Homework 1

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1 Question 1

1.1 Part a: Show that $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$

$$(1+x)^n = 1 + nx + o(x)$$
$$(1+x)^n - 1 - nx = o(x)$$

$f(x) \in o(g(x))$ as $x \to 0$ if and only if $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

$$\lim_{x\to 0} \frac{(1+x)^n - 1 - nx}{x} = \lim_{x\to 0} n(1+x)^{n-1} - n = 0$$

1.2 Part b: Show that $(x \sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$

To prove that a function $f(x)$ is in $O(g(x))$ we need that $\exists$K s.t.$|f(x)| \leq K|g(X)|$ as $x \to 0$. Let K = 1.

$$|x \sin(\sqrt{x}) - 0| = |x||\sin(\sqrt{x})| \leq |x||\sqrt{x}| = |x^{3/2}|$$

1.3 Part c: Show that $e^{-t} = o\left(\frac{1}{t^2}\right)$ as $t \to \infty$

$f(x) \in o(g(x))$ as $x \to \infty$ if and only if $\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$

$$\lim_{t\to\infty} \frac{e^{-t}}{1/t^2} = \lim_{t\to\infty} \frac{t^2}{e^{-t}} = 0$$

1.4 Part d: Show that $\int_0^\epsilon e^{-x^2} dx = O(\epsilon)$ as $\epsilon \to 0$

To prove that a function $f(x)$ is in $O(g(x))$ we need that $\exists$K s.t.$|f(x)| \leq K|g(X)|$ as $x \to 0$. Let K = 1.

$$\left| \int_0^\epsilon e^{-x^2} dx \right| \leq \int_0^\epsilon \left| e^{-x^2} \right| dx \leq \int_0^\epsilon 1 dx = \epsilon$$

2 Question 2

Consider Solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \frac{1}{2} \left(\begin{smallmatrix} 1 & 1 \\ 1+10^{10} & 1-10^{10} \end{smallmatrix} \right)$ and $\mathbf{b} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right)$. The exact solution is $\mathbf{x} = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right)$ and the inverse of $\mathbf{A} = \left(\begin{smallmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{smallmatrix} \right)$. In this problem we will investigate a perturbation in b of $\left(\begin{smallmatrix} \Delta b_1 \\ \Delta b_2 \end{smallmatrix} \right)$ and the numerical effects of the condition number.

2.1 Part a: Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx.

We have two equations to start with and with a little algebra we can get:

$$\begin{aligned}
Ax &= b \\
A\Delta x &= \Delta b \\
Ax - A\Delta x &= b - \Delta b \\
A(x - \Delta x) &= b - \Delta b \\
x - \Delta x &= A^{-1}(b - \Delta b) \\
x - \Delta x &= \begin{pmatrix} 1 - 10^{10} & 10^{10} \\ 1 + 10^{10} & -10^{10} \end{pmatrix} \begin{pmatrix} 1 - \Delta b_1 \\ 1 - \Delta b_2 \end{pmatrix} \\
x - \Delta x &= \begin{pmatrix} (1 - 10^{10})(1 - \Delta b_1) + 10^{10}(1 - \Delta b_2) \\ (1 + 10^{10})(1 - \Delta b_1) + (-10^{10})(1 - \Delta b_2) \end{pmatrix} \\
x - \Delta x &= \begin{pmatrix} 1 - \Delta b_1 - 10^{10} + 10^{10}\Delta b_1 + 10^-1010^{10}\Delta b_2 \\ 1 - \Delta b_1 + 10^{10} - 10^{10}\Delta b_1 - 10^{10} + 10^{10}\Delta b_2 \end{pmatrix} \\
x - \Delta x &= \begin{pmatrix} 1 - \Delta b_1 + 10^{10}(\Delta b_1 - \Delta b_2) \\ 1 - \Delta b_1 + 10^{10}(\Delta b_2 - \Delta b_1) \end{pmatrix} \\
\Delta x &= \begin{pmatrix} \Delta b_1 + 10^{10}(\Delta b_2 - \Delta b_1) \\ \Delta b_1 + 10^{10}(\Delta b_1 - \Delta b_2) \end{pmatrix}
\end{aligned}$$

