Boring but important disclaimers:

- If you are not getting this from the GitHub repository or the associated Canvas page (e.g. CourseHero, Chegg etc.), you are probably getting the substandard version of these slides Don't pay money for those, because you can get the most updated version for free at

  `https://github.com/julianmak/academic-notes`

  The repository principally contains the compiled products rather than the source for size reasons.
- Associated Python code (as Jupyter notebooks mostly) will be held on the same repository. The source data however might be big, so I am going to be naughty and possibly just refer you to where you might get the data if that is the case (e.g. JRA-55 data). I know I should make properly reproducible binders etc., but I didn't...
- I do not claim the compiled products and/or code are completely mistake free (e.g. I know I don't write Pythonic code). Use the material however you like, but use it at your own risk.
- As said on the repository, I have tried to honestly use content that is self made, open source or explicitly open for fair use, and citations should be there. If however you are the copyright holder and you want the material taken down, please flag up the issue accordingly and I will happily try and swap out the relevant material.

OCES 2003 : Descriptive Physical Oceanography

(a.k.a. physical oceanography by drawing pictures)

Lecture 4: forces and some mathematical background

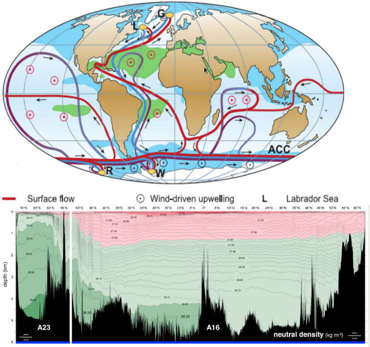
Outline

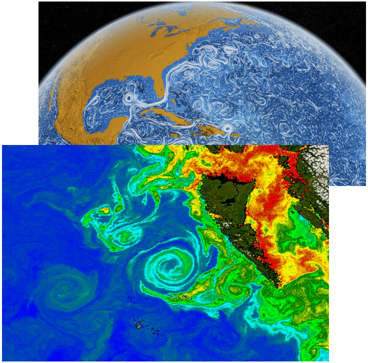
- concept of forces and forces acting on the ocean
  $\rightarrow$ thermodynamic (Solar, EmP, freshwater)
  $\rightarrow$ mechanical (wind, gravity, rotation, pressure etc.)
  $\rightarrow$ contrasts to the atmosphere

- (quick) review of some vector calculus concepts
  $\rightarrow$ scalars (e.g. p), vectors (e.g. $\boldsymbol{u}$), dot ($\cdot$) and cross ($\times$) product
  $\rightarrow$ derivatives ∇ (gradients, think rate of change)
  $\rightarrow$ integral $\int$ (think sum)
  $\rightarrow$ divergence $\nabla\cdot$ (think di/convergence)
  $\rightarrow$ curl $\nabla\times$ (think spin)

Key terms: forces (thermodynamic + mechanical), gradients, grad/div/curl

Recap: features in ocean





- highlighted features in the ocean previously, but how/why do they arise?

  $\rightarrow$ focus on dynamical links and consequences

  $\rightarrow$ effectively classical mechanics + fluid mechanics

Forces + Newton's laws

Intuitively, things move when there are forces acting on it

- more precisely, objects are in steady state (at rest or steady speed) unless there is a net force (or imbalance of forces)

  (this is **Newton's first law essentially**)

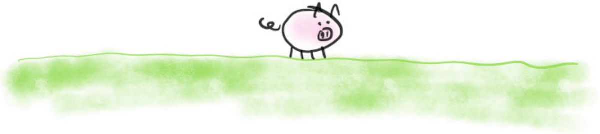


Figure: Forces acting on a (physicist joke: uniform point-mass, spherical) pig (not in a vaccum because we have air resistance + abuse of animal rights).

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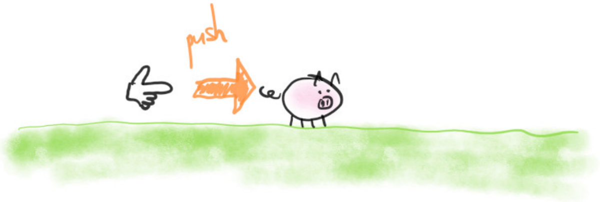

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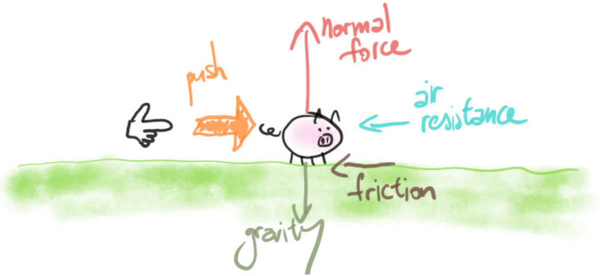

