QF620: Stochastic Modelling in Finance

Group Project Report

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1.0 Types of Options

Vanilla Call/Put Options: enable buying or selling an asset at a future price.

Digital Cash-or-Nothing Options: offer a fixed payout if the asset hits the strike price at expiration.

Digital Asset-or-Nothing Options: pay out the asset itself if it exceeds the strike price at expiration.

Black-Scholes	Call	Put
Vanilla	$C = S \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2)$	$P = K \cdot e^{-rT} \cdot N(-d_2) - S \cdot N(-d_1)$
Digital Cash-or-Nothing	$C_{DCN} = e^{-rT} \cdot \text{cash} \cdot N(d_2)$	$P_{DCN} = e^{-rT} \cdot \text{cash} \cdot N(-d_2)$
Digital Asset-or-Nothing	$C_{DAN} = S \cdot N(d_1)$	$P_{DAN} = S \cdot N(-d_1)$

Bachelier	Call	Put
Vanilla	$\mathrm{C} = e^{-rT} \cdot ((S-K) \cdot N(-d) + \sigma\sqrt{T} \cdot n(-d))$	$\mathrm{P} = e^{-rT} \cdot ((K-S) \cdot N(d) + \sigma\sqrt{T} \cdot n(d))$
Digital Cash-or-Nothing	$\mathrm{C} = e^{-rT} \cdot \text{cash} \cdot N(-d)$	$\mathrm{P} = e^{-rT} \cdot \text{cash} \cdot N(d)$
Digital Asset-or-Nothing	$\mathrm{C} = e^{-rT} \cdot (S \cdot N(-d) + \sigma\sqrt{T} \cdot n(-d))$	$\mathrm{P} = e^{-rT} \cdot (S \cdot N(d) - \sigma\sqrt{T} \cdot n(d))$

In the formulas for Digital Cash-or-Nothing Option Values, C and P represent the value of the call and put options, respectively, and the value here is a fixed cash payment (cash).

Four Models

1.1 Black-Scholes Model

(Developed by Fischer Black and Myron Scholes in 1973, further extended by Robert Merton)

Theory: Based on the principle of no-arbitrage and the assumption of log-normal distribution, this model proposes a partial differential equation for estimating the theoretical prices of European options.

Key Formulas: differential equation $dS_t = rS_t dt + \sigma S_t dW_t$

$C = S_0 N(d_1) - Ke^{-rT} N(d_2)$	$P = Ke^{-rT} N(-d_2) - S_0 N(-d_1)$

let us derive the option pricing formula for a European call option under Black-Scholes model.

$$V_c = e^{-rT} E\left[(S_T - K)^+\right] = e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S_0 e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x - K} \cdot e^{-\frac{x^2}{2}} dx$$

The terms in the $(\cdot)+$ operator will need to be $S_T - K > 0$ positive:

$$S_T - K > 0 \Rightarrow x > \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = x^*$$

$V_c = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{x^*}^{\infty} S_0 e^{\left(r-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x-\frac{x^2}{2}} dx - \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{x^*} Ke^{-\frac{x^2}{2}} dx = \frac{S_0}{\sqrt{2\pi}} \int_{x^*}^{\infty} e^{-(x-\sigma\sqrt{T})^2/2} dx - \frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{x^*} e^{-\frac{x^2}{2}} dx$
$V_c = S_0 \left(\Phi(\infty) - \Phi(x^* - \sigma\sqrt{T})\right) - Ke^{-rT}\Phi(-x^*) = S_0\Phi(-x^* + \sigma\sqrt{T}) - Ke^{-rT}\Phi(-x^*) = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$

Where $d_1 = \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$, and N is the cumulative normal distribution function. The subsequent three models, the Bachelier model, the Black76 model, and the Displaced-diffusion model, employ the same method for calculating d1 and d2. It is suitable for options pricing in stock, index, currency, and some commodity markets, particularly in markets with relatively stable volatility and risk-free interest rates. This model is directly used to calculate the prices of vanilla call and put options and can be adapted for pricing digital options through appropriate mathematical transformations.