2.2 Part b: What is the condition number of A?

The condition number of A is: 19999973849.225224 from np.linalg.cond(A)

2.3 Part c: Let Δb_1 and Δb_2 be of magnitude 10^{-5}; not necessarily the same value. What is the relative error in the solution? What is the relationship between the relative error, the condition number, and the perturbation. Is the behavior different if the perturbations are the same? Which is more realistic: same value of perturbation or different value of perturbation?

The relative error with $\Delta b_1 = a10^{-5}$ and $\Delta b_2 = c10^{-5}$:

$$\begin{aligned}
\frac{||\Delta x||}{||x||} &= \frac{\sqrt{(\Delta b_1 + 10^{10}(\Delta b_2 - \Delta b_1))^2 + (\Delta b_1 + 10^{10}(\Delta b_1 - \Delta b_2))^2}}{\sqrt{1^2 + 1^2}} \\
&= \frac{1}{\sqrt{2}}\sqrt{(a10^{-5} + 10^{10}(c10^{-5} - a10^{-5}))^2 + (a10^{-5} + 10^{10}(a10^{-5} - c10^{-5})^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{(a10^{-5} + (c - a)10^5)^2 + (a10^{-5} + (a - c)10^5)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{(a10^{-5} - (a - c)10^5)^2 + (a10^{-5} - (c - a)10^5)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{a10^{-10} - (a - c)10^{10} + a10^{-10} - (c - a)10^{10}} \\
&= \frac{1}{\sqrt{2}}\sqrt{2a10^{-10} + (c - a)10^{10} + (a - c)10^{10}} \\
&= \frac{1}{\sqrt{2}}\sqrt{2a10^{-10} + 2(a - c)10^{10}} \\
&=
\end{aligned}$$

$$sqrta10^{-10} + (a - c)10^{10}$$

When the Condition Number is high the system is more sensitive to small perturbations. And then the higher the relative error is for these perturbations. When the perturbations are the same, the relative error goes down since the term being multiplied by (a-c) is now being multiplied by 0. We would expect perturbations not to be identical since all of the steps that cause these perturbations (computing coefficients, floating point arithmetic, etc) is independent for each value.

3 Question 3

Recall the concept of *relative condition number* $\kappa_f(x)$ for a function f(x). For $\tilde{x} = x + \delta x$, and $\delta x \to 0$, it gives an upper bound on the relative error output$\tilde{y} = f(\tilde{x})$. That is:

$$\frac{|f(x) - f(\tilde{x}|}{|f(x)|} \leq \kappa_f(x)\frac{|x - \tilde{x}|}{|x|}$$

For a differentiable function f(x), there is a formula for the relative condition number:

$$\kappa_f(x) = \left|\frac{xf'(x)}{f(x)}\right|$$

Let $f(x) = e^x - 1$.

3.1 Part a: What is the relative condition number $\kappa_f(x)$? Are there any values of x for which this is ill-conditioned (for which $\kappa_f(x)$ is very large)?

The relative condition number is the above function substituting f for our function:

$$\kappa_f(x) = \left|\frac{xf'(x)}{f(x)}\right|$$
$$\kappa_f(x) = \left|\frac{x(e^x)}{e^x - 1}\right|$$
$$\kappa_f(x) = \frac{xe^x}{e^x - 1}$$

$\kappa_f(x)$ is only large when x is either very large, or it is undefined at x = 0.

3.2 Part b: Consider computing f(x) via the following algorithm: 1. y = math.e^x 2. return y-1. Is this algorithm stable. Justify your answer

Yes this algorithm is stable. This is because the relative error grows linearly.