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- classical mechanics encapsulated by Newton's laws, which for our purposes is

$$\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$$

$\rightarrow$ mass m (in units of kg)

$\rightarrow$ velocity $\boldsymbol{u}$ (units of m s^{-1})

$\rightarrow$ acceleration $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$ (units of m s^{-2})

$\rightarrow$ **net** force $\boldsymbol{F}$ (units of Newtons, N = kg m s^{-2})

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- forces directly affecting momentum I am going to call mechanical forcing
- thermodynamic forcing affects density, which has consequences for momentum

Forces acting on the ocean

What external forces are acting on the ocean?

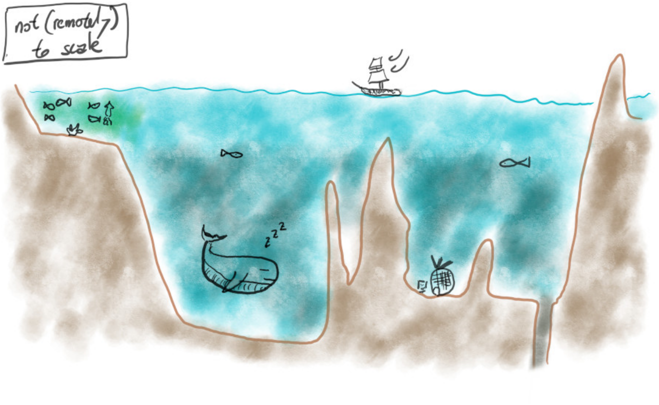
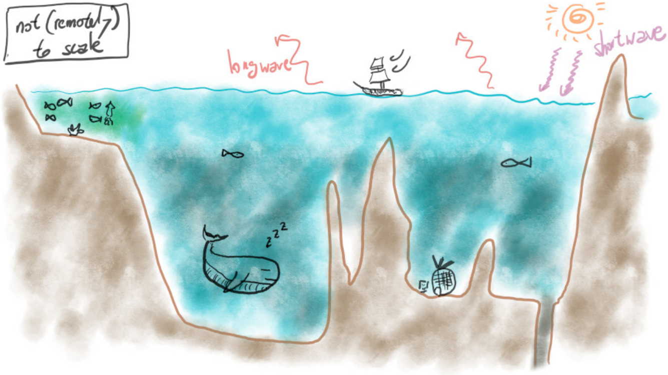


Figure: Schematic of ocean forcing.

Forces acting on the ocean

What external forces are acting on the ocean?

- temperature: sun + radiation (see Lec. 5 + OCES 4001)



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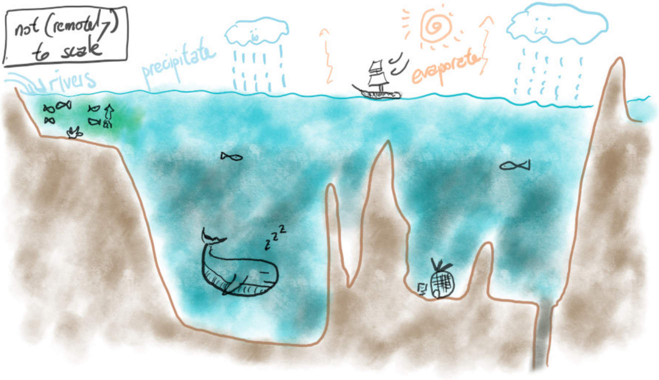
What external forces are acting on the ocean?

- salinity: river runoff, evaporation, precipitation (see Lec. 5)



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Forces acting on the ocean

What external forces are acting on the ocean?

- momentum + vorticity: wind (see Lec. 9)

