Functional Analysis

Daniele Zago

October 20, 2021

Contents

Lecture 1: Introduction **1**
1.1 Review of prerequisites 1
1.2 Measure theory 3

Lecture 1: Introduction

2021-10-19

The course contents will be mostly on measure theory with some basic functional analysis notions about Banach and Hilbert spaces.

1.1 Review of prerequisites

Def. (Eigenvalue)
Let $A \in M^{n\times n}(\mathbb{R})$ a $n \times n$ symmetric matrix of real numbers, such that $A^\top = A$. An *eigenvalue* $\lambda \in \mathbb{C}$ of A is a number such that there exists $v \in \mathbb{R}^n$, $v \neq \mathbf{0}$ we have

$$Av = \lambda v.$$

Remark

- v is called an *eigenvector* associated to λ.
- Eigenvalues are the solution in λ to the following equation, called *characteristic polynomial*

$$p(\lambda) = \det\big(A - \lambda I_n\big) = \mathbf{0}.$$

The equation has at most n solution when counting multiplicities.

Prop.
0 is an eigenvalue of A $\iff$ $\det A = 0$.

Since by the Binet formula we have $\det(AB) = \det A \det B$, then we find that A is invertible $\iff$ $\det A \neq 0$:

$$\det I = 1 = \det A \det A^{-1}.$$

In general, we could define the eigenvalues as those numbers λ such that $A - \lambda I$ is not invertible. This definition is useful when considering linear operators defined on Hilbert spaces instead of classical spaces such as $\mathbb{R}^n$.

In fact, a matrix $A \in M^{n\times n}(\mathbb{R})$ is associated to linear functions $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $v \mapsto Av$. This function is linear since

$$\lambda v + \mu w \implies A(\lambda v + \mu w) = \lambda Av + \mu Aw.$$

Def. (Invertible matrix)
A is *invertible* with inverse A^{-1} if the associated linear function T admits an inverse, i.e.

$$\begin{aligned} T : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ v &\longmapsto Av \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} T^{-1} : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ v &\longmapsto A^{-1}v \end{aligned}$$

Def. (Positive definite matrix)
A matrix A is *positive definite* if A is symmetric and all its eigenvalues $\lambda_1, \dots, \lambda_k$ are strictly positive.

Remark

A linear operator T will accordingly be called positive definite when the eigenvalues of the associated matrix A are positive definite.

In particular, a positive definite matrix is such that there exists a $c > 0$ for which

$$v^\top Av \geq c|v|^2 \quad \forall v \in \mathbb{R}^n,$$

and in particular $c = \min\{\lambda_1, \lambda_2, \dots, \lambda_k\}$.

Def. (Random variable)
A random variable is a measurable function on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined by

$$\begin{aligned} X : \Omega &\longrightarrow \mathbb{R} \\ \omega &\longmapsto X(\omega) \end{aligned}$$

Most of the time, instead of looking at the random variable itself we consider the law of X, which is a function μ_X defined on all Borel sets of Ω,

$$\mu_X(A) = \mathbb{P}\big(\omega \in \Omega : X(\omega) \in A\big).$$

We characterize the random variables by the measure μ_X that it is induced on the real numbers. The gaussian distribution is the distribution of a r.v. with mean 0 and variance 1 such that

$$\mu_X(A) = \frac{1}{C} \int_A e^{-|x|/2} dx$$

For convergence since sums of integers are very hard to compute we can instead look at the corresponding function integral:

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \longleftrightarrow \int_1^\infty \frac{1}{x^\alpha} dx.$$

The sum of the areas of all the rectangles corresponding to the integers $\big(n, f(n)\big)$ is such that

$$\text{Area} = f(1) + f(2) + \ldots + f(n) + \ldots,$$

and for sure the sum of all rectangles is controlled by the area of the curve $f(x)$,

$$\sum_{i=1}^{\infty} \frac{1}{n^\alpha} - f(1) \leq \int_1^{+\infty} \frac{1}{x^\alpha} dx.$$

Therefore, if the integral is finite we automatically know that the series is convergent, and we also have an upper bound on the value of the series. However, if the integral is infinite we can take the upper summation of rectangles and control the integral by the upper summation S_n

