SET THEORY

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CONTENTS

1. Introduction 2
2. The axioms 2
2.1. Other Axioms 2
3. Construction of objects in Set Theory 2
3.1. Cartesian Products 2
3.2. Functions 3
3.3. Numbers 3
3.4. Exercises 3
4. Peano Arithmetic in Set Theory 3
4.1. Axioms of PA 3
4.2. ω as a model of Peano Arithmetic 4
4.3. Exercises 4
5. Order in ω 4
5.1. Exercises 4
References 4

Date: December 2015.

1. Introduction

The objective of this classes is to show an example of a formal theory of mathematics expressive enough to express the theorems of number theory. Following Zermelo and Fraenkel, we will use Set Theory as our framework of formalization.

We will begin examining the axioms of set theory, followed by the construction of some mathematical objects indispensable to accomplish our goal. We will then turn to examine the properties of one of those objects, ω, and see that it satisfies the Peano Axioms.

2. The axioms

Axiom 1 (Extension)**.** $A = B \iff \forall a \in A\ a \in b \wedge \forall b \in B\ b \in A$

Axiom 2 (Specification)**.** *If A is a set and P a well formed predicate of First Order Logic, then* $\{a \in A : P(a)\}$ *is a well formed set.*

As an application, note that if A is a set, $\varnothing = \{a \in A : \perp\}$; that is, $\varnothing$ is a well formed set given the existence of any set.

Axiom 3 (Pairing)**.** *If A and B are sets,* $\{A,B\}$ *is a set.*

Note: we can weaken this axiom so that $\{A,B\}$ is contained in a set C, and then filter C to the desired set $\{A,B\}$ using the axiom of Specification. This can be also done in the axiom of Union, Powers, and Infinity.

Axiom 4 (Unions)**.** *If C is a collection of sets,* $\cup C$ *is a well formed set.*

Axiom 5 (Powers)**.** *If A is a set, then the set of subsets of A, denoted* $\mathscr{P}(A)$*, is a set.*

Note that the construction of $\mathscr{P}(A)$ if A is finite follows from the axioms of pairing and specification. The axiom of Powers is needed however for the infinite cases.

Axiom 6 (Infinity)**.** *There exists a successor set A such that* $\varnothing \in A$*.*

2.1. Other Axioms.

The previous are not all the axioms used in ZFC, the standard approach to Set Theory.

If we wanted the whole set, we would have to add the axiom of Choice, the axiom of Foundation and the axiom of Substitution. Those axioms are however not needed to construct the arithmetic theory we are aiming for. The curious reader is invited to research them if desired.

3. Construction of objects in Set Theory

It is interesting to see how the sets that are valid according to the axioms of the previous section can be used to model mathematical objects with several properties.

3.1. Cartesian Products.

An ordered pair is a set $(a,b) = \{\{a\},\{a,b\}\}$. It is easy to see from the definition that the order of a,b is preserved uniquely.

The Cartesian Product of two sets is defined as the set of all pairs whose first element is in A and second element is in B. The axiom of powers together with the axiom of specification guarantee that such a set is always well-formed.

$$A \times B = \{(a,b)\ for\ a \in A\ and\ b \in B\} =$$
$$= \{x \in \mathscr{P}(\mathscr{P}(\cup\{A,B\}) : \exists a \in A \wedge b \in B\ such\ that\ (a,b) = x\}$$

3.2. **Functions.** A function f is a subset of $A \times B$ such that if $(x,y) \in f$ *and* $(x,y') \in f$ then $y = y'$.

Functions can be onto and one-to-one. A function that is both onto and one-to-one is called a correspondence.

A family is a function $f : I \to X$ where the range is more important than the function itself. I is called the index set, and X the indexed set.

3.3. **Numbers.** We identify the number 0 with the empty set, and the rest of the numbers are implicitly defined by the following successor operation on arbitrary sets.

Successor operation: $n^+ = n \cup \{n\}$

A set that contains the 0 and the successor of every set it contains is called a successor set. The existence of such a set is guaranteed by the Axiom of Infinity.

Let A be a successor set. We define $\omega = \bigcap\{S \subset A : S \text{is a successor set}\}$ as the set of natural numbers.

ω turns out to behave almost as expected from the natural numbers we are used to, with a few quirks. In the rest of the document we will turn to examine its properties.

3.4. **Exercises.**

(1) Show that functions are well-formed sets according to the axioms of Set Theory. (Remember, a function from A to B is defined as a subset of $A \times B$ satisfying certain properties)
(2) Show that $A^{B \times C}$ (the set of all functions from $B \times C$ to A) is a well-formed set.
(3) We define $\times_{i \in I} A_i$ as the cartesian product of every member of a family A_i indexed by the set I. Show that $\times_{i \in I} A_i$ is a well-formed set.
(4) Show that if I is a non zero natural number and none of the A_i is the empty set then $\times_{i \in I} A_i$ is not empty. Can we prove this for $I = \omega$?

4. Peano Arithmetic in Set Theory

4.1. **Axioms of PA.** The axioms of Peano Arithmetic are five properties that Giussepe Peano believed were needed by a model of the natural numbers to prove all the theorems in number theory. Nowadays we know that it is incomplete, in the sense that there are true theorems in number theory that are not decidable using the Peano Axioms, but it will be enough for our purposes.

The axioms are the following:

(1) $0 \in \omega$
(2) $n \in \omega \to n^+ \in \omega$
(3) If $S \subset \omega$, and S is a successor set, then $S = \omega$
(4) $\forall n \in \omega 0 \neq n^+$
(5) $n^+ = m^+ \to n = m$

4.2. ω as a model of Peano Arithmetic.

Proposition 1. *ω is the minimal successor set; ie, it satisfies 1,2,3.*

Proposition 2. *ω satisfies 4.*

Lemma 3 (Transitivity of the natural numbers)**.** *If $m \in n$ then $m \subset n$*

Corollary 4. *If $m \in n$ and $n \in p$ then $m \in p$*

Lemma 5. *If $m \in n$ then $n \not\subset m$*

Corollary 6. *$n \notin n$*

Proposition 7. *ω satisfies axiom 5.*

Theorem 8 (Recursion)**.** *If f is a function from X into X and $a \in X$ there exists a function $g : \omega \to X$ such that $g(0) = a$ and $g(n^+) = f(g(n))$.*

Proof. Pick the collection C of subsets S of $\omega \times X$ such that $(0,a) \in S$ and $(n,b) \in S \implies (n^+, f(b)) \in S$.

C is not empty as $\omega \times X \in C$. We prove by induction that $g = \cap C$. ■

4.3. Exercises.

Show that the function $+ : \omega^2 \to \omega$ such that $+(n,0) = n$ and $+(n,m^+) = +(n,m)^+$ is associative and commutative.

5. Order in ω

We define order in the natural numbers using the relation of belonging. That is, $n < m \iff n \in m$.

Proposition 9 (Total order of ω)**.** *If $m,n \in \omega$ then $n \in m$ or $n = m$ or $m \in n$. These options are exclusive.*

Proposition 10. *The relation $<$ is a total order over ω.*

5.1. Exercises.

(1) Prove that if $a + b < a + c$ then $b < c$.
(2) Prove that ω is transitive; that is, if $n \in \omega$ then $n \subset \omega$.
(3) Prove that ω is well-ordered; that is, that each nonempty subset has a least element.

References

[1] P. Halmos, *Naive Set Theory*