Exersice Sheet 4

Sample Solution

Task 1: Chain Complete Partial Orders

(a): TRUE

Let $d \sqsubseteq_1 d'$. Then $\{d,\ d'\}$ is a chain.
Thus $f(d') = f\big(\underbrace{\sqcup_1 \{d,\ d'\}}_{=d'}\big) \overset{f \text{ continous}}{=} \sqcup_2 \{f(d),\ f(d')\} \overset{\text{def. L.U.B}}{\sqsupseteq_2} f(d)$.
Therefore $f(d) \sqsubseteq_2 f(d')$ holds.
Alternatively: True by Definition 7.15.

(b): FALSE

Let $S = \left\{x \in \mathbb{Q} \mid x \leq \sqrt{2}\right\}$.
Then S is a chain, but $\sqcup S = \sqrt{2} \notin \mathbb{Q}$.

(c): FALSE

Let $(D_1,\ \sqsubseteq) = (\mathbb{N} \cup \{\infty\},\ \leq)$ and

$$f:\ D_1 \to D_1,\ x \mapsto \begin{cases} 0, & x < \infty \\ \infty & x = \infty \end{cases}.$$

f is monotonic, because $x \leq y \Rightarrow f(x) = \begin{cases} 0 & \leq f(y) & \text{if } x < \infty \\ \infty & \leq f(y) & \text{if } x = y = \infty \end{cases}$.

However, $\underbrace{f(\sqcup \mathbb{N})}_{=\infty} \not\leq \underbrace{\sqcup f(\mathbb{N})}_{=0}$.

(d): TRUE

Since $f(p) \sqsubseteq p$, it suffices to prove $p \sqsubseteq f(p)$.
First note that $f(p)$ implies $f(f(p)) \sqsubseteq f(p)$ $\qquad (f(p) = p')$
Since p is the least element with $f(p) \sqsubseteq p$, we have $p \sqsubseteq p' = f(p)$.
Thus $p = f(p)$ holds.

Task 2: repeat-until Loops

Task 3: Closed Sets