Exersice Sheet 4

Sample Solution

Task 1: Chain Complete Partial Orders

(a): TRUE

Let $d \sqsubseteq_1 d'$. Then $\{d,\ d'\}$ is a chain.
Thus $f(d') = f\left(\underbrace{\sqcup_1 \{d,\ d'\}}_{=d'}\right) \overset{f \text{ continous}}{=} \sqcup_2 \{f(d),\ f(d')\} \overset{\text{def. L.U.B}}{\sqsupseteq_2} f(d)$.
Therefore $f(d) \sqsubset_2 f(d')$ holds.
Alternatively: True by Definition 7.15.

(b): FALSE

Let $S = \left\{x \in \mathbb{Q} \mid x \leq \sqrt{2}\right\}$.
Then S is a chain, but $\sqcup S = \sqrt{2} \notin \mathbb{Q}$.

(c): FALSE

Let $(D_1,\ \sqsubseteq) = (\mathbb{N} \cup \{\infty\},\ \leq)$ and
$f:\ D_1 \to D_1,\ x \mapsto \begin{cases} 0, & x < \infty \\ \infty & x = \infty \end{cases}$.
f is monotonic, because $x \leq y \Rightarrow f(x) = \begin{cases} 0 & \leq f(y) & \text{if } x < \infty \\ \infty & \leq f(y) & \text{if } x = y = \infty \end{cases}$.
However, $\underbrace{f(\sqcup \mathbb{N})}_{=\infty} \nleq \underbrace{\sqcup f(\mathbb{N})}_{=0}$.

(d): TRUE

Since $f(p) \sqsubseteq p$, it suffices to prove $p \sqsubseteq f(p)$.
First note that $f(p)$ implies $f(f(p)) \sqsubseteq f(p)$ $\quad (f(p) = p')$
Since p is the least element with $f(p) \sqsubseteq p$, we have $p \sqsubseteq p' = f(p)$.
Thus $p = f(p)$ holds.

Task 2: repeat-until Loops

(a)

$$
\begin{aligned}
&\mathfrak{C}[\![\text{repeat } c \text{ until } b]\!]\\
=&\mathfrak{C}[\![c;\ \text{if } b \text{ then skip else repeat } c \text{ until } b]\!]\\
=&\mathfrak{C}[\![\text{if } b \text{ then skip else repeat } c \text{ until } b]\!] \circ \mathfrak{C}[\![c]\!]\\
=&\text{cond}\left(\mathfrak{B}[\![b]\!],\ \mathfrak{C}[\![\text{skip}]\!],\ \mathfrak{C}[\![\text{repeat } c \text{ until } b]\!]\right) \circ \mathfrak{C}[\![c]\!]\\
=&\text{cond}\left(\mathfrak{B}[\![b]\!],\ id_\Sigma,\ \mathfrak{C}[\![\text{repeat } c \text{ until } b]\!]\right) \circ \mathfrak{C}[\![c]\!]
\end{aligned}
$$

Then $\mathfrak{C}[\![\text{repeat } c \text{ until } b]\!]$ is the least fixed point of F given by:

$$F\left(f\right) = \text{cond}\left(\mathfrak{B}[\![b]\!],\ id_\Sigma,\ \mathfrak{C}[\![\text{repeat } c \text{ until } b]\!]\right) \circ \mathfrak{C}[\![c]\!]$$

(b)

$$
\begin{aligned}
\hat{F}\left(f\right) &= \text{cond}\left(\mathfrak{B}[\![\text{false}]\!],\ id_\Sigma,\ f\right) \circ \mathfrak{C}[\![\text{skip}]\!]\\
&= f \circ \mathfrak{C}[\![\text{skip}\\
&= f \circ id_\Sigma\\
&= f
\end{aligned}
$$

$\hat{F}$ is the identity transformer.

(c)

Since $\hat{F}\left(f\right) = f$ for all $f : \Sigma \dashrightarrow \Sigma$ and $f_\emptyset \sqsubseteq f$ for all $f : \Sigma \dashrightarrow \Sigma$, we have $\text{fix}\left(\hat{F}\right) = f_\emptyset$

Task 3: Closed Sets

(a)

Apply Tarski-Knaster Theorem:

$$\text{fix}\left(f\right) = \sqcup \left\{f^n\left(\sqcup\emptyset\right) | n \geq 0\right\}$$

Since $\emptyset$ is a chain and $\emptyset \subseteq C$, we have $\sqcup\emptyset \in C$.
By definition, $f\left(\sqcup\emptyset\right) \in C$.
By complete induction, $\forall n. f^n\left(\sqcup\emptyset\right) \in C$.
Hence, $G = \left\{f^n\left(\sqcup\emptyset\right) | n \geq 0\right\} \subseteq C$ is a chain.
By definition, $\text{fix}\left(f\right) = \sqcup G \in C$.

(b)

Let $C := \{y \in D | y \sqsubseteq x\}$. C is closed.
If $y \in D$ then $f(y) \sqsubseteq f(x) \sqsubseteq x$.
Thus $f(y) \in C$. By (a), $\mathrm{fix}(f) \in C$.
By definition of C, $\mathrm{fix}(f) \sqsubseteq x$.