1.2 Bachelier Model

Theory: Assumes that the price changes of the underlying asset follow a normal distribution, as opposed to the log-normal distribution assumed in the Black-Scholes model.

Key Formulas

$C = (S_0 - K)N(d) + \sigma\sqrt{T}n(d)$	$P = (K - S_0)N(-d) + \sigma\sqrt{T}n(-d)$

Where $d = \frac{K - S_0}{\sigma\sqrt{T}}$ Since the Bachelier model directly deals with price changes of the asset (rather than log changes), it is particularly suitable for pricing interest rate options, bond options, and other fixed-income products. For markets where asset prices may be close to zero or have limited fluctuations, the Bachelier model provides a more appropriate pricing approach.

1.3 Black76 Model

Theory: Designed specifically for pricing futures options, it takes into account the characteristics of futures contracts, especially the cost and profit structure of futures.

Key Formulas: Its form is similar to the Black-Scholes model, but it uses the futures price F instead of the spot price S.

$C = e^{-rT}[FN(d_1) - KN(d_2)]$	$P = e^{-rT}[KN(-d_2) - FN(-d_1)]$
$\text{Black 76}(F, K, r, \sigma, T) = \text{Black-Scholes}\left(F \cdot e^{-rT}, K, r, \sigma, T\right)$	

Where d1 and d2 are intermediate variables in the model, calculated similarly to the Black-Scholes model, but using the futures price F instead of the spot price S. The formulas for calculating them are as follows:

$d_1 = \frac{\ln(F/K) + (0.5 \cdot \sigma^2)T}{\sigma\sqrt{T}}$	$d_2 = d_1 - \sigma\sqrt{T}$

The Black76 model is particularly suitable for pricing options in commodity and financial futures markets, such as options on oil, metals, agricultural products, and interest rate futures. It can adapt to markets where futures prices and spot prices are not always closely linked.

1.4 Displaced-diffusion Model

An extension of the Black-Scholes model used to capture price behavior more accurately, especially when the market exhibits skewed or fat-tailed price distributions.

Theory: It introduces a displacement factor β to adjust the underlying asset's price, allowing for skewed and fat-tailed price distributions.

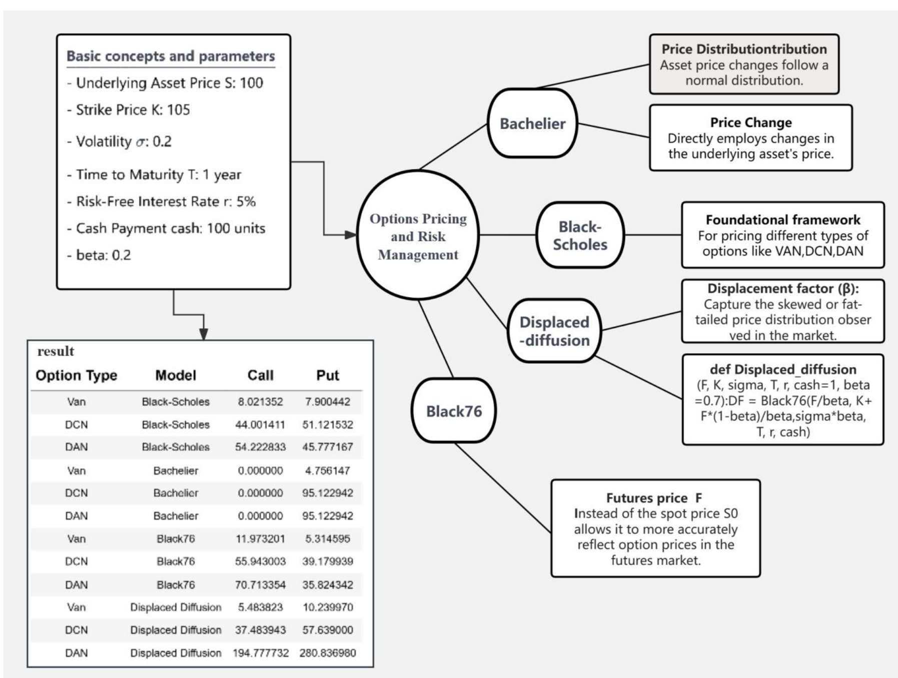
Key Formulas:

The pricing formula for a call option in the displaced-diffusion model is:

$C = e^{-rT}\left[FN(d_1) - (K + \beta(F-K))N(d_2)\right]$	$P = e^{-rT}\left[(K + \beta(F-K))N(-d_2) - FN(-d_1)\right]$
$d_1 = \dfrac{\ln\left(\frac{F}{K+\beta(F-K)}\right) + \left(0.5\cdot(\sigma\beta)^2\right)T}{\sigma\beta\sqrt{T}}$	$d_2 = d_1 - \sigma\beta\sqrt{T}$
$\text{Displaced-Diffusion}(F, K, r, \sigma, T, \beta) = \text{Black-76}\left(\frac{F}{\beta}, \frac{K+(1-\beta)F}{\beta}, r, \sigma\beta, T\right)$	

The displaced-diffusion model is particularly useful when market data shows that the logarithmic changes in the underlying asset's price are not symmetric or when the price distribution's tails are thicker than those of a standard normal distribution. It is widely used in markets exhibiting significant volatility smiles, a phenomenon commonly observed in forex and commodity markets.

Let's examine the relationships among the four models through an example:



Option Type	Model	Call	Put
Van	Black-Scholes	8.021352	7.900442
DCN	Black-Scholes	44.001411	51.121532
DAN	Black-Scholes	54.222833	45.777167
Van	Bachelier	0.000000	4.756147
DCN	Bachelier	0.000000	95.122942
DAN	Bachelier	0.000000	95.122942
Van	Black76	11.973201	5.314595
DCN	Black76	55.943003	39.179939
DAN	Black76	70.713354	35.824342
Van	Displaced Diffusion	5.483823	10.239970
DCN	Displaced Diffusion	37.483943	57.639000
DAN	Displaced Diffusion	194.777732	280.836980

2.0 Model Calibration

2.1 Functions to calibrate

Black-Scholes model:

Call option: $C = S\phi(d_1) - Ke^{-rT}\phi(d_2)$

Put option: $P = -S\phi(-d_1) + Ke^{-rT}\phi(-d_2)$

$hint: d_1 = \frac{log\left(\frac{S}{k}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, d_2 = d_1 - \sigma\sqrt{T}$

SABR Model: $\sigma_{SABR}(F_0, K, \alpha, \beta, \rho, \upsilon) = \frac{\alpha}{(F_0K)^{\frac{1-\beta}{2}}\left\{1+\frac{(1-\beta)^2}{24}\log^2\left(\frac{F_0}{K}\right)+\frac{(1-\beta)^4}{1920}\log^4\left(\frac{F_0}{K}\right)+\ldots\right\}} \times \frac{z}{x(z)}$

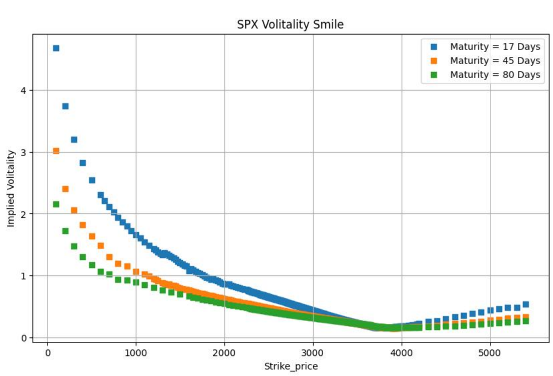
$$\times\left\{1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(F_0K)^{1-\beta}}+\frac{1}{4}\frac{\alpha\beta\rho\upsilon}{(F_0K)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\upsilon^2\right]T+\ldots\right\}$$

