}^{\mathrm{i}\pi^2}$.

   $$|z_n| = \left|\frac{\sqrt{n^3+1}}{n^2+2\mathrm{i}}\right| \cdot 1 = \frac{\sqrt{1/n+1/n^4}}{|1+2\mathrm{i}/n^2|} \to 0$$

   as $n \to 0$, so $z_n \to 0$.

4. $z_n = (2+\mathrm{e}^n)^{-1} \exp[n + 3\mathrm{i}n\pi/(\sqrt{n^2+1}+\sqrt{n^2-1})]$.

   $$z_n = \frac{\mathrm{e}^n}{2+\mathrm{e}^n} \exp\left(\frac{3\mathrm{i}\pi}{\sqrt{1+1/n^2}+\sqrt{1-1/n^2}}\right) = \frac{1}{2\mathrm{e}^{-n}+1} \exp\left(\frac{3\mathrm{i}\pi}{\sqrt{1+1/n^2}+\sqrt{1-1/n^2}}\right) \to \mathrm{e}^{3\pi\mathrm{i}/2} = -\mathrm{i}$$

   as $n \to \infty$.

Theorem II.8 *Let $\{z_n\}$ be a complex sequence, then since $z_n = x_n + \mathrm{i}y_n$, $\{x_n\}$ and $\{y_n\}$ are real sequences. For $c = ax + \mathrm{i}b$, $z_n \to c$ as $n \to \infty$ iff $x_n \to a$ and $y_n \to b$ as $n \to \infty$.*

Proof Assuming $x_n \to a$ and $y_n \to b$ as $n \to \infty$, we have, by the triangle inequality

$$|z_n - c| = \sqrt{(x_n-a)^2 + (y_n-b)^2} \le |x_n - a| + |y_n - b| \to 0,$$

so $z_n \to c$ by squeezing. Conversely, assuming $z_n \to c$, since

$$0 \le |x_n - a| \le |z_n - c|, \qquad 0 \le |y_n - b| \le |z_n - c|$$

again by triangle inequality, $x_n \to a$ and $y_n \to b$ as $n \to \infty$ by squeezing.

Theorem II.9 *If $z_n = x_n + \mathrm{i}y_n$, then $\sum_{n=1}^{\infty} z_n$ converges iff $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$ converges.*

Proof Apply the previous theorem to the individual partial sums.

Theorem II.10 *If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=1}^{\infty} z_n$ converges.*

Proof Since $0 \le |x_n| \le |z_n|$ and $0 \le |y_n| \le |z_n|$ by the triangle inequality, if $\sum_{n=1}^{\infty} |z_n|$ converges, this implies that $\sum_{n=1}^{\infty} |x_n|$ and $\sum_{n=1}^{\infty} |y_n|$ converges, and so by the absolute convergence theorem (on x_n and y_n) and the previous theorem, this implies the convergence of $\sum_{n=1}^{\infty} z_n$.

Theorem II.11 *If $\sum_{n=1}^{\infty} z_n$ converges, then $z_n \to 0$ as $n \to \infty$.*

Proof This is analogous to the one for the real case.

Example Determine the convergence of the series associated with the following sequences:

1. $z_n = c^n$, $c \in \mathbb{C}$, the complex geometric series.

   Analogous to the real case, the partial sum is $S_k = (1 - c^{k+1})/(1 - c)$ for $c \neq 1$. Observe that $|c^{k+1}| = |c|^{k+1} \to 0$ iff $|c| < 1$, so the series converges iff $|c| < 1$, and converges to $1/(1 - c)$ when it does converge.

2. $z_n = 1/(1 = \mathrm{i}n^2)$.

   Observe that
   $$\left|\frac{1}{1 + \mathrm{i}n^2}\right| = \frac{1}{\sqrt{1 + n^4}} < \frac{1}{n^2},$$
   and by comparison and the absolute convergence test, the series converges.

3. $z_n = 1/(1 + \mathrm{i}\sqrt{n})$.
   $$z_n = \frac{1}{1 + n} - \mathrm{i}\frac{\sqrt{n}}{1 + n},$$
   and the sequence associated with the real part may be shown to diverge by comparing to $1/2n$ for example, so the series diverges.

Theorem II.12 (Ratio test) *If $|z_{n+1}/z_n| \to L$ as $n \to \infty$, then if $L < 1$, $\sum_{n=1}^{\infty} z_n$ converges, whilst it diverges when $L > 1$, and it is inconclusive if $L = 1$.*

Proof If $L < 1$, then $\sum_{n=1}^{\infty} z_n$ converges by the absolute convergence test. If $L > 1$, $|z_n| \not\to 0$, so we do not have convergence.

Example For $z_n = (n + \mathrm{i})/(2^n + \mathrm{i})$,
$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{n + 1 + \mathrm{i}}{2^{n+1} + \mathrm{i}}\right| \cdot \left|\frac{n + \mathrm{i}}{2^n + \mathrm{i}}\right| = \sqrt{\frac{(n + 1)^2 + 1}{n^2 + 1}}\sqrt{\frac{4^n + 1}{4^{n+1} + 1}} \to \frac{1}{2}$$
as $n \to \infty$, so the series converges by the ratio test.

A power series in z is a series of the form $\sum_{n=0}^{\infty} a_n x^n$, where a_n are constant complex coefficients. Associated with any such series is the radius of convergence $R \ge 0$. The series converges if $|z| < R$, and it diverges for $|z| > R$. The two special cases are when $R = 0$ (so we have convergence iff $z = 0$) and R being infinite, which means the series converges for any z.

Sometimes the ratio test gives us R, through
$$\left|\frac{a_{n+1} z^{n+1}}{a_n z^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |z_n|.$$
Supposing $|a_{n+1}/a_n| \to L$, the power series converges if $L|z| < 1$ and diverges if $L|z| > 1$, so the radius of convergence is $R = 1/L$.

Example Find the radius of convergence for the following power series:

1. $\sum_{n=1}^{\infty} (2^n/n) z^n$.
   $$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{2^{n+1}}{n + 1}\frac{n}{2^n}\right| \to 1 \cdot 2 = 2$$
   as $n \to \infty$, so $R = 1/2$.

