# NETWORK FLOW MODELING

LECTURE 4

# INTRODUCTION

A critical function of a communication network is to carry or flow the *volume* of user traffic.

The traffic volume or demand volumes can impact routing and routing decisions, which are also influenced by the goal or objective of the network.

The volume of traffic or demand, to be referred to as *traffic volume* or *demand volume*, is an important entity in a communication network that can impact routing.

In general, traffic volume will be associated with traffic networks while demand volume will be associated with transport networks; for example, in regard to IP networks or the telephone network, we will use the term traffic volume.

# INTRODUCTION

However, for transport networks such as DS3-cross-connect, SONET, or WDM networks where circuits are deployed on a longer term basis, we will use the term demand volume.

Similarly, routing in a traffic network is sometimes referred to as *traffic routing* while in a transport network it is referred to as *transport routing*, *circuit routing*, or *demand routing*.

The measurement units can also vary depending on the communication network of interest.

For example, in IP networks, traffic volume is measured often in terms of *Megabits per sec (Mbps)* or *Gigabits per sec (Gbps)*, while in the telephone network, it is measured in *Erlangs*.

When we consider telecommunications transport networks, the demand volume is measured in terms of number of digital signals such as DS3, OC-3, and so on.

# INTRODUCTION

For any demand volume between two nodes in a network, one or more paths may need to be used to carry it.

Any amount of demand volume that uses or is carried on a path is referred to as *flow*; this is also referred to as *path flow*, or *flowing demand volume on a path*, or even *routing demand volume on a path*.

A path is one of the routes possible between two end nodes with or without positive flows.

Since a network consists of nodes and links, we will also use the term *link flow* to refer to the amount of flow on a link regardless of which end nodes the demand volume is for.

# INTRODUCTION

A given network may not always be able to carry all its demand volume; this can be due to limits on network capacity but also can be dictated by the stochastic nature of traffic.

If the network capacity is given, then we call such a network a *capacitated network*.

Typically, traffic engineering refers to the best way to flow the demand volume in a capacitated network—this is where network flow models are helpful in determining routing or flow decisions.

A communication network can be represented as a *directed network*, or an *undirected network*.

A directed network is one in which the flow is directional from one node to another and the links are considered as directional links.

# INTRODUCTION

An undirected network is a network in which there is no differentiation between the direction of flow; thus, it is common to refer to such flows as bidirectional and links as bidirectional links.

For example, an IP network that uses OSPF protocol is modeled as a directed network with directional links.

However, a telephone network is an undirected network in which a link is bidirectional and where calls from either end can use the undirected link.

In this chapter, we present network flow models assuming networks to be undirected since this allows small models to be explained in a fairly simple way.

# INTRODUCTION

For instance, for a three-node network we need to consider only three links in an undirected network while six links are required to be considered in a directed network.

A pair of demand nodes will be referred to as a *node pair* or a *demand pair*.

A node pair in a network will be denoted by $i$:$j$ where $i$ and $j$ are the end nodes for this pair; if it is a directed network, the first node $i$ should be understood as the *origin* or *source* while the second node should be understood as the *destination* or *sink*.

For an undirected network, $i$ and $j$ are interchangeable while we will typically write the smaller numbered node first; for example, the demand pair with end nodes 2 and 5 will be written as 2:5.

# INTRODUCTION

A link directly connecting two nodes *i* and *j* in general will be denoted as *i-j* ; in case we need to illustrate a point about a directional link, we will specifically use $i \rightarrow j$ to denote the directional link from node *i* to node *j*.

