Complex Analysis 2H

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- *Last compiled: May 2021*
- Blended from notes of R. Gregory and J. Bolton, Durham
- This was part of the Durham core second year modules. Involves more things to do with analysis in the complex plane, involving holomorphic functions, contour integrals, residue theorems, conform mappings, etc.
- The original course does not have geometry of complex numbers since that was covered in Core A (Geometry 1A), but for consistency reasons this has been moved here.

- **TODO!** Diagrams to do

1 Geometry of complex numbers

1.1 Complex numbers and the Argand diagram

We define $\sqrt{-1} = \mathrm{i}$, which is the basic unit imaginary number. A **complex number** is then a combination of real and imaginary parts $z = a + b\mathrm{i}$, with $a, b \in \mathbb{R}$. The complex numbers $\mathbb{C}$ then obeys the same axioms for addition and multiplication as $\mathbb{R}$ (both are **fields**).

Consider instead $\mathbb{C}$ as a vector space $z = (x, y)$, where multiplication is defined on $\mathbb{R}^2$ as

$$z_1 \times z_2 = (x_1x_2 - y_1y_2, x_1y_2 - x_2y_1),$$

and this is commutative. $1 = (1, 0)$ is the identity. So we see that $\mathbb{R}^2$ with this multiplication is a concrete visualisation of $\mathbb{C}$, and is called the **Argand diagram**.

Given $z = x + \mathrm{i}y$, the **conjugate** of z is defined to be $\overline{z} = x - \mathrm{i}y$. Geometrically, this represents a reflection of z in the 'real' axis. The **real** and **imaginary** part of z is given respectively by

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2}.$$

In polar form, $z = r(\cos\theta + \mathrm{i}\sin\theta)$. r is called the **modulus** of z and is denoted $|z|$, whilst θ is called the **argument** of z, denoted $\arg(z)$.

1.2 Geometry of addition and multiplication in $\mathbb{C}$

Addition is as in $\mathbb{R}^2$. From this, we can deduce the **triangle inequality**.

Lemma 1.2.1 *For $z_1, z_2 \in \mathbb{C}$, $|z_1 + z_2| \leq |z_1| + |z_2|$, and we have an equality iff $\arg(z_1) = \arg(z_2)$. By corollary, we have $|z_2 + z_2| \geq ||z_1| - |z_2||$.*

Proof Without loss of generalisation, let $|z_1| > |z_2|$, then $|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |z_2|$ by the triangle inequality for real numbers. So $|z_1| - |z_2| \leq |z_1 + z_2|$, and since $|z_1| > |z_2|$, we have the corollary of the result as required. ■

For multiplication, we observe that $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. Geometrically, this is a spiral scaling.

We can use the $\mathbb{C}$-plane to describe various geometrical objects.

Example A circle may be described by $|z - z_0| = a$, where z_0 is the centre of the circle and a is the radius; expanding this accordingly, we see that $a^2 = (x - x_0)^2 + (y - y_0)^2$.

Example The equation $|z - x_0| + |z + x_0| = 2r$ describes an ellipse, where $r > |x_0|$. This may be done via expansion in (x, y). Alternatively, in polar form, we observe that, for $z = a + ib$, $|z \pm x_0|^2 = (a^2 - b^2)\cos^2\theta \pm 2ax_0\cos\theta + (x_0^2 + b^2)$. If $x_0^2 = (a^2 - b^2)$, then this may be simplified to $|z \pm x_0| = a \pm x_0 \cos\theta$ since $a > x_0$. With this, we obtain $|z - x_0| + |z + x_0| = 2a$, thus, with $x = a\cos\theta$ and $y = b\sin\theta$, this describes an ellipse.

Example The locus of $|z - z_1| = |z - z_2|$ describes the line that is equidistant to the points z_1 and z_2. To see this, expanding everything in x and y and we obtain the equality

$$x(x_2 - x_1) + y(y_2 - y_1) = \frac{y_2^2 - y_1^2}{2} + \frac{x_2^2 - x_1^2}{2},$$

and the normal to the line is $z_2 - z_1$.