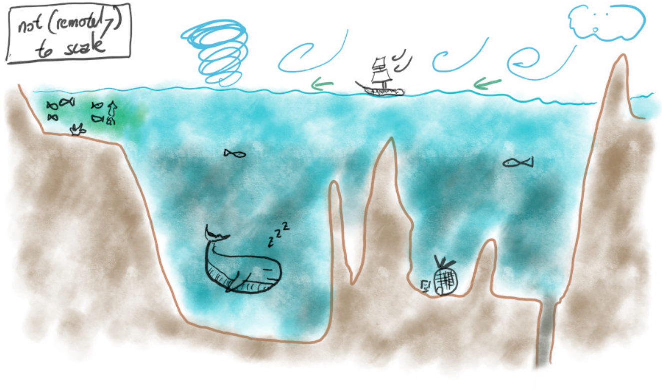
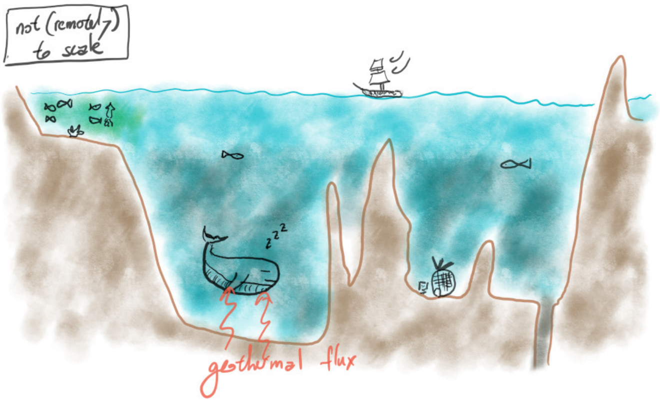


Figure: Schematic of ocean forcing.

Forces acting on the ocean

What external forces are acting on the ocean?

- geothermal flux (mostly quite small, but see Lec. 13 + 14)



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How are these represented in models?

Equations of Motion (EOM)

Denoting $\boldsymbol{u} = (u, v)$ and $\boldsymbol{u}_3 = (u, v, w)$, to numerous approximations (!!!) (see OCES 3203) ocean dynamics is governed by

$$\rho_0 \left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} \right) = -\nabla p + \boldsymbol{F}_{\boldsymbol{u}} + \boldsymbol{D}_{\boldsymbol{u}} \tag{1}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{2}$$

$$\nabla \cdot \boldsymbol{u}_3 = 0 \tag{3}$$

$$\left(\frac{\partial T}{\partial t} + \boldsymbol{u}_3 \cdot \nabla T \right) = F_T + D_T \tag{4}$$

$$\left(\frac{\partial S}{\partial t} + \boldsymbol{u}_3 \cdot \nabla S \right) = F_S + D_S \tag{5}$$

$$\rho = \rho(T, S, p) \tag{6}$$

Respectively, (1) momentum equation, (2) hydrostatic balance, (3) incompressibility, (4) temperature equation, (5) salinity equation, and (6) equation of state (EOS)

Equations of Motion (EOM)

Without vector calculus notation:

$$\rho_0 \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - 2\Omega v \right) = -\frac{\partial p}{\partial x} + F_x + D_u \tag{7}$$

$$\rho_0 \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2\Omega u \right) = -\frac{\partial p}{\partial y} + F_y + D_v \tag{8}$$

$$\frac{\partial p}{\partial z} = -\rho g \tag{9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10}$$

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \right) = F_T + D_T \tag{11}$$

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Decipher these throughout the course...

Vector calculus crash course

Terms in equations tend to have geometric meanings

- can understand them by drawing pictures
- encoded best through vector calculus

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Disclaimer!!!

- focus on the geometric meanings of vector calculus
- you are **not examined** on computing integrals/derivatives, but you are expected to understand/interpret their meanings

Vector calculus concepts: scalars and vectors

Scalars

- are just numbers and only have a magnitude, e.g.

  $\rightarrow g = 9.8\text{m s}^{-2}$

  $\rightarrow$ speed $|u| = 10\text{m s}^{-1}$

  $\rightarrow$ pressure $p = p(x, y, z)$

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Vectors (normally denoted with boldface $\boldsymbol{u}_3$, underline $\underline{u}_3$ or arrow $\vec{u}_3$)

- have a direction and magnitude, e.g.

  $\rightarrow$ weight is mg **acting towards centre of gravity**

  $\rightarrow$ velocity $\boldsymbol{u} = 10\text{m s}^{-1}$ **going East**

  $\rightarrow$ pressure **gradient** ∇p **acting from South**

Vector calculus concepts: scalars and vectors

Remember this guy?

Figure: Victor Perkins (aka **Vector**) from Despicable Me 1, because he is "*committing crimes with both direction and magnitude*". From Minion Rush, copyright with Universal Studios.

