# MIE 1622: Assignment 1 – Mean-Variance Portfolio Selection Strategies

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## Introduction

The purpose of this assignment was to gain an introduction into mean-variance portfolio optimization by using a prepared Jupyter notebook and data from 20 stocks to test different portfolio strategies and compare results. The CPLEX python API was implemented to solve convex optimization problems subject to constraints for minimum variance and maximum Sharpe ratio optimization.

The four strategies examined were buy and hold, equally weighted, minimum variance, and maximum Sharpe ratio. An additional strategy was tested at the end to improve performance from the pre-defined strategies, which will be discussed further. No short selling was allowed for this strategy, and a 0.5% transaction fee was not Included in the optimization but worked around to keep free cash non-zero. A total of 12 rebalancing periods (bi-monthly over 2 years) were used to reassess portfolio weights for most of the strategies.

## Transaction fee adherence strategy

For all implemented strategies apart from “buy and hold”, a static fractional transaction fee needed to be addressed by re-balancing the calculated portfolio weights after solving the optimization problems at each rebalancing period. The strategy I chose to implement was simple: After the optimal portfolio weights were found, if not enough cash was leftover to pay for transaction fees, we would reduce all weights evenly by a small amount (99% their original value), then recalculate how many stocks could be bought before checking if there was enough cash leftover, then the cycle would repeat. Note that we were also left to decide how to handle the fractional stock purchases which would result from the optimization. I chose a simple but effective approach, which was to implement the numpy.floor method, which effectively just rounds down any floating value to the nearest integer. Since the weights would likely need to be reduced afterwards in order to account for the transaction costs, It felt unnecessary to try and implement a more complicated method to round some stocks up and some down while keeping a non-negative cash amount. Overall, this strategy worked, though my approach to simply reducing the weights evenly until the cash amount was non-negative is not optimal, in the future I would have simply included transaction costs in the optimization problem using CPLEX.

## Mathematical Formulations

As their name suggests, the “Buy and Hold” and “Equal Weight” strategies work by keeping the same stocks through the rebalancing periods, and by rebalancing to ensure roughly equal weights respectively. The Minimum Variance and Max Sharpe Ratio portfolios require the following mathematical formulations to implement in CPLEX:

**Minimum Variance Portfolio (MVP)**

The objective of the Minimum Variance Portfolio is to minimize the portfolio's overall variance, given a set of assets and their covariances, without considering the expected returns. The optimization problem can be formulated as:
Minimize: $\sigma_P^2 = w^T \Sigma w$
Subject to:
- $\sum_{i=1}^{n} w_i = 1$
- $w \geq 0$ (no short selling)

where:
- $\sigma_P^2$ is the portfolio variance,
- $w$ is the vector of portfolio weights,
- $\Sigma$ is the covariance matrix of asset returns, and
- $n$ is the number of assets in the portfolio.

**Maximum Sharpe Ratio Portfolio (MSRP)**
The Maximum Sharpe Ratio Portfolio optimization aims to maximize the Sharpe ratio of the portfolio, which is the ratio of the portfolio's excess return over its standard deviation. The optimization can be expressed as:
Maximize: $S = \frac{w^T(\mu - r_f)}{w^T \Sigma w}$
Subject to:
- $\sum_{i=1}^{n} w_i = 1$
- $w \geq 0$ (no short selling)

Where:
- $r_f$ is the risk-free rate set to 2.5% annually

Note that the constraints for this problem are the same as that of minimum variance, however the objective is significantly more complicated as it is an optimization over a non-linear, possibly non-convex problem. This equation can be reframed to introduce a new variable:
- $w^* = \frac{y}{\kappa}$ Where $\kappa$ is an optimization parameter.