1.3 *de Moivre's theorem*

Theorem 1.3.1 (de Moivre's theorem) *For all $n \in \mathbb{Z}^+$ and angle θ, $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$.*

Proof We do this by induction. The $n = 1$ case is trivial, so, assuming it is true for n, then we observe that

$$\begin{aligned}
&\cos(n+1)\theta + \mathrm{i}\sin(n+1)\theta \\
&\qquad = \cos n\theta\cos\theta + \mathrm{i}^2\sin\theta\sin n\theta + \mathrm{i}\sin n\theta\cos\theta + \mathrm{i}\sin\theta\cos n\theta \\
&\qquad = (\cos n\theta + \mathrm{i}\sin n\theta)(\cos\theta + \mathrm{i}\sin\theta) \\
&\qquad = (\cos\theta + \mathrm{i}\sin\theta)^{n+1}.
\end{aligned}$$

■

Example Since

$$\cos 2\theta + \mathrm{i}\sin 2\theta = (\cos\theta + \mathrm{i}\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + \mathrm{i}(2\sin\theta\cos\theta),$$

and remembering the double angle formulae, the equality agrees.

From de Moivre's theorem, we see that

$$\cos n\theta = \mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^n, \qquad \sin n\theta = \mathrm{Im}(\cos\theta + \mathrm{i}\sin\theta)^n.$$

We can also use the theorem to find sin or cos of rational multiples of π.

Example Express $\sin 4\theta / \cos\theta$ as a polynomial in $\sin\theta$, and hence find $\sin(\pi/4)$.

$$\begin{aligned}\sin 4\theta = \operatorname{Im}(\cos\theta + \mathrm{i}\sin\theta)^4 &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta \\ &= 4\cos\theta(\sin\theta - 2\sin^3\theta),\end{aligned}$$

so $\sin 4\theta/\cos\theta = 4\sin\theta(1-2\sin^{\theta})$. Evaluating this $\pi/4$, we see that the LHS is zero. Now, $4\sin(\pi/4) > 0$, so we conclude that $\sin(\pi/4) = 1/\sqrt{2}$, as expected.

Example Find $\cos(k\pi/6)$ for $k = 1, 2, 3, 4, 5$.

Letting $c = \cos\theta$ and $s = \sin\theta$, observe that

$$\sin 6\theta = sc(6c^4 + 6s^4 - 20s^2c^2) = sc(32c^4 - 32c^2 + 6) = 2sc(4c^3 - 3)(4c^2 - 1).$$

Now, $\sin(k\pi) = 0$, so LHS is zero, and since $\sin(k\pi/6) \neq 0$, we have

$$\cos^2(k\pi/6) = 3/4, \quad \cos^2(k\pi/6) = 1/4, \quad \cos\theta = 0,$$

which implies that

$$\cos(k\pi/6) = \pm\sqrt{3}/2,\ \pm 1/2,\ 0.$$

Since $\cos\theta$ is a decreasing function in $[0,\pi]$, we have

$$\begin{aligned}\cos(\pi/6) = \sqrt{3}/2, \quad \cos(2\pi/6) = 1/2, \quad \cos(\pi/2) = 0,\\ \cos(2\pi/3) = -1/2, \quad \cos(5\pi/6) = -\sqrt{3}/2.\end{aligned}$$

1.4 *Imaginary exponentials*

de Moivre's theorem hints at a deeper geometric significance of cosine and sine functions and a way of encoding multiplication by imaginary numbers. Suppose $f(\theta) = \cos\theta + \mathrm{i}\sin\theta$, then we notice that $f'(\theta) = \mathrm{i}f(\theta)$, and, more generally, $f^{(n)}(\theta) = \mathrm{i}^n f(\theta)$. We know that also that the n-th derivative of $\mathrm{e}^{\lambda x}$ is $\lambda^n \mathrm{e}^{\lambda x}$, so this suggests a link with exponential functions; indeed, we have **Euler's formula**

$$\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}. \tag{1.1}$$

By de Moivre's theroem then,

$$r(\cos n\theta + \mathrm{i}\sin n\theta) = r(\cos\theta + \mathrm{i}\sin\theta)^n = r\mathrm{e}^{\mathrm{i}n\theta}.$$

Lemma 1.4.1 (Euler identity) $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$. □

Example Find all the roots of $z^6 + 4z^3 + 8 = 0$.