$hint: z = \frac{\upsilon}{\alpha}(F_0K)^{\frac{1-\beta}{2}}\log\left(\left(\frac{F_0}{K}\right), x(z) = \log\left[\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right]\right.$

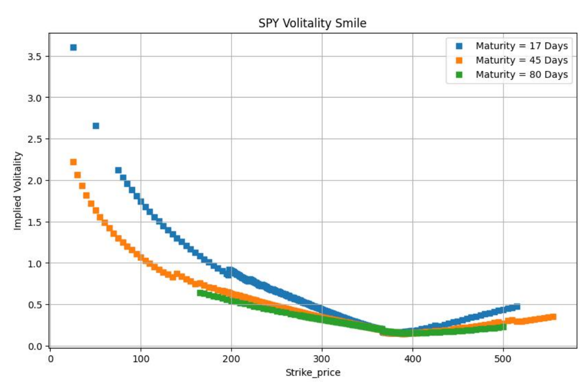
Displaced-Diffusion model: $\text{Displaced} - \text{Diffusion} = \text{Black}\left(\frac{F_0}{\beta}, K + \frac{1-\beta}{\beta}F_0, \sigma\beta, T\right)$

Implied Volatility is defined as the value of σ such that the Black Scholes Call option formula produces the same price as the observable option price.

First, we calculate and extract the implied volatility from the given options data and organize the results into a list of data frames with different expirations. The volatility curves for these different maturities are then plotted, forming what is known as the "Volatility Smile".



2-1 SPX Volatility Smile



2-2 SPY Volatility Smile

2.2 Calibrate models to match the option

Displaced-diffusion model

Before we calibrate the DD model, we calculate ATM volatility. We use σ, which is implied volatility calculated by Black-Scholes model and β, which is solved by least-square method using black-sholes implied volatility calculation on Displaced-Diffusion price. The result is:

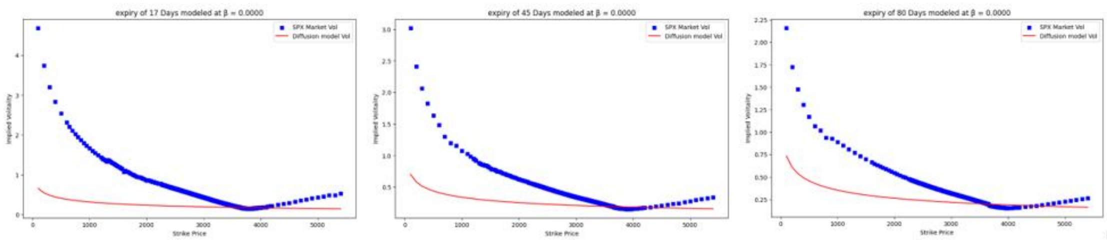
Days to expiry	SPX_σ	SPX_β
17	0.178215	1.890812e-07
45	0.188860	8.050313e-12
80	0.197174	5.006780e-07

2-3 DD of SPX

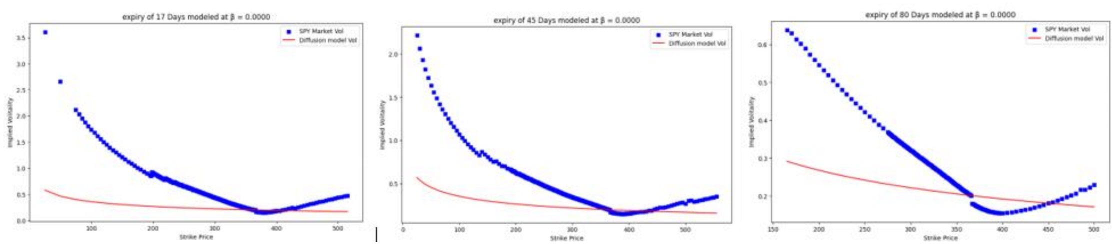
Days to expiry	SPY_σ	SPY_β
17	0.201551	5.420452e-07
45	0.197767	6.791159e-11
80	0.200653	5.988021e-10

