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<https://github.com/julianmak/academic-notes>

The repository principally contains the compiled products rather than the source for size reasons.

- ▶ Associated Python code (as Jupyter notebooks mostly) will be held on the same repository. The source data however might be big, so I am going to be naughty and possibly just refer you to where you might get the data if that is the case (e.g. JRA-55 data). I know I should make properly reproducible binders etc., but I didn't...
- ▶ I do not claim the compiled products and/or code are completely mistake free (e.g. I know I don't write Pythonic code). Use the material however you like, but use it at your own risk.
- ▶ As said on the repository, I have tried to honestly use as content that is self made, open source or explicitly open for fair use, and citations should be there. If however you are the copyright holder and you want the material taken down, please flag up the issue accordingly and I will happily try and swap out the relevant material.

OCES 2003 : Descriptive Physical Oceanography

(a.k.a. physical oceanography by drawing pictures)

Lecture 4: forces and some mathematical background

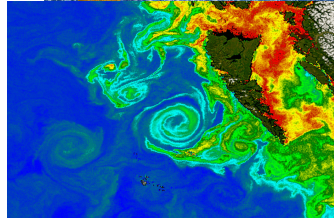
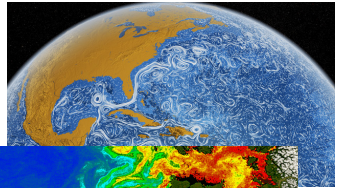
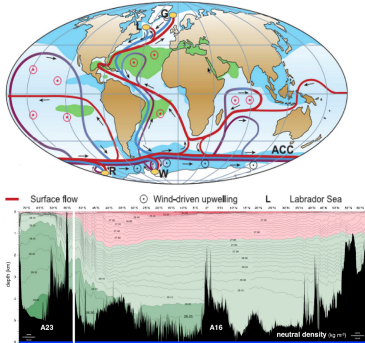
Thur 11th Feb

Outline

- ▶ concept of **forces** and forces acting on the ocean
 - **thermodynamic** (Solar, EmP, freshwater)
 - **mechanical** (wind, gravity, rotation, pressure etc.)
 - contrasts to the **atmosphere**
- ▶ (quick) review of some vector calculus concepts
 - **scalars** (e.g. p), **vectors** (e.g. \mathbf{u}), **dot** (\cdot) and **cross** (\times) product
 - **derivatives** ∇ (**gradients**, think rate of change)
 - **integral** \int (think sum)
 - **divergence** $\nabla \cdot$ (think di/convergence)
 - **curl** $\nabla \times$ (think spin)

Key terms: forces (thermodynamic + mechanical), gradients, grad/div/curl

Recap: features in ocean



- highlighted features in the ocean previously, but how/why do they arise?
 - focus on **dynamical** links and consequences
 - effectively **classical mechanics** + **fluid mechanics**

Forces + Newton's laws

Intuitively, things **move** when there are **forces** acting on it

- ▶ more precisely, objects are in **steady state** (at **rest** or **steady speed**) unless there is a **net force** (or **imbalance of forces**)

(this is **Newton's first law** essentially)



Figure: Forces acting on a (physicist joke: uniform point-mass, spherical) pig (not in a vacuum because we have air resistance + abuse of animal rights).

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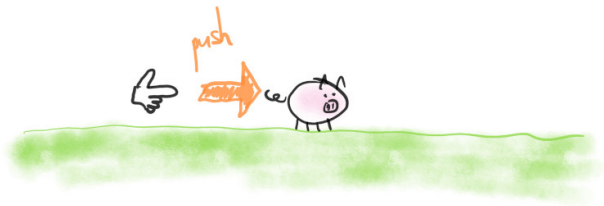


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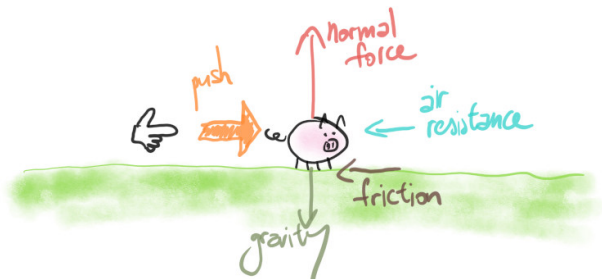


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- ▶ classical mechanics encapsulated by **Newton's laws**, which for our purposes is

$$F = m \frac{du}{dt}$$

- mass m (in units of kg)
- **velocity** u (units of m s^{-1})
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- ▶ forces directly affecting momentum I am going to call **mechanical forcing**
- ▶ **thermodynamic forcing** affects **density**, which has consequences for **momentum**

Forces acting on the ocean

What **external** forces are acting on the ocean?