Vector calculus concepts: scalars and vectors

Scalars are just numbers so can do stuff to them as normal

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Vectors on the hand you can only

- add/subtract vectors to vectors (e.g. $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$)
- multiply/divide vector by scalar (e.g. $k\boldsymbol{a}$)

  $\rightarrow$ note the resulting things are **vectors**

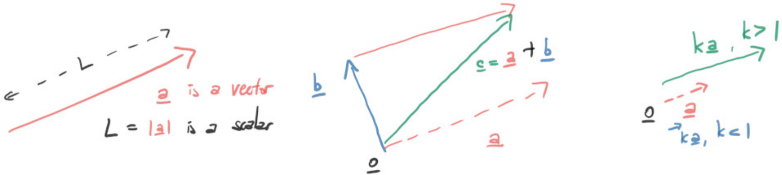

Figure: Schematics of elementary vector operations: (*a*) vector vs. scalar; (*b*) addition/subtraction of vector by vector; (*c*) multiplication of vector by scalar.

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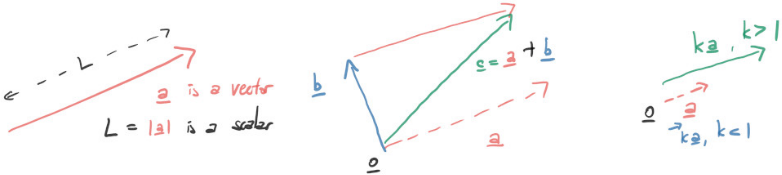

Figure: Schematics of elementary vector operations: (*a*) vector vs. scalar; (*b*) addition/subtraction of vector by vector; (*c*) multiplication of vector by scalar.

You <u>do not</u> multiply/divide vectors by vectors!

Vector calculus concepts: scalars and vectors

Representing a vector with a basis, e.g. the standard basis

$$\boldsymbol{e}_x = (1, 0, 0), \quad \boldsymbol{e}_y = (0, 1, 0), \quad \boldsymbol{e}_z = (0, 0, 1)$$

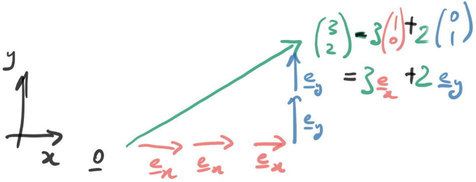

Figure: 2d example of representing a vector with the “standard” basis.

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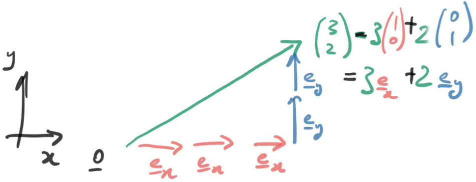

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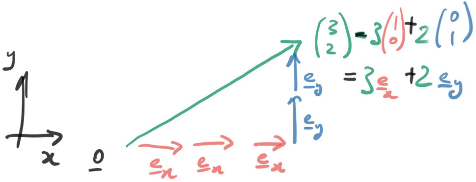

Figure: 2d example of representing a vector with the "standard" basis.

- "e.g." here because this is not the only choice of a basis (though it is the most convenient)
- length of vector $\boldsymbol{a} = (a_1, a_2, a_3)$ is then (just Pythagoras' theorem...)

$$|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vector calculus concepts: dot product ($\cdot$)

Two other things you can do to vectors $\boldsymbol{a}$ and $\boldsymbol{b} = (b_1, b_2, b_3)$

- dot/scalar product

  $\rightarrow$ takes two vectors and returns a scalar as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

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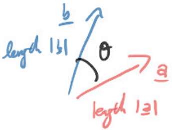
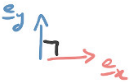


$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$



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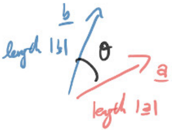

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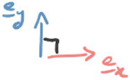

$\underline{e}_x \cdot \underline{e}_y = 0$

  $\rightarrow$ note that length of vector is $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$

  $\rightarrow$ if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then the two vectors are perpendicular

(recall $\cos 90^\circ = \cos \pi/2 = 0$)

Vector calculus concepts: cross product ($\times$)

- the cross product takes two vectors and returns a third vector $\boldsymbol{c} = (c_1, c_2, c_3)$ with

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_3 \end{pmatrix}$$

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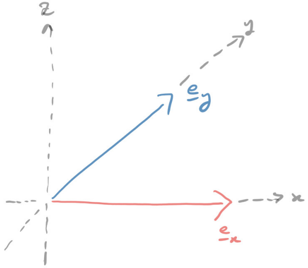

Figure: Right-hand-screw convention in $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$: resulting vector $\boldsymbol{c}$ points in direction of trying to turn/screw $\boldsymbol{a}$ into $\boldsymbol{b}$ using the right hand (clockwise motion).