3.3 Part c: Let x have the value $9.999999995000000 \times 10^{-10}$, in which case the true value for f(x) is equal to 10^{-9} up to 16 decimal places. How many correct digits does the algorithm listed above give you? Is this expected?

The algorithm listed above will only give us 8 digits of accuracy. This is expected since we are subtracting two numbers close to each other so we expect catastrophic cancellation. And our actual answer is 0 up to 8 decimal places.

3.4 Part d: Find a polynomial approximation of f(x) that is accurate to 16 digits for $x = 9.999999995000000 \times 10^{-10}$. Hint: use Taylor series, and remember that 16 digits of accuracy is a relative error, not an absolute one.

let us consider a taylor series centered at $x_0 = 0$

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1} < 10^{-16}$$

$$\frac{e^{(\xi(x))}}{(n+1)!}x^{n+1} < 10^{-16}$$

$$\frac{x^{(n+1)}}{(n+1)!} < 10^{-16}$$

$$\frac{(9.999999995000000 \times 10^{-10})^{(n+1)}}{(n+1)!} < 10^{-16}$$

$$n \geq 1$$

so our Taylor Polynomial is: $P_1(x) = x$

3.5 Part e: Verify that your answer from part (d) is correct.

Let us re-write our value in normal decimal notation: 0.0000000009999999—995000000. That bar is at the 10^{-16} place and we can see that what we have is that value is 10^{-9} at 16 digits of accuracy. And since our function is just x then our output is also 10^{-9} to 16 digits of accuracy.

3.6 Part f: [Optional] How many digits of precision do you have if you do a simpler Taylor series?

The only Simpler Taylor Series is $P_0(x) = 0$. This Taylor series is still accurate to 8 decimal places.

3.7 [Fact; no work required] Matlab provides expm1 and Python provides numpy.expm1 which are special-purpose algorithms to compute ex - 1 for $x \approx 0$. You could compare your Taylor series approximation with expm1.

4 Question 4: Practicing Python

4.1 Part a: Create a vector t with entries starting at 0 incrementing by $\frac{\pi}{30}$ to π. Then create the vector $y = \cos(t)$. Write a code that evaluates the following sum: $S = \sum_{k=1}^{N} t(k)y(k)$. Print the statement "the sum is: S" with the numerical value of S.

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The sum is: -20.68685236434684
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4.2 Part b: Wavy circles. In one figure, plot the parametric curve: $x(\theta) = R(1 + \delta r \sin(f\theta + p))\cos(\theta), y(\theta) = R(1 + \delta r \sin(f\theta + p))\sin(\theta)$ **for** $0 \leq \theta \leq 2\pi$**, R = 1.2,** δr **= 0.1, f = 15, and p = 0. Make sure to adjust the scale so that the axis have the same scale. In a second figure, use a for loop to plot 10 curves and let with R = i,** δr **= 0.05, f = 2 + i for the ith curve. Let the value of p be a uniformly distributed random number (look up random.uniform) between 0 and 2.**