Finally, we use the term *unit cost of flow* on a link or *unit link cost* of flow in regard to carrying demand volume; this term should not be confused with *link cost* or *distance cost* of a link used earlier

# SINGLE-COMMODITY NETWORK FLOW

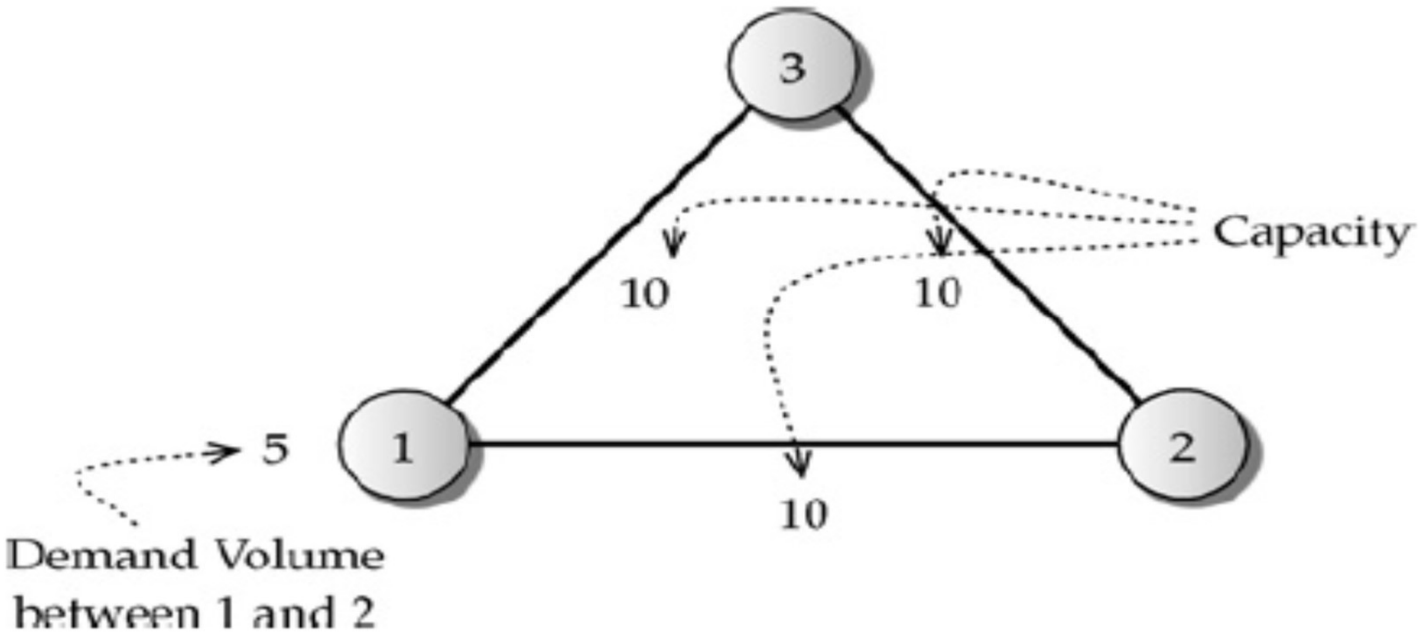
We start with the single-commodity network flow problem.

This means that only a node pair has positive demand volume, thus, the name *single-commodity* where the term *commodity* refers to a demand.

For illustration of network flow models, we will use a three-node network in this section.

# SINGLE-COMMODITY NETWORK FLOW

A Three-Node Illustration



# SINGLE-COMMODITY NETWORK FLOW

Consider a three-node network where 5 units of demand volume need to be carried between node 1 and node 2 (see Figure 1); we assume that the demand volume is a deterministic number.

We are given that all links in the network are bidirectional and have a capacity of 10 units each.

It is easy to see that the direct link 1-2 can easily accommodate the 5 units of demand volume since there the direct link can handle up to 10 units of capacity; this remains the case as long as the demand volume between node 1 and node 2 is 10 units or less.

As soon as the demand volume becomes more than 10 units, it is clear that the direct link path cannot carry all of the demand volume between node 1 and node 2.

# SINGLE-COMMODITY NETWORK FLOW

In other words, any demand in *excess* of 10 units would need to be carried on the second path 1-3-2.