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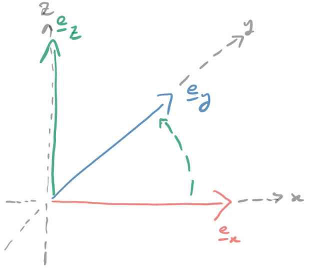

$e_x \times e_y = e_z$

(look from below, right hand turn clockwise)

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- resulting $\boldsymbol{c}$ follows right-hand-screw convention

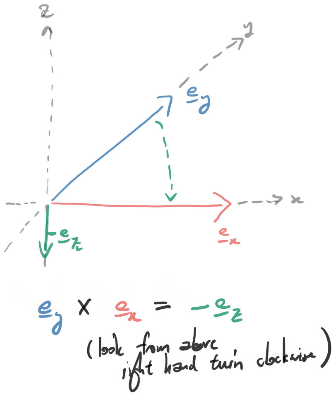

Figure: Right-hand-screw convention in $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{c}$: resulting vector $\boldsymbol{c}$ points in direction of trying to turn/screw $\boldsymbol{a}$ into $\boldsymbol{b}$ using the right hand (clockwise motion).

Vector calculus concepts: scalar/vector fields

Scalar/vector field is when the scalar/vector is a function, e.g.

$$p = p(x, y, z) = xy^2z^3, \qquad \boldsymbol{u}_3 = (u, v, w) = (x, y, 1) = x\boldsymbol{e}_x + y\boldsymbol{e}_y + 1\boldsymbol{e}_z,$$

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In terms of default notation for this course:

- vectors will be **bold** (e.g. $\boldsymbol{u}_3$)

  $\rightarrow$ if you need to write it by hand, use underlined (e.g. $\underline{u}_3$)

  $\rightarrow$ $\boldsymbol{u}_3$ is a vector field

  $\rightarrow$ u is a scalar field and the x-component of $\boldsymbol{u}_3$

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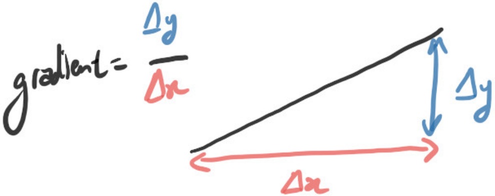
  $\rightarrow$ called zonal, meridional and vertical direction
- $x, y, z > 0$ is East, North and up
  $\rightarrow$ $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ points East, North and up
  $\rightarrow$ $u, v, w > 0$ is East, North and upward velocity

Vector calculus concepts: gradients

Gradient of something **straight** (linear) is simply



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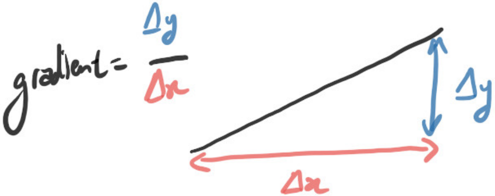


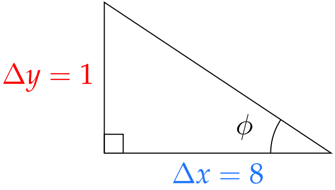
- gradient $> 0 \leftrightarrow$ "up"-slope
- gradient $< 0 \leftrightarrow$ "down"-slope

- think **rate of change**

Vector calculus concepts: gradients

Figure: Image from HK transport department (`www.td.gov.hk`), the kind of sign you see around Clear Water Bay Road quite a bit...

- what 1 : 8 here means is



$\rightarrow$ every 8 steps across move up/down 1

Vector calculus concepts: gradients

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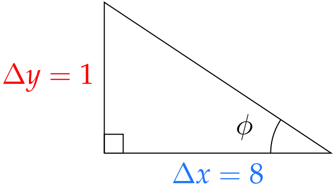
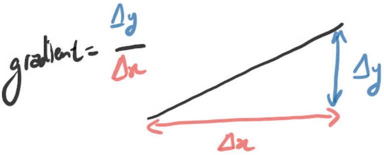


$\rightarrow$ every 8 steps across move up/down 1

- for completeness,

$$\phi = \arctan\frac{1}{8} \approx 7^\circ$$

Vector calculus concepts: derivatives (d/dx)



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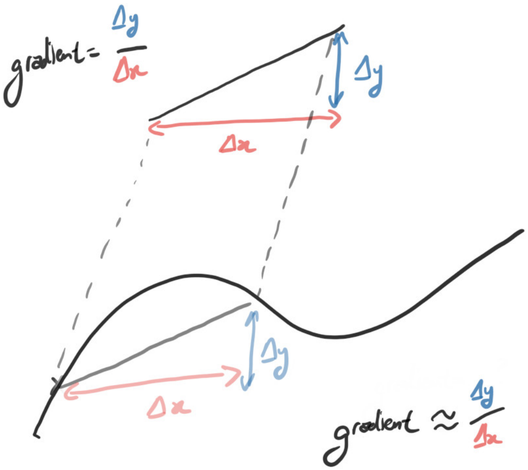

