EQUINUMERABILITY

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Now we present some basic results of equinumerability. We will later apply some of those results to computability.

Definition 1. Two sets A and B are **equivalent** or **equinumerable** if there exists a correspondence (bijection) between them. This is denoted $A \sim B$.

Proposition 2. *The relation of equivalence is an equivalence relation.*

Theorem 3 (Schröder-Bernstein)**.** *If $X \preceq Y$ and $X \succeq Y$ then $X \sim Y$.*

Lemma 4. *Let f be a total function from X to Y and g be a total function from Y to X. Then there exists $A \subset X$ and $B \subset Y$ such that $f(A) = B$ and $g(Y - B) = X - A$.*

Proof Sketch of lemma 1. The lemma is equivalent to the affirmation of the existence of a fixed point of the equation $A = X - g(Y - f(A))$.
Consider the chain $C = \{X, X - g(Y - f(X)), \ldots, h(n), X - g(Y - f(h(n)))\}$.
This chain is ordered by inclusion $\supset$.
We can check that $A = \cap C$ is the desired fixed point. ■

Proof Sketch of Schröder-Bernstein. From the hypothesis we have two one to one functions, f from X to Y and g from Y to X. Using lemma 1, there exists $A \subset X$ and $B \subset Y$ such that $f(A) = B$ and $g(Y - B) = X - A$. Now we define $h : X \to Y$ as

$$h(z) = \begin{cases} f(z) & \text{if } z \in A \\ g^{-1}(z) & \text{if } z \notin A \end{cases}$$

It can be shown that h is a correspondence. ■

Definition 5. A set is **enumerable** if it is equivalent with the set of natural numbers $\mathbb{N}$.

Proposition 6. *The following sets are all enumerable:*

(1) The set of integers $\mathbb{Z}$.
(2) The set of rational numbers $\mathbb{Q}$.
(3) The set of n-uples of natural numbers for all n $\mathbb{N}^n$.
(4) The set of finite words over an enumerable alphabet Σ^*.
(5) The set of finite subsets of an enumerable set.

Proposition 7. *The set $\mathscr{P}(\mathbb{N})$ is not enumerable.*

Proposition 8. *The interval $(0\ 1)$ is not enumerable.*

Proposition 9. *The following sets are all equivalent:*

(1) The set of subsets of natural numbers $\mathscr{P}(\mathbb{N})$.

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(2) The set of boolean functions over the natural numbers, $2^{\mathbb{N}}$.
(3) The interval $(0\ 1)$.
(4) The set of real numbers $\mathbb{R}$.
(5) The interval $[0\ 1]$.
(6) $\mathbb{R}^2$
(7) $\mathbb{R}^n$ *for all n.*
(8) $\mathbb{R} \times \mathbb{N}$
(9) $\mathbb{R}^{\mathbb{N}}$

Proof of $2^{\mathbb{N}} \sim (0\ 1)$. Every real number in (0 1) can be represented in binary as $0.1010...a_n a_{n+1}...$, and therefore as a function $f : \mathbb{N} \to 2$ such that $f(n) = 1$ iff $a_n = 1$. This is not an equivalence, since for example $0.1111... \notin (0\ 1)$. Therefore we have shown that $(0\ 1) \preceq 2^{\mathbb{N}}$.

Conversely, every function $f : \mathbb{N} \to 2$ can be represented as a real number $a = 0.f(0)0f(1)1f(2)0....$. Thus $(0\ 1) \succeq 2^{\mathbb{N}}$.

We conclude by Schöder-Bernstein that $2^{\mathbb{N}} \sim (0\ 1)$. ■

Exercises

(1) Show that the set of cofinite subsets of an enumerable set is enumerable.
(2) Show that the points inside of the inferior semicircle of radius 1 and center (1,1) are equivalent to $\mathbb{R}$.

References

[1] George S. Boolos, John P. Burgess, Richard C. Jeffrey, *Computability and Logic*
[2] Paul Halmos, *Naive Set Theory*