Academic notes: 2H Complex Analysis

J. Mak (May 14, 2021) [Blended from notes of R. Gregory and J. Bolton, Durham]*

MISC. NOTES

- This was part of the Durham core second year modules. Involves more things to do with analysis in the complex plane, involving holomorphic functions, contour integrals, residue theorems, conform mappings, etc.
- The original course does not have geometry of complex numbers since that was covered in Core A (Geometry 1A), but for consistency reasons this has been moved here.

- (to be fixed?) Diagrams to do

I. GEOMETRY OF COMPLEX NUMBERS

A. Complex numbers and the Argand diagram

We define $\sqrt{-1} = \mathrm{i}$, which is the basic unit imaginary number. A complex number is then a combination of real and imaginary parts $z = a + b\mathrm{i}$, with $a, b \in \mathbb{R}$. The complex numbers $\mathbb{C}$ then obeys the same axioms for addition and multiplication as $\mathbb{R}$ (both are fields).

Consider instead $\mathbb{C}$ as a vector space $z = (x, y)$, where multiplication is defined on $\mathbb{R}^2$ as

$$z_1 \times z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 - x_2 y_1),$$

and this is commutative. $1 = (1, 0)$ is the identity. So we see that $\mathbb{R}^2$ with this multiplication is a concrete visualisation of $\mathbb{C}$, and is called the Argand diagram.

Given $z = x + \mathrm{i}y$, the conjugate of z is defined to be $\overline{z} = x - \mathrm{i}y$. Geometrically, this represents a reflection of z in the 'real' axis. The real and imaginary part of z is given respectively by

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \overline{z}}{2}.$$

In polar form, $z = r(\cos\theta + \mathrm{i}\sin\theta)$. r is called the modulus of z and is denoted $|z|$, whilst θ is called the argument of z, denoted $\arg(z)$.

B. Geometry of addition and multiplication in $\mathbb{C}$

Addition is as in $\mathbb{R}^2$. From this, we can deduce the triangle inequality.

Lemma I.1 *For $z_1, z_2 \in \mathbb{C}$, $|z_1 + z_2| \leq |z_1| + |z_2|$, and we have an equality iff $\arg(z_1) = \arg(z_2)$. By corollary, we have $|z_2 + z_2| \geq ||z_1| - |z_2||$.*

Proof Wlog, let $|z_1| > |z_2|$, then $|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |z_2|$ by the triangle inequality for real numbers. So $|z_1| - |z_2| \leq |z_1 + z_2|$, and since $|z_1| > |z_2|$, we have the corollary of the result as required.

For multiplication, we observe that $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. Geometrically, this is a spiral scaling. We can use the $\mathbb{C}$-plane to describe various geometrical objects.

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Example A circle may be described by $|z - z_0| = a$, where z_0 is the centre of the circle and a is the radius; expanding this accordingly, we see that $a^2 = (x - x_0)^2 + (y - y_0)^2$.

Example The equation $|z - x_0| + |z + x_0| = 2r$ describes an ellipse, where $r > |x_0|$. This may be done via expansion in (x, y). Alternatively, in polar form, we observe that, for $z = a + \mathrm{i}b$, $|z \pm x_0|^2 = (a^2 - b^2)\cos^2\theta \pm 2ax_0\cos\theta + (x_0^2 + b^2)$. If $x_0^2 = (a^2 - b^2)$, then this may be simplified to $|z \pm x_0| = a \pm x_0\cos\theta$ since $a > x_0$. With this, we obtain $|z - x_0| + |z + x_0| = 2a$, thus, with $x = a\cos\theta$ and $y = b\sin\theta$, this describes an ellipse.

Example The locus of $|z - z_1| = |z - z_2|$ describes the line that is equidistant to the points z_1 and z_2. To see this, expanding everything in x and y and we obtain the equality

$$x(x_2 - x_1) + y(y_2 - y_1) = \frac{y_2^2 - y_1^2}{2} + \frac{x_2^2 - x_1^2}{2},$$

and the normal to the line is $z_2 - z_1$.

C. de Moivre's theorem

Theorem I.2 (de Moivre's theorem) *For all $n \in \mathbb{Z}^+$ and angle θ,* $\cos n\theta + \mathrm{i}\sin n\theta = (\cos\theta + \mathrm{i}\sin\theta)^n$.

Proof We do this by induction. The $n = 1$ case is trivial, so, assuming it is true for n, then we observe that

$$\begin{aligned}\cos(n+1)\theta + \mathrm{i}\sin(n+1)\theta &= \cos n\theta\cos\theta + \mathrm{i}^2\sin\theta\sin n\theta + \mathrm{i}\sin n\theta\cos\theta + \mathrm{i}\sin\theta\cos n\theta \\ &= (\cos n\theta + \mathrm{i}\sin n\theta)(\cos\theta + \mathrm{i}\sin\theta) \\ &= (\cos\theta + \mathrm{i}\sin\theta)^{n+1}.\end{aligned}$$

Example

$$\cos 2\theta + \mathrm{i}\sin 2\theta = (\cos\theta + \mathrm{i}\sin\theta)^2 = (\cos^2\theta - \sin^2\theta) + \mathrm{i}(2\sin\theta\cos\theta),$$

and remembering the double angle formulae, the equality agrees. From de Moivre's theorem, we see that

$$\cos n\theta = \mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^n, \qquad \sin n\theta = \mathrm{Im}(\cos\theta + \mathrm{i}\sin\theta)^n.$$

We can also use the theorem to find $\sin$ or $\cos$ of rational multiples of π.

