Computational Methods in Finance, Lecture 2,   
Diffusions and Diffusion Equations.

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1. Introduction

This lecture and the next are about finite difference methods for solving the diffusion equations that arise in financial applications. Before get to finite differences, we review stochastic differential equations. As in Lecture 1, we discuss the forward and backward equations and the differences between them.

2. Diffusions and Diffusion Equations

Lecture 1 discusses Markov processes in discrete state space and discrete time. Now we turn to continuous time and continuous state space. The state at time is a vector, consisting of components, or “factors”, . The dynamics are given by the Ito differential equation

|  |  |  |
| --- | --- | --- |
|  |  | (1) |

Here is a vector of independent standard Brownian motions. For each , there is a “drift”, , and an matrix , that is related to the volatility. There is no reason that , the number of noise sources, should equal , the number of factors, but there is no reason ever to have more noises than factors. The columns of are vectors in . Column gives the influence of noise on the dynamics. If these columns do not span , then the diffusion is “degenerate”. Otherwise, it is nondegenerate. Both types come up in financial applications.

As in Lecture 1, there are forward and backward evolution equations that are dual to each other. The forward equation is for , the probability density for . This is the “diffusion equation”

|  |  |  |
| --- | --- | --- |
|  |  | (2) |

The matrix of diffusion coefficients, , is related to by

|  |  |  |
| --- | --- | --- |
|  |  | (3) |

We write for the transpose of a matrix, . The coefficients, , in (2) are the components of in (1).

A strict mathematical derivation of (2) from (1) or vice versa is outside the scope of this course. However, some aspects of (2) that are natural. First, because is a probability density, the integral of over should be independent of . That will happen if all the terms on the right of (2) are derivatives of something. This is sometimes called “conservation form”. The second term on the right of (2) involves two derivatives. Someone used to physical modeling might expect it to take the form

|  |  |  |
| --- | --- | --- |
|  |  | (4) |

The actual form (2) has the “martingale” property that, if there is no drift, then the expected value of does not change with time. To see this, use (2) with and compute

The last line follow from the one above if you integrate by parts twice to put the two derivatives on the . The result would generally not be zero using (4) instead of (2).

Here is an equally rigorous derivation of the relation (3) between volatility and diffusion coefficients. Suppose first that . The number should depend on in some way. Observe that the diffusion governed by (1) will be unchanged of is replaced by ; is indistinguishable from . This suggests the formula . In general, we need a matrix analogue of that produces an matrix, , from an matrix, . The simplest possibility is (3). For constant and , we can match moments. Use (1) and assume and [[2]](#footnote-0000000210) and get . From this it follows that

On the other hand, from (2) we compute

which again agrees with (3).

The drift term in (2), , corresponds to the drift term in (1), . It is easy to see what the term would be if were constant (independent of and ) and were zero. In that case the solution of (1) would be . For this reason, the probability density is also simply shifted with speed : . This function satisfies (2) (if the signs are right).

As in Lecture 1, the simplest backward equation is for

More complicated expectations satisfy more complicated but related equations. The backward equation for is

|  |  |  |
| --- | --- | --- |
|  |  | (5) |

Still following Lecture 1, this is supplemented with “initial data” given at the final time, , , and determines for . Again, we can express unconditional expectation in terms of conditional expectation starting from time and the probability density for :

|  |  |  |
| --- | --- | --- |
|  |  | (6) |

The fact that the right side of (6) is independent of allows us to derive (5) from (2) or vice versa. Finally, satisfies a “maximum principle”:

The probability interpretation of makes this obvious; the expected reward cannot be less than the least possible reward nor larger than the largest.

1. goodman@cims.nyu.edu, or http://www.math.nyu.edu/faculty/goodman, I retain the copyright to these notes. I do not give anyone permission to copy the computer files related to them (the .tex files, .dvi files, .ps files, etc.) beyond downloading a personal copy from the class web site. If you want more copies, contact me. [↑](#footnote-ref-0000000009)
2. The Wiener process is usually defined to that . [↑](#footnote-ref-0000000210)