This simple illustration illustrates that not all demand volume can always be carried on a single path or the shortest, hop-based path; the capacity limit on a link along a path matters.

In addition, we have made an *implicit* assumption up to this point that the direct link path 1-2 is less costly per unit of demand flow than the two-link alternate path 1-3-2.

However, in many networks, this may not always be true.

# SINGLE-COMMODITY NETWORK FLOW

If we instead suppose that the per-unit cost of the two-link path 1-3-2 is 1 while the per-unit cost on the direct link 1-2 is 2, then it would be more natural or optimal to route demand volume first on the alternate path 1-3-2 for up to the first 10 units of demand volume, and then route any demand volume above the first 10 units on the direct link path 1-2.

The above illustration helps us to see that the actual routing decision should depend on the goal of routing, irrespective of the hop count.

This means that we need a generic way to represent the problem so that various situations can be addressed in a capacitated network in order to find the best solution.

# SINGLE-COMMODITY NETWORK FLOW

## Formal Description and Minimum Cost Routing Objective

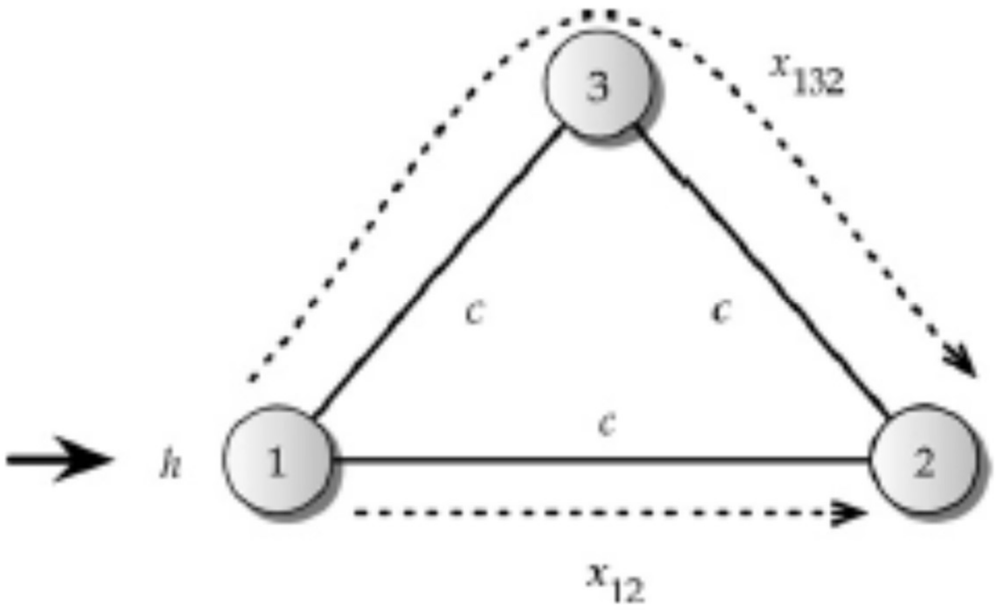
We are now ready to present the above discussion in a formal manner using *unknowns* or *variables*.

We assume here that the capacity of each link is the same and is given by $c$. Let the demand volume for node pair 1:2 be denoted by $h$.

For example, in the above illustration capacity $c$ was set to 10.

Since the demand volume for the node pair 1:2 can possibly be divided between the direct link path 1-2 and the two-link path 1-3-2 , we can use two unknowns or variables to represent this aspect.

# SINGLE-COMMODITY NETWORK FLOW



# SINGLE-COMMODITY NETWORK FLOW

Let $x_{12}$ be the amount of the total demand volume $h$ to be routed on direct link path 1-2 , and let $x_{132}$ be any amount of the demand volume to be routed on the alternate path 1-3-2 (see Figure 2).

Note the use of subscripts so that it is easy to track a route with flows.

