Inverse Laplace Transform

Defn –

$$L^{-1}\{X(s)\} = x(t).$$

Case 1) – When there are simple pole:

Obtain inverse L.T. of $X(s) = \dfrac{s - 1/2}{s^2 + 3/4\, s + 1/8}$

ROC : $\sigma > -1/4$

⇒ **Soln** – Obtain the roots (poles) of denominator

$$X(s) = \frac{s - 1/2}{s^2 + 3/4\, s + 1/8} = \frac{s - 1/2}{(s + 1/2)(s + 1/4)}$$

In partial fraction expansion $X(s)$ is

$$X(s) = \frac{A_1}{s + 1/2} + \frac{A_2}{s + 1/4}$$

$$A_1 = (s - P_1)\, X(s)\Big|_{s = P_1}$$

$$= \left.\frac{\cancel{(s + 1/2)}\,(s - 1/2)}{\cancel{(s + 1/2)}(s + 1/4)}\right|_{s = -1/2}$$

$$= \frac{-1}{-1/2 + 1/4} = \frac{-1}{-1/4} = 4$$

$$\boxed{A_1 = 4}$$

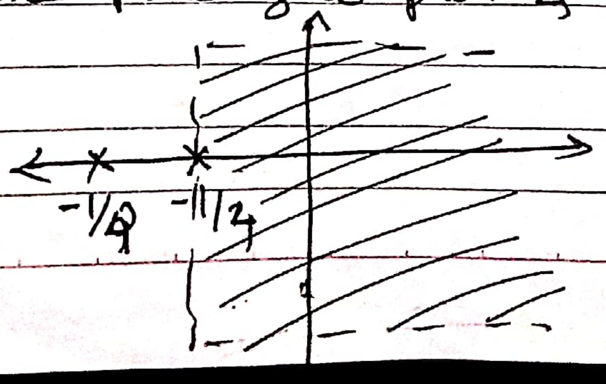
$$A_2 = (s - P_2)\, X(s)\Big|_{s = P_2}$$

$$= \left.\frac{\cancel{(s + 1/4)}\,(s - 1/2)}{(s + 1/2)\cancel{(s + 1/4)}}\right|_{s = -1/4}$$

$$= \frac{-1/4 - 1/2}{-1/4 + 1/2} = \frac{-3/4}{+1/4} = -3$$

$$X(s) = \frac{4}{s + 1/2} - \frac{3}{s + 1/4}$$

The pole-zero plot of given ROC is shown in fig.



Here both poles are on left side of ROC. Both signal corresponds to right handed signal (multiplied by $u(t)$).

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We have standard L.T. pair

$$e^{-at}u(t) \xleftrightarrow{L.T.} \frac{1}{s+a} \quad ; \text{ROC } \sigma > -a$$

ie. $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t)$; ROC $\sigma > -a$.

$$L^{-1}\left\{\frac{4}{s+1/2}\right\} = 4e^{-1/2t}u(t)$$

$$L^{-1}\left\{\frac{3}{s+1/4}\right\} = 3e^{-1/4t}u(t).$$

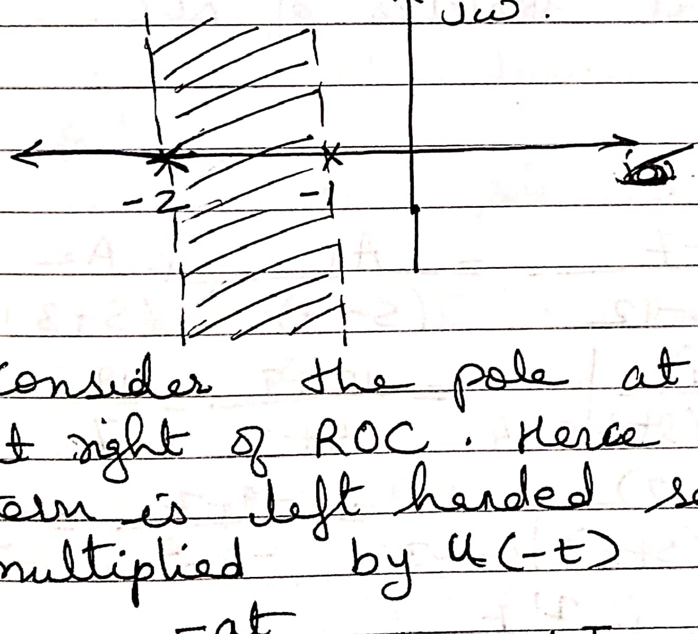
$$x(t) = 4e^{-1/2t}u(t) - 3e^{-1/4t}u(t).$$