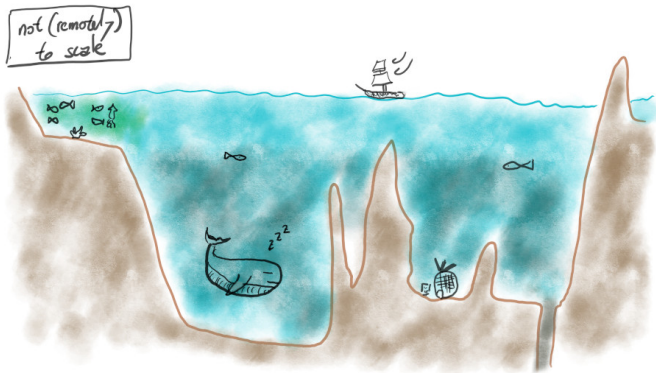


Figure: Schematic of ocean forcing.

Forces acting on the ocean

What **external** forces are acting on the ocean?

- temperature: **sun** + **radiation** (see Lec. 5 + OCES 4001)

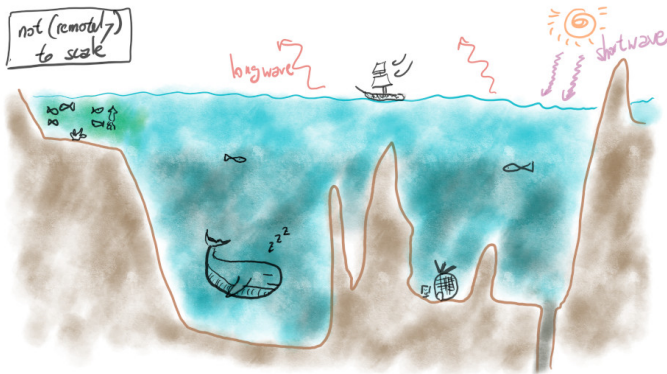


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Forces acting on the ocean

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- salinity: **river runoff**, **evaporation**, **precipitation** (see Lec. 5)

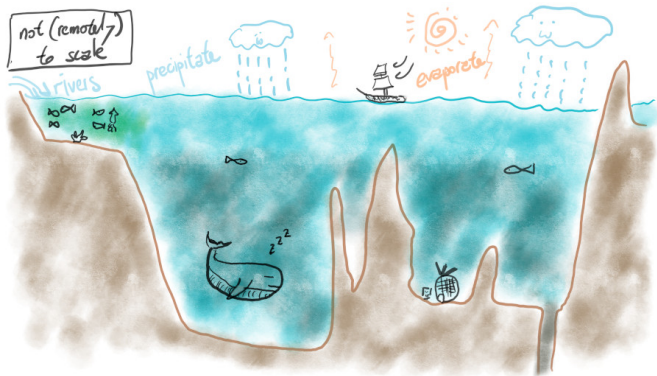


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What **external** forces are acting on the ocean?

- **momentum + vorticity: wind** (see Lec. 9)

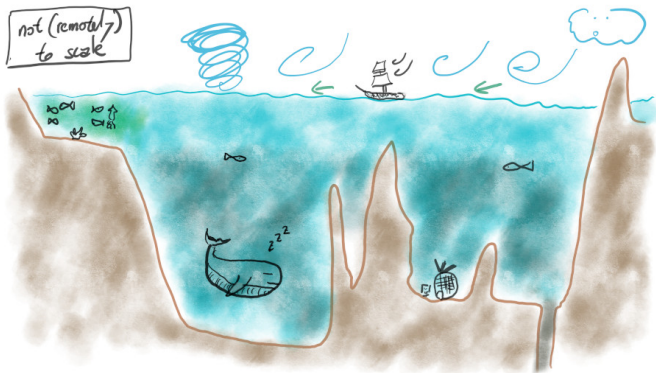


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Forces acting on the ocean

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- **geothermal flux** (mostly quite small, but see Lec. 13 + 14)

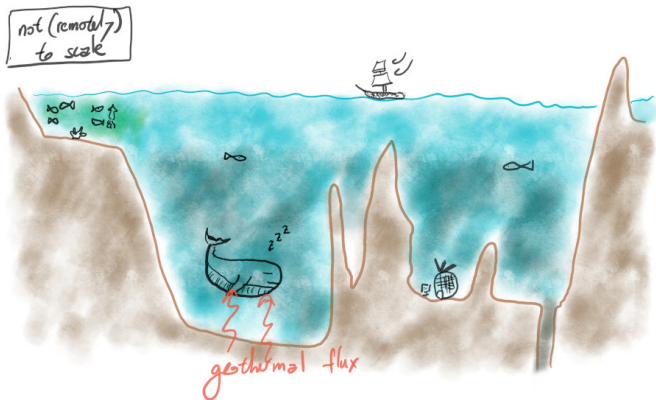


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How are these represented in models?

