Exersice Sheet 5

Sample Solution

Task 1: Fusion Lemma

(a)

$$
\begin{tikzcd}
D_1 \arrow[r, "H"] \arrow[d, "F"'] & D_2 \\
D_1 \arrow[r, "H"'] & D_2 \arrow[u, "G"']
\end{tikzcd}
$$

1 We first prove that $G^n(\perp_2) = H \circ F^n(\perp_1)$ by induction over n.

Induction Base: $n = 0$

$$G^0(\perp_2) = \perp_2 \overset{H \text{ strict}}{=} H(\perp_1) = H\left(F^0(\perp_1)\right)$$

Induction Hypothesis:

$G^n(\perp_2) = H(F^n(\perp_1))$ holds for any arbitrary but fixed $n \in \mathbb{N}$.

Induction Step: $n \mapsto n+1$

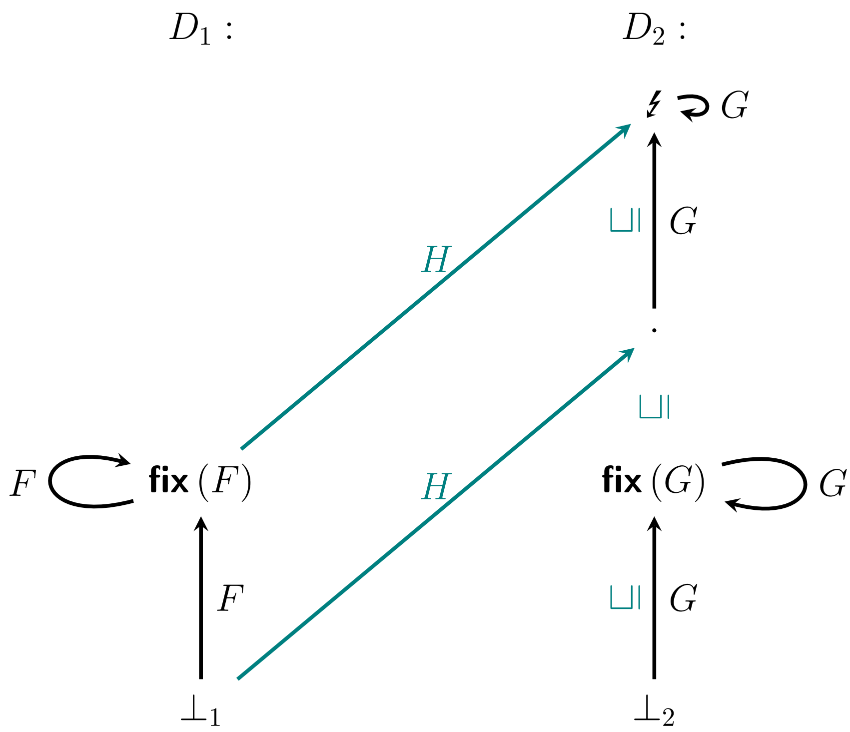
$$
\begin{aligned}
G^{n+1}(\perp_2) &= G \circ G^n(\perp_2) \\
&\overset{I.H.}{=} G \circ H \circ F^n(\perp_1) \\
&\overset{G \circ H = H \circ F}{=} H \circ F \circ F^n(\perp_1) \\
&= H\left(F^{n+1}(\perp_1)\right)
\end{aligned}
$$

2 Now we prove that $\text{fix}(G) = H(\text{fix}(F))$.

$$
\begin{aligned}
\text{fix}(G) &= \sqcup\{G^n(\perp_2) \mid n \in \mathbb{N}\} && \mid \text{by Tarski-Knaster} \\
&= \sqcup\{H \circ F^n(\perp_1) \mid n \in \mathbb{N}\} && \mid \text{by } (1) \\
&= H(\sqcup\{F^n(\perp_1) \mid n \in \mathbb{N}\}) && \mid \text{by } H \text{ continuous} \\
&= H(\text{fix}(F)) && \mid \text{by Tarski-Knaster}
\end{aligned}
$$

(b)

Wrong!



G, F, H continuous ✓

$H \circ F = G \circ H$ ✓

$H(\text{fix}(F)) = \lightning \neq \text{fix}(G)$

Task 2: Tarski-Kantorovich Principle

Prove or disprove: Let $(D, \sqsubseteq)$ be a CCPO and let $F: D \to D$ be continuous. Moreover, let $d \in D$, such that $d \sqsubseteq F(d)$.
Then F has at least one fixpoint larger than d and the least of those fixpoints is given by

$$\sqcup \{F^n(d) \mid n \in \mathbb{N}\}$$

We disprove this assumption by specifying a counterexample.
We define F in such a way that $d = F(d)$ but $d \neq \text{fix}(F)$.
Utilising Tarski-Knaster and the definition of this partial order we get $d \sqsubseteq F(d)$ and $\sqcup \{F^n(d) \mid n \in \mathbb{N}\} = d \sqsupsetneq d$. $\lightning$

Prove or disprove: Let $(D, \sqsubseteq)$ be a CCPO and let $F: \; D \to D$ be continuous. Moreover, let $d \in D$, such that $d \sqsubseteq F(d)$.
Then F has at least one fixpoint larger or equal than d and the least of those fixpoints is given by

$$\sqcup \{F^n(d) \,|n \in \mathbb{N}\}$$

Let $\hat{d} := \sqcup \{F^n(d) \,|n \in \mathbb{N}\}$.
$\hat{d}$ exists, since $d \sqsubseteq F(d) \sqsubseteq F^2(d) \sqsubseteq \cdots$ is a chain and D is a CCPO.
$\hat{d}$ is a fixpoint since

$$\begin{array}{rcll} F\left(\hat{d}\right) & = & F\left(\sqcup \{F^n(d)\, n \in \mathbb{N}\}\right) & \\ & = & \sqcup \left\{F^{n+1}(d) \,|n \in \mathbb{N}\right\} & \mid \text{by continuity of } F \\ & = & \sqcup \{F^n(d) \,|n \in \mathbb{N}\} & \mid d \sqsubseteq F(d) \\ & = & \hat{d} & \end{array}$$

$\hat{d}$ is least fixpoint "above" d
Let x be another fixpoint "above" d "smaller" than $\hat{d}$, i.e.

$$\begin{array}{ll} (1) & f(x) = x \\ (2) & d \sqsubseteq x \\ (3) & x \sqsubseteq \hat{d} \end{array}$$

<u>But:</u>

$$\begin{array}{rcll} d \sqsubseteq x & \Rightarrow & F(d) \sqsubseteq F(x) & \mid F \text{ monoton} \\ & \Rightarrow & \forall n \in \mathbb{N} \;\left(F^n(d) \sqsubseteq F^n(x)\right) & \mid \\ & \Rightarrow & \sqcup \{F^n(d) \,|n \in \mathbb{N}\} \sqsubseteq \sqcup \{F^n(x) \,|n \in \mathbb{N}\} & \\ & \Rightarrow & \hat{d} \sqsubseteq \sqcup \{F^n(x) \,|n \in \mathbb{N}\} & \\ & \Rightarrow & \hat{d} \sqsubseteq \sqcup \{x|n \in \mathbb{N}\} = \{x\} & \mid \text{by (1)} \\ & \Rightarrow & \hat{d} \sqsubseteq x & \\ & \Rightarrow & \hat{d} = x & \mid \text{by (3) and anti-symmetry of CCPO} \end{array}$$

Task 3: Complete Lattice

Every chain is a subset
Thus: Complete Lattices are chain complete partial orders (**CCPO**)
$\Rightarrow$ Tarski-Knaster is applicable.