We then re-write the objective in terms of only the denominator (Maximizing the variance reciprocal), which is the same as minimizing the variance:
Minimize: $\sigma_P^2 * \kappa = y^T \Sigma y$
Subject to:
- $\sum_{i=1}^{n} (\mu_i - r_f) * y_i = 1$
- $\sum_{i=1}^{n} y_i = \kappa$
- $y_i \geq 0$
- $\kappa \geq 0$

This reformulation is useful as we ensure that the objective can be minimized and is convex, while introducing the original numerator of the objective as a constraint, which is essentially the excess return on the risk-free rate.

## Results
The 4 strategies produced combined portfolio and free cash values which have been amalgamated in the table below according to the start and end dates of each periods.

*Table 1: Portfolio value for all strategies at period start and end dates*

| Period | Buy and Hold | | Equal Weight | |
|---|---|---|---|---|
| | Start | End | Start | End |
| 1 | $ 1,000,013 | $ 893,957 | $ 995,013 | $ 995,013 |
| 2 | $ 945,076 | $ 949,228 | $ 990,063 | $ 916,799 |
| 3 | $ 937,917 | $ 913,415 | $ 883,420 | $ 992,908 |
| 4 | $ 905,420 | $ 994,693 | $ 986,088 | $ 1,127,476 |
| 5 | $ 993,195 | $ 971,914 | $ 1,135,569 | $ 1,062,069 |
| 6 | $ 983,801 | $ 1,004,436 | $ 1,071,457 | $ 1,269,331 |
| 7 | $ 1,005,601 | $ 956,244 | $ 1,254,951 | $ 1,346,776 |
| 8 | $ 957,791 | $ 1,019,731 | $ 1,379,051 | $ 1,486,626 |
| 9 | $ 1,022,205 | $ 987,843 | $ 1,485,454 | $ 1,550,888 |
| 10 | $ 993,283 | $ 975,250 | $ 1,558,752 | $ 1,612,699 |
| 11 | $ 974,520 | $ 949,068 | $ 1,608,192 | $ 1,661,139 |
| 12 | $ 951,350 | $ 932,471 | $ 1,683,780 | $ 1,749,511 |
| Period | Minimum Variance | | Max Sharpe Ratio | |
| | Start | End | Start | End |
| 1 | $ 992,777 | $ 916,378 | $ 990,065 | $ 922,164 |
| 2 | $ 955,960 | $ 851,310 | $ 961,985 | $ 1,017,139 |
| 3 | $ 827,098 | $ 854,278 | $ 974,303 | $ 1,175,235 |
| 4 | $ 856,664 | $ 980,804 | $ 1,219,185 | $ 1,600,973 |
| 5 | $ 982,538 | $ 942,289 | $ 1,634,035 | $ 1,548,455 |
| 6 | $ 950,670 | $ 1,004,966 | $ 1,547,124 | $ 1,784,352 |
| 7 | $ 1,003,004 | $ 974,545 | $ 1,733,161 | $ 1,846,790 |
| 8 | $ 974,867 | $ 1,086,711 | $ 1,894,849 | $ 2,051,822 |
| 9 | $ 1,086,511 | $ 1,075,476 | $ 2,043,020 | $ 2,005,963 |
| 10 | $ 1,075,502 | $ 1,085,258 | $ 2,004,665 | $ 2,105,263 |
| 11 | $ 1,079,770 | $ 1,056,129 | $ 2,085,469 | $ 2,127,437 |
| 12 | $ 1,053,535 | $ 1,047,649 | $ 2,097,101 | $ 2,198,638 |

Over the 2-year stretch, the best performing strategy was the Sharpe ratio at 120% return, followed by equally weighted, then minimum variance, and then buy and hold. By maximizing the Sharpe ratio on the efficient frontier, we are effectively maximizing our return to risk ratio, and I would expect this optimization to yield better results than other naïve strategies. If I were to implement one of these strategies in my portfolio, it would be a Sharpe ratio optimization. The equally weighted portfolio also did quite well at 75% ROI. The equally weighted strategy is naturally diverse and creating a lower risk without performing any optimization, this also allows it to outperform the minimum variance strategy because it can keep some riskier stocks with higher expected return. There's not much to say about the buy and hold strategy except that it allows one to ignore transaction costs, which is good. I would say that if I were to use buy and hold, I'd maintain a diverse portfolio, like implementing equally weighted but then never re-adjusting. This is likely a much better strategy than only holding 2 stocks.

