Theory and Methods of Inference

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Contents

Lecture 3: Lecture 9

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Lecture 4: Lecture 10

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??. Efficient estimators

Def.
An estimator $\tilde{\theta}$ of θ is called **uniformly minimum variance unbiased** (UMVU) if

$$\mathbb{E}_\theta[\tilde{\theta}] = \theta$$

$$\mathbb{V}_\theta[\tilde{\theta}] = \inf_{\widehat{\theta}} \mathbb{V}_\theta[\widehat{\theta}]$$

Theorem 1
In a one-parameter model with regular likelihood such that derivation and integration may be exchanged, if $\tilde{\theta}$ is an unbiased estimator of θ with finite variance,

$$\mathrm{Cov}_\theta\left(\tilde{\theta}, l_*(\theta)\right) = 1.$$

Theorem 2
If $p = 1$ and $\tilde{\theta}$ be an unbiased estimator of θ, then under the regularity conditions needed for the previous theorem and of the information identity,

$$\mathbb{V}_\theta[\tilde{\theta}] \geq \frac{1}{i(\theta)}.$$

If $p > 1$, the inequality becomes

$$\mathbb{V}_\theta[\tilde{\theta}] - i(\theta)^{-1} \succeq 0,$$

with $\succeq$ is understood as meaning "positive definiteness".

Remark. If $\theta = (\tau, \zeta)$ and ζ is a nuisance parameter, then the above equalities for $\tilde{\tau}$ are valid by replacing $i(\theta)^{-1}$ with $i_{\tau\tau}(\theta)^{-1}$.

Remark. An estimator $\tilde{\theta}$ might have variance lower than the Cramér-Rao lower bound on a set of θ values having Lebesgue measure zero.

Theorem 3
Let $q(y, \theta)$ be an estimating equation for a scalar parameter θ, then

$$\frac{\mathbb{V}_\theta[q(Y;\theta)]}{\mathbb{E}_\theta[\frac{\partial q(Y;\theta)}{\partial \theta}]^2} \geq \frac{\mathbb{V}_\theta[l_*(Y;\theta)]}{\mathbb{E}_\theta[\frac{\partial l_*(Y;\theta)}{\partial \theta}]^2},$$

moreover under regularity conditions, the left hand side equals $J(\theta)/H(\theta)^2$.

Def.

An estimator $\tilde{\theta}$ is said to be second-order efficient if

$$\mathbb{E}_\theta[\tilde{\theta}] = \theta + O(n^{-2}), \qquad \mathbb{V}_\theta[\tilde{\theta}] = \mathbb{V}_\theta[\widehat{\theta}_n] + O(n^{-3}),$$

where $\widehat{\theta}$ is the maximum likelihood estimator of θ.

Remark. The estimator $\tilde{\theta}$ may be constructed by applying a first-order bias correction to the maximum likelihood estimator,

$$\tilde{\theta} = \widehat{\theta}_n - \frac{b(\widehat{\theta}_n)}{n}.$$

Theorem 4 (Rao-Blackwell-Lehmann-Scheffé)

Let θ be a scalar parameter and s be a sufficient statistic. Let $\tilde{\theta} = \tilde{\theta}(Y)$ be an unbiased estimator of θ with $\mathbb{V}_\theta[\tilde{\theta}] < \infty$. Then an unbiased estimator $\widehat{\theta} = \widehat{\theta}(S)$ exists and $\mathbb{V}_\theta[\widehat{\theta}(S)] \leq \mathbb{V}_\theta[\tilde{\theta}(Y)]$ for each $\theta \in \Theta$. Moreover, such an estimator is

$$\widehat{\theta} = \mathbb{E}_\theta[\tilde{\theta}|S = s].$$

Theorem 5

Let s be a sufficient and complete statistic for F, and let $\tilde{\theta}(Y)$ be an unbiased estimator of θ, then the UMVU estimator is unique (a.s.) and is equal to $\mathbb{E}_\theta[\tilde{\theta}(Y)|S = s]$.

Theorem 6

$\tilde{\theta}(Y)$ is the UMVU estimator of θ if and only if

$$\mathrm{Cov}(\tilde{\theta}, \widehat{\theta}) = 0$$

for all estimators $\widehat{\theta}$ such that $\mathbb{E}_\theta[\widehat{\theta}] = \theta$.

Def.

