Exersice Sheet 5

Sample Solution

Task 1: Fusion Lemma

(a)

$$
\begin{tikzcd}
D_1 \arrow[r, "H"] \arrow[d, "F"'] & D_2 \\
D_1 \arrow[r, "H"'] & D_2 \arrow[u, "G"']
\end{tikzcd}
$$

1 We first prove that $G^n(\perp_2) = H \circ F^n(\perp_1)$ by induction over n.

Induction Base: $n = 0$

$$G^0(\perp_2) = \perp_2 \overset{H \text{ strict}}{=} H(\perp_1) = H\left(F^0(\perp_1)\right)$$

Induction Hypothesis:

$G^n(\perp_2) = H(F^n(\perp_1))$ holds for any arbitrary but fixed $n \in \mathbb{N}$.

Induction Step: $n \mapsto n+1$

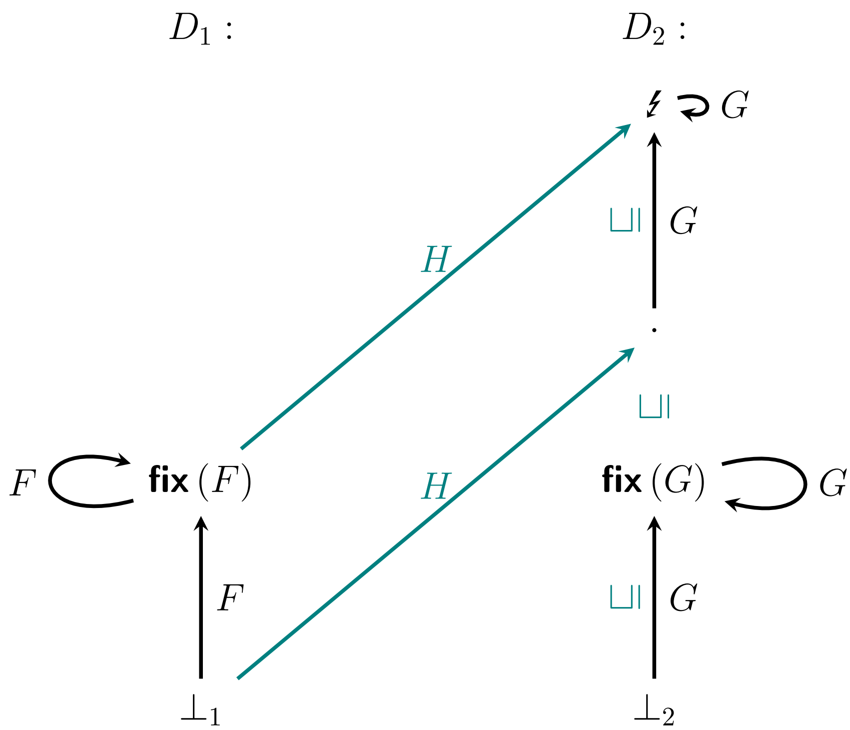
$$
\begin{aligned}
G^{n+1}(\perp_2) &= G \circ G^n(\perp_2) \\
&\overset{I.H.}{=} G \circ H \circ F^n(\perp_1) \\
&\overset{G \circ H = H \circ F}{=} H \circ F \circ F^n(\perp_1) \\
&= H\left(F^{n+1}(\perp_1)\right)
\end{aligned}
$$

2 Now we prove that $\operatorname{fix}(G) = H(\operatorname{fix}(F))$.

$$
\begin{aligned}
\operatorname{fix}(G) &= \sqcup\{G^n(\perp_2) \mid n \in \mathbb{N}\} && \mid \text{by Tarski-Knaster} \\
&= \sqcup\{H \circ F^n(\perp_1) \mid n \in \mathbb{N}\} && \mid \text{by } (1) \\
&= H(\sqcup\{F^n(\perp_1) \mid n \in \mathbb{N}\}) && \mid \text{by } H \text{ continuous} \\
&= H(\operatorname{fix}(F)) && \mid \text{by Tarski-Knaster}
\end{aligned}
$$

(b)

Wrong!



G, F, H continuous ✓

$H \circ F = G \circ H$ ✓

$H\,(\text{fix}\,(F)) = ↯ \neq \text{fix}\,(G)$

Task 2: Tarski-Kantorovich Principle

Task 3: Complete Lattice

Every chain is a subset
Thus: Complete Lattices are chain complete partial orders (**CCPO**)
$\Rightarrow$ Tarski-Knaster is applicable.