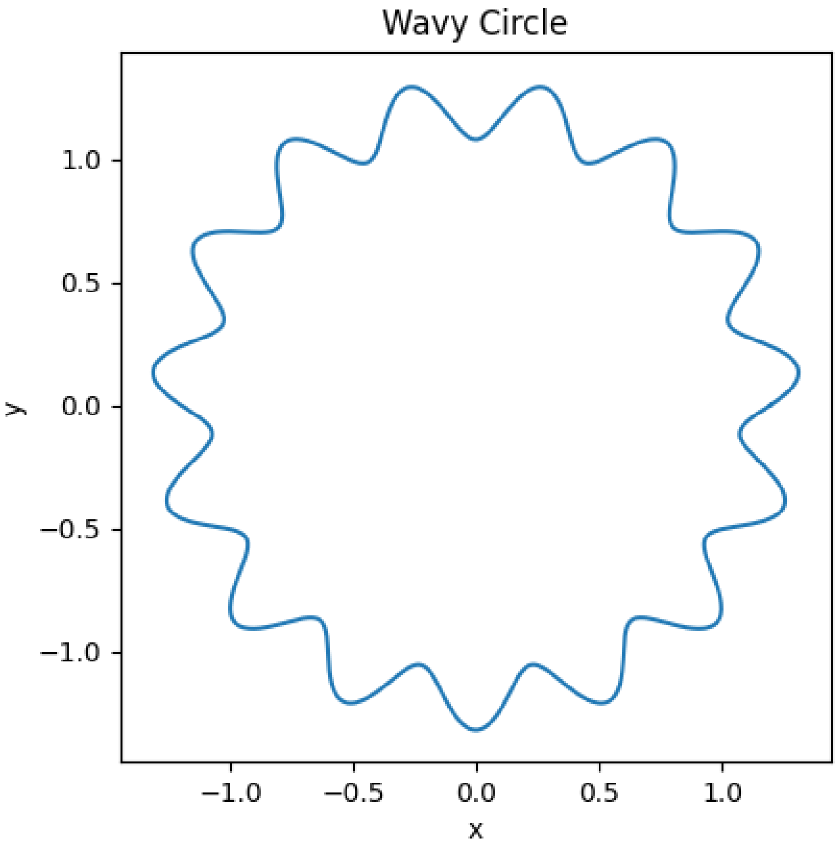

Figure 1: R = 1.2, δr = 0.1, f = 15, and p = 0.

[H]

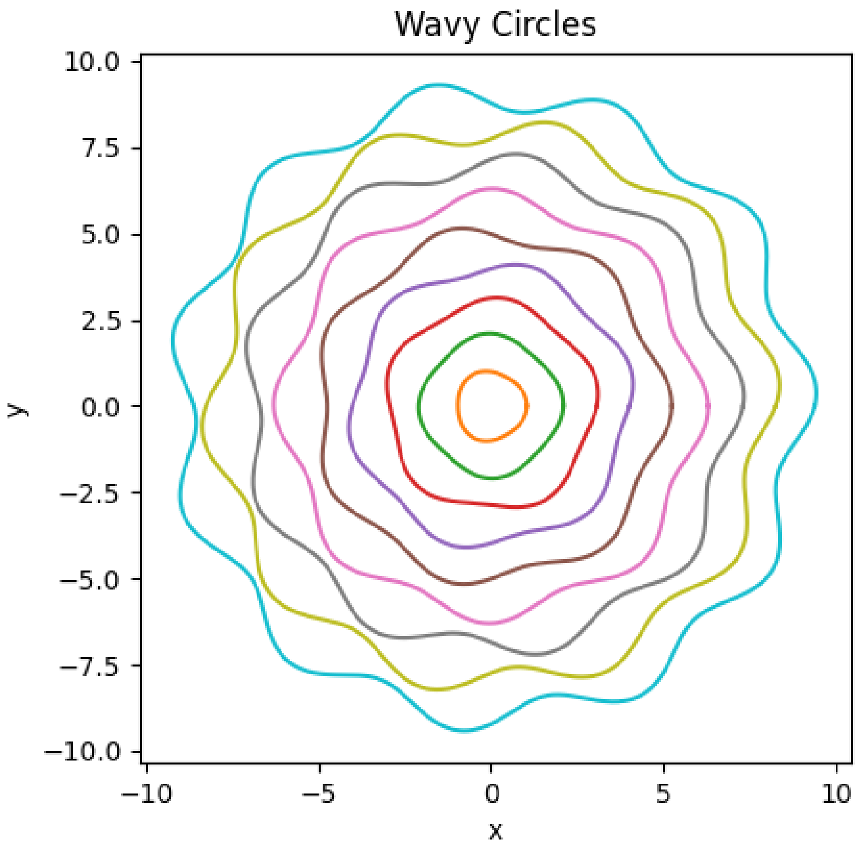

Figure 2: R = i, δr = 0.05, f = 2 + i, p ~Uniform(0,2)