Factorising the above gives $z^3 = -2 \pm 2\mathrm{i}$. So since $|z^3| = 2\sqrt{2}$, we have $|z| = \sqrt{2}$. Now,

$$\arg(-2+2\mathrm{i}) = \frac{3\pi}{4}, \qquad \arg(-2-2\mathrm{i}) = \frac{5\pi/4}{,}$$

and the argument of the roots z satisfies

$$\arg(z) = \frac{3\pi/4 + 2n\pi}{3}, \qquad \arg(z) = \frac{5\pi/4 + 2n\pi}{3},$$

where the division by 3 is to take into account the cube root, and the $2n\pi$ factors is to account for all the roots. This eventually yields

$$z = \sqrt{2}(\mathrm{e}^{\mathrm{i}\pi/4}, \mathrm{e}^{5\mathrm{i}\pi/4}, \mathrm{e}^{11\mathrm{i}\pi/12}, \mathrm{e}^{13\mathrm{i}\pi/12}, \mathrm{e}^{19\mathrm{i}\pi/12}, \mathrm{e}^{21\mathrm{i}\pi/21}).$$

2 Basics of complex functions

A real function can for example be once differentiable, but not twice. One example is $f(x) = x|x|$, where $f'(x)$ is not differentiable at $x = 0$.

Theorem 2.0.1 *If a complex function is once differentiable, it is differentiable as many times as you like.*

It is possible for two real functions to agree on an interval but not everywhere, assuming they are differentiable. One example is $f(x) = x|x|$ and $g(x) = x^2$ for $x > 0$.

Theorem 2.0.2 *If two complex differentiable functions agree on any disc in the complex plane, then they agree everywhere (subject to certain conditions...)*

Recall that a real function assigns any real number x to at most one real number (i.e. it is injective). A **complex function** therefore assigns any complex number z to at most one complex number. These include standard polynomials, rational functions, transcendental functions, trigonometric functions, hyperbolic functions, where the argument is in z. Some examples have already been given above.

Example Solve $e^z = 1$.

Writing $z = x + iy$ and using Euler's formula,

$$e^x(\cos y + i \sin y) = 1,$$

and equating real and imaginary parts lead to

$$e^x \cos y = 1, \qquad e^x \sin y = 0.$$

Considering the imaginary part, since $e^x \neq 0$, $y = n\pi$ for $n \in \mathbb{Z}$, but from the real part, since $e^x > 0$ and $\cos n\pi = \pm 1$, we should only have $y = 2n\pi$ for $n \in \mathbb{Z}$. The real part then additionally implies that $x = 0$ since $\cos 2n\pi = 1$, so $z = 2in\pi$ for $n \in \mathbb{Z}$.

Note that $|e^{iz}| \geq 0$ for all $z \in \mathbb{C}$.

Example Solve $\sin z = 0$.

With the standard identity for sine with complex arguments, we have

$$\frac{e^{iz} - e^{-iz}}{2i} = 0.$$

Equating real and imaginary parts lead to $z = m\pi$, $m \in \mathbb{Z}$.

The (natural) **logarithm** we define by

$$\log z = \log |z| + i\arg z \tag{2.1}$$

to give a complex version of the log function that satisfies the usual rules of

$$\log z = \log re^{i\theta} = \log r + i\theta = \log |z| + i\arg z.$$

Here we need to choose a **branch**, and we take $\theta \in (-\pi, \pi)$ (the **principal branch**) to preserve the continuity property, so that $\log z$ is undefined on the negative real axis, coinciding with the real case.

Example $\log(1 - i) = \log \sqrt{2} - i(\pi/4)$

We use $\log z$ to define powers of complex numbers. Recall that for real numbers we have $x^a = e^{a \log a}$ for $a > 0$, so for $z, w \in \mathbb{C}$, we analogously define

$$z^w = e^{w \log z}, \tag{2.2}$$

choosing the principal branch unless otherwise stated.

Example

$$\begin{aligned}
(1 + i\sqrt{3})^{1/2} &= \exp\left[\frac{1}{2}\log(1 + i\sqrt{3})\right] \\
&= \exp\left[\frac{1}{2}\left(\log 2 + i\frac{\pi}{3}\right)\right] \\
&= e^{\log\sqrt{2}} e^{i(\pi/6)} \\
&= \sqrt{2}e^{i(\pi/6)},
\end{aligned}$$

which in this case is could have been gotten from $(1 + i\sqrt{3}) = 2e^{i(\pi/3)}$.