Equations of Motion (EOM)

Denoting $\mathbf{u} = (u, v)$ and $\mathbf{u}_3 = (u, v, w)$, to numerous approximations (!!!) (see OCES 3203) ocean dynamics is governed by

$$\rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} \right) = -\nabla p + \mathbf{F}_u + \mathbf{D}_u \quad (1)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad (2)$$

$$\nabla \cdot \mathbf{u}_3 = 0 \quad (3)$$

$$\left(\frac{\partial T}{\partial t} + \mathbf{u}_3 \cdot \nabla T \right) = F_T + D_T \quad (4)$$

$$\left(\frac{\partial S}{\partial t} + \mathbf{u}_3 \cdot \nabla S \right) = F_S + D_S \quad (5)$$

$$\rho = \rho(T, S, p) \quad (6)$$

Respectively, (1) **momentum equation**, (2) **hydrostatic balance**, (3) **incompressibility**, (4) **temperature equation**, (5) **salinity equation**, and (6) **equation of state (EOS)**

Equations of Motion (EOM)

Without **vector calculus** notation:

$$\rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - 2\Omega v \right) = -\frac{\partial p}{\partial x} + F_x + D_u \quad (7)$$

$$\rho_0 \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + 2\Omega u \right) = -\frac{\partial p}{\partial y} + F_y + D_v \quad (8)$$

$$\frac{\partial p}{\partial z} = -\rho g \quad (9)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (10)$$

$$\left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = F_T + D_T \quad (11)$$

$$\left(\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial y} + w \frac{\partial S}{\partial z} \right) = F_S + D_S \quad (12)$$

$$\rho = \rho(T, S, p) \quad (13)$$

Decipher these throughout the course...

Vector calculus crash course

Terms in equations tend to have geometric meanings

- ▶ can understand them by drawing pictures
- ▶ encoded best through **vector calculus**

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Disclaimer!!!

- ▶ focus on the **geometric meanings** of vector calculus
- ▶ you are **not examined** on computing integrals/derivatives, but you are expected to **understand/interpret** their meanings

Vector calculus concepts: scalars and vectors

Scalars

- are just numbers and only have a **magnitude**, e.g.

→ $g = 9.8 \text{ m s}^{-2}$

→ speed $|u| = 10 \text{ m s}^{-1}$

→ pressure $p = p(x, y, z)$

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Vectors (normally denoted with boldface \mathbf{u}_3 , underline \underline{u}_3 or arrow \vec{u}_3)

- ▶ have a **direction** and **magnitude**, e.g.
 - weight is mg **acting towards centre of gravity**
 - velocity $\mathbf{u} = 10\text{m s}^{-1}$ **going East**
 - pressure **gradient** ∇p **acting from South**

Vector calculus concepts: scalars and vectors

Remember this guy?



Figure: Victor Perkins (aka **Vector**) from Despicable Me 1, because he is “committing crimes with both direction and magnitude”. From Minion Rush, copyright with Universal Studios.

Vector calculus concepts: scalars and vectors

Scalars are just numbers so can do stuff to them as normal

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Vectors on the hand you can only

- ▶ add/subtract vectors to vectors (e.g. $\mathbf{a} + \mathbf{b} = \mathbf{c}$)
 - ▶ multiply/divide vector by scalar (e.g. $k\mathbf{a}$)
- note the resulting things are **vectors**

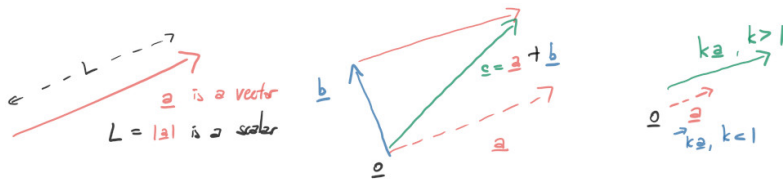


Figure: Schematics of elementary vector operations: (a) vector vs. scalar; (b) addition/subtraction of vector by vector; (c) multiplication of vector by scalar.

