Exersice Sheet 2

—— Sample Solution ——

Task 1:

Task 2: Operational Semantics of other Statements

For $c \in Cmd$, $\sigma, \sigma^{'}, \sigma^{''} \in \Sigma$ and $b \in BExp$ the repeat until relation $\langle \textbf{repeat}\ c\ \textbf{until}\ b,\ \sigma \rangle \rightarrow \sigma^{''}$ is defined by:

$$\frac{\langle \mathrm{c},\ \sigma \rangle \rightarrow \sigma^{''} \quad \left\langle \mathrm{b},\ \sigma^{''} \right\rangle \rightarrow \textbf{true}}{\langle \textbf{repeat}\ \mathrm{c}\ \textbf{until}\ \mathrm{b},\ \sigma \rangle \rightarrow \sigma^{''}}\ (\textbf{repeat-true})$$

$$\frac{\langle \mathrm{c},\ \sigma \rangle \rightarrow \sigma^{'} \quad \left\langle \mathrm{b},\ \sigma^{'} \right\rangle \rightarrow \textbf{false} \quad \left\langle \textbf{repeat}\ \mathrm{c}\ \textbf{until}\ \mathrm{b},\ \sigma^{'} \right\rangle \rightarrow \sigma^{''}}{\langle \textbf{repeat}\ \mathrm{c}\ \textbf{until}\ \mathrm{b},\ \sigma \rangle \rightarrow \sigma^{''}}\ (\textbf{repeat-false})$$

Task 3: Termination

Prove that $\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma \rangle \rightarrow \sigma^{'}$ implies that $\left\langle \mathrm{b},\ \sigma^{'} \right\rangle \rightarrow \textbf{false}$.
This will be proven by induction on the height h of derivation trees.

Induction Base: (h=1)

If the derivation tree has hight 1 only one derivation is possible, namely

$$\frac{\langle \mathrm{b},\ \sigma \rangle \rightarrow \mathrm{false}}{\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma \rangle \rightarrow \sigma^{'}}(\textbf{while}\text{-false})$$

Since this rule is unambiguous the induction base holds trivially.

Induction Hypothesis:

$$\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma \rangle \rightarrow \sigma^{'} \text{ implies } \left\langle \mathrm{b},\ \sigma^{'} \right\rangle \rightarrow \textbf{false}$$

holds for all derivations of an arbitrary, but fixed height h and for all states $\sigma,\ \sigma^{'}$.

Induction Step: $(h \mapsto h+1)$

For all derivations of height $h+1$ $(h \geq 1)$, we have

$$\frac{\langle \mathrm{b},\ \sigma\rangle \rightarrow \textbf{true} \quad \langle \mathrm{c},\ \sigma\rangle \rightarrow \sigma^{'} \quad \dfrac{\cdots \ \text{(derivation tree of height h)}}{\left\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma^{'}\right\rangle \rightarrow \sigma^{''}}}{\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma\rangle \rightarrow \sigma^{''}}$$

By Induction Hypothesis $\left\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma^{'}\right\rangle \rightarrow \sigma^{''}$ implies $\left\langle \mathrm{b},\ \sigma^{'}\right\rangle \rightarrow$ **false**.

Due to the propagating characteristics of the derivation trees we also know that $\langle \textbf{while}\ \mathrm{b}\ \textbf{do}\ \mathrm{c}\ \textbf{end},\ \sigma\rangle \rightarrow \sigma^{''}$ implies $\left\langle \mathrm{b},\ \sigma^{''}\right\rangle \rightarrow \mathrm{false}$. ■

Task 4: Variables that do not matter

(a)

$\textbf{mod}: \quad \textbf{Cmd} \rightarrow 2^{\textbf{Var}},$
$\textbf{skip} \mapsto \emptyset$
$x := a \mapsto \{x\}$
$c_1; c_2 \mapsto \textbf{mod}\,(c_1) \cup \textbf{mod}\,(c_2)$
$\textbf{repeat}\ c\ \textbf{until}\ b \mapsto \textbf{mod}\,(c)$

(b)

$\textbf{dep}: \quad \textbf{Cmd} \rightarrow 2^{\textbf{Var}},$
$\textbf{skip} \mapsto \emptyset$
$x := a \mapsto \textbf{FV}\,(a)$
$c_1; c_2 \mapsto \textbf{dep}\,(c_1) \cup \textbf{dep}\,(c_2)$
$\textbf{repeat}\ c\ \textbf{until}\ b \mapsto \textbf{dep}\,(c) \cup \textbf{FV}\,(b)$