Example

$$(1 - i)^i = e^{i\log(1-i)} = e^{i(\log\sqrt{2} - i\pi/4)} = e^{\pi/4} e^{i\log\sqrt{2}}.$$

We say a complex function $f(z)$ is **complex differentiable at** $z = z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, or that

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists at $z = z_0$. The derivative is denoted $f'(z)$ as usual.

Example For $f(z) = z^2$,

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{z^2 + 2hz + h^2 - z^2}{h} = \lim_{h\to 0} 2z + h = 2z.$$

$f(z)$ is differentiable everywhere.

The usual rules for differentiation hold (linearity, product rule, chain rule etc.)

Note that $f(x) = x|x|$ is real differentiable everywhere. $f(z) = z|z|$ on the other hand is differentiable on the real axis, and complex differentiable at the origin.

Complex differentiation is a much stronger condition. Recall that for the limit to exist in the real case, the limit only needs to be equal when approached from above or below on the real line. In the complex plane however there are an infinite numbers of cases the limit can be approach, and thus a infinite number of cases to check. We see that a necessary condition for complex differentiability is that the limit needs to exist when z_0 is approached in the lines parallel to the real and imaginary axis. If we set $f(z)$ to be

$$f(z) = u(x,y) + \mathrm{i}v(x,y)$$

for some real functions u and v, then it turns out that

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y},$$

when we take the limit in the direction parallel to the real and imaginary axis respectively. It follows that a *necessary* conditions for a function to be complex differentiable is that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{2.3}$$

These are known as the **Cauchy–Riemann equations**, and we actually have the following theorem.

Theorem 2.0.3 *If $f(z)$ is complex differentiable at $z = z_0$, then the Cauchy–Riemann equations hold at (x_0, y_0) for $z_0 = x_0 + \mathrm{i}y_0$, and that*

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)\bigg|_{(x_0,y_0)} = \left(\frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}\right)\bigg|_{(x_0,y_0)}.$$

Proof If we approach z_0 in a line parallel to the real axis, we have

$$\begin{aligned}
f'(z_0) &= \lim_{x\to x_0} \frac{u(x,y_0) + \mathrm{i}v(x,y_0) - u(x_0,y_0) - \mathrm{i}v(x_0,y_0)}{x - x_0} \\
&= \lim_{x\to x_0} \left(\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0} + \mathrm{i}\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}\right) \\
&= \frac{\partial u}{\partial x}\bigg|_{(x_0,y_0)} + \mathrm{i}\,\frac{\partial v}{\partial x}\bigg|_{(x_0,y_0)}.
\end{aligned}$$

We have the analogous result when approaching z_0 in a line parallel to the imaginary axis. ■

In actual fact, the Cauchy–Riemann equation holding is a necessary *and* sufficient condition for complex differentiability.

Theorem 2.0.4 *Let $f(z) = u(x,y) + \mathrm{i}v(x,y)$. If the partial derivatives of u and v exist in some disk centered on (x_0, y_0) and are continuous at $z_0 = x_0 + \mathrm{i}y_0$, and u and v satisfy the Cauchy–Riemann equation, then $f(z)$ is complex differentiable at z_0.* □

A function is said to be **holomorphic** (or **analytic**) at z_0 if it is complex differentiable on some disk centred at z_0. A function is holomorphic if it is analytic at all points where it is defined.

Example If $f(z) = y^3 - 3\mathrm{i}xy^2$, fine where $f(z)$ is complex differentiable, and compute $f'(z)$.

Note that for $u = y^3$ and $v = -3xy^2$,

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = -3y^2, \quad \frac{\partial u}{\partial y} = -3y^2, \quad \frac{\partial v}{\partial y} = -6xy,$$

so it is differentiable if $-6xy = 0$ and $3y^2 = 3y^2$, which is only satisfied at $x = 0$ or $y = 0$, i.e. at the co-ordinate axes. In this case $f'(z) = -\mathrm{i}3y^2$, and that $f(z)$ is nowhere holomorphic.