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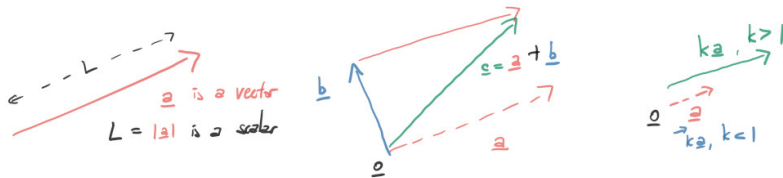


Figure: Schematics of elementary vector operations: (a) vector vs. scalar; (b) addition/subtraction of vector by vector; (c) multiplication of vector by scalar.

You do not multiply/divide vectors by vectors!

Vector calculus concepts: scalars and vectors

Representing a vector with a **basis**, e.g. the **standard** basis

$$\mathbf{e}_x = (1, 0, 0), \quad \mathbf{e}_y = (0, 1, 0), \quad \mathbf{e}_z = (0, 0, 1)$$

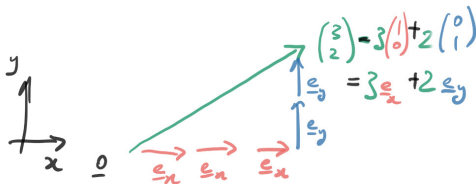


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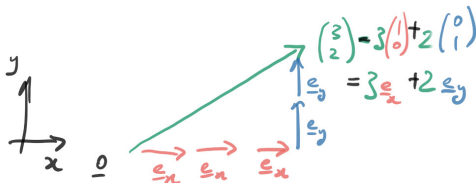


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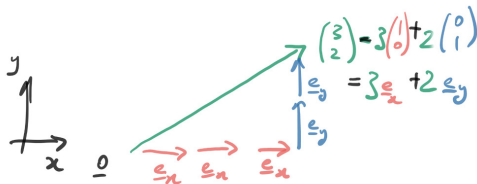


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- ▶ “e.g.” here because this is not the only choice of a basis (though it is the most convenient)
- ▶ **length** of vector $\mathbf{a} = (a_1, a_2, a_3)$ is then (just Pythagoras’ theorem...)

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vector calculus concepts: dot product (\cdot)

Two other things you can do to vectors \mathbf{a} and $\mathbf{b} = (b_1, b_2, b_3)$

► dot/scalar product

→ takes two vectors and returns a scalar as

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

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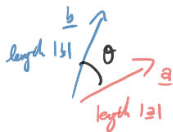
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$$\underline{\mathbf{a}} \cdot \underline{\mathbf{b}} = |\underline{\mathbf{a}}| |\underline{\mathbf{b}}| \cos \theta$$



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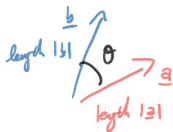
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$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$



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→ note that **length** of vector is $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$

→ if $\mathbf{a} \cdot \mathbf{b} = 0$ then the two vectors are **perpendicular**

(recall $\cos 90^\circ = \cos \pi/2 = 0$)

Vector calculus concepts: cross product (\times)

- ▶ the **cross** product takes two vectors and returns a third vector $\mathbf{c} = (c_1, c_2, c_3)$ with

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_3 \end{pmatrix}$$

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- ▶ resulting \mathbf{c} follows **right-hand-screw convention**

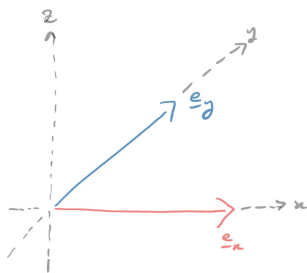


Figure: Right-hand-screw convention in $\mathbf{a} \times \mathbf{b} = \mathbf{c}$: resulting vector \mathbf{c} points in direction of trying to turn/screw \mathbf{a} into \mathbf{b} using the right hand (clockwise motion).

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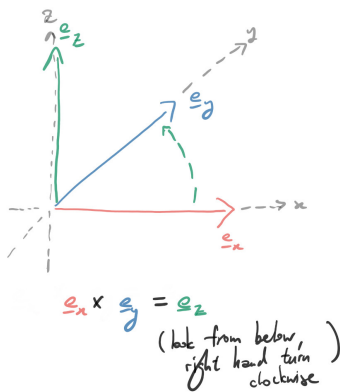


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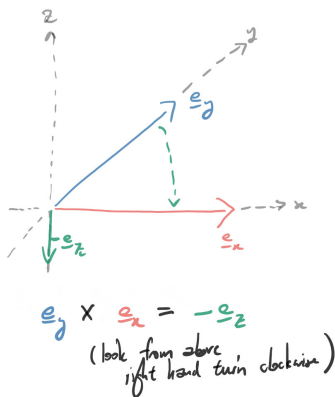