2. $\sum_{n=0}^{\infty}(n^2/3^n)z^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^2}{n^2}\frac{3^n}{3^{n+1}}\right| \to 1 \cdot \frac{1}{3} = \frac{1}{3}$$

as $n \to \infty$, so $R = 3$.

3. $\sum_{n=1}^{\infty}[(n!)^3 2^n/(3n)!]z^n$.

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{[(n+1)!]^3}{(n!)^3}\frac{2^{n+1}}{2^n}\frac{(3n)!}{(3n+3)!}\right| = \left|\frac{2(n+1)^3}{(3n+3)(3n+2)(3n+1)}\right| \to \frac{2}{27}$$

as $n \to \infty$, so $R = 27/2$.

Lemma II.13 *Suppose $\sum_{n=1}^{\infty} a_n c^n$ converges and $c \neq 0$, then if $|z| < |c|$, $\sum_{n=1}^{\infty} a_n z^n$ converges absolutely.*

Proof Since $\sum a_n c^n$ converges, $a_n c^n \to 0$ as $n \to \infty$, so there exists M such that $|a_n c^n| \leq M$ for all n. Then

$$|a_n z^n| = \left|a_n \left(\frac{z}{c}\right)^n c^n\right| \leq M \left|\frac{z}{c}\right|^n.$$

Since $|z| < |c|$ by assumption, $|z/c| < 1$, so the geometric series converges, and by comparison, $\sum a_n z^n$ converges absolutely.

Theorem II.14 *For any power series $\sum_{n=1}^{\infty}$, one of the following possibilities hold:*

1. *$\sum a_n z^n$ converges for $z = 0$, and $R = 0$;*
2. *$\sum a_n z^n$ converges absolutely for all $z \in \mathbb{C}$, so $R = \infty$;*
3. *There exists $R > 0$ for which the power series converges absolutely if $|z| < R$, and diverges if $|z| > R$.*

Proof Let $S = \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n w^n \text{ converges for some } w,\ |w| = n\}$. Then:

1. the first case is the trivial case where for $S = \{0\}$, and $\sup S = R = 0$;
2. if S is unbounded, then R is infinite, and $\sum a_n z^n$ converges for all $z \in \mathbb{C}$. Observe that $|z| \in \mathbb{R}$, but this cannot be an upper bound since S is unbounded, so there exists $x \in S$ such that $|z| < x$, and hence there exists $w \in \mathbb{C}$ such that $|z| < |w|$, and $\sum a_n w^n$ converges, so $\sum a_n z^n$ converges absolutely by comparison;
3. if S is bounded and $R = \sup S$, then for $R > 0$, $|z| < R$, and there exists $x \in S$ with $|z| < x$ such that $\sum a_n w^n$ converges, and so by previous lemme, $\sum a_n z^n$ converges absolutely. If $|z| > R$, then $|z| \notin S$, and by the definition of S, the power series diverges.

F. Taylor series

Suppose we have a sequence $\{f_n(x)\}_{n=0}^{\infty}$, $x \in \mathbb{R}$. Let $f(x) = \sum_{n=0}^{\infty} f_n(x)$. One special case of this is when $f_n(x) = a_n x^n$, where a_n are constants, and we have a power series. Then it follows also from the previous lemma that is $\sum a_n c^n$ converges for $c > 0$, $\sum a_n x^n$ converges absolutely for all $x \in (-c, c)$. So then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad x \in (-c, c),$$

and

$$\int_a^b \sum_{n=0}^{\infty} a_n x^n \,\mathrm{d}x = \sum_{n=0}^{\infty} \int_a^b a_n x_n \,\mathrm{d}x, \qquad -c < a < b < c.$$

This is only true for infinite power series (note: and the integration is possible because we have uniform convergence in this case).

For certain functions $f(x)$ and certain ranges of z, we can write

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_n = \frac{1}{n!} f^{(n)}(0).$$

This says that the power series converges and, furthermore, to $f(x)$ for certain ranges of x. Some notable examples of Taylor series are:

1. if $f(x)$ is a polynomial then trivially the Taylor series is $f(x)$ for all $x \in \mathbb{R}$;
2. $$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots, \qquad x \in \mathbb{C};$$
3. Remembering that $\cosh x + \sinh x = \mathrm{e}^x$, we have that
$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \qquad \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
4. using $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$, we have that
$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \qquad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$
5. $$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad |x| < 1$$
(note that $x = 1$ is actually defined, with $\log 2 = 1 - 1/2 + 1/3 + \cdots$);
6. $$(1+x)^c = 1 + \sum_{n=1}^{\infty} \binom{c}{n} x^n, \qquad \binom{c}{n} = \frac{c(c-1)\cdots(c-n+1)}{n!}.$$

Example The energy of an object with mass m and speed v is, in Einstein's model of energy and relativistic kinetic energy,
$$E = mc^3(c^2 - v^2)^{-1/2} = mc^2(1 - v^2/c^2)^{-1/2},$$
where c is the speed of light, and $|v/c| < 1$ (an object cannot travel faster than the speed of light). Then a Taylor expansion of this gives, in powers of v^2/c^2,
$$E = m\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots\right)c^2 = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}\frac{mv^4}{c^2} + \cdots, \qquad |v| < c.$$
We know that $E = mc^2$ is the Einstein rest energy equation, and $E = mv^2/2$ is the Newtonian energy equation for moving mass; the rest are relativistic corrections which are only significant when $v \lesssim c$.

Example Using $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$, derive the first two terms of the Taylor series for $g(x) = (1 + 2x^3)^{-2}$.
Let $x = -2x^3$. Then $g(x) = (1 - (2x^3) + (2x^3)^2 + \cdots)^2 = 1 - 4x^3 + 12x^6 + \cdots$.

III. INTEGRATION

There are different definitions of the integral, with different properties.