2) If $X(s) = \frac{-2}{s+1} + \frac{4}{s+2}$ ROC: $-2 < s < -1$

then find $x(t)$

⇒ Here poles are are at $P_1 = -1$ & $P_2 = -2$

$X(s)$ is already in P.F.E.



Consider the pole at -1. This pole is at right of ROC. Hence corressponding term is left handed sequence. ie. it is multiplied by $u(-t)$

$$-e^{-at}u(-t) \xleftrightarrow{L.T} \frac{1}{s+a} \quad \text{ROC. } \sigma < -a$$

$$L^{-1}\left\{\frac{1}{s+a}\right\} = -e^{-at}u(-t)$$

$$L^{-1}\left\{\frac{2}{s+1}\right\} = -(2)e^{-t}u(-t) = 2e^{-t}u(-t)$$

Consider pole at -2. This pole is to the left of ROC. Hence corresponding time domain term is right handed seq^n. i.e. multiply by $u(t)$.

$$e^{-at}u(t) \xleftrightarrow{L.T.} \frac{1}{s+a} \qquad \text{ROC } \sigma > -a$$

$$L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t).$$

$$L^{-1}\left\{\frac{4}{s+2}\right\} = 4e^{-2t}u(t).$$

$$\boxed{x(t) = 2e^{-t}u(t) + 4e^{-2t}u(t)}$$

3) Obtain I.L.T. of fun^n $X(s) = \frac{3s+7}{s^2-s-12}$ for foll. ROC. Also comment on stability & causality of the system for each of ROC cond^{n}s. Support your answer with appropriate sketches of ROC.

1) $Re(s) > 4$
2) $Re(s) < -3$
3) $-3 < Re(s) < 4$

$$\Rightarrow X(s) = \frac{3s+7}{s^2-s-12} = \frac{A_1}{(s-4)} + \frac{A_2}{(s+3)}$$

$$A_1 = \left.\frac{(3s+7)(s-4)}{(s-4)(s+3)}\right|_{s=4} = \frac{12+7}{4+3} = \frac{19}{7}.$$

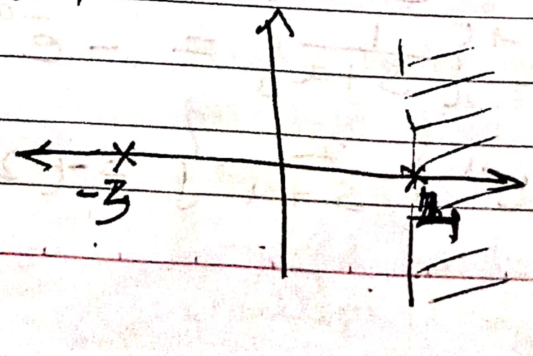
$$A_2 = \left.\frac{(s+3)(3s+7)}{(s-4)(s+3)}\right|_{s=-3} = \frac{-9+7}{-3-4} = \frac{-2}{-7} = \frac{2}{7}$$

$$X(s) = \frac{19/7}{(s-4)} + \frac{2/7}{(s+3)}$$

1) $Re(s) > 4$

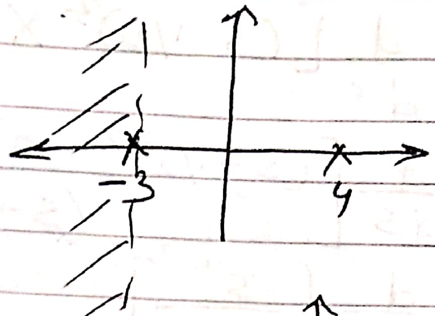
Signal is causal

ROC lies on right side, but unstable system.