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Scalar/vector **field** is when the scalar/vector is a function, e.g.

$$p = p(x, y, z) = xy^2z^3, \quad \mathbf{u}_3 = (u, v, w) = (x, y, 1) = x\mathbf{e}_x + y\mathbf{e}_y + 1\mathbf{e}_z,$$

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 - \mathbf{u}_3 is a **vector field**
 - u is a **scalar field** and the x -component of \mathbf{u}_3
- ▶ x, y, z you can/should think as **EW**, **NS** and **up/down**
 - called **zonal**, **meridional** and **vertical** direction

Vector calculus concepts: scalar/vector fields

Scalar/vector **field** is when the scalar/vector is a function, e.g.

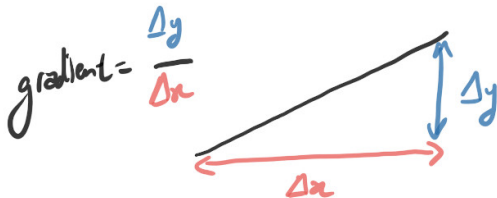
$$p = p(x, y, z) = xy^2z^3, \quad \mathbf{u}_3 = (u, v, w) = (x, y, 1) = x\mathbf{e}_x + y\mathbf{e}_y + 1\mathbf{e}_z,$$

In terms of default notation for this course:

- ▶ vectors will be **bold** (e.g. \mathbf{u}_3)
 - if you need to write it by hand, use underlined (e.g. \underline{u}_3)
 - \mathbf{u}_3 is a **vector field**
 - u is a **scalar field** and the x -component of \mathbf{u}_3
- ▶ x, y, z you can/should think as **EW, NS** and **up/down**
 - called **zonal**, **meridional** and **vertical** direction
- ▶ $x, y, z > 0$ is East, North and up
 - $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ points East, North and up
 - $u, v, w > 0$ is East, North and upward velocity

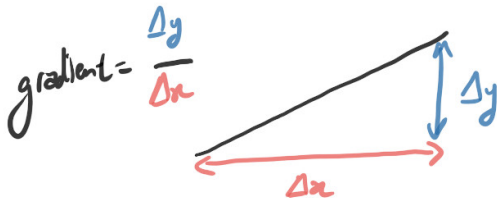
Vector calculus concepts: gradients

Gradient of something **straight** (linear) is simply



Vector calculus concepts: gradients

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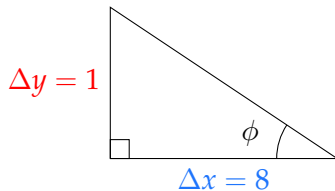
- ▶ gradient $> 0 \leftrightarrow$ “up”-slope
- ▶ gradient $< 0 \leftrightarrow$ “down”-slope
- ▶ think **rate of change**

Vector calculus concepts: gradients



Figure: Image from HK transport department (www.td.gov.hk), the kind of sign you see around Clear Water Bay Road quite a bit...

- what 1 : 8 here means is



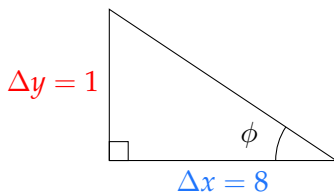
→ every 8 steps across
move up/down 1

Vector calculus concepts: gradients



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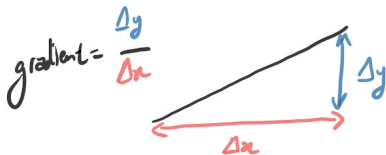


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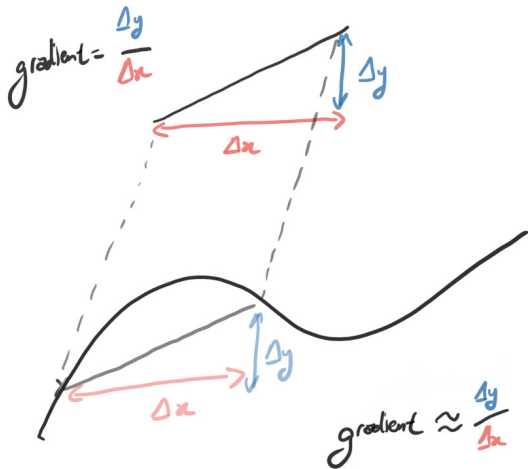
- ▶ for completeness,