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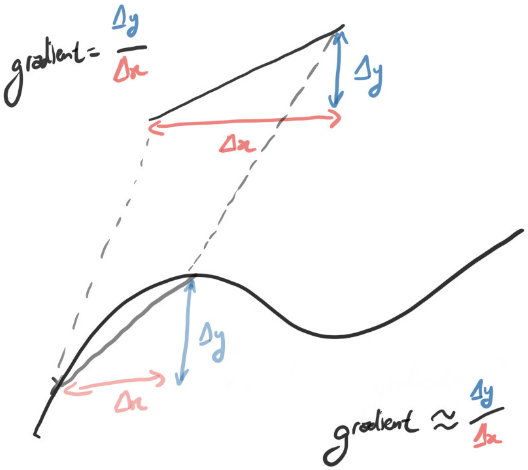

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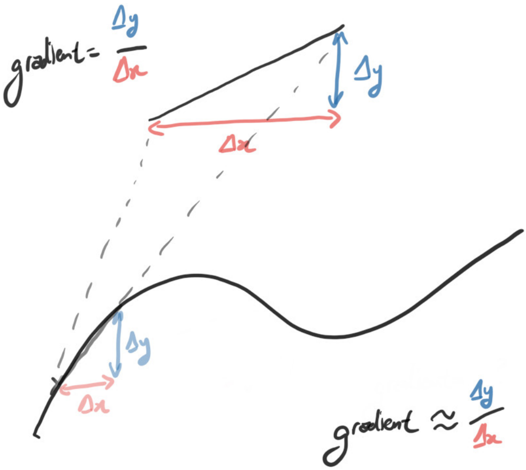

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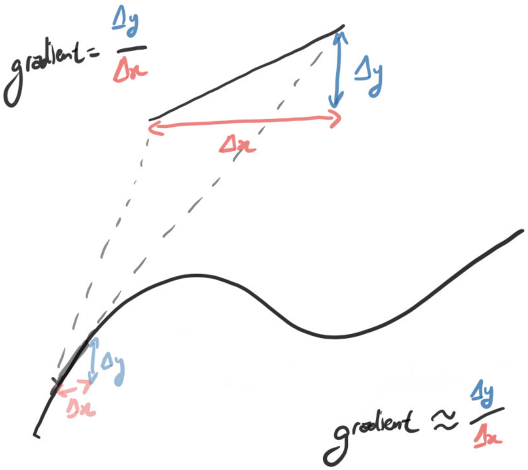

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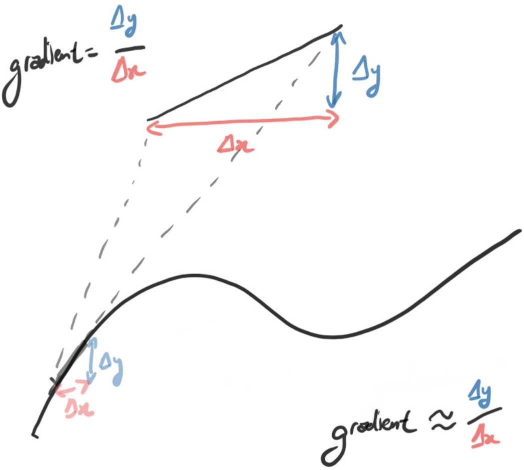

$$\frac{\Delta y}{\Delta x} \to \frac{dy}{dx} \text{ as } \Delta x \to 0$$

Vector calculus concepts: derivatives ($\mathrm{d}/\mathrm{d}x$)

The derivative

$$\frac{\mathrm{d}}{\mathrm{d}(\text{something})}$$

is really just the gradient (again, think **rate of change**)

Some examples:

- for p the pressure depending on only the depth z,

$$p = z^3, \qquad \Rightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}z} = 3z^2$$

  means "rate of change of pressure with depth is $3z^2$"

- rate of change of $p\mathrm{CO}_2$ concentration with (in-situ) temperature T would be

$$\frac{\mathrm{d}}{\mathrm{d}T}[p\mathrm{CO}_2]$$

Vector calculus concepts: partial derivatives ($\partial/\partial x$)

If something depends on multiple variables (e.g. $p = p(x, y, z)$ or $[CO_2] = [pCO_2](T, p, \text{fish}, \ldots)$ then sometimes we talk about the partial derivative

$$\frac{\partial}{\partial(\text{something})}$$

(again these are just gradients)