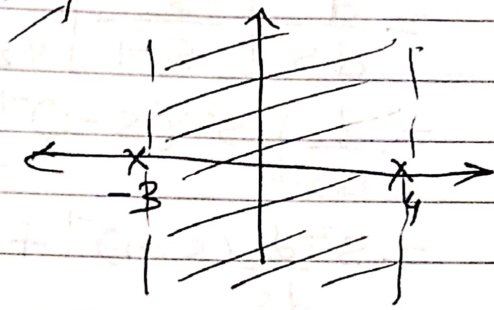
2) $Re(s) < -3$

signal is left sided anti-causal & unstable.



3) $-3 < Re(s) < 4$

The signal is non-causal and stable.



Case-II When there are repeated poles.

We have to add extra terms in the eqn of $X(s)$. For such extra terms the coefficients A are calculated by taking derivatives of $X(s)$.

1) Find I.L.T. of $X(s) = \dfrac{s^2}{(s-1)(s-1/2)^2}$; for the causal time domain signal.

$\Rightarrow$ There is a single pole at $s=1$, repeated poles at $+1/2$.

In ~~partial~~ Partial Fraction Expansion

$$X(s) = \frac{A_1}{(s-1)} + \frac{A_2}{(s-1/2)} + \frac{A_3}{(s-1/2)^2}$$; 2nd term is added term.

The coeff. A_1 & A_3 are calculated using PFE.

$$A_1 = \cancel{(s-1)}\frac{s^2}{\cancel{(s-1)}(s-1/2)^2}\bigg|_{s=1} = \frac{1}{(1-1/2)^2} = \frac{1}{(1/2)^2} = 4$$

$$A_3 = \cancel{(s-1/2)^2}\frac{s^2}{(s-1)\cancel{(s-1/2)^2}}\bigg|_{s=1/2} = \frac{1/4}{(1/2-1)} = \frac{1}{4} \times \frac{-2}{1}$$

$$A_3 = -\frac{1}{2}$$

$$A_1 = 4,\ A_3 = -\frac{1}{2}$$

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$$A_2 = \frac{d}{ds}\left[(s-1/2)^2 X(s)\right|_{s=\frac{1}{2}}$$

$$= \frac{d}{ds}\left[\cancel{(s-1/2)^2}\frac{s^2}{(s-1)\cancel{(s-1/2)^2}}\right]_{s=1/2}$$

$$= \frac{d}{ds}\left[\frac{s^2}{s-1}\right]_{1/2=s.}$$

$$= \frac{d}{ds}\left[s^2(s-1)^{-1}\right]_{s=1/2}$$

$$= s^2\frac{d}{ds}(s-1)^{-1} + (s-1)^{-1}\frac{d}{ds}s^2\Big|_{s=1/2}$$

$$= \left[s^2(-1)(s-1)^{-2} + (s-1)^{-1}.2s\right|_{s=1/2}$$

$$= \frac{1}{4}(-1)\frac{1}{(1/2-1)^2} + \frac{1}{(1/2-1)}2.1/2.$$

$$= \frac{-4}{4} - \frac{2}{1} = -1-2 = -3$$

$$A_2 = -3$$

$$X(s) = \frac{4}{(s-1)} - \frac{3}{(s-1/2)} - \frac{1/2}{(s-1/2)^2}$$

$$L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}u(t)$$

$$L^{-1}\left\{\frac{1}{s-1/2}\right\} = e^{1/2t}u(t)$$

$$t^n e^{at} \xleftrightarrow{L.T.} \frac{n!}{(s-a)^{n+1}}$$

$$L^{-1}\left\{\frac{1}{(s-1/2)^2}\right\} = L^{-1}\left\{\frac{1!}{(s-1/2)^{1+1}}\right\} = t e^{1/2t}$$

$$x(t) = 4e^{t}u(t) - 3e^{1/2t}u(t) - \frac{1}{2}te^{1/2t}.$$

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27 Obtain I.L.T. of $X(s) = \frac{8}{(s+2)^3(s+4)}$