Example Express $\sin 4\theta / \cos\theta$ as a polynomial in $\sin\theta$, and hence find $\sin(\pi/4)$.

$$\sin 4\theta = \mathrm{Im}(\cos\theta + \mathrm{i}\sin\theta)^4 = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta = 4\cos\theta(\sin\theta - 2\sin^3\theta),$$

so $\sin 4\theta/\cos\theta = 4\sin\theta(1 - 2\sin^\theta)$. Evaluating this $\pi/4$, we see that the LHS is zero. Now, $4\sin(\pi/4) > 0$, so we conclude that $\sin(\pi/4) = 1/\sqrt{2}$, as expected.

Example Find $\cos(k\pi/6)$ for $k = 1, 2, 3, 4, 5$.

Letting $c = \cos\theta$ and $s = \sin\theta$, observe that

$$\sin 6\theta = sc(6c^4 + 6s^4 - 20s^2c^2) = sc(32c^4 - 32c^2 + 6) = 2sc(4c^3 - 3)(4c^2 - 1).$$

Now, $\sin(k\pi) = 0$, so LHS is zero, and since $\sin(k\pi/6) \neq 0$, we have

$$\cos^2(k\pi/6) = 3/4, \quad \cos^2(k\pi/6) = 1/4, \quad \cos\theta = 0 \quad \Rightarrow \qquad \cos(k\pi/6) = \pm\sqrt{3}/2,\ \pm 1/2,\ 0.$$

Since $\cos\theta$ is a decreasing function in $[0, \pi]$, we have

$$\cos(\pi/6) = \sqrt{3}/2, \quad \cos(2\pi/6) = 1/2, \quad \cos(\pi/2) = 0, \quad \cos(2\pi/3) = -1/2, \quad \cos(5\pi/6) = -\sqrt{3}/2.$$

D. Imaginary exponentials

de Moivre's theorem hints at a deeper geometric significance of cosine and sine functions and a way of encoding multiplication by imaginary numbers. Suppose $f(\theta) = \cos\theta + \mathrm{i}\sin\theta$, then we notice that $f'(\theta) = \mathrm{i}f(\theta)$, and, more generally, $f^{(n)}(\theta) = \mathrm{i}^n f(\theta)$. We know that also that the n-th derivative of $\mathrm{e}^{\lambda x}$ is $\lambda^n \mathrm{e}^{\lambda x}$, so this suggests a link with exponential functions; indeed, we have Euler's formula

$$\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}. \tag{1}$$

By de Moivre's theroem then,

$$r(\cos n\theta + \mathrm{i}\sin n\theta) = r(\cos\theta + \mathrm{i}\sin\theta)^n = r\mathrm{e}^{\mathrm{i}n\theta}.$$

Lemma I.3 (Euler identity) $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$.

Example Find all the roots of $z^6 + 4z^3 + 8 = 0$.
Factorising the above gives $z^3 = -2 \pm 2\mathrm{i}$. So since $|z^3| = 2\sqrt{2}$, we have $|z| = \sqrt{2}$. Now,

$$\arg(-2+2\mathrm{i}) = \frac{3\pi}{4}, \qquad \arg(-2-2\mathrm{i}) = \frac{5\pi/4}{,}$$

and the argument of the roots z satisfies

$$\arg(z) = \frac{3\pi/4 + 2n\pi}{3}, \qquad \arg(z) = \frac{5\pi/4 + 2n\pi}{3},$$

where the division by 3 is to take into account the cube root, and the $2n\pi$ factors is to account for all the roots. This eventually yields

$$z = \sqrt{2}(\mathrm{e}^{\mathrm{i}\pi/4}, \mathrm{e}^{5\mathrm{i}\pi/4}, \mathrm{e}^{11\mathrm{i}\pi/12}, \mathrm{e}^{13\mathrm{i}\pi/12}, \mathrm{e}^{19\mathrm{i}\pi/12}, \mathrm{e}^{21\mathrm{i}\pi/21}).$$

II. BASICS OF COMPLEX FUNCTIONS

A real function can for example be once differentiable, but not twice. One example is $f(x) = x|x|$, where $f'(x)$ is not differentiable at $x = 0$.

Theorem II.1 *If a complex function is once differentiable, it is differentiable as many times as you like.*

It is possible for two real functions to agree on an interval but not everywhere, assuming they are differentiable. One example is $f(x) = x|x|$ and $g(x) = x^2$ for $x > 0$.