$$\phi = \arctan \frac{1}{8} \approx 7^\circ$$

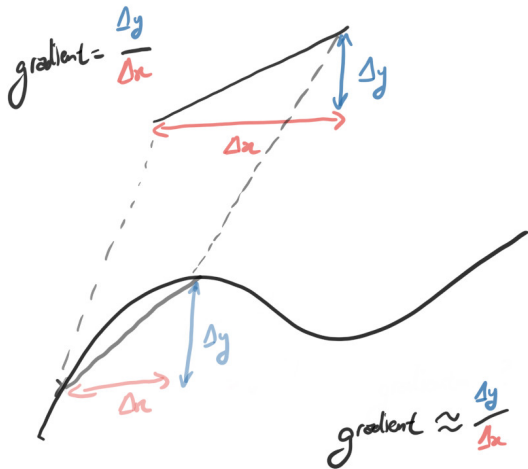
Vector calculus concepts: derivatives (d/dx)



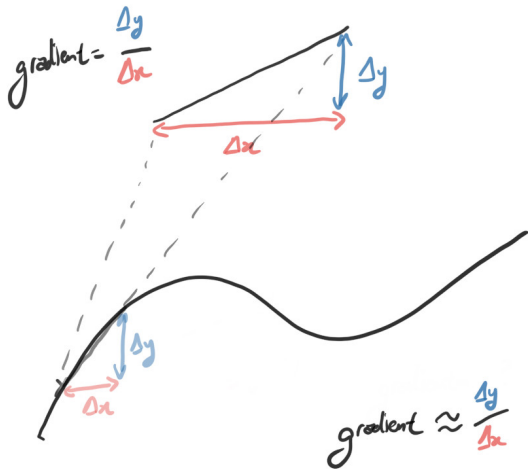
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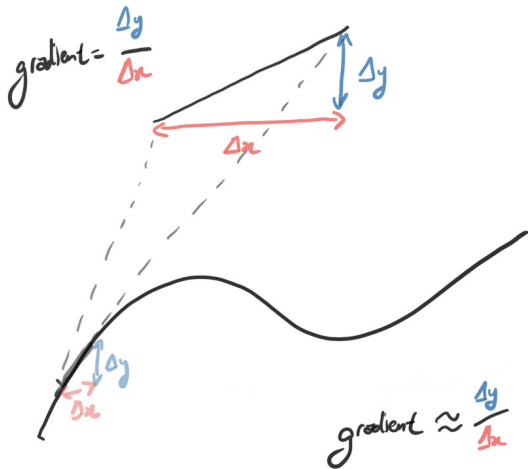
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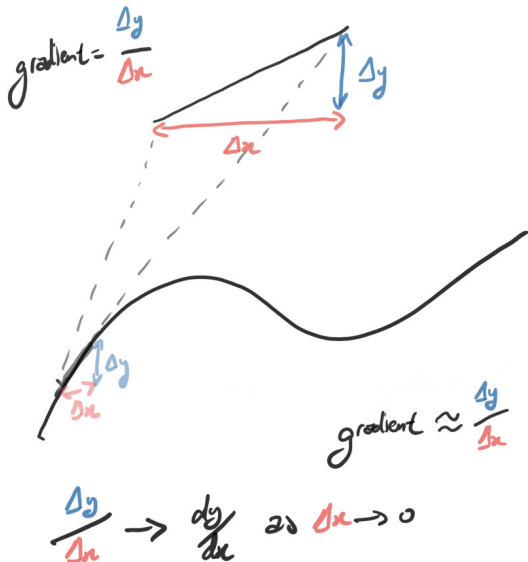
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Vector calculus concepts: derivatives (d/dx)



Vector calculus concepts: derivatives (d/dx)

The **derivative**

$$\frac{d}{d(\text{something})}$$

is really just the **gradient** (again, think **rate of change**)

Some examples:

- ▶ for p the pressure depending on only the depth z ,

$$p = z^3, \quad \Rightarrow \quad \frac{dp}{dz} = 3z^2$$

means “**rate of change of pressure with depth** is $3z^2$ ”

- ▶ rate of change of $p\text{CO}_2$ concentration with **(in-situ)** temperature T would be

$$\frac{d}{dT}[p\text{CO}_2]$$

Vector calculus concepts: partial derivatives ($\partial/\partial x$)

If something depends on multiple variables (e.g. $p = p(x, y, z)$ or $[\text{CO}_2] = [p\text{CO}_2](T, p, \text{fish}, \dots)$) then sometimes we talk about the **partial derivative**

$$\frac{\partial}{\partial(\text{something})}$$

(again these are just gradients)

Some examples:

- for $p = p(x, y, z)$ the pressure,

$$\frac{\partial p}{\partial x}, \quad \frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial z},$$

are respectively “**the rate of change of pressure with $x/y/z$ keeping the other variables fixed**”

Vector calculus concepts: partial derivatives ($\partial/\partial x$)

Some examples:

- ▶ for $p = p(x, y, z) = xy^2z^3$ the pressure and x, y the horizontal co-ordinate, then

$$\frac{\partial p}{\partial x} = y^2z^3 \frac{d}{dx}x = y^2z^3, \quad \frac{\partial p}{\partial y} = xz^3 \frac{d}{dy}y^2 = 2xyz^3,$$

$$\frac{\partial p}{\partial z} = xy^2 \frac{d}{dz}z^3 = 3xy^2z^2,$$

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→ the partial derivative hits the variable of interest and leaves others alone

- ▶ if (say) $[p\text{CO}_2] = T^3 \cos(p)$ then (assuming neither T nor p depend on fish at all...)

$$\frac{\partial [p\text{CO}_2]}{\partial (\text{fish})} = \dots?$$

Vector calculus concepts: integrals (\int)

The **integral** \int can be thought of as the opposite of the **derivative**, e.g.,

$$p(z) = z^3, \quad \frac{dp}{dz} = 3z^2, \quad \int 3z^2 \, dz = z^3 + \underline{\text{constant}}$$

A **definite integral** evaluates difference of integrated quantity at the limits, e.g.,

$$\int_0^3 3z^2 \, dz = [z^3]_0^3 = 3^3 - 0^3 = 27$$

- no constant because $\text{constant}(z = 3) - \text{constant}(z = 0) = 0$

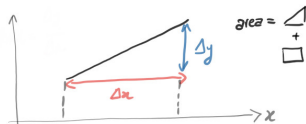
Vector calculus concepts: integrals (\int)

- ▶ think **integral** \int as a **sum**
→ “hence” elongated S
- ▶ sum the function over a particular interval, e.g.

$$\int_{z_1}^0 \rho \, dz$$

is the sum of density (as a **scalar** field) from z_1 to sea surface denoted $z = 0$

→ see this again in
hydrostatic balance (see Lec 7)



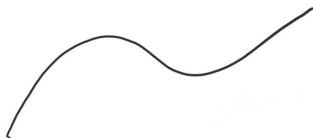
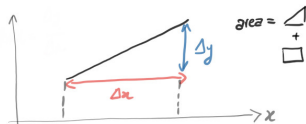
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area = ?

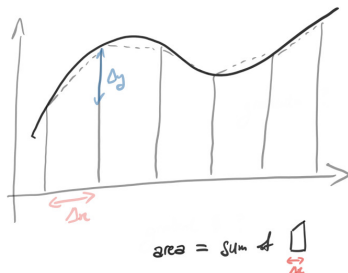
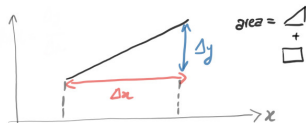
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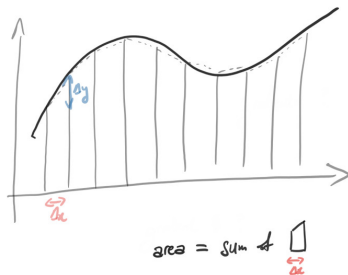
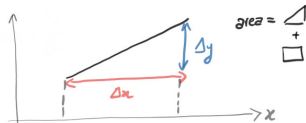
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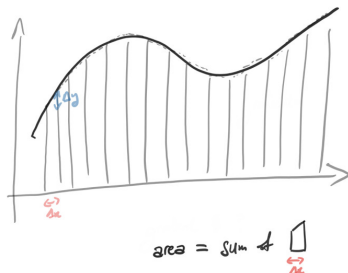
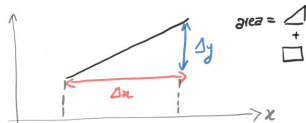
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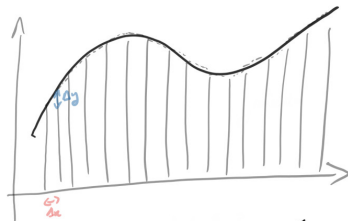
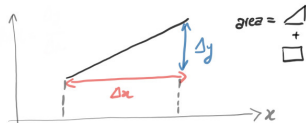
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$$\text{area} = \text{sum of } \square$$

$$\text{area} \rightarrow \int y(x) \, dx \quad \text{as } \Delta x \rightarrow 0$$