A. Riemann integral

A partition of a closed interval $[a, b]$ is a finite set of real numbers $\{x_0, x_1, \cdots x_n\}$ where
$$a = x_0 < x_1 < x_2 \cdots < x_n = b.$$

Let $f(x)$ be defined and bounded on $[a,b]$, and $\mathcal{P}$ be a partition of $[a,b]$, i.e., $\mathcal{P} = \{x_0, x_1, \cdots x_n\}$. Then the upper Riemann sum $\mathcal{U}(\mathcal{P})$ and lower Riemann sum $\mathcal{L}(\mathcal{P})$ of $f(x)$ relative to $\mathcal{P}$ are, respectively,

$$\mathcal{U}(\mathcal{P}) = \sum_{i=1}^{n}(x_i - x_{i-1}) \cdot \sup_{x\in[x_i,x_{i-1}]} f(x), \qquad \mathcal{L}(\mathcal{P}) = \sum_{i=1}^{n}(x_i - x_{i-1}) \cdot \inf_{x\in[x_i,x_{i-1}]} f(x).$$

Geometrically, $(x_i - x_{i-1})$ is the base of the rectangle, and the $\sup$ and $\inf$ part are the height of the rectangles that bounds the graph $f(x)$ just above and just below respectively.

Example For $f(x) = x$ on $[0,1]$, let $\mathcal{P} = \{i/n\}_{i=0}^{n}$. Then since $x_i - x_{i+1} = 1/n$ for all i, we have

$$\mathcal{U}(\mathcal{P}) = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} \cdots + \frac{n}{n}\right) = \frac{1}{n^2}(1 + 2 \cdots + n) = \frac{1}{n^2}\frac{1}{2}n(n+1) = \frac{n+1}{2n}$$

(the sum of the first n integers is the n^{th} triangle number), and

$$\mathcal{L}(\mathcal{P}) = \frac{1}{n}\left(0 + \frac{1}{n} \cdots + \frac{n-1}{n}\right) = \frac{1}{n^2}\frac{1}{2}(n-1)n = \frac{n-1}{2n}.$$

Additionally, notice that $\mathcal{U}(\mathcal{P}) < 1/2$ and $\mathcal{L}(\mathcal{P}) > 1/2$, with $\mathcal{U}(\mathcal{P}) - \mathcal{L}(\mathcal{P}) \to 0$ as $n \to \infty$.

Let f be defined and bounded on $[a,b]$, and

$$\mathcal{U} = \inf\{\mathcal{U}(\mathcal{P})\}, \qquad \mathcal{L} = \sup\{\mathcal{L}(\mathcal{P})\}$$

for all possible partitions $\mathcal{P}$. Then f is Riemann integrable if $\mathcal{U} = \mathcal{L}$, and we write

$$\int_a^b f(x)\,\mathrm{d}x = \mathcal{U} = \mathcal{L}.$$

Example For

$$f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Q}, \\ 1, & \text{if } x \in \mathbb{Q}, \end{cases}, \qquad x \in [0,1],$$

f is defined and bounded, but $\mathcal{U}(\mathcal{P}) = 1$ and $\mathcal{L}(\mathcal{P}) = 0$ for all $\mathcal{P}$, so f is not Riemann integrable.

Lemma III.1 *If $\mathcal{P}$ is a partition of $[a,b]$ and we add one extra point to give $\mathcal{P}'$, then $\mathcal{L}(\mathcal{P}) \le \mathcal{L}(\mathcal{P}') \le \mathcal{U}(\mathcal{P}') \le \mathcal{L}(\mathcal{P})$.*

Proof Suppose we add x' into $(x_i - x_{i-1})$, then in $\mathcal{L}(\mathcal{P}')$, $(x_i - x_{i-1})\inf\ f$ is replaced by

$$(x' - x_{i-1}) \inf_{[x_{i-1},x']} f + (x_i - x') \inf_{[x',x_i]} f,$$

which is bigger than or equation, so $\mathcal{L}(\mathcal{P}) \le \mathcal{L}(\mathcal{P}')$. Similarly, we have $\mathcal{U}(\mathcal{P}') \le \mathcal{L}(\mathcal{P})$.

Lemma III.2 *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are two partitions of $[a,b]$, with $\mathcal{P}_1 \subseteq \mathcal{P}_2$, then $\mathcal{L}(\mathcal{P}_1) \le \mathcal{L}(\mathcal{P}_2) \le \mathcal{U}(\mathcal{P}_2) \le \mathcal{L}(\mathcal{P}_1)$.*

Proof Apply previous lemma as many times as required.

Theorem III.3 *Let f be defined and bounded on $[a,b]$. Then f is Riemann integrable iff for any $\epsilon > 0$, there exists a partition $\mathcal{P}$ of $[a,b]$ such that $\mathcal{U}(\mathcal{P}) - \mathcal{L}(\mathcal{P}) < \epsilon$.*

Proof We first show that we always have $\mathcal{L} \le \mathcal{U}$. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be any two partitions, and put $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. By the lemma, we have

$$\mathcal{L}(\mathcal{P}_1) \le \mathcal{L}(\mathcal{P}) \le \mathcal{U}(\mathcal{P}) \le \mathcal{L}(\mathcal{P}_2),$$

so taking the $\sup$ of the left hand side and the $\inf$ of the right hand side, we have $\mathcal{L} \le \mathcal{U}$.

Suppose now f is Riemann integrable, so $\mathcal{L} = \mathcal{U}$. Let $\epsilon > 0$ by given, then, by definition of $\mathcal{L}$, there exists a $\mathcal{P}_2$ such that $\mathcal{L}(\mathcal{P}_2) \geq \mathcal{L} - \epsilon/2$ (since $\mathcal{L}$ is the supremum over all partitions). Similarly, we have some $\mathcal{P}_1$ where $\mathcal{U}(\mathcal{P}_1) \leq \mathcal{U} + \epsilon/2$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then, by previous lemma, we have

$$\mathcal{L} - \frac{\epsilon}{2} < \mathcal{L}(\mathcal{P}_1) < \mathcal{L}(\mathcal{P}) < \mathcal{U}(\mathcal{P}) < \mathcal{U}(\mathcal{P}_2) < \mathcal{U} + \frac{\epsilon}{2},$$

and since $\mathcal{L} = \mathcal{U}$, we have $\mathcal{U}(\mathcal{P}) - \mathcal{L}(\mathcal{P}) < \epsilon$ after rearranging.