Theorem 2.0.5 *Let $f(z)$ be holomorphic and $f(z) = u(x,y) + \mathrm{i}v(x,y)$. Then u and v are solutions to Laplace's equation in two dimensions.*

Proof By Cauchy–Riemann equations and the holomorphic property,

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x \partial y} = \frac{\partial v}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right) = -\frac{\partial^2 u}{\partial y^2},$$

so that $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. Similarly for v. ■

Recall that if $f(x)$ is an infinitely differentiable real function, that its Taylor series about $x = x_0$ is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

The complex counterpart is then if $f(z)$ is an infinitely complex differentiable complex function, its Taylor series about $z = z_0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k.$$

Since derivatives of standard functions are the same in the complex case, their Taylor series are the same too.

Theorem 2.0.6 *Let $f(z)$ be complex differentiable. Then its Taylor series converges to $f(z)$ for all z where it converges.* □

This implies any complex differentiable function is just a power series.

If we let $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ be a power series centred on $z = z_0$, then there exists some $R \in [0, \infty]$ where the power series

- converges for $|z - z_0| < R$,
- diverges for $|z - z_0| > R$,
- inconclusive for $|z - z_0| = R$.

R is called the **radius of convergence**, and $\{z : |z - z_0| < R\}$ is the **disk of convergence**.

To find the disk of convergence we can often use the ratio test.

Example Find the radius of convergence for $f(z) = (1-z)^{-1}$ around $z_0 = 0$.

Recall that $f(z) = \sim_{n=0}^{\infty} z^n$, then we note that $\lim_{n\to\infty} |z^{n+1}/z^n| = |z|$, hence we have convergence if $|z| < 1$ by the ratio test, and the radius of convergence is $R = 1$.

Example For $f(z) = \sum_{n=0}^{\infty} n^2(z-\mathrm{i})^{2n}/2^n$ as a power series around $z_0 = \mathrm{i}$, by the ratio test,

$$\lim_{n\to\infty} \left| \frac{(n+1)^2(z-\mathrm{i})^{2n+2}}{2^{n+1}} \frac{2^n}{n^2(z-\mathrm{i})^{2n}} \right| = \frac{|z-\mathrm{i}|^2}{2},$$

so we have convergence if $|z - \mathrm{i}| < \sqrt{2}$, and the radius of convergence is $R = \sqrt{2}$.

Theorem 2.0.7 *If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R and converges to $f(z)$ of its disk of convergence D, then $f(z)$ is complex differentiable, and $\sum_{n=0}^{\infty} a_n n(z-z_0)^{n-1}$ converges to $f'(z)$ in D.* □

Theorem 2.0.8 *If $\sum_{n=0}^{\infty} a_n(z-z_0)^n \to f(z)$ in its disk of convergence, then $f(z)$ is complex differentiable an infinite number of times, and $f^{(n)}(z_0) = n!a_n$.*

Proof By previous theorem, we have

$$\begin{aligned} f(z) &= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots \\ f'(z) &= a_1 + 2a_2(z-z_0) + \dots \\ f''(z) &= 2 \cdot 1a_2 + \dots \end{aligned}$$

and so on. Hence the function is infinitely complex differentiable, and $f^{(n)}(z_0)$ is as required. ■

Example Find the Taylor series of $(1-z)^{-2}$ about $z=0$.

We see that since $\mathrm{d}/\mathrm{d}z(1-z)^{-1} = (1-z)^{-2}$,

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1}, \qquad |z| < 1.$$

Example Find the Taylor series for $\cosh(4z^3)$ about $z=0$.

Recall that

$$\cosh y = 1 + \frac{y^2}{2!} + \ldots = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!},$$

so that

$$\cosh(4z^3) = \sum_{n=0}^{\infty} \frac{(4z^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{16^n z^{6n}}{(2n)!}, \qquad \forall z \in \mathbb{C}.$$

Example Find the Taylor series of $z^3/(1-5z)^2$ about $z=0$.

Using the identity from two examples ago,

$$\frac{1}{(1-5z)^2} = \sum_{n=1}^{\infty} n(5z)^{n-1},$$

so that

$$\frac{z^3}{(1-5z)^2} = \sum_{n=1}^{\infty} n5^{n-1}z^{n+2}, \qquad |z| < \frac{1}{5}.$$

Example Find the Taylor series for $3z(z+1)^{-1}(z-2)^{-1}$ about $z=0$.