Vector calculus concepts: grad (∇)

The **gradient operator** ∇ (called “grad” or “nabla”)

- ▶ acts on a **scalar** field and returns a **vector** field as, e.g.,

$$\nabla p = \frac{\partial p}{\partial x} \mathbf{e}_x + \frac{\partial p}{\partial y} \mathbf{e}_y + \frac{\partial p}{\partial z} \mathbf{e}_z = \begin{pmatrix} \partial p / \partial x \\ \partial p / \partial y \\ \partial p / \partial z \end{pmatrix}$$

- ▶ encodes how a **scalar** field (e.g. p) changes in physical space

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again, really just gradients

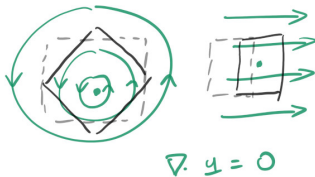
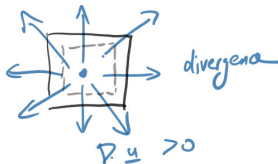
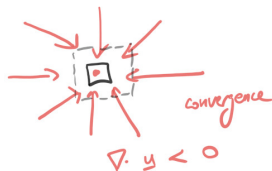
Vector calculus concepts: divergence ($\nabla \cdot$)

The **divergence** of a vector field $\mathbf{u}_3 = (u, v, w)$ is

$$\nabla \cdot \mathbf{u}_3 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

→ note $\nabla \cdot \mathbf{u}$ is a **scalar** field

- measures **con/divergence** (cf. compression/expansion) of a vector field
- strongly linked to **up/downwelling** (see Lec 9)



Vector calculus concepts: curl ($\nabla \times$)

The **curl** of a vector field

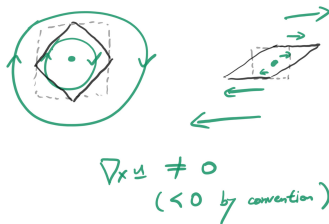
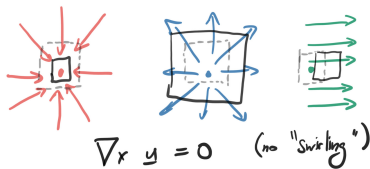
$\mathbf{u}_3 = (u, v, w)$ is denoted

$\nabla \times \mathbf{u}_3 =$ see Wikipedia

→ note $\nabla \times \mathbf{u}_3$ is a
vector field

- ▶ measures direction and strength of **swirl/spin**-iness
- ▶ important concept for **rotating systems**, **wind-driven gyre circulation**, **eddies** etc...

(see basically Lec 8 onwards...)



Summary

- ▶ at heart of physical oceanography is some form of

$$F = ma = m \frac{du_3}{dt}$$

→ residual force \sim mass times acceleration = mass times rate of change of velocity

→ force and acceleration/velocity are vector fields here, mass is a scalar

→ understand the forces contribute = understand how/why the fluid behaves the way it does (in principle, doesn't mean it's easy...)

Summary

- ▶ dealing with scalars and scalar/vector fields
→ **vector calculus**: the language to talk about these objects

term	note	symbols
scalars	magnitude only	$u, \mathbf{u} p$
vectors	magnitude and direction	$\mathbf{u}_3 = (u, v, w), \nabla p$
dot product	angles, lengths	\cdot
cross product	generates a 3rd vector	\times
derivative	gradients / rate of change	d, ∂, ∇
integral	sum	\int
div(ergence)	di/convergence	$\nabla \cdot (\cdot)$
curl	swirl/spin-iness	$\nabla \times (\cdot)$

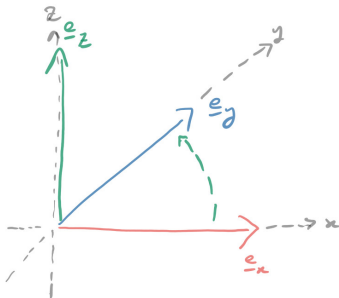
ability to do vector calculus not examined in this course

- ▶ but understanding/interpreting them is part of the course

Summary

Default notation for this course:

- ▶ x, y, z you can/should think as **E-W**, **N-W** and **up-down**
→ **zonal**, **meridional** and **vertical**
- ▶ $x, y, z > 0$ is East, North and up
→ $\underline{e_x}, \underline{e_y}, \underline{e_z}$ points East, North and up
→ $u, v, w > 0$ is East, North and upward velocity



$$\underline{e_x} \times \underline{e_y} = \underline{e_z}$$

(look from below, right hand turn clockwise)