An estimator is said to be **median unbiased** if

$$\mathbb{P}_\theta(\tilde{\theta}(Y) \leq \theta) = 1/2 \quad \text{for all } \theta \in \Theta.$$

Remark. If $\tilde{\theta}$ is median unbiased for θ and $\psi : \Theta \to \Psi$ is a monotone reparametrization of θ, then $\tilde{\psi} = \psi(\tilde{\theta})$ is median unbiased for ψ.

Def.
The **risk** of a test $H_0 : \theta \in \Theta_0$ vs $H_1 : \theta \in \Theta_1$ is defined as

$$\text{Risk} = \begin{cases} \beta(\theta) & \text{if } \theta \in \Theta_0 \\ 1 - \beta(\theta) & \text{if } \theta \in \Theta_1 \end{cases}$$

where $\beta(\theta) = \mathbb{P}_\theta(H_0 \text{ is rejected})$ is the **power function**.

Def.
A test has **exact size** $\boldsymbol{\alpha}$ if $\sup_{\theta \in \Theta_0} \beta(\theta) = \alpha$.

Def.
A test is said to be **unbiased** if $\beta(\theta) \geq \alpha$ for all $\theta \in \Theta_1$.

Theorem 7
Let $\Theta = \{\theta_0, \theta_1\}$ and consider testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, then the most powerful test of level α has rejection region of the form

$$R(y) = \left\{ y \in \mathcal{Y} : \frac{p(y;\theta_1)}{p(y;\theta_0)} > c_\alpha \right\}, \tag{1}$$

for some c_α. Moreover, this test is also unbiased.

Remark. If the hypotheses are composite and unidirectional, e.g. $H_0^{sx} : \theta \leq \theta_0$ vs $H_1^{dx} : \theta > \theta_0$ and the likelihood ratio

$$t^*(y; \theta', \theta'') = \frac{p(y;\theta'')}{p(y;\theta')}$$

is monotone in the scalar statistic $t(y)$, then the UMP test of level α exists and has the same form as (1)

Remark. Examples of monotone likelihood ratio families are: one-parameter exponential families, hypergeometric, non-central t, chi-square, F with given degrees of freedom.

Def.
A **locally most powerful** test maximizes power for alternatives closer to H_0 among all level α tests, that is,

$$\text{Locally most powerful test} \iff \beta'(\theta_0) \text{ is maximized.}$$

Theorem 8

If the conditions for the Neyman-Pearson theorem 7 can be applied, then the LMP test has critical region

$$R(y) = \left\{ y \in \mathcal{Y} : \frac{p'(y;\theta_0)}{p(y;\theta_0)} > c_\alpha \right\}, \tag{2}$$

hence it is based on the score function $l_*(\theta_0) = p'(y;\theta_0)/p(y;\theta_0)$.

Def.

A test is said to be **uniformly most powerful unbiased** (UMPU) if the power is maximized among all test that are unbiased at level α.

Theorem 9

In a one-parameter exponential family with monotone likelihood ratio with respect to a statistic t*, the UMPU test for a hypothesis test* $H_0 : \theta = \theta_0$ *vs* $H_1 : \theta \neq \theta_0$ *has rejection region*

$$R^* = R^*_{sx} \cup R^*_{dx} = \{y \in \mathcal{Y} : t(y) < t_{\alpha''}\} \cup \{y \in \mathcal{Y} : t(y) > t_{1-\alpha'}\},$$

where $\alpha' + \alpha'' = \alpha$.

Remark. The reasoning behind Theorem 9 is that the critical region for the left and right alternative hypotheses is in general are different. Therefore, a UMP test does not exist unless $t(Y)$ has a symmetric null distribution.

Def.

A test of level α $H_0 : \theta \in \Theta_0$ is said to be **uniformly most powerful similar on the boundary** (UMPS) if

$$\mathbb{P}_\theta(H_0 \text{ is rejected}) = \alpha \quad \text{for all } \theta in \Gamma,$$

where Γ is the boundary of Θ_0 and Θ_1.

Remark. This definition is useful whenever $\theta = (\tau, \zeta)$ and we want to test a hypothesis of the form $H_0 : \tau \leq \tau_0$, since

$$\mathbb{P}_{(\tau,\zeta)}(H_0 \text{ is rejected}) = \alpha \quad \text{for all } (\tau,\zeta) \text{ such that } \tau = \tau_0$$

Def.

A confidence region based on the inversion of a UMP (UMPU, UMPS), confidence interval is called **uniformly most accurate (unbiased, similar)**.

Def.

If $\mathcal{F}$ is a composite group family with shape parameter τ, a UMP statistical test on τ based on the reduced model **TODO: scrivere quale** is termed **uniformly most powerful invariant** (MPI).