Suppose instead $\mathcal{U}(\mathcal{P}) - \mathcal{L}(\mathcal{P}) < \epsilon$ for all $\epsilon > 0$, then, by definition,

$$0 \leq \mathcal{U} - \mathcal{L} \leq \mathcal{U}(\mathcal{P}) - \mathcal{L}(\mathcal{P}) < \epsilon.$$

For arbitrary small ϵ errors, we have $\mathcal{U} = \mathcal{L}$, so f is Riemann integrable.

Example Consider $f(x) = x^2$ on $[0, 1]$. Let $\mathcal{P}_n = \{i/n\}_{i=0}^{n}$. Then observing that all intermediate terms cancel, $\mathcal{U}(\mathcal{P}_n) - \mathcal{L}(\mathcal{P}_n) = (1 - 0)/n$. For all $\epsilon > 0$, we take $1/n < \epsilon$, and this shows $f(x)$ is Riemann integrable for $x \in [0, 1]$. In fact, recalling the formula for the sum of the first n square numbers,

$$\mathcal{U}(\mathcal{P}_n) = \frac{1}{n}\left(\frac{1^2}{n^2} + \frac{2^2}{n^2} \cdots + \frac{n^2}{n^2}\right) = \frac{1}{n^3}\frac{n(2n+1)(n+1)}{6} \to \frac{1}{3}$$

as $n \to \infty$, so $\int_0^1 x^2 \,\mathrm{d}x = 1/3$, which we know already.

Theorem III.4 *If f is an increasing function on $[a, b]$ then f is Riemann integrable. (Similarly, if it is decreasing, consider $-f$.)*

Proof Since f is increasing, on $[x_{i-1}, x_i]$, sup $f(x) = f(x_i)$ and inf $f(x) = f(x_{i-1})$. Let $\epsilon > 0$ be given, and take $\mathcal{P}_n = \{x_i = a + ih/n\}_{i=0}^{n}$, where $h = b - a$ is the width of the rectangle. Then

$$\mathcal{U}(\mathcal{P}_n) = \frac{h}{n}\sum_{i=1}^{n} f(x_i), \qquad \mathcal{L}(\mathcal{P}_n) = \frac{h}{n}\sum_{i=1}^{n} f(x_{i-1}),$$

and

$$\mathcal{U}(\mathcal{P}_n) - \mathcal{L}(\mathcal{P}_n) = \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a)),$$

so choosing n big enough gives us $\mathcal{U}(\mathcal{P}_n) - \mathcal{L}(\mathcal{P}_n) < \epsilon$ as required.

Theorem III.5 *If f is continuous on $[a, b]$, then f is Riemann integrable.*

Some properties of the integral:

- if $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,\mathrm{d}x \geq 0$ (because both the supremum and infimum are both positive, so $\mathcal{U}$ and $\mathcal{L}$ are positive);
- if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b f(x)\,\mathrm{d}x$ (since $f(x) - g(x) \geq 0$ for all $x \in [a, b]$);
- $|\int_a^b f(x)\,\mathrm{d}x| \leq \int_a^b |f(x)|\,\mathrm{d}x$ (since $-f(x) \leq |f(x) \leq f(x)$);
- $\int [Af(x) + Bg(x)]\,\mathrm{d}x = A\int f(x)\,\mathrm{d}x + B\int g(x)\,\mathrm{d}x$ (linearity);
- $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$ for $c \in (a, b)$.

B. Improper integrals

Let $f(x)$ be continuous for $x \geq c$, and define $F(m) = \int_c^m f(x)\,\mathrm{d}x$. If $f(m) \to L$ as $m \to \infty$, then we say that $\int_c^\infty f(x)\,\mathrm{d}x$ converges to L, otherwise the integral diverges.

Example Determine the convergence of the following integrals:

1. $\int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x$. We have $F(m) = 1 - \mathrm{e}^{-m} \to 1$ as $m \to \infty$, so it converges.

2. $\int_0^\infty \sin x \,\mathrm{d}x$. We have $F(m) = 1 - \cos m$, and this does not converge, so integral diverges.

Proposition III.6 *If* $\int_0^\infty f(x)\,\mathrm{d}x = L$ *and* $\int_0^\infty g(x)\,\mathrm{d}x = K$*, then* $\int_0^\infty [Af(x) + Bg(x)]\,\mathrm{d}x = AL + BK$*.*

Proposition III.7 *If* $\int_c^\infty f(x)\,\mathrm{d}x$ *converges and* $a > c$*, then* $\int_a^\infty f(x)\,\mathrm{d}x$ *converges.*

Theorem III.8 (Comparison test) *Suppose* $f(x)$ *and* $g(x)$ *are continuous, with* $0 \le f(x) \le g(x)$*. If* $\int_c^\infty g(x)\,\mathrm{d}x$ *converges, then* $\int_c^\infty f(x)\,\mathrm{d}x$ *converges.*

Proof Defining $F(m) = \int_c^m f(x)\,\mathrm{d}x$ and $G(m) = \int_c^m g(x)\,\mathrm{d}x$, then $F(m) \le G(m)$, and both are increasing functions since $f, g \ge 0$. Since $G(m) \to K$ as $m \to \infty$, then $F(m) \le G(m) \le K$. $F(m)$ is thus bounded, and by completeness axiom, $F(m)$ tends to a limit as $m \to \infty$.

Example Determine the convergence of the following integrals:

1. $\int_1^\infty x^{-2} log x^2 \,\mathrm{d}x$.