$\Rightarrow$ Case of repeated roots,

$$X(s) = \frac{A_1}{(s+2)} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{A_4}{(s+4)}.$$

$$A_4 = (s+4) \cdot \frac{8}{(s+2)^3(s+4)}\Big|_{s=-4} = \frac{8}{-8} = -1.$$

$$A_3 = (s+2)^3 \frac{8}{(s+2)^3(s+4)}\Big|_{s=-2} = \frac{8}{2} = 4.$$

$$\boxed{A_3 = 4}$$

$$A_1 = \frac{d^2}{ds^2}\left[(s+2)^3 \cdot X(s)\right]\Big|_{s=-2}$$

$$= \frac{d^2}{ds^2}\left[(s+2)^3 \cdot \frac{8}{(s+2)^3 \cdot (s+4)}\right]\Big|_{s=-2}$$

$$= \frac{d^2}{ds^2}\left[\frac{8}{(s+4)}\right]_{s=-2} = \frac{d}{ds}\left[\frac{(s+4)0 - 8(1)}{(s+4)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-8}{(s+4)^2}\right]_{s=-2} = \left[\frac{0+8(2(s+4))}{(s+4)^4}\right]_{s=-2}$$

$$= \frac{+16}{(s+4)^3}\Big|_{s=-2} = \frac{+16}{8}$$

$$A_1 = +2$$

$$A_2 = \frac{d}{ds}\left[(s+2)^3 X(s)\right]_{s=-2} = \left[\frac{-8}{(s+4)^2}\right]_{s=-2}$$

$$A_2 = -2$$

$$L^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = t^n e^{at} u(t).$$

$$X(s) = \frac{2}{(s+2)} - \frac{2}{(s+2)^2} + \frac{4}{(s+2)^3} - \frac{1}{(s+4)}$$

$$x(t) = 2e^{-2t}u(t) - 2t\,e^{-2t}u(t) + t^2 8 e^{-2t}u(t) - e^{-4t}u(t)$$

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Case III – when there are complex roots. Here the poles are complex conjugate of each other. Transformation in time domain is same as in case I.

1) Find the inverse L.T. of $X(s) = \frac{1}{(s^2+1)(s+1)}$

$\Rightarrow$ $X(s) = \frac{1}{(s-j)(s+j)(s+1)}$

$\sin\omega t = \frac{\omega}{s^2+\omega^2}$

$e^{-at}\sin\omega t = \frac{\omega}{(s+a)^2+\omega^2}$

By P.F.E

$X(s) = \frac{A_1}{s-j} + \frac{A_2}{s+j} + \frac{A_3}{s+1}$

$A_1 = (s-j)\frac{1}{(s+j)(s-j)(s+1)}\Big|_{s=j} = \frac{1}{2j(j+1)}$

$A_1 = \frac{1}{2j^2+2j} = \frac{1}{-2+2j} = \frac{1}{2(j-1)}$

$A_2 = (s+j)\frac{1}{(s+j)(s-j)(s+1)}\Big|_{s=-j} = \frac{1}{-2j(-j+1)}$

$= \frac{1}{2j^2-2j} = \frac{1}{-2-2j} = \frac{1}{-2(1+j)}$

$A_3 = \frac{1}{(s-j)(s+j)}\Big|_{s=-1} = \frac{1}{(-1-j)(-1+j)} = \frac{1}{1-j^2} = \frac{1}{2}$

$\therefore X(s) = \frac{1}{2(j-1)(s-j)} + \frac{1}{-2(1+j)(s+j)} + \frac{1}{2(s+1)}$

$= \frac{1}{\cancel{2j}(2j(j+1))} e^{jt}u(t) - \frac{1}{2(1+j)} e^{-jt}u(t) + \frac{1}{2}e^{-t}u(t)$