First note that the radius of convergence cannot be greater than 1. By partial fractions,

$$\frac{3z}{(z+1)(z-2)} = \frac{1}{z+1} + \frac{2}{z-2},$$

so that the Taylor series is

$$\sum_{n=0}^{\infty} (-z)^n - \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty} \left[(-1)^n - \frac{1}{2^n}\right] z^n, \qquad |z| < 1.$$

3 Integration in the complex plane

Recall that in the real case we have the **indefinite integral** with

$$\int f(x)\,\mathrm{d}x = F(x),$$

where $F(x)$ is the primitive of f. We also have the **definite integral** where, by the fundamental theorem of calculus, gives

$$\int_b^a f(x)\,\mathrm{d}x = F(b) - F(a).$$

Although we can generalise the indefinite integral to the complex case, the definite integral doesn't generalise directly, because we are essentially trying to talk about a 2-dimensional surface in 4-space. So instead we integrate complex functions along curves, or contours, in the complex plane.

3.1 Curves in $\mathbb{C}$

A differentiable curve in $\mathbb{C}$ is a function $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(t) = \gamma_1(t) + \mathrm{i}\gamma_2(t)$, where γ_1 and γ_2 are real differentiable functions in t.

Example One way to generate the unit circle is with

$$\gamma : [0, 1] \to \mathbb{C}, \qquad \gamma(t) = \mathrm{e}^{2\pi \mathrm{i} t}.$$

Notice here that γ is closed, and has a direction characterised by how t is parameterised (in this case it is in the positive sense, or in the anti-clockwise). In general, a circle centred at z_0 with radius r has the associated curve

$$\gamma : [0, 1] \to \mathbb{C}, \qquad \gamma(t) = z_0 + r\mathrm{e}^{2\pi \mathrm{i} t}.$$

Example Consider two curves

$$\gamma(t) = t + \mathrm{i}t, \quad 0 \le t \le 1, \qquad \beta(t) = \begin{cases} t, & 0 \le t \le 1, \\ 1 + (t-1)\mathrm{i} & 1 \le t \le 2. \end{cases}$$

TODO! diagram

Both curves connect the origin to $z = 1 + \mathrm{i}$, but the path is difference. $\beta(t)$ here is piecewise differentiable. One question of course is whether the path matters (see later). In general, a vector from z_0 to z_1 may be parameterised as $\gamma(t) = z_0 + t(z_1 - z_0)$, for $t \in [0, 1]$.

3.1.1 *Contour integrals*

To integrate along the curve $z = \gamma(t)$ with $t \in [a, b]$, we have from chain rule that $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$, so that

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t,$$

where the latter is as before since we are dealing with a function of a real variable.

Example Compute the contour integrals of the following:

1. $f(z) = z^2$, $\gamma(t) = \mathrm{e}^{\mathrm{i}\pi t}$, $t \in [0, 1]$
   The path is the upper unit semi-circle, and we have
   $$\int_\gamma f(z)\,\mathrm{d}z = \mathrm{i}\pi \int_0^1 \mathrm{e}^{3\mathrm{i}\pi t}\,\mathrm{d}t = -\frac{2}{3}.$$
2. $f(z) = z^2$, $\gamma(t) = \mathrm{e}^{-\mathrm{i}\pi t}$, $t \in [0, 1]$
   The path is the lower unit semi-circle, and we have
   $$\int_\gamma f(z)\,\mathrm{d}z = -\mathrm{i}\pi \int_0^1 \mathrm{e}^{-3\mathrm{i}\pi t}\,\mathrm{d}t = -\frac{2}{3}.$$
   Notice here the integral has the same value as the previous part, which in this case is not a coincidence.
3. $f(z) = \bar{z}$, $\gamma(t) = 1 + \mathrm{i}t$, $t \in [0, 2]$
   We have
   $$\int_\gamma f(z)\,\mathrm{d}z = \mathrm{i} \int_0^2 (1 - \mathrm{i}t)\,\mathrm{d}t = 2 + 2\mathrm{i}.$$

A **contour** is a continuous curve made up a finite number of differentiable curves. The contour itself does not need to be differentiable although the individual pieces should. The integral of $f(z)$ along a contour is then the sum of integrals along each individual differentiable curve.