Since the total demand volume is required to be carried over these two paths, we can write

$$x_{12} + x_{132} = h \qquad (1a)$$

This requirement is known as the *demand flow constraint*, or simply the *demand constraint*.

It is clear that the variables cannot take negative values since a path may not carry any negative demand; this means the lowest value that can be taken is zero.

# SINGLE-COMMODITY NETWORK FLOW

Thus, we include the following additional conditions on the variables:

$$x_{12} \geq 0, x_{132} \geq 0 \qquad \text{(1b)}$$

In addition, we need to address the capacity limit on each link.

Certainly, any flow on a path due to routing cannot exceed the capacity on any of the links that this path uses.

An implicit assumption here is that the flow and the capacity are using the same measurement units.

Since we assume that the capacity limit is the same on all links in this three-node network, we can write $x_{12} \leq c$ , $x_{132} \leq c$ (1c)

# SINGLE-COMMODITY NETWORK FLOW

The first one addresses the flow on the direct link 1-2 being less than its capacity; flow $x_{132}$ uses two links 1-3 and 2-3, and we can use only a single condition here since the capacity is assumed to be the same on each link.

Constraints (1c) are called *capacity constraints*.

From the above discussion, we can see that we need conditions (1a), (1b), and (1c) to define the basic system.

It is important to note that it is not a system of equations; while the first one, i.e., (1a), is an equation, the second and the third ones, i.e., (1b) and (1c), are inequalities.

Together, the system of equations and inequalities given by Eq. (1), which consists of conditions (1a), (1b), and (1c), is referred to as *constraints* of the problem.

# SINGLE-COMMODITY NETWORK FLOW

Even when all the constraints are known, our entire problem is not complete since we have not yet identified the *goal* of the problem.

In fact, without defining a goal, system (1) has infinite numbers of solutions since an infinite combination of values can be assigned to $x_{12}$ and $x_{132}$ that satisfies constraints (1a), (1b), and (1c).

As the first goal, we consider the cost of routing flows.

To do that, we introduce a generic nonnegative cost *per unit* of flow on each path: $\xi_{12}$ ($\geq 0$) for direct path 1-2 and $\xi_{132}$ ($\geq 0$) for alternate path 1-3-2 .

Thus, the total cost of the demand flow can be written as

$$\text{Total cost} = \xi_{12}\, x_{12} + \xi_{132}\, x_{132} \qquad (2)$$

# SINGLE-COMMODITY NETWORK FLOW

The total cost is referred to as the *objective function.*

In general, the objective function will be denoted by *F*.

If the goal is to minimize the total cost of routing, we can write the complete problem as follows:

$$\begin{array}{ll} \textbf{\textit{Minimize}}\{x_{12}, x_{132}\} & F = \xi_{12} x_{12} + \xi_{132} x_{132} \\ \textbf{\textit{subject to}} & x_{12} + x_{132} = h \qquad (3) \\ & x_{12} \le c,\ x_{132} \le c \\ & x_{12} \ge 0,\ x_{132} \ge 0. \end{array}$$

# SINGLE-COMMODITY NETWORK FLOW

The problem presented in Eq. (3) is a *single-commodity network flow* problem; it is also referred to as a *linear programming problem* since the requirements given by Eq. (1) are all linear, which are either equations or inequalities, and the goal given by Eq. (2) is also linear.

In general, a representation as given in Eq. (3) is referred to as the *formulation* of an optimization problem.

The system given by Eq. (1) is referred to as *constraints*.

To avoid any confusion, we will identify the variables in any formulation by marking them as subscripts with ***minimize*****.**

# SINGLE-COMMODITY NETWORK FLOW

Thus, in the above problem, we have noted that $x_{12}$ and $x_{132}$ are variables by indicating so as subscripts with ***minimize*.**

Often, the list of variables can become long; thus, we will also use a short notation such as $x$ in the subscript with ***minimize*** to indicate that all $x$s are variables.