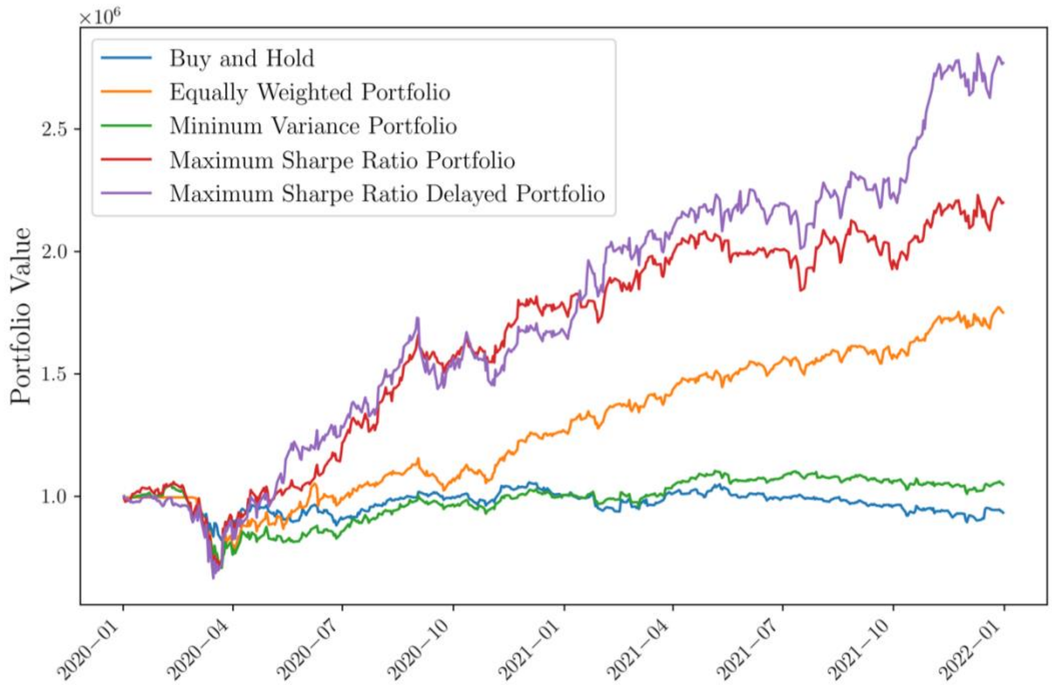


*Figure 1: Portfolio value over time for all strategies*

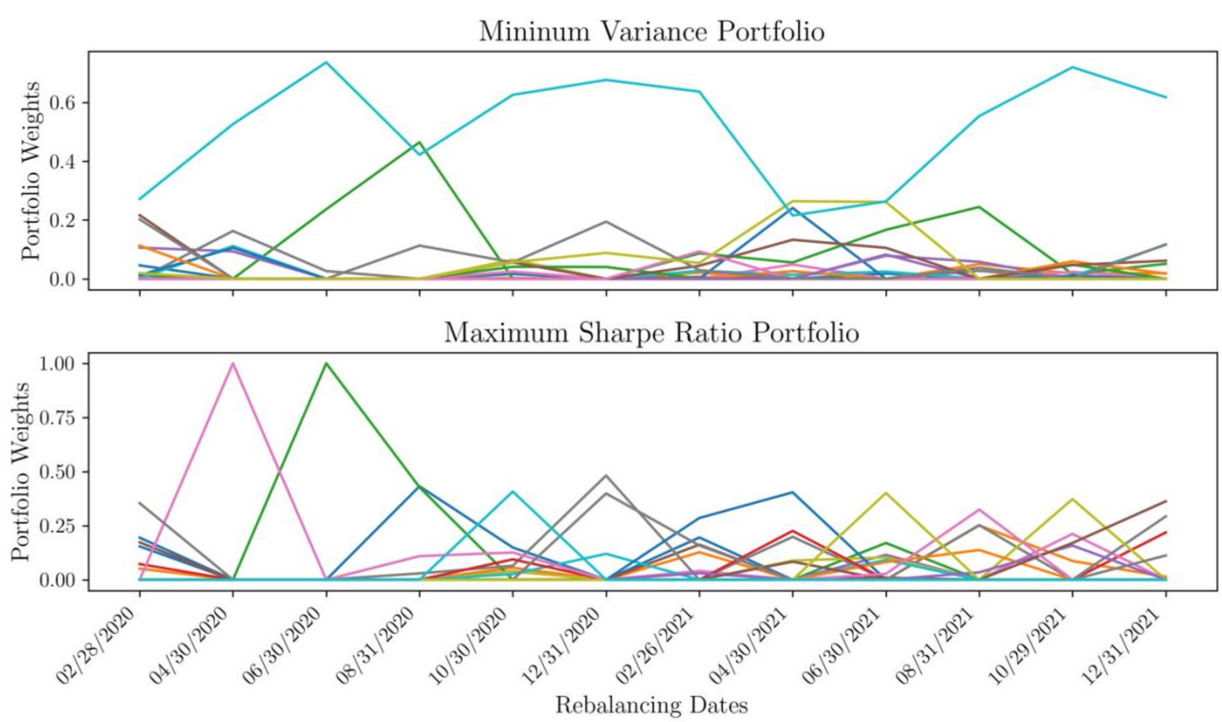


*Figure 2: Portfolio weights over time for Maximum Sharpe Ratio and Minimum Variance Portfolios*

Note in Figure 2, the second graph *only plots portfolio value at the end-dates from each period*. We notice when looking at Figure 2 above that the minimum variance strategy tended to favor certain stocks and rebalance less frequently when compared to the max Sharpe ratio strategy. I speculate that this is because there are certain stocks (light blue) which tend to have lower variance in general, which allows the minimum variance strategy to also spend less on transaction costs.

## Improvements

We notice in Figure 2, graph 2, that the max Sharpe ratio strategy tends to completely rebalance the portfolio very frequently, leading to higher transaction costs compared to the other strategies. To address this issue and improve on the baseline approach, I'm going to introduce a new function "strat_max_Sharpe_delayed". The only difference between this function and the normal Sharpe strategy is that it only applies rebalances every *other* period, which allows us to keep transaction