Proposition 3.1.1 *We have the following properties for contour integrals:*

1. *Linearity, where*
   $$\int_\gamma (\alpha f(z) + \beta g(z))\,\mathrm{d}z = \alpha \int_\gamma f(z)\,\mathrm{d}z + \beta \int_\gamma g(z))\,\mathrm{d}z.$$

2. *If contours γ_1 and γ_2 have the same track in $\mathbb{C}$ and transverse it in the same direction, then*

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_{\gamma_2} f(z)\,\mathrm{d}z.$$

3. *If $\gamma : [a,b] \to \mathbb{C}$ and $\mu : [-b,-a] \to \mathbb{C}$ where $\mu(t) = \gamma(-t)$, i.e. μ is the 'reverse' of γ, then*

$$\int_{\mu} f(z)\,\mathrm{d}z = -\int_{\gamma} f(z)\,\mathrm{d}z.$$

4. *We have the inequality*

$$\left|\int_{\gamma} f(z)\,\mathrm{d}z\right| \leq \int_{\gamma} |f(\gamma(t))| \cdot |\gamma'(t)|\,\mathrm{d}t \leq \mathrm{length}(\gamma) \cdot \max_{\gamma} |f(\gamma(t))|.$$

5. *(Fundamental Theorem of Calculus) Let $F(z)$ be holomorphic on an open set $D \subset \mathbb{C}$, and $F'(z) = f(z)$. Then for any contour $\gamma : [a,b] \to D$ with end points $z_0 = \gamma(a)$ and $z_1 = \gamma(b)$, we have*

$$\int_{\gamma} f(z)\,\mathrm{d}z = F(z_1) - F(z_0).$$

Proof 1. Since we have linearity when the integrals are real, this one is just by definition:

$$\begin{aligned}\int_{\gamma} (\alpha f(z) + \beta g(z))\,\mathrm{d}z &= \int_a^b [\alpha f(\gamma(t)) + \beta g(\gamma(t))]\gamma'(t)\,\mathrm{d}t \\ &= \alpha \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t + \beta \int_a^b g(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \alpha \int_{\gamma} f(z)\,\mathrm{d}z + \beta \int_{\gamma} g(z))\,\mathrm{d}z.\end{aligned}$$

2. Let $\gamma_k : [a_k, b_k] \to \mathbb{C}$ with $k = 1,2$, and assume that $\gamma_2(h(t)) = \gamma_1(t)$. Then taking a substitution $u = h(t)$ and judicious use of chain rule gives

$$\begin{aligned}\int_{\gamma_2} f(z)\,\mathrm{d}z &= \int_{a_2}^{b_2} f(\gamma_2(u))\gamma_2'(u)\,\mathrm{d}u \\ &= \int_{a_1}^{b_1} f(\gamma_2(h(t)))\gamma_2'(h(t))h'(t)\,\mathrm{d}t \\ &= \int_{a_1}^{b_1} f(\gamma_1(t))\gamma_1'(t)\,\mathrm{d}t \\ &= \int_{\gamma_1} f(z)\,\mathrm{d}z.\end{aligned}$$

3. As in previous case but use different limits.

4. Let $\theta = \arg \int_\gamma f(z)\, \mathrm{d}z$, then

$$\begin{aligned}
\left| \int_\gamma f(z)\, \mathrm{d}z \right| &= \mathrm{e}^{-\mathrm{i}\theta} \int_\gamma f(z)\, \mathrm{d}z \\
&= \int_\gamma \mathrm{e}^{-\mathrm{i}\theta} f(z)\, \mathrm{d}z \\
&= \mathrm{Re} \left(\int_\gamma \mathrm{e}^{-\mathrm{i}\theta} f(z)\, \mathrm{d}z \right) \\
&= \mathrm{Re} \left(\int_a^b \mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(t)\, \mathrm{d}t \right) \\
&\leq \int_a^b \left| \mathrm{e}^{-\mathrm{i}\theta} f(\gamma(t))\gamma'(t) \right| \, \mathrm{d}t \\
&= \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)|\, \mathrm{d}t \\
&\leq \mathrm{length}(\gamma) \cdot \max_\gamma |f(\gamma(t))|.
\end{aligned}$$

5. Let $F(\gamma(t)) = u(t) + \mathrm{i}v(t)$, where u and v are real functions. By the chain rule, $u'(t) + \mathrm{i}v'(t) = F'(\gamma(t))\gamma'(t)$, so

$$\int_\gamma f(z)\, \mathrm{d}z = \int_a^b F'(\gamma(t))\gamma'(t)\, \mathrm{d}t = \int_a^b [u'(t) + \mathrm{i}v'(t)]\, \mathrm{d}t = F(b) - F(a).$$

Example Let $\gamma(t) = R\mathrm{e}^{\mathrm{i}t}$, $t \in [0, 2\pi]$, then

$$\int_\gamma \frac{1}{z}\, \mathrm{d}z = \int_0^{2\pi} \frac{1}{R\mathrm{e}^{\mathrm{i}t}} R\mathrm{i}\mathrm{e}^{\mathrm{i}t}\, \mathrm{d}t = 2\pi\mathrm{i},$$

and this is because the primitive is not well-defined at $z = 0$.