Because of the way the goal is described in Eq. (3), the problem is also known as the *minimum cost routing* or *minimum cost network flow* problem.

An optimal solution to an optimization problem is a solution that satisfies the constraints of the problem, i.e., it is a *feasible solution* and the objective function value attained is the lowest (if it is a minimization problem) possible for any feasible solution.

# SINGLE-COMMODITY NETWORK FLOW

For clarity, the optimal solution to a problem such as Eq. (3) will be denoted with asterisks in the superscript, for example, $x^*_{12}$ and $x^*_{132}$.

INSTANCE 1:

We now consider some specific cases discussed earlier in Section 1 to obtain solutions to problem (1). First, we consider the capacity to be 10, i.e., $c = 10$.

If the unit cost is based on a unit flow per link, then we can clearly write cost components as $\xi_{12} = 1$ (since it is a direct link path) and $\xi_{132} = 2$ (due to two links making a path).

This will then correspond to the first case discussed in Section 1. In this case, optimal flows can be written as:

# SINGLE-COMMODITY NETWORK FLOW

$x^*_{12} = 10,\ \ x^*_{132} = 0$ when $0 \leq h \leq 10$

$x^*_{12} = 10,\ \ x^*_{132} = h - 10$ when $h \geq 10$, and $h \leq 20$ (4)

If $h > 20$, it is clear that the network does not have enough capacity to carry all of the demand volume—this is referred to as an *infeasible* situation and the problem is considered to be infeasible.

INSTANCE 2:

Consider the alternate case where per unit cost on the alternate path is 1 while on the direct path it is 2, i.e., $\xi_{12} = 2$ and $\xi_{132} = 1$.

In this case, optimal flows can be written as:

# SINGLE-COMMODITY NETWORK FLOW

$x^*_{12} = 0, \; x^*_{132} = 10$ when $0 \leq h \leq 10$

$x^*_{12} = h - 10, \; x^*_{132} = 10$ when $h \geq 10$, and $h \leq 20$ (5)

We now consider the general solution to Problem (3) when the demand volume is less than the capacity of a link, i.e., $h \leq c$.

With two unknowns, problem (3) can be solved by using substitutions, i.e., by setting $x_{132} = h - x_{12}$ and using it back in the objective.

Then, the objective becomes

$F = \xi_{12}\, x_{12} + \xi_{132}\,(h - x_{12}) = (\xi_{12} - \xi_{132})\, x_{12} + \xi_{132}h.$

Note that the last term, $\xi_{132}h$, remains constant for a specific problem instance.

# SINGLE-COMMODITY NETWORK FLOW

Thus, we need to consider the minimization of the rest of the expression, i.e.,

***Minimize***$_{\{x\}}$ $(\xi_{12} - \xi_{132})x_{12}$

subject to appropriate constraints.

We can easily see that if $\xi_{12} < \xi_{132}$, then the problem is at minimum when $x^*_{12} = h$; however, if $\xi_{12} > \xi_{132}$ , then the minimum is observed when $x^*_{12} = 0$.

When $\xi_{12} = \xi_{132}$ , then $x_{12}$ can take any value in the range $[0, h]$, that is, the problem has *multiple* optimal solutions.

Consider now the case in which demand volume, $h$, is more then c but the problem is still feasible, i.e., $h > c$, but $h \leq 2c$.

# SINGLE-COMMODITY NETWORK FLOW

In this case, we need to take the bounds into account properly; thus, if $\xi 12 < \xi 132$, then $x^*_{12} = \min\{h, c\}$; similarly, if $\xi_{12} > \xi_{132}$, then the minimum is observed when $x^*_{12}$

$= \max\{0, h - c\}$.

Thus, for values of $h$ ranging from 0 to 2c, we can see that optimal flows are as we have already identified in (4) and (5), corresponding to $\xi_{12} < \xi_{132}$ and $\xi_{12} > \xi_{132}$, respectively.