Exersice Sheet 2

——— Sample Solution ———

Task 1: Operational Semantics & Derivation Trees

Task 2: Operational Semantics of other Statements

For $c \in Cmd,\ \sigma, \sigma^{'}, \sigma^{''} \in \Sigma$ and $b \in BExp$ the repeat until relation $\langle \textbf{repeat}\ c\ \textbf{until}\ b,\ \sigma\rangle \to \sigma^{''}$ is defined by:

$$\frac{\langle \mathrm{c},\ \sigma\rangle \to \sigma^{''} \qquad \langle \mathrm{b},\ \sigma^{''}\rangle \to \textbf{true}}{\langle \textbf{repeat}\ \mathrm{c}\ \textbf{until}\ \mathrm{b},\ \sigma\rangle \to \sigma^{''}}\ (\textbf{repeat-true})$$

$$\frac{\langle \mathrm{c},\ \sigma\rangle \to \sigma^{'} \qquad \langle \mathrm{b},\ \sigma^{'}\rangle \to \textbf{false} \qquad \langle \textbf{repeat}\ \mathrm{c}\ \textbf{until}\ \mathrm{b},\ \sigma^{'}\rangle \to \sigma^{''}}{\langle \textbf{repeat}\ \mathrm{c}\ \textbf{until}\ \mathrm{b},\ \sigma\rangle \to \sigma^{''}}\ (\textbf{repeat-false})$$

Task 3: Termination

Prove that $\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma\rangle \to \sigma^{'}$ implies that $\langle \mathrm{b},\ \sigma^{'}\rangle \to \textbf{false}$.
This will be proven by induction on the height h of derivation trees.

Induction Base: (h=1)

If the derivation tree has height 1 only one derivation is possible, namely

$$\frac{\langle \mathrm{b},\ \sigma\rangle \to \mathrm{false}}{\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma\rangle \to \sigma^{'}}(\textbf{while}\text{-false})$$

Since this rule is unambiguous the induction base holds trivially.

Induction Hypothesis:

$$\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma\rangle \to \sigma^{'} \text{ implies } \langle \mathrm{b},\ \sigma^{'}\rangle \to \textbf{false}$$

holds for all derivations of an arbitrary, but fixed height h and for all states $\sigma,\ \sigma^{'}$.

Induction Step: $(h \mapsto h+1)$

For all derivations of height $h+1$ $(h \geq 1)$, we have

$$\frac{\langle \text{b},\ \sigma\rangle \to \textbf{true} \qquad \langle \text{c},\ \sigma\rangle \to \sigma^{'} \qquad \dfrac{\cdots \text{ (derivation tree of height h)}}{\left\langle \textbf{while } \text{b} \textbf{ do } \text{c} \textbf{ end},\ \sigma^{'}\right\rangle \to \sigma^{''}}}{\langle \textbf{while } \text{b} \textbf{ do } \text{c} \textbf{ end},\ \sigma\rangle \to \sigma^{''}}$$

By Induction Hypothesis $\left\langle \textbf{while } \text{b} \textbf{ do } \text{c} \textbf{ end},\ \sigma^{'}\right\rangle \to \sigma^{''}$ implies $\left\langle \text{b},\ \sigma^{'}\right\rangle \to$ **false**.

Due to the propagating characteristics of the derivation trees we also know that $\langle \textbf{while } \text{b} \textbf{ do } \text{c} \textbf{ end},\ \sigma\rangle \to \sigma^{''}$ implies $\left\langle \text{b},\ \sigma^{''}\right\rangle \to \text{false}$. ■

Task 4: Variables that do not matter

(a)

$\textbf{mod}: \quad \textbf{Cmd} \to 2^{\textbf{Var}},$
$\textbf{skip} \mapsto \emptyset$
$x := a \mapsto \{x\}$
$c_1; c_2 \mapsto \textbf{mod}\,(c_1) \cup \textbf{mod}\,(c_2)$
$\textbf{repeat}\ c\ \textbf{until}\ b \mapsto \textbf{mod}\,(c)$

(b)

