COMP 202. Introduction to Electronics

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Fundamentals of Boolean Algebra

Basic Postulates

The basic description of the Boolean algebra formulation is based on concepts from **set theory**.

In 1849, George Boole presented an algebraic formulation of the processes of logical thought and reason.

It is what has come to be known as **Boolean algebra**.

Postulate 1

Definition

A Boolean algebra is a closed algebraic system containing a set K of two or more elements and the two operators $\cdot$ and $+$; alternatively, for every a and b in set K, $a \cdot b$ belongs to K and $a + b$ belongs to K ($+$ is called OR and $\cdot$ is called AND)

Postulate 2

Existence of 1 and 0 elements: There exist unique elements 1 (one) and 0 (zero) in set K such that for every a in K

(a) $a + 0 = a$,

(b) $a \cdot 1 = a$,

where 0 is the identity element for the $+$ operation and 1 is the identity element for the $\cdot$ operation.

Postulate 3

Commutativity of the $+$ and $\cdot$ operations: For every a and b in K

(a) $a + b = b + a$,

(b) $a \cdot b = b \cdot a$.

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Postulate 4

Associativity of the $+$ and $\cdot$ operations: For every a, b and c in K

(a) $a + (b + c) = (a + b) + c$,

(b) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Postulate 5

Distributivity of $+$ over $\cdot$ and $\cdot$ over $+$: For every a, b, and c in K

(a) $a + (b \cdot c) = (a + b) \cdot (a + c)$,

(b) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Postulate 6

Existence of the complement: For every a in K there exists a unique element called $\overline{a}$ (*complement* of a) in K such that

(a) $a + \overline{a} = 1$,

(b) $a \cdot \overline{a} = 0$.

Venn Diagrams for Postulates

The postulates may be graphically presented in the form of Venn diagrams.

This graphical description is possible since the algebra of sets is a Boolean algebra in which the sets correspond to elements, the intersection operation corresponds