costs down while still choosing stocks using the Sharpe ratio. The results are good, improving over the previously implemented max Sharpe ratio strategy with a return of 177%. The results from this change can also be seen in Figure 1 as the purple line.

*Table 2: Portfolio value for maximum Sharpe ratio 'delayed' approach.*

| Period | Max Sharpe Ratio Delayed | |
|---|---|---|
| | Start | End |
| 1 | $ 1,000,013 | $ 893,957 |
| 2 | $ 935,673 | $ 989,313 |
| 3 | $ 957,163 | $ 1,286,460 |
| 4 | $ 1,279,395 | $ 1,681,043 |
| 5 | $ 1,728,012 | $ 1,473,468 |
| 6 | $ 1,453,468 | $ 1,676,505 |
| 7 | $ 1,642,139 | $ 1,966,863 |
| 8 | $ 1,994,813 | $ 2,160,491 |
| 9 | $ 2,170,716 | $ 2,194,908 |
| 10 | $ 2,190,827 | $ 2,300,627 |
| 11 | $ 2,299,010 | $ 2,585,854 |
| 12 | $ 2,640,641 | $ 2,768,426 |

This improvement is very significant, but there are more to make. One obvious method would be to include transaction costs in the CPLEX optimization rather than using a more naïve heuristic function like mine. Another might be to include more diverse or nuanced asset features, such as momentum. Overall, I would say the concept of utilizing the maximum Sharpe ratio on the efficient frontier offers excellent results for trading.