$\cancel{= -\frac{1}{2}\left[\frac{e^{jt} - e^{-jt}}{2j(1+j)}\right]u(t) + \frac{1}{2}e^{-t}u(t)}$

$\cancel{\boxed{X(s) = -\frac{1}{2}\sin t\, u(t) + \frac{1}{2}e^{-t}u(t)}}$

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$$= \frac{1}{2(j-1)} e^{jt} u(t) - \frac{1}{2(j+1)} e^{-jt} u(t)$$

$$= \frac{1}{2} u(t) \left[\frac{e^{jt}}{(j-1)} - \frac{e^{-jt}}{(j+1)} \right]$$

$$= \frac{1}{2} u(t) \left[\frac{je^{jt} + e^{jt} - je^{-jt} - e^{-jt}}{(j^2-1) \Rightarrow (-2)} \right]$$

$$= \frac{-1}{2} u(t) \left[\frac{je^{jt} - je^{-jt}}{2} + \frac{e^{jt} - e^{-jt}}{2} \right]$$

$$= \frac{-1}{2} u(t) \left[-\sin t + \cos t \right]$$

$$x(t) = -\frac{1}{2} [-\sin t + \cos t] u(t) + \frac{1}{2} e^{-t} u(t)$$

Application of L.T. to LTI System Analysis
Solution of Differential Equations

1) Solve the differential eqⁿ

$$\frac{d(y(t))}{dt} + 2y(t) = x(t)$$ for input $x(t) = e^{-3t}u(t)$

Assume zero initial condⁿ.

$\Rightarrow$ $[sY(s) - y(0^-)] + 2Y(s) = X(s)$ is the L.T. of differential eqⁿ

$y(0^-) = 0$

$\therefore sY(s) + 2Y(s) = X(s)$

$Y(s)(s+2) = X(s)$.

$x(t) = e^{-3t}u(t) \quad \therefore X(s) = \dfrac{1}{(s+3)}$

$$Y(s) = \frac{1}{(s+2)(s+3)}$$

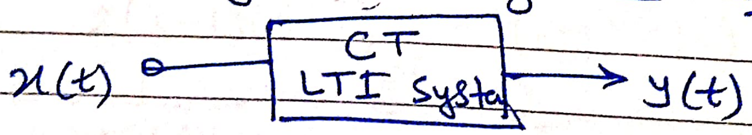
$$A_1 = (s+2)\frac{1}{(s+2)(s+3)}\Big|_{s=-2} = 1$$

$$A_2 = \frac{1}{s+2}\Big|_{s=-3} = -1$$

$$Y(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

$$y(t) = e^{-2t}u(t) - e^{-3t}u(t).$$

System Analysis through Evaluation of T.F.



$$y(t) = x(t) * h(t)$$

L.T. $\quad Y(s) = X(s) . H(s)$.

$$\therefore H(s) = \frac{Y(s)}{X(s)}.$$

H(s) - System Transfer function

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1) Find the T.F. of the system governed by foll. differential eqn.

$$\frac{d^3y(t)}{dt^3} + 8\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 11y(t) = x(t-2)$$

$\Rightarrow$ $s^3 y(s) + 8 s^2 Y(s) + 6s Y(s) + 11 Y(s) = e^{-2s} X(s)$

Now, T.F is

$$Y(s) = [s^3 + 8s^2 + 6s + 11] = e^{-2s} X(s).$$

$$\therefore H(s) = \frac{Y(s)}{X(s)} = \frac{e^{-2s}}{[s^3+8s^2+6s+11]}$$

2) A stable system has i/p $x(t)$ & o/p $y(t)$. using L.T. determine T.F. & impulse response $h(t)$ of the system. $x(t) = e^{-2t}u(t)$

$y(t) = -2e^{-t}u(t) + 2e^{-3t}u(t)$.