Theorem 3.1.2 (Path Independent Theorem) *Let f be continuous on an open connected set $D \subset \mathbb{C}$. Then the following statements are equivalent to each other:*

1. *integrals are path independent;*

2. *if γ is a closed curve in D, then $\oint_\gamma f(z)\, \mathrm{d}z = 0$;*

3. *there exists a primitive $F(z)$ of $f(z)$ where $F'(z) = f(z)$, defined globally on D.*

Proof We show that 1 is equivalent to 2, and 2 is equivalent to 3, so 1 is then equivalent to 3 by default.

(1 ⇔ 2) Suppose Γ is a closed curve consisting of some arbitrary closed simple curves $\gamma_{0,1}$ as illustrated **TODO!** diagram

Then

$$\oint_\Gamma f(z)\, \mathrm{d}z = \left(\int_{\gamma_0} + \int_{-\gamma_1} \right) f(z)\, \mathrm{d}z = \left(\int_{\gamma_0} - \int_{\gamma_1} \right) f(z)\, \mathrm{d}z.$$

Since the integrals are path independent, we have $\int_\Gamma f(z)\ \mathrm{d}z = 0$. Conversely, if the integral is zero by assumption, since $\gamma_{0,1}$ are arbitrary, this implies path independence.

($2 \Leftrightarrow 3$) Assuming there is a primitive, then the fundamental theorem of calculus implies that since we have the existence of the primitive, we have $\int_\gamma f(z)\ \mathrm{d}z = F(z_1) - F(z_0)$ regardless of path, so if $z_1 = z_0$ then $\oint_\gamma f(z)\ \mathrm{d}z = 0$.

Conversely, let z_0 be any fixed point on D, and z be any other point on D. Since D is open and connected, the contour γ joining z_0 to z exists. Defining then $F(z) = \int_\gamma f(\zeta)\ \mathrm{d}\zeta$, by the assumption of of path independence, $F(z)$ is well-defined, and by the estimation property, $F'(z) = f(z)$, so there exists a primitive. ■

3.1.2 *Cauchy's theorem and residue theorem*

Cauchy's theorem is one of the centre pieces of complex analysis. Before the statement, we need an extra tool from topology regarding simple closed curves.

Theorem 3.1.3 (Jordan curve theorem) *Let γ be a simple closed contour, i.e. no self-intersections except at the end points. Then the compliment of γ in $\mathbb{C}$ is the disjoint union of exactly two sets, where exactly one is bounded.* □

Intuitively this says that a simple closed curve splits the space into an outside and inside (trivial as it may sound rigourously proofing this is not so obvious...)

Theorem 3.1.4 (Cauchy's theorem) *Let $f(z)$ be holomorphic on and inside a simple closed curve γ. Then $\oint_\gamma f(z)\ \mathrm{d}z = 0$.*

Proof Let $f = u + \mathrm{i}v$ for real u and v, then using Green's theorem (since the resulting integrands are real)

$$\begin{aligned}\oint_\gamma f(z)\ \mathrm{d}z &= \oint_\gamma (u+\mathrm{i}v)(\mathrm{d}x + \mathrm{i}\mathrm{d}y)\\ &= \oint_\gamma [(u\ \mathrm{d}x - v\ \mathrm{d}y) + \mathrm{i}(u\ \mathrm{d}y + v\ \mathrm{d}x)]\\ &= \iint_A \left[\left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \mathrm{i}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] = 0.\end{aligned}$$

The latter quality to zero is because $f(z)$ is holomorphic, so u and v satisfies the Cauchy–Riemann equations, and thus the partials are continuous and equal. ■

Example

3.2 Residue theorem

3.3 Applications for real integrals

4 More analysis topics

5 Conformal mapping