Some examples:

- for $p = p(x, y, z)$ the pressure,

$$\frac{\partial p}{\partial x}, \quad \frac{\partial p}{\partial y}, \quad \frac{\partial p}{\partial z},$$

  are respectively "the rate of change of pressure with $x/y/z$ keeping the other variables fixed"

Vector calculus concepts: partial derivatives ($\partial/\partial x$)

Some examples:

- for $p = p(x, y, z) = xy^2z^3$ the pressure and x, y the horizontal co-ordinate, then

$$\frac{\partial p}{\partial x} = y^2z^3\frac{\mathrm{d}}{\mathrm{d}x}x = y^2z^3, \qquad \frac{\partial p}{\partial y} = xz^3\frac{\mathrm{d}}{\mathrm{d}y}y^2 = 2xyz^3,$$

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$\rightarrow$ the partial derivative hits the variable of interest and leaves others alone

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  $\rightarrow$ the partial derivative hits the variable of interest and leaves others alone

- if (say) $[p\mathrm{CO}_2] = T^3\cos(p)$ then (assuming neither T nor p depend on fish at all...)

$$\frac{\partial[p\mathrm{CO}_2]}{\partial(\mathrm{fish})} = \ldots?$$

Vector calculus concepts: integrals ($\int$)

The integral $\int$ can be thought of as the opposite of the derivative, e.g.,

$$p(z) = z^3, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = 3z^2, \qquad \int 3z^2 \,\mathrm{d}z = z^3 + \underline{\text{constant}}$$

A definite integral evaluates difference of integrated quantity at the limits, e.g.,

$$\int_0^3 3z^2 \,\mathrm{d}z = \left[z^3\right]_0^3 = 3^3 - 0^3 = 27$$

- no constant because constant$(z = 3) -$ constant$(z = 0) = 0$

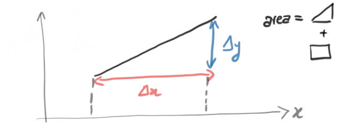
Vector calculus concepts: integrals ($\int$)

- think integral $\int$ as a sum
  → "hence" elongated S
- sum the function over a particular interval, e.g.

$$\int_{z_1}^{0} \rho \, \mathrm{d}z$$

is the sum of density (as a **scalar** field) from z_1 to sea surface denoted $z = 0$

→ see this again in hydrostatic balance (see Lec 7)



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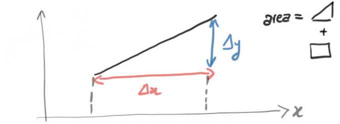

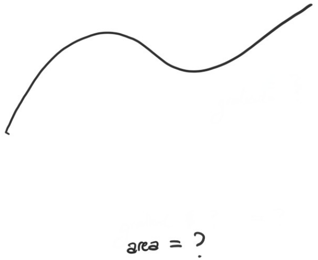

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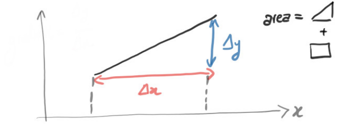

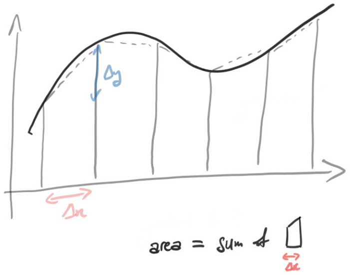

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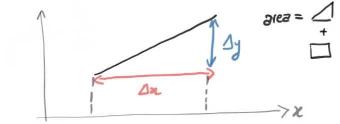

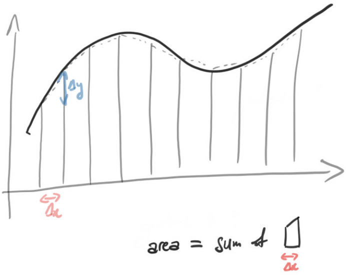

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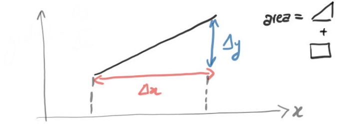

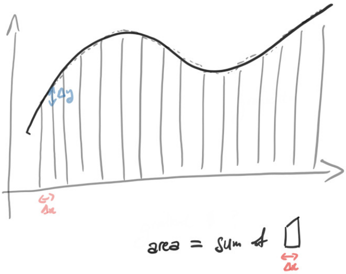

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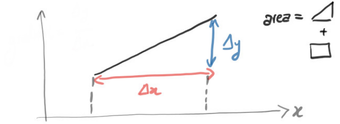