2-4 DD of SPY

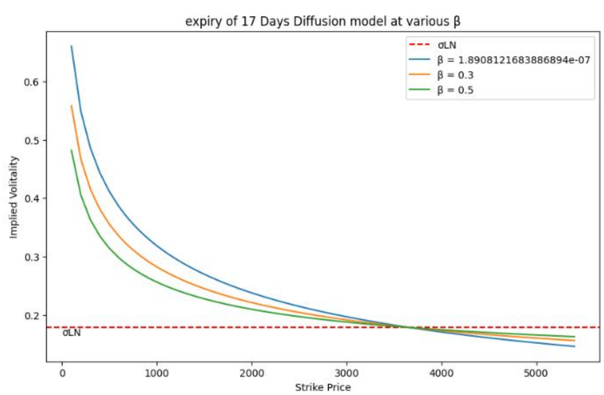
And then, we plot the result:



2-5 Volatility curves for different maturities-SPX



2-6 Volatility curves for different maturities-SPY



2-7 various β

The conclusion is that: β range from (0,1) controls the slope. Bigger β means more similarity to Log-Normal option pricing

SABR model (fix β=0.7).

We use the parameters α, ρ and ν(The reason to choose them is that the output of the SABR model matches the implied volatilities observed in the market, so called model calibration).we get the result:

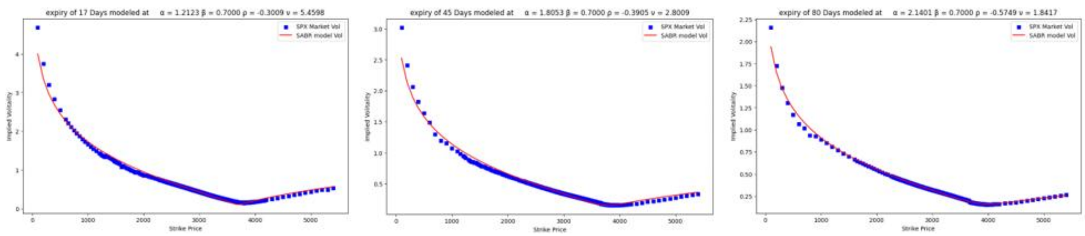
Days to expiry	SPX_α	SPX_β	SPX_ρ	SPX_v
17	1.212295	0.7	-0.300898	5.459778
45	1.805269	0.7	-0.390537	2.800936
80	2.140142	0.7	-0.574916	1.841737

2-8 SABR-SPX

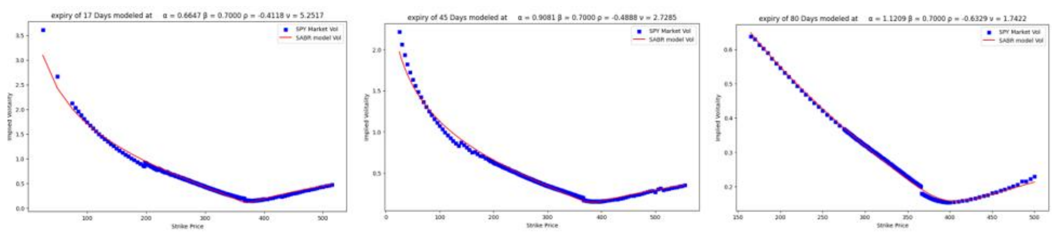
Days to expiry	SPY_α	SPY_β	SPY_ρ	SPY_v
17	0.664672	0.7	-0.411839	5.251738
45	0.908142	0.7	-0.488776	2.728522
80	1.120922	0.7	-0.632938	1.742224