   By comparing with $x^{-1/2}\log x$, we observe that $x^{-1/2}\log x \to 0$ as $x \to \infty$, so g is bounded above by some K, with
   $$0 \le x^{-2}\log x = x^{-3/2}(x^{-1/2}\log x) \le x^{-3/2}K.$$
   By comparison, since $\int_1^\infty x^{-3/2}\,\mathrm{d}x$ converges, the integral converges.

2. $\int_1^\infty t/\sqrt{t^4+1}\,\mathrm{d}t$.

   This integral roughly goes like t^{-1} for large t, so we expect divergence. Indeed, for $t \ge 1$, $t^4 + 1 \le t^4 + t^4 \le 2t^4$, so
   $$0 \le \int_1^\infty \frac{1}{\sqrt{2}}\frac{1}{t}\,\mathrm{d}t \le \int_1^\infty \frac{t}{\sqrt{t^4+1}}\,\mathrm{d}t,$$
   so the integral diverges by comparison.

If $\int_0^\infty |f(x)|\,\mathrm{d}x$ converges, we say $\int_0^\infty f(x)\,\mathrm{d}x$ is <u>absolutely convergent</u>. If $\int_0^\infty f(x)\,\mathrm{d}x$ converges but $\int_0^\infty |f(x)|\,\mathrm{d}x$ diverges, then $\int_0^\infty f(x)\,\mathrm{d}x$ is <u>conditionally convergent</u>.

Theorem III.9 (Absolute convergence theorem) *If* $\int_a^\infty f(x)\,\mathrm{d}x$ *is absolutely convergent, then* $\int_a^\infty f(x)\,\mathrm{d}x$ *converges.*

Proof Given that $\int_a^\infty |f(x)|\,\mathrm{d}x$ converges, we have, for all $x \ge a$,
$$0 \le f(x) + |f(x)| \le 2|f(x)|,$$
and so by comparison and linearity of the integral, $\int_a^\infty f(x)\,\mathrm{d}x$ converges.

Example Determine the convergence of the following integrals:

1. $\int_\pi^\infty x^{-2}\cos x\,\mathrm{d}x$.

   Since $|x^{-2}\cos x| = x^{-2}$, the integral converges absolutely, so the integral converges.

2. $\int_\pi^\infty x^{-1}\sin x\,\mathrm{d}t$.

   Doing an integration by parts, we have
   $$\int_\pi^m \frac{\sin x}{x}\,\mathrm{d}x = \left[-\frac{\cos x}{x}\right]_\pi^m - \int_\pi^m \frac{\cos x}{x^2}\,\mathrm{d}x = -\frac{\cos m}{m} - \frac{1}{\pi} - \int_\pi^m \frac{\cos x}{x^2}\,\mathrm{d}x.$$
   From the previous example, all terms are finite as $m \to \infty$, so the integral converges.

   We note however the integral is only conditionally convergent. Denoting
   $$I_n = \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,\mathrm{d}x,$$
   note that $\int_\pi^\infty f(x)\,\mathrm{d}x = \sum_{n=1}^\infty I_n$. Let $x = n\pi + y$, then $y + n\pi \le (n+1)\pi$ as $y \le \pi$, so doing a change of variables and using the double angle formulae, we have
   $$I_n = \int_0^\pi \frac{\sin y}{n\pi + y}\,\mathrm{d}y \ge \int_0^\pi \frac{\sin y}{(n+1)\pi}\,\mathrm{d}y = \frac{2}{(n+1)\pi},$$
   and the series on the RHS diverges, so $\sum I_n$ diverges by comparison, and hence the integral is only conditionally convergent.

3. $\int_1^\infty (x^2+\log x)^{-1}\sin \pi x\,\mathrm{d}x$.

   Since $0 \le |f(x)| \le x^{-2}$ on this domain, the integral converges absolutely by comparison.

Suppose f is continuous on $(a,b]$. Writing $F(\epsilon) = \int_\epsilon^b f(x)\,\mathrm{d}x$, if $F(\epsilon) \to L$ as $\epsilon \to a$, the integral converges to L.

Example Determine the convergence of the following integrals:

1. $\int_0^1 1/x^p \cos x\,\mathrm{d}x$, $p > 0$.

$$F(\epsilon) = \int_\epsilon^1 \frac{1}{x^p}\,\mathrm{d}x = \begin{cases} -\log\epsilon, & p = 1, \\ \dfrac{1-\epsilon^{1-p}}{1-p}, & p \neq 1. \end{cases}$$

   This has a limit iff $1-p > 0$ as $\epsilon \to 0$, i.e., iff $p < 1$. So, for example, $\int_0^1 1/x^2\,\mathrm{d}x$ diverges but $\int_0^1 1/\sqrt{x}\,\mathrm{d}x$ converges.

2. $\int_0^1 \log x\,\mathrm{d}x$.

$$F(\epsilon) = \int_\epsilon^1 \log x\,\mathrm{d}x = -1 - \epsilon\log\epsilon + \epsilon.$$

   With $\epsilon = 1/y$, $\epsilon \to 0$ is equivalent to $y \to \infty$, and so $-\epsilon\log\epsilon = y^{-1}\log y \to 0$ as $\epsilon \to 0$, thus $F(\epsilon) \to -1$ as $\epsilon \to 0$, and hence the integral converges.

3. $\int_0^1 x^{-3/2}\mathrm{e}^{-x}\,\mathrm{d}x$.

   Note that $1/\mathrm{e} \le \mathrm{e}^{-x} \le 1$ for $x \in [0,1]$, so $x^{-3/2}\mathrm{e}^{-x} \ge x^{-3/2}\mathrm{e}^{-1}$. The integral $\int_0^1 x^{-3/2}\,\mathrm{d}x$ diverges from the previous example, so the integral diverges by comparison.

4. $\int_0^1 x^{-1/2}\cos x\,\mathrm{d}x$.

   For $x \in (0,1]$, $0 < \cos x \le 1$, so $0 < x^{-1/2}\cos x \le x^{-1/2}$, so the integral converges by comparison.

5. $\int_0^1 x^{-1/2}\cos 2x\,\mathrm{d}x$.

   We note that $0 \le |x^{-1/2}\cos 2x| \le x^{-1/2}$, so the integral converges absolutely by comparison.