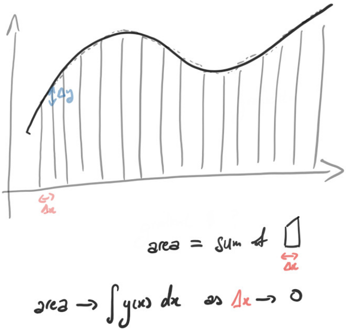

Vector calculus concepts: grad (∇)

The gradient operator ∇ (called "grad" or "nabla")

- acts on a **scalar** field and returns a **vector** field as, e.g.,

$$\nabla p = \frac{\partial p}{\partial x}\boldsymbol{e}_x + \frac{\partial p}{\partial y}\boldsymbol{e}_y + \frac{\partial p}{\partial z}\boldsymbol{e}_z = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \\ \partial p/\partial z \end{pmatrix}$$

- encodes how a **scalar** field (e.g. p) changes in physical space

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again, really just gradients

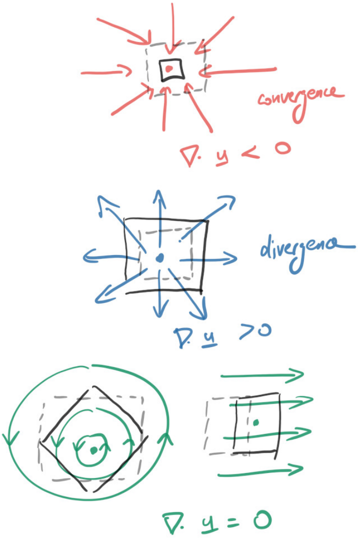
Vector calculus concepts: divergence ($\nabla\cdot$)

The divergence of a vector field $\boldsymbol{u}_3 = (u, v, w)$ is

$$\nabla \cdot \boldsymbol{u}_3 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$\rightarrow$ note $\nabla \cdot \boldsymbol{u}$ is a **scalar** field

- measures con/divergence (cf. compression/expansion) of a vector field
- strongly linked to up/downwelling (see Lec 9)



Vector calculus concepts: curl ($\nabla\times$)

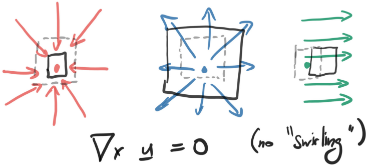
The curl of a vector field $\boldsymbol{u}_3 = (u, v, w)$ is denoted

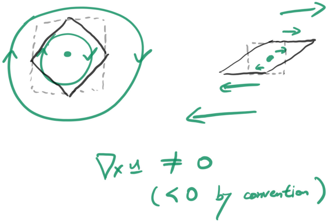
$$\nabla \times \boldsymbol{u}_3 = \text{see Wikipedia}$$

$\to$ note $\nabla \times \boldsymbol{u}_3$ is a **vector** field

- measures direction and strength of swirl/spin-iness
- important concept for rotating systems, wind-driven gyre circulation, eddies etc...

(see basically Lec 8 onwards...)





Summary

- at heart of physical oceanography is some form of

$$\boldsymbol{F} = m\boldsymbol{a} = m\frac{\mathrm{d}\boldsymbol{u}_3}{\mathrm{d}t}$$

$\rightarrow$ residual force $\sim$ mass times acceleration = mass times rate of change of velocity

$\rightarrow$ force and accleration/velocity are vector fields here, mass is a scalar

$\rightarrow$ understand the forces contribute $=$ understand how/why the fluid behaves the way it does (in principle, doesn't mean it's easy...)

Summary

- dealing with scalars and scalar/vector fields
  $\rightarrow$ vector calculus: the language to talk about these objects

term	note	symbols
scalars	magnitude only	u, $\lvert\boldsymbol{u}\rvert$ p
vectors	magnitude and direction	$\boldsymbol{u}_3 = (u, v, w)$, ∇p
dot product	angles, lengths	$\cdot$
cross product	generates a 3rd vector	$\times$
derivative	gradients / rate of change	$\mathrm{d}, \partial, \nabla$
integral	sum	$\int$
div(ergence)	di/convergence	$\nabla \cdot (\cdot)$
curl	swirl/spin-iness	$\nabla \times (\cdot)$

ability to <u>do</u> vector calculus not examined in this course

- but understanding/interpreting them is part of the course

Summary

Default notation for this course:

- x, y, z you can/should think as E-W, N-W and up-down

  $\rightarrow$ zonal, meridional and vertical
- $x, y, z > 0$ is East, North and up

  $\rightarrow$ $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ points East, North and up

  $\rightarrow$ $u, v, w > 0$ is East, North and upward velocity