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} - 1 \leq \int_1^{\infty} \frac{1}{x^\alpha} dx \leq \sum_{n=1}^{+\infty} \frac{1}{n^\alpha}.$$

If instead we want to control the summation we can majorize and minorize it with respect to the integral by considering

$$\int_1^{+\infty} \frac{1}{x^\alpha} dx \leq \sum_{n=1}^{\infty} \frac{1}{n^\alpha} - 1 \leq \int_1^{+\infty} \frac{1}{x^\alpha} dx + 1.$$

1.2 Measure theory

Let X be a set and Σ a collection of subsets of X. Σ is called a σ-algebra if it is closed wrt to complement and countable unions:

i $A \in \Sigma \implies X \setminus A \in \Sigma$.

ii $A_1, A_2, \ldots \in \Sigma \implies \bigcup_{n\in\mathbb{N}} A_i \in \Sigma$.

The smallest σ-algebra is $\{\emptyset, X\}$, whereas the biggest is the family of all subsets of X, $\mathscr{P}(X)$. In principle we need to consider σ-algebras which are smaller than $\mathscr{P}(X)$, since coherent measures are difficult to define on it.

If $C \subseteq \mathscr{P}(X)$ is a family of subsets of X, then the *σ-algebra generated by C* is the smallest σ-algebra which contains C,

$$\sigma(C) =$$

If we consider $X = \mathbb{R}$, then the *Borelian σ-algebra $\mathscr{B}$* is the σ-algebra generated by all open intervals (a, b), $a, b \in \mathbb{R}$:

$$C = \{(a, b) : a < b, a, b \in \mathbb{R}\}.$$

$$\mathscr{B} = \sigma(C).$$

In any set X where we can define a topology we can also define its associated Borel σ-algebra.

Under the above σ-algebra, all intervals of the form

$$[a, b], [a, b), (a, b], (a, \infty), [a, \infty), (-\infty, b], (-\infty, b)$$

are all contained in B. This is easy to prove by considering countable unions and complements

$$(a, +\infty) = \bigcup_{i=1}^{\infty} (a, a+i) \in \mathscr{B}$$

$$\mathbb{R} \setminus (a, +\infty) = (-\infty, a] \in \mathscr{B}.$$

Moreover, $\mathscr{B}$ is equivalently

$$\sigma\big(\{(a,b) : \dots\}\big)$$

$$\sigma\big(\{(a,b] : \dots\}\big)$$

$$\sigma\big(\{[a,b] : \dots\}\big)$$

$$\sigma\big(\{[a,b) : \dots\}\big)$$

We can construct some subsets A of $\mathbb{R}$ such that $A \notin \mathscr{B}$, for example the Vitali set. The more general proof that $\mathscr{B} \subsetneq \mathscr{P}(\mathbb{R})$ takes into account the cardinality of the two sets, which can be shown to be different.

(X, Σ) then $\mu : \Sigma \to [0, +\infty]$ is a measure if

- $\mu(\emptyset) = 0$
- $\forall A_i \in \Sigma$, $A_i \cap A_j = \emptyset$, then $\mu\big(\bigcup_{i=1}^{\infty} A_i\big) = \sum_i \mu(A_i)$.

Consequence

We can prove some properties

a) μ is monotone, i.e. $A \subseteq B \implies \mu(A) \leq \mu(B)$

b) μ is "continuous", i.e. if $A_i \subseteq A_{i+1} \subseteq A_{i+2}$ then

$$\mu\big(\bigcup_i A_i\big) = \lim_{i \to \infty} \mu(A_i).$$

c) If $A_{i+1} \subseteq A_i$ at each i we have

$$\mu\big(\bigcap_i A_i\big) = \lim_{i \to \infty} \mu(A_i).$$

Def. (Finiteness)
μ is said to be *finite* if $\mu(X) < \infty$, and *σ-finite* if there is a sequence of $A_i \in \Sigma$ such that $X = \bigcup_{i=1}^{\infty} A_i$ with $\mu(A_i) < \infty$.