6. $\int_0^1 1/\sqrt{1-x^2}\,\mathrm{d}x$.

   Notice that $(1-x^2)^{-1/2} = (1-x)^{-1/2}(1+x)^{-1/2}$, so we can split this accordingly as partial fractions. A substitution with $y = 1-x$ gives $\int_0^1 y^{-1/2}\,\mathrm{d}y$ and converges by comparison. The second integral is finite and well-defined, so the total integral converges. (Alternatively, spot that we can use $x = \sin u$, and the integral has the value $\pi/2$.)

7. $\int_0^\infty (\log x)/(1+x^4)\,\mathrm{d}x$.

   We split this integral into $\int_0^1 + \int_1^\infty$. For $x \in (0,1]$, $1 \le 1 + x^4 \le 2$, so

$$1 \le \frac{1}{1+x^4} \le \frac{1}{2} \qquad \Rightarrow \qquad \left|\frac{\log x}{1+x^4}\right| \le |\log x|,$$

   so the first part of the integral converges. For the second integral, since $x^{-1}\log x \to 0$, $x^{-1}\log x \le K < \infty$, and so

$$0 \le \frac{\log x}{1+x^4} \le \frac{\log x}{x}\frac{x}{1+x^4} \le K\frac{x}{x^4} = \frac{K}{x^3},$$

   so the second integral converges by comparison, and the whole integral converges.

8. $\int_0^\infty (x^3+\sqrt{x})^{-c}\,\mathrm{d}x$, $c \in \mathbb{R}$.

   Let $f(x) = (x^3+\sqrt{x})^{-c}$ and consider $\int_0^1 f(x)\,\mathrm{d}x$ and $\int_1^\infty f(x)\,\mathrm{d}x$. for $0 \le x \le 1$, $\sqrt{x} \le x^3 + \sqrt{x} \le 2\sqrt{x}$, so $x^{-c/2} \ge f(x) \ge 2x^{-c/2}$, and the first integral converges iff $c/2 < 1$, or $c < 2$. On the other hand, $x^3 \le x^3 + \sqrt{x} \le 2x^3$, so $x^{-3c} \ge f(x) \ge 2x^{-3/2}$, and the second integral converges iff $3c > 1$, or $c > 1/3$. Thus the whole integral converges iff $1/3 < c < 2$.

IV. LIMITS, CONTINUITY AND DIFFERENTIABILITY OF FUNCTIONS

A. Limits and continuity

Suppose $a < c < b$, and $f(x)$ is defined in (a, b) except possibly at c. Then we say $f(c) \to L$ if, given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $\delta > |x - c|$ (i.e., we make arbitrary small errors if we choose are close enough to $x = c$).

Example Show whether the limit exists or not in these examples:

1. $f(x) = 2x$ at $c = 1$.

   We expect $f(x) = 2$ for $x \to 1$. Given $\epsilon > 0$, we have to $|f(x) - 2| < \epsilon$, which results in $2|x - 1| < \epsilon$, and we need to choose an appropriate δ. Taking $\delta = \epsilon/2$, we have

   $$|x - 1| < \delta = \frac{\epsilon}{2} \qquad \Leftrightarrow \qquad 2|x - 1| < \epsilon$$

   as required. We could also choose $\delta = \epsilon/4$, which will give the same thing.

2. $f(x) = H(x)$, the Heaviside function, with $H(x) = 1$ for $x > 0$ and zero otherwise, at $x = 0$.

   In this case there is a jump at $x = 0$ and we claim the limit does not exist at $x = 0$. Suppose otherwise, and take $\epsilon = 1/2$. We need to choose $\delta > 0$ such that $|f(x) - L| < \epsilon$ for $|x - 0| < \delta$, i.e.,$|f(x) - L| < 1/2$ if $|x| < \delta$.

   If $x \in (-\delta, 0)$, $f(x) = 0$, so $|L| < 1/2$, and $-1/2 < L < 1/2$. If $x \in (0, \delta)$, then $f(x) = 1$, then $|1 - L| < 1/2$, and so $1/2 < L < 3/2$. The two regions however are not over-lapping, so there we have a contradiction, and $f(x)$ has no limit as $x \to 0$.

Suppose $f(x)$ is defined for (a, b) and $a < c < b$, then $f(x)$ is <u>continuous</u> at c if $f(x) \to f(x)$ if $x \to c$. More formally, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x)| < \epsilon \qquad \text{for all} \qquad |x - c| < \delta.$$

Theorem IV.1 *Let $f(x)$ be continuous at $x = c$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence with $\lim_{x\to\infty} = c$. Then $f(x_n) \to f(x)$ as $n \to \infty$.*

Proof Let $\epsilon > 0$ be given, and we seek an N such that $|f(x_n) - f(x)| < \epsilon$ for all $n \geq N$. First, by assumption, there exists $\delta > 0$ such that $|f(x) - f(x)| < \epsilon$ for all $|x - c| < \delta$. Then, there exists N such that $|x_n - c| < \delta$ for all $n \geq$ also by assumption, so we can always choose δ for given ϵ since we can choose a big enough n.

Example Determine the continuity of the following functions:

1. $f(x) = x^2$ at $x = 1$ using ϵ-δ.

   For continuity, we want to show

   $$|f(x) - f(1)| = |x^2 - 1| = |x - 1| \cdot |x + 1| < \epsilon$$

   for small enough δ. Taking $\delta < 1$, if $|x - 1| < \delta$, then

   $$|x - 1| < 1 \qquad \Rightarrow \qquad 1 < x + 1 < 3.$$

   So we could take $\delta = \min\{\epsilon/3, 1\}$, i.e., $\delta \leq \epsilon/3$ and $\delta \leq 1$. Then we have $|x + 1| < 3$ and $|x - 1| > \epsilon/3$, so

   $$|x + 1| \cdot |x - 1| < 3(\epsilon/3) = \epsilon.$$

$f(x) = 1/x$ at $x = 1$ using ϵ-δ.
We want

$$|f(x) - f(1)| = \left|\frac{1}{x} - 1\right| = \left|\frac{x - 1}{x}\right| < \epsilon$$

given $|x - 1| < \delta$. Let $\delta \leq 1/2$, then if $|x - 1| < \delta$, we have $2 > 1/x > 2/3$, so we take $\delta = \min\{1/2, \epsilon/2\}$. This gives

$$\frac{1}{|x|} < 2 \qquad \text{and} \qquad |x - 1| < \epsilon/2,$$

so $|x - 1|/|x| < 2(\epsilon/2) = \epsilon$, so function is continuous at $x = 1$. (If we instead chose $\delta \leq 1$, we would have $|f(x) - f(1)| < \tilde{\epsilon} = \epsilon/2$, and the same conclusion holds.)