2-9 SABR-SPY

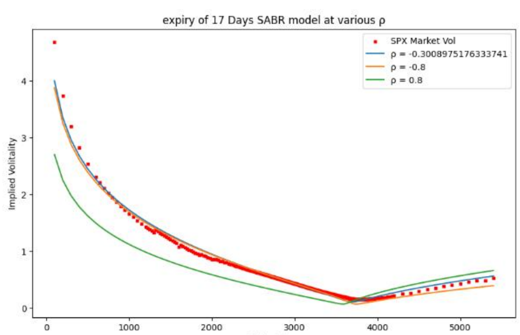
The results' plot:



2-10 SABR-SPX-plot

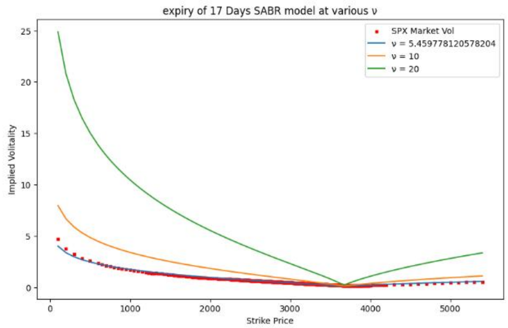


2-11 SABR-SPY-plot



2-12 Various ρ

The conclusion is that ρ range from (-1, 1), measures stock return and volatility. Negative correlation increases the price of out-of-the-money put options and decreases the price of out-of-the-money call options. Thus, increasing ρ, the implied volitality seems rotating anti-clockwise.



The reason is that υ range from $(0, \infty)$, controls the curvature. Bigger υ means more bending on the implied volatility graph.

2-13 Various υ

2.3 Conclusion

In general, Black-Scholes model assumes that sigma is a constant value, but we have smile, so it is not accurate. The beta of DD is basically 0, so one degree of freedom as beta is not accurate. In addition, beta = 0 in DD is equivalent to Bachelier model, which also means that Bachelier cannot fit market data. SABR model has four degrees of freedom, which perfectly matches smile, indicating that the four variables can accurately describe the market.

3.0 Static Replication

To solve this question, we need to introduce theoretical frameworks that we just inferred before: Black-Scholes Model, Bashier Model and SABR Model.

We first calculate the implied volatility of option data, mainly through OTM, that is, the option strike price is higher than the current price of the underlying asset for a call option, or lower than the current price of the underlying asset for a put option, which means Therefore, if the option is executed immediately, there will be no profit.

Secondly, based on the given S_0, the implied volatility corresponding to the closest execution prices is calculated, using the Black-Scholes Model and Bachelier model mentioned above. At the same time, according to part2, the SABR model was also obtained. The reasons are:

a. Volatility surface calibration

The implied volatility of ATM is often an important point in the volatility surface, and by comparing the ATM volatility calculated by different models, the model can be better calibrated to fit the volatility surface observed in the market.

b. Model selection and parameter adjustment

If the volatility calculated by different models differs significantly from the market observed OTM implied volatility, this may indicate that the model needs to be adjusted, or that other more complex forms need to be considered. By comparing the results, you can choose the model that best suits specific market conditions or adjust model parameters.

	SPX T=45	SPY T=45
sigma_BS	0.188860	0.197767
sigma_Bach	688.146018	72.049480

3-1 Sigma

(using Black-Scholes Model and Bachelier model)

	SPX T=45	SPY T=45
α	1.805269	0.908142
β	0.700000	0.700000
ρ	-0.390537	-0.488776
ν	2.800936	2.728522

3-2 Sigma (using SABR model)

The following is our topic. The first step is to solve the first question, which is Payoff function. We need to use this formula: $S_T^{1/3} + 1.5\log(S_T) + 10.0$

The result of the Static-replication of European payoff is:

	Black-Scholes	Bachelier	SABR
SPX T=45	37.704449	37.704501	37.700448
SPY T=45	25.994212	25.994234	25.992678

3-3 question 1

We need to solve the second problem, which is "Model-free" integrated variance. A formula needs to be used here: $\sigma_{MF}^2 T = E[\int_0^T \sigma_t^2\, dt]$

The result obtained is:

	Black-Scholes	Bachelier	SABR
SPX T=45	0.188860	0.188515	0.226413
SPY T=45	0.197767	0.197563	0.220904

3-4 question 2

It can be found that the value of SABR is meaningful because it makes use of all market data. It is because the implied volatility smile curve is constantly fitted. And the model free variance calculated by integrated variance is much different from Bachelier model, and is closer to the sigma of Black-Scholes Model, so Black-Scholes Model is reasonable as a reporting model.