$\Rightarrow$ $X(s) = \frac{1}{s+2}$ $\quad Y(s) = \frac{-2}{(s+1)} + \frac{2}{(s+3)}$

$$H(s) = \frac{Y(s)}{X(s)} = \left[\frac{-2}{(s+1)} + \frac{2}{s+3}\right](s+2)$$

$$H(s) = \left[\frac{-2s-6+2s+2}{(s+1)(s+3)}\right](s+2)$$

$$H(s) = \frac{-4s-8}{(s+1)(s+3)}$$

Perform P.F.E

$$H(s) = \frac{A_1}{(s+1)} + \frac{A_2}{s+3}$$

$A_1 = -2$ $\quad \therefore H(s) = \frac{-2}{(s+1)} - \frac{2}{(s+3)}$

$A_2 = -2$

$$\therefore h(t) = -2e^{-t}u(t) - 2e^{-3t}u(t).$$

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Pr The differential eqn of system is given by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

with $y(\bar{0}) = 3$, $\left.\frac{dy(t)}{dt}\right|_{t=\bar{0}} = -5$. Determine o/p of system $x(t) = 2u(t)$

$$\Rightarrow \frac{d^2y(t)}{dt^2} = s^2 \cancel{X}(s) - \left.\frac{dy(t)}{dt}\right|_{t=\bar{0}} - sy(\bar{0})$$

$$= s^2 y(s) - sy(\bar{0}) - \frac{dy(t)}{dt}$$

Taking LT on both sides

$$\left[s^2 y(s) - sy(0) - \frac{dy(t)}{dt}\right] + 3\left[sY(s) - y(\bar{0})\right] + 2Y(s) = X(s)$$

$x(t) = 2u(t)$

$\therefore X(s) = \frac{2}{s}$ $\quad y(\bar{0}) = 3$, $\left.\frac{dy(t)}{dt}\right|_{t=\bar{0}} = -5$

$$\left[s^2 Y(s) - 3s + 5\right] + 3sY(s) - 9 + 2Y(s) = 2/s.$$

$$Y(s)\left[s^2 + 3s + 2\right] - 3s - 4 = 2/s$$

$$Y(s)\left[s^2 + 3s + 2\right] = 2/s + 3s + 4$$

$$\therefore Y(s) = \frac{3s^2 + 4s + 2}{s(s^2 + 3s + 2)} = \frac{3s^2 + 4s + 2}{s(s+2)(s+1)}$$

$$= \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+1}$$

$$A_1 = s.Y(s)\big|_{s=0} = \frac{2}{2} = 1$$

$$A_2 = \left.\frac{(3s^2 + 4s + 2)}{s(s+1)}\right|_{s=-2} = \frac{12 - 8 + 2}{(-2)(-1)} = \frac{6}{2} = 3$$

$$A_3 = \left.\frac{3s^2 + 4s + 2}{s(s+2)}\right|_{s=-1} = \frac{3 - 4 + 2}{(-1)(1)} = \frac{1}{-1} = -1$$

$$Y(s) = \frac{1}{s} + \frac{3}{s+2} - \frac{1}{s+1}$$

Taking I.L.T.

$$y(t) = u(t) + 3e^{-2t}u(t) - e^{-t}u(t).$$

Properties of System using T.F & ROC.

⇒ Pole/zero of ROC of system T.F. H(S) provide foll. inform^n

1. Freq^n response
2. Causality
3. Stability etc.

1.

1. **Freq^n response** – The freq^n response of LTI system is directly obtained by $s = j\omega$ in T.F.

2. **Causality** – If the ROC of H(S) of LTI system must be 'entire region' in the S-plane to the right of right most pole then that system is causal.

3. **Stability** –

- If ROC of H(S) include 'jω' axis then that LTI system is stable.
- If all the poles of H(S) must lie in the left half of S-plane then the system is causal and stable.
- The system is marginally stable if poles of H(S) are on the jω axis.

- For a stable LTI, CT causal system, the poles should lie on the left half of s-plane & the imaginary axis should be included in the ROC.
- For a stable LTI, CT non-causal system, the imaginary axis should be included in the ROC.