Theorem IV.2 *Let $f(x)$ and $g(x)$ be continuous at $x = c$, then:*

1. *$Af(x) + Bg(x)$ is continuous at $x = c$ for $A, B \in \mathbb{R}$;*
2. *$f(x)g(x)$ is continuous at $x = c$;*
3. *$f(x)/g(x)$ is continuous at $x = c$ for $g(x) \neq 0$ for all $x \in (c-k, c+k)$;*
4. *if $h(y)$ is continuous at $d = f(c)$, then $(h \circ f)(x) = h(f(x))$ is continuous at $x = c$.*

Proof All of these are fairly obvious except perhaps for the last one. Let $\epsilon > 0$ be given. We want to find an α such that

$$|h(y) - h(d)| < \epsilon \qquad \text{given} \qquad |y - d| < \alpha,$$

for $y = f(x)$, $d = f(x)$. So we want to find δ such that

$$|f(x) - f(c)| < \alpha \qquad \text{given} \qquad |x - c| < \delta.$$

We have $|x - c| < \delta$, so

$$|y - d| < \alpha \qquad \Rightarrow \qquad |h(f(x)) - h(f(c))| < \epsilon$$

as required.

Theorem IV.3 (Intermediate Value Theorem) *For $f : [a, b] \to \mathbb{R}$ that is continuous, if $f(a) < 0$ and $f(b) > 0$, there exists $c \in [a, b]$ such that $f(c) = 0$. (This may be obviously shifted for $f(c) = k$.)*

Proof We proceed by the bisection algorithm. Suppose we start with $a_1 = a$, and $b_1 = b$, then we look at $c = (b_1 - a_1)/2$. If $f(c) = 0$, we output this c. Else, if $f(c) > 0$, then we take $a_2 = a_1$, $b_2 = c$ and iterate, otherwise if $f(c) < 0$, we take $a_2 = c$, $b_2 = b_1$, and iterate; by construction, the initial assumptions are still satisfied at the iteration.

Either way, we either find c such that $f(c) = 0$ after a finite number of iterations, or we get $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$. By construction, these sequences satisfy the following properties:

1. $\{a_n\}$ and $\{b_n\}$ are increasing and decreasing sequences respectively;
2. $f(a_n) < 0$ and $f(b_n) > 0$ by construction;
3. by completeness of $\mathbb{R}$, $\{a_n\}$ is bounded above and $\{b_n\}$ is bounded below;
4. $b_n - a_n = (b - a)2^{1-n}$.

Then we clearly have $a_n \to L$ and $b_n \to K$, and

$$|L - K| = |L - a_n + a_n - b_n + b_n - K| \leq |L - a_n| + |b_n - a_n| + |b_n - K| \to 0$$

as $n \to \infty$, so $L = k$. Then since $f(a_n) \to f(L)$ and $f(b_n) \to f(K)$ as $n \to \infty$ by continuity, but $f(a_n) < 0$ and $f(b_n) > 0$, $f(L) \leq 0$ and $f(K) \geq 0$, with $L = K$, we have $f(L) = 0$, and we output this L as our c.

Theorem IV.4 *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.*

Proof We proceed again by bisection to look for a contradiction. suppose f satisfies the assumptions but is not bounded, the, without loss of generality, suppose f is not bounded above on $[a, b]$. Let $a_1 = a$ and $b_1 = b$ and bisect to find the $c \in [a, b]$ value where $f(c)$ is unbounded. Doing the iteration, we obtain sequences $a_n, b_n \to c$ where a_n and b_n satisfies the conditions above. If the function is not bounded above, there exists $x_n \in [a_n, b_n]$ such that $f(x_n) > n$ for all n. With this, $\{f(x_n)\}$ cannot have a limit, but since f is continuous, $f(x_n) \to f(c)$ as $n \to \infty$, which is a contradiction. A similar argument for boundedness below shows that f is bounded.

Theorem IV.5 *If $f : [a, b] \to \mathbb{R}$ is continuous, then $\sup\ f$ is attained, i.e., there exists $c \in [a, b]$ such that $f(c) = \sup\ f(x)$. (Similarly for $\inf\ f$.)*

Proof There exists $\sup\ f$ by the previous theorem. Suppose that $\sup\ f$ is not attained, therefore $f(x) < \sup\ f$ for all $x \in [a, b]$. Let $g(x) = (\sup\ f - f(x))^{-1}$. Since $f(x)$ is continuous, $\sup\ f - f(x)$ is continuous and positive, so $g(x)$ is continuous and well-defined. Applying the previous theorem to g shows that g is bounded also, so for $g(x) < K$ for all $x \in [a, b]$, we have

$$\sup\ f - f(x) > \frac{1}{K} \qquad \Leftrightarrow \qquad f(x) < \sup\ f - \frac{1}{K}.$$

This violates the definition of the supremum, so we have a contradiction, and the supremum is attained.

Let $f(x)$ be defined on $x \in (b, c)$. Then we say $f(x) \to L$ as $x \searrow b$ (x tending to b from the right/above) if, given $\epsilon > 0$, there exists δ such that $|f(x) - L| < \epsilon$ for $x \in (b, b + \delta)$. Similar,y we have $x \nearrow c$ (x tending to b from the left(below).

