Exersice Sheet 3

——— Sample Solution ———

Task 1: Operational Equivalence

Prove or disprove:

$$\textbf{repeat } c \textbf{ until } b \sim c;\ \textbf{while } b \textbf{ do } c \textbf{ end}$$

The claim will be disproved using a counter example.

Lets assume $b := \textbf{true}$ and $c := \textbf{skip}$.
The **repeat** statement will terminate after the first iteration with $\langle \textbf{repeat } c \textbf{ until } b,\ \sigma\rangle \to \sigma$ while the **while** statement will never terminate as its condition is always satisfied.

Prove or disprove:

$$\underbrace{\textbf{repeat } c \textbf{ until } b}_{=r} \sim c;\ \underbrace{\textbf{while } \neg b \textbf{ do } c \textbf{ end}}_{=w}$$

To prove this statement we claim that
$\langle r,\ \sigma\rangle \to \sigma' \Leftrightarrow \langle c;\ w,\ \sigma\rangle \to \sigma'$

" $\Rightarrow$ "

Proof by induction on the structure of proof trees
$\langle \textbf{repeat } c \textbf{ until } b,\ \sigma\rangle \to \sigma'$

Induction Base: (b = true)

$$\cfrac{\cfrac{\cdots}{\langle c,\ \sigma\rangle \to \sigma'} \quad \cfrac{\cdots}{\langle b,\ \sigma 1'\rangle \to \textbf{true}}}{\langle r,\ \sigma\rangle \to \sigma'}\ \text{(repeat-true)}$$

Applying this assignment to the **while**-sequence we receive:

$$\cfrac{\langle c,\ \sigma\rangle \to \sigma' \quad \cfrac{\cfrac{\langle b,\ \sigma'\rangle \to \textbf{true}}{\langle \neg b,\ \sigma'\rangle \to \textbf{false}}}{\langle w,\ \sigma'\rangle \to \sigma'}\ \text{(while-false)}}{\langle c;\ w,\ \sigma\rangle \to \sigma'}\ \text{(seq)}$$

Thus the claim holds if $b = \textbf{true}$.

Induction Hypothesis:

$\left\langle r,\ \sigma^{'}\right\rangle \rightarrow \sigma^{''} \Rightarrow \left\langle c;\ w,\ \sigma^{'}\right\rangle \rightarrow \sigma^{''}$ holds.

Induction step: (b = false)

$$\dfrac{\dfrac{\textbf{assumption}}{\langle c,\ \sigma\rangle \rightarrow \sigma^{'}} \quad \left\langle b,\ \sigma^{'}\right\rangle \rightarrow \textbf{false} \quad \dfrac{\textbf{by assumption}}{\left\langle r,\ \sigma^{'}\right\rangle \rightarrow \sigma^{''}}}{\langle r,\ \sigma\rangle \rightarrow \sigma^{''}}\ \text{(repeat-false)}$$

Applying the same assignment to the **while**-sequence we receive:

$$\dfrac{\dfrac{\textbf{assumption}}{\langle c,\ \sigma\rangle \rightarrow \sigma^{'}} \quad \dfrac{\dfrac{\left\langle b,\ \sigma^{'}\right\rangle \rightarrow \textbf{false}}{\left\langle \neg b,\ \sigma^{'}\right\rangle \rightarrow \textbf{true}} \quad \dfrac{\dfrac{\textbf{I.H.}}{\left\langle c;\ w,\ \sigma^{'}\right\rangle \rightarrow \sigma^{''}}}{\left\langle c,\ \sigma^{'}\right\rangle \rightarrow \sigma^{'''} \quad \left\langle w,\ \sigma^{'''}\right\rangle \rightarrow \sigma^{''}}\ !}{\left\langle w,\ \sigma^{'}\right\rangle \rightarrow \sigma^{''}}}{\langle c;\ w,\ \sigma\rangle \rightarrow \sigma^{''}}$$

Thus the claim also holds for $b = \textbf{false}$.

" $\Leftarrow$ " analogously.

Task 2: Translation of Statements

$$\mathfrak{T}_c[\![\textbf{repeat}\ c\ \textbf{until}\ b]\!] = \mathfrak{T}_c[\![c]\!];\ \mathfrak{T}_b[\![b]\!];\ \textbf{JMPF}\,(-|\mathfrak{T}_c[\![c]\!]| + |\mathfrak{T}_b[\![b]\!]|)$$

Task 3: loop Loops

(a)

$$\dfrac{\langle x > 0,\ \sigma\rangle \rightarrow \textbf{false}}{\langle \textbf{loop}\ x\ \textbf{begin}\ c\ \textbf{end},\ \sigma\rangle \rightarrow \sigma}$$

$$\dfrac{\langle x > 0,\ \sigma\rangle \rightarrow \textbf{true} \quad \langle c,\ \sigma\rangle \rightarrow \sigma^{'} \quad \left\langle z := x - 1,\ \sigma^{'}\right\rangle \rightarrow \sigma^{''} \quad \left\langle \textbf{loop}\ z\ \textbf{begin}\ c\ \textbf{end},\ \sigma^{''}\right\rangle \rightarrow \sigma^{'''}}{\langle \textbf{loop}\ x\ \textbf{begin}\ c\ \textbf{end},\ \sigma\rangle \rightarrow \sigma^{'''}}$$

(b)

$$\begin{aligned}&\mathfrak{T}_c[\![\textbf{loop}\ x\ \textbf{begin}\ c\ \textbf{end}]\!] = \textbf{LOAD}(x);\ \textbf{STO}(\xi);\\ &\textbf{LOAD}(\xi);\ \textbf{PUSH}(0);\ \textbf{GT};\ \textbf{JMPF}(|\mathfrak{T}_c[\![c]\!]| + 6);\\ &\mathfrak{T}_c[\![c]\!];\ \textbf{LOAD}(\xi);\ \textbf{PUSH}(1);\ \textbf{SUB};\ \textbf{STO}(\xi);\ \textbf{JMP}(-(|\mathfrak{T}_c[\![c]\!]| + 8))\end{aligned}$$