4.0 Dynamic hedging

The insight behind the Black-Scholes formula for options valuation is the recognition that, if you know the future volatility of a stock, you can replicate an option payoff exactly by a continuous rebalancing of a portfolio consisting of the underlying stock and a risk-free bond.

The hedger now follows a Black-Scholes hedging strategy, rehedging at discrete, evenly spaced time intervals as the underlying stock changes. At expiration, the hedger delivers the option payoff to the option holder, and unwinds the hedge. We are interested in understanding the final profit or loss of this strategy:

Final profit&loss = Value of Black-Scholes hedge at T -Final option payoff

If the hedger had followed the exact Black-Scholes replication strategy, rehedging continuously as the underlying stock evolved towards its final value at expiration, then, no matter what path the stock took, the final P&L would be exactly zero. This is the essence of the statement that the Black-Scholes formula provides the ‘‘fair’’ value of the option.

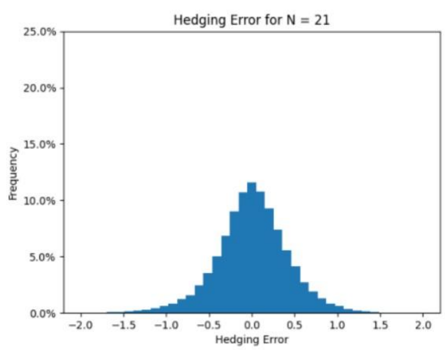
When the replication strategy deviates from the exact Black-Scholes method, the final P&L may deviate from zero. This deviation is called the replication error

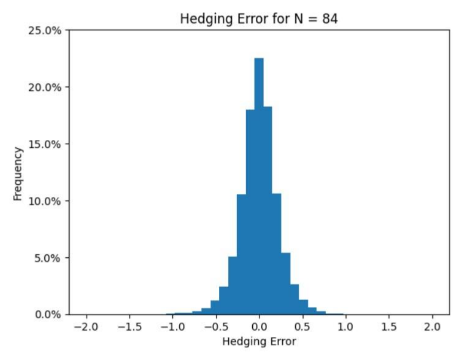
However, in many cases, our hedge is not perfect, which means that there will be replication errors, which are sometimes conducive to profit and loss, and sometimes not conducive to profit and loss.

To test the range, we ran 50000 random paths by same hedging strategy. We assume a volatility of 20% and a discount rate of 5%, and we make N rehedging trades evenly spaced throughout the life of the Black-Scholes model.

First we choose N=21, and here is its profit and loss histogram

Second we choose N=84, and here is its profit and loss histogram





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To make this more intuitive, I've created a chart to show statistical summary of the simulated profit/loss.

	Mean P&L	Standard Dev. of P&L	StDev of P&L as a % of option premium
Number of trades			
21	0.004117	0.423816	16.871198
84	0.001180	0.217767	8.668840

We can summarize the following conclusions: First, within the statistical error range of the simulation, the average profit and loss is 0. Second, more frequent hedging will reduce the standard deviation of the final profit and loss, as shown in the table above, as part of the option premium, when N=21, the standard deviation is 16.871, while when N=84, the standard deviation is only 8.667, which is nearly half lower than that of N=21.

So we can conclude that the relationship between the frequency of rebalancing in hedging trading strategies and the final profit and loss standard deviation typically indicates that as the frequency of rebalancing increases, the standard deviation of the final profit and loss decreases.