Academic notes: Differential Geometry

J. Mak (January 6, 2017) [From notes of J. R. Parker, Durham]*

Geometry is the study of curves and surfaces, and there are two points of view one usually take. The intrinsic point of view is where we describe the objects in terms of how we see it inside the same space the objects are nested in, whilst the extrinsic point of view is opposite. In differential geometry, we do this by using calculus to either describe the object locally, or globally.

I. THE GEOMETRY OF CURVES

The intrinsic point of view of curves is not interesting, so we consider the extrinsic geometry of curves sitting in $\mathbb{R}^2$ or $\mathbb{R}^3$. A curve in $\mathbb{R}^n$ is a continuous map $\boldsymbol{\alpha} : I \to \$\mathbb{R}^n$, where I is an interval, i.e., for all $u \in I$, there exists $\boldsymbol{\alpha}(u) = (x_i(u))$ corresponding to a point on the curve $\boldsymbol{\alpha}$. The image $\{\boldsymbol{\alpha}(u) : u \in I\}$ is the trace of $\boldsymbol{\alpha}$ and u is the parameter along the curve.

A curve $\boldsymbol{\alpha}$ is smooth is for all i, $x_i(u)$ are each infinitely differentiable for all u, with

$$\boldsymbol{\alpha}'(u) = \frac{\mathrm{d}}{\mathrm{d}u}\boldsymbol{\alpha}(u) = (x_1'(u), \ldots, x_n'(u)).$$

$\boldsymbol{\alpha}$ is regular is $\boldsymbol{\alpha}'(u) \neq \mathbf{0}$ for all $u \in I$. If $\boldsymbol{\alpha}'(u) \neq \mathbf{0}$ then $\boldsymbol{\alpha}'(u)$ is a tangent vector to$\boldsymbol{\alpha}$ at u, and

$$\hat{\boldsymbol{t}}_\alpha(u) = \frac{\boldsymbol{\alpha}'(u)}{\|\boldsymbol{\alpha}'(u)\|}$$

is the tangent unit vector of $\boldsymbol{\alpha}$ at u.

Example

1. The circle of radius r in $\mathbb{R}^2$ is described by $\boldsymbol{\alpha}(u) = (r\cos u, r\sin u)$, which is clearly smooth and regular, with $\hat{\boldsymbol{t}}_\alpha(u) = \boldsymbol{\alpha}'(u)/r$.
2. $\boldsymbol{\alpha}(u) = (u^3, u^2)$ is smooth but not regular.
3. $\boldsymbol{\alpha}(u) = (u^3 - 4u, u^2 - 4)$ is smooth and regular. Notice however that $\boldsymbol{\alpha}(-2) = \boldsymbol{\alpha}(2)$ so there is a self intersection.
4. $\boldsymbol{\alpha} : (-\pi, \pi) \to \mathbb{R}^2$ with

$$\boldsymbol{\alpha}(u) = \begin{cases} (\cos u, \sin u), & u \geq 0, \\ (1, u), & u < 0, \end{cases}$$

has a continuous first derivative but not second derivative, so $\boldsymbol{\alpha}$ is not smooth.

Suppose $\boldsymbol{\alpha} : I \to \mathbb{R}^n$ and $\tilde{I}$ is another interval in $\mathbb{R}$, and $\phi : \tilde{I} \to I$. We can define a new smooth regular to be $\alpha\phi : \tilde{I} \to \mathbb{R}^n$ for smooth ϕ. Notice that the trace of $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}\phi$ are the same, the only difference being the parameter has changed. This is called a smooth change of parameter. In principle this process can be reversed, by having $\boldsymbol{\alpha}\phi = \boldsymbol{\beta}$ and so $\boldsymbol{\beta}\phi = \boldsymbol{\alpha}$ assuming $\boldsymbol{\beta}$ is invertible.

Example

1. $I = \tilde{I} = [-1, 1]$ and $\phi(t) = t^3$ has $\phi^{-1}(t) = t^{1/3}$ which is not smooth at zero, so the change of parameter is not smooth.
2. $I = (a, b)$ and $\tilde{I} = (-b, -a)$ with $\phi(t) = -t$ is a smooth change of parameter.

A canonical choice of parameter is the arc length s. For $\boldsymbol{\alpha} : I \to \mathbb{R}^n$ is a smooth regular curve and $t_0 \in I$, the arc length from $t_0$is given by

$$s(t) = \int_{t_0}^{t} \|\boldsymbol{\alpha}'(u)\| \,\mathrm{d}u.$$

Notice that $s(t)$ is a smooth function of t and $s'(t) = \|\boldsymbol{\alpha}'(t)\| \neq 0$ and so it is also regular. The arc length is a useful theoretical parameter for parameterising curves.

* julian.c.l.mak@googlemail.com

Example 1. The caternary is described by $\boldsymbol{\alpha}(t) = (t, \cosh t)$, and we see that $\|\boldsymbol{\alpha}'(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t$. Let$t_0 = 0$, then $s(t) = \sinh t$. Thus the caternary is parameterised by arc length as $\boldsymbol{\alpha}(s) = (\mathrm{arcsinh} s, \sqrt{1 + s^2})$.

2. For a circle of radius r, $\boldsymbol{\alpha} : [0, 2\pi) \to \mathbb{R}^2$, $\boldsymbol{\alpha} = (r \cos t, r \sin t)$, $\|\boldsymbol{\alpha}'\| = r$ and so $s = rt$, thus $\boldsymbol{\alpha}(s) : [0, 2\pi) \to \mathbb{R}^2$ with $\boldsymbol{\alpha}(s) = (r \cos(s/r), r \sin(s/r))$.

3. For an ellipse with instead $\boldsymbol{\alpha}(t) = (a \cos t, b \sin t)$, it may be seen that

$$s(t) = \int_0^t \sqrt{a^2 \cos^2 u + b^2 \sin^2 u}\, \mathrm{d}u.$$

The solution to this integrate are written in terms of elliptic functions, which makes arch a useful but theoretical choice of parameter.

Suppose $\boldsymbol{\alpha}$ is smooth, regular and parameterised by arc lenth, then $\|\boldsymbol{\alpha}'(s)\| = 1$, so that $\boldsymbol{t}_\alpha(s) = \boldsymbol{\alpha}'(s)$.

II. SECTION

III. SECTION