$\textbf{dep}: \quad \textbf{Cmd} \to 2^{\textbf{Var}},$
$\textbf{skip} \mapsto \emptyset$
$x := a \mapsto \textbf{FV}\,(a)$
$c_1; c_2 \mapsto \textbf{dep}\,(c_1) \cup \textbf{dep}\,(c_2)$
$\textbf{repeat}\ c\ \textbf{until}\ b \mapsto \textbf{dep}\,(c) \cup \textbf{FV}\,(b)$

(c)

Show for every program c and states $\sigma_1,\ \sigma_2$ with

- $\sigma_1 =_{\textbf{dep}} (c)\sigma_2$
- $\langle c,\ \sigma_1\rangle \to \sigma_1^{'}$ and
- $\langle c,\ \sigma_2\rangle \to \sigma_2^{'}$

that $\sigma_1^{'} =_{\textbf{mod}} (c)\sigma_2^{'}$.
This will be shown by induction on the height h of derivation trees.

Induction Base: (h=1)

If the derivation tree has height 1 only two derivations are possible, namely the skip and the assignment derivations.

case: $c =$ **skip**

This case is trivial due to the definition of **mod** and that the empty set is identical in any two arbitrary but fixed states σ_1 and σ_2.

case: $c = \boldsymbol{x := a}$

Following the definitions of **dep** and **mod** we get $\mathbf{mod}(c) = \{x\}$ and $\mathbf{dep}(c) = \mathbf{FV}(a)$.

Furthermore we have

$$\frac{\langle \mathrm{a},\ \sigma_i\rangle \rightarrow z_i}{\langle x := a,\ \sigma_i\rangle \rightarrow \sigma_i\,[x \mapsto z_i]},\ i \in \{1,\ 2\}$$

Since $\sigma_1 =_{\mathbf{FV}(a)} \sigma_2$ (assumption) it holds that $\langle a,\ \sigma_1\rangle \rightarrow z \Leftrightarrow \langle a,\ \sigma_2\rangle \rightarrow z$ (Lemma 2.6, Chapter 2, Slide 17). Thus $z_1 = z_2$ and moreover $\sigma_1' =_{\mathbf{mod}(c)} \sigma_2'$.

Induction Hypothesis:

$$\sigma_1 =_{\mathbf{dep}} (c)\sigma_2,\ \langle c,\ \sigma_1\rangle \rightarrow \sigma_1' \text{ and } \langle c,\ \sigma_2\rangle \rightarrow \sigma_2' \text{ imply that } \sigma_1' =_{\mathbf{mod}} (c)\sigma_2'$$

holds for all derivations of an arbitrary, but fixed height h and for all states $\sigma,\ \sigma'$.

Induction Step: ($\boldsymbol{h \mapsto h+1}$)

case: $c =\mathbf{c_1; c_2}$

Following the definition of **dep** and **mod** we get
$\mathbf{mod}(c) = \mathbf{mod}(c_1) \cup \mathbf{mod}(c_2)$ and $\mathbf{dep}(c) = \mathbf{dep}(c_1) \cup \mathbf{dep}(c_2)$.

Furthermore we have

$$\frac{\langle c_1,\ \sigma_i\rangle \rightarrow \sigma_i^* \qquad \langle c_2,\ \sigma_i^*\rangle \rightarrow \sigma_i'}{\langle c_1; c_2,\ \sigma_i\rangle \rightarrow \sigma_i'},\ i \in \{1,\ 2\}$$

By induction hypothesis it holds that $\sigma_1^* =_{\mathbf{mod}(c_1)} \sigma_2^*$.
Now let $R = \mathbf{dep}(c_2)\backslash\mathbf{mod}(c_1)$. Then we get two additional coherences:

(1) $\sigma_1 =_R \sigma_2$ (because $R \subseteq \mathbf{dep}(c)$)

(2) $\sigma_i =_R \sigma_i^*,\ i \in \{1,\ 2\}$ (by auxiliary (a))

Thus it holds that $\sigma_1^* =_R \sigma_2^*$ and therefore $\sigma_1^* =_{\mathbf{mod}(c_1)\cup\mathbf{dep}(c_2)} \sigma_2^*$
Applying the induction hypothesis we then get that $\sigma_1' =_{\mathbf{mod}(c_2)} \sigma_2'$