Theorem II.2 *If two complex differentiable functions agree on any disc in the complex plane, then they agree everywhere (subject to certain conditions...)*

Recall that a real function assigns any real number x to at most one real number (i.e. it is injective). A complex function therefore assigns any complex number z to at most one complex number. These include standard polynomials, rational functions, transcendental functions, trigonometric functions, hyperbolic functions, where the argument is in z. Some examples have already been given above.

Example Solve $\mathrm{e}^z = 1$.
Writing $z = x + \mathrm{i}y$ and using Euler's formula,

$$\mathrm{e}^x(\cos y + \mathrm{i}\sin y) = 1,$$

and equating real and imaginary parts lead to

$$\mathrm{e}^x \cos y = 1, \qquad \mathrm{e}^x \sin y = 0.$$

Considering the imaginary part, since $\mathrm{e}^x \neq 0$, $y = n\pi$ for $n \in \mathbb{Z}$, but from the real part, since $\mathrm{e}^x > 0$ and $\cos n\pi = \pm 1$, we should only have $y = 2n\pi$ for $n \in \mathbb{Z}$. The real part then additionally implies that $x = 0$ since $\cos 2n\pi = 1$, so $z = 2\mathrm{i}n\pi$ for $n \in \mathbb{Z}$.

Note that $|e^{iz}| \geq 0$ for all $z \in \mathbb{C}$.

Example Solve $\sin z = 0$.

With the standard identity for sine with complex arguments, we have

$$\frac{e^{iz} - e^{-iz}}{2i} = 0.$$

Equating real and imaginary parts lead to $z = m\pi$, $m \in \mathbb{Z}$.

The (natural) logarithm we define by

$$\log z = \log |z| + i\arg z \tag{2}$$

to give a complex version of the log function that satisfies the usual rules of

$$\log z = \log re^{i\theta} = \log r + i\theta = \log |z| + i\arg z.$$

Here we need to choose a branch, and we take $\theta \in (-\pi, \pi)$ (the principal branch) to preserve the continuity property, so that $\log z$ is undefined on the negative real axis, coinciding with the real case.

Example $\log(1 - i) = \log \sqrt{2} - i(\pi/4)$

We use $\log z$ to define powers of complex numbers. Recall that for real numbers we have $x^a = e^{a \log a}$ for $a > 0$, so for $z, w \in \mathbb{C}$, we analogously define

$$z^w = e^{w \log z}, \tag{3}$$

choosing the principal branch unless otherwise stated.

Example

$$(1 + i\sqrt{3})^{1/2} = \exp\left[\frac{1}{2}\log(1 + i\sqrt{3})\right] = \exp\left[\frac{1}{2}\left(\log 2 + i\frac{\pi}{3}\right)\right] = e^{\log\sqrt{2}}e^{i(\pi/6)} = \sqrt{2}e^{i(\pi/6)},$$

which in this case is could have been gotten from $(1 + i\sqrt{3}) = 2e^{i(\pi/3)}$.

Example

$$(1 - i)^i = e^{i\log(1-i)} = e^{i(\log\sqrt{2} - i\pi/4)} = e^{\pi/4}e^{i\log\sqrt{2}}.$$

We say a complex function $f(z)$ is complex differentiable at $z = z_0$ if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, or that

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists at $z = z_0$. The derivative is denoted $f'(z)$ as usual.

Example For $f(z) = z^2$,

$$\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = \lim_{h \to 0} \frac{z^2 + 2hz + h^2 - z^2}{h} = \lim_{h \to 0} 2z + h = 2z.$$

$f(z)$ is differentiable everywhere.

The usual rules for differentiation hold (linearity, product rule, chain rule etc.)

Note that $f(x) = x|x|$ is real differentiable everywhere. $f(z) = z|z|$ on the other hand is differentiable on the real axis, and complex differentiable at the origin.

Complex differentiation is a much stronger condition. Recall that for the limit to exist in the real case, the limit only needs to be equal when approached from above or below on the real line. In the complex plane however there are an infinite numbers of cases the limit can be approach, and thus a infinite number of cases to check. We see that a necessary condition for complex differentiability is that the limit needs to exist when z_0 is approached in the lines parallel to the real and imaginary axis. If we set $f(z)$ to be

$$f(z) = u(x, y) + \mathrm{i}v(x, y)$$

for some real functions u and v, then it turns out that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y},$$

when we take the limit in the direction parallel to the real and imaginary axis respectively. It follows that a *necessary* conditions for a function to be complex differentiable is that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{4}$$

These are known as the Cauchy–Riemann equations, and we actually have the following theorem.

Theorem II.3 *If $f(z)$ is complex differentiable at $z = z_0$, then the Cauchy–Riemann equations hold at (x_0, y_0) for $z_0 = x_0 + \mathrm{i}y_0$, and that*

$$f'(z_0) = \left(\frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}\right)\bigg|_{(x_0,y_0)} = \left(\frac{\partial v}{\partial y} - \mathrm{i}\frac{\partial u}{\partial y}\right)\bigg|_{(x_0,y_0)}.$$

III. INTEGRATION IN THE COMPLEX PLANE

IV. CONFORM MAPPING