Theorem IV.6 *If $f(x)$ is defined on $x \in (a, b) \cup (b, c)$, then*

$$\lim_{x \to b} f(x) = L \qquad \textit{iff} \qquad \lim_{x \searrow b} f(x) = \lim_{x \nearrow b} f(x) = L.$$

Proof Essentially follows by definition.

Theorem IV.7 *If $f(x)$ is an increasing function which is bounded above on (a, b), then $f(x) \to L = \sup\ f$ as $x \nearrow b$.*

Proof There exists $\sup\ f$ by completeness of $\mathbb{R}$, with $f(x) \le \sup\ f = L$ for all $x \in (a, b)$. Let $\epsilon > 0$ be given. Since $L - \epsilon$ is not an upper bound, there exists $x_0 \in (a, b)$ such that $f(x_0) > L - \epsilon$. Taking $\delta = b - x_0$, then for $x \in (b - \delta b)$, we have

$$L - \epsilon < f(x_0) \le f(x) \le g < L + \epsilon,$$

so $|f(x) - L| < \epsilon$, as required.

B. Differentiation

A function $f(x)$ is <u>differentiable</u> at $x = c$ if there exists

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

or, equivalently,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

In terms of ϵ-δ, $f(x)$ is differentiable at $x = c$ with derivative $f'(c)$ there if, given $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon \qquad \text{for all} \qquad |x - c| < \delta.$$

Theorem IV.8 *If f is differentiable at $x = c$, then f is continuous at $x = c$.*

Proof

$$|f(x) - f(c)| = |x - c| \left| \frac{f(x) - f(c)}{x - c} \right| \to 0 \cdot f'(c) = 0$$

as $x \to c$, so f is continuous at $x = c$ since $f'(c)$ is finite by assumption.

Note that continuity does not imply differentiability.

Theorem IV.9 (Fundamental theorem of calculus) *If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\, \mathrm{d}t$ is differentiable for $x \in (a, b)$ and $F'(x) = f(x)$.*

Proof Let $c \in (a, b)$ and f be continuous. Then, by definition, given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $|x - c| < \delta$. For $0 < h < \delta$,

$$\left| \frac{F(c + h) - F(c)}{h} - f(c) \right| = \left| \frac{1}{h} \left(\int_a^{c+h} f(t)\, \mathrm{d}t - \int_a^c f(t)\, \mathrm{d}t \right) - f(c) \right| = \left| \frac{1}{h} \int_c^{c+h} f(t)\, \mathrm{d}t - f(c) \right|.$$

Now, $\int_c^{c+h} f(c)\, \mathrm{d}t = hf(t)$, so by absorbing the term into one integral, we have the inequality

$$\left| \frac{1}{h} \int_c^{c+h} (f(t) - f(c))\, \mathrm{d}t \right| \le \frac{1}{h} \int_c^{c+h} |f(t) - f(c)|\, \mathrm{d}t < \frac{1}{h} \int_c^{c+h} \epsilon\, \mathrm{d}t = \epsilon,$$

as required. A similar argument holds for $-\delta < h < 0$, so $F(x)$ is differentiable for $|h| < \delta$, and $F'(c) = f(c)$.

Remark Notice that we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^x f(t)\,\mathrm{d}t\right) = f(x),$$

but

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^b f(x,y)\,\mathrm{d}y\right) = \int_a^b \left(\frac{\partial}{\partial x} f(x,y)\right)\,\mathrm{d}y.$$

Theorem IV.10 (Leibniz rule) *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x).$$

Proof By definition,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}\\ &= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right]\\ &\to f(x)g'(x) + g(x)f'(x)\end{aligned}$$

because the functions are assumed to be differentiable.

A function f has a local maximum at $x = a$ if there exists $h > 0$ such that $f(x) \le f(a)$ for all $x \in (a-h, a+h)$. A similar definition holds for local minimum.

Theorem IV.11 *If f is differentiable at $x = a$ and f has a local extrema at $x = a$, then $f'(a) = 0$.*

Proof Suppose f has a local maximum at $x = a$, then

$$R(k) = \frac{f(a+k) - f(a)}{k} \to f'(a)$$

as $k \to 0$. For $k > 0$, $f(a+k) \le f(a)$ by assumption, so $R(k) \le 0$. Let $\{k_n\}$ be a positive decreasing sequence with $k_n \to 0$ as $n \to \infty$. So we have $R(k_n) \to f'(a) \le 0$ as $n \to \infty$. Similarly, for $k < 0$, $R(k) \ge 0$ because now the denominator is negative. With $\{k_n\} \to 0$ a negative increasing sequence, we have $R(k_n) \to f'(a) \ge 0$ as $n \to \infty$, so $f'(a) = 0$ for the individual one-sided limits to agree. A similar argument assuming for a local minimum shows at local extrema, $f'(a) = 0$.

Theorem IV.12 (Rolle's theorem) *Let f be continuous on $[a,b]$ and differentiable on (a,b), with $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof since f is continuous, f is bounded and the bounds are attained, i.e., there exists $c_1, c_2 \in [a,b]$ such that $f(c_2) \le f(x) \le f(c_1)$ for all $x \in [a,b]$. If $f(c_1) = f(c_2)$, then f is constant and $f'(x) = 0$ for all $x \in [a,b]$, and the conclusion holds trivially. If $f(c_1) > f(c_2)$, then c_1 and c_2 cannot simultaneous be end points (because $f(a) = f(b)$ by assumption). Taking c be one of thec_1 or c_2 which is in (a,b). If $c = c_1$, then $f(c)$ is the local maximum, otherwise if $c = c_2$, then $f(c)$ is a minimum minimum. In either case, $f'(c) = 0$ as required.

Theorem IV.13 (Mean value theorem) *Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ such that $f'(c) = (f(b) - f(a))/(b-a)$.*

Proof With $g(x) = f(x) - [(f(b) - f(a))/(b-a)]x$ we have $g(a) = g(b)$, and g satisfies the condition of Rolle's theorem, so $g'(c) = f'(c) - (f(b) - f(a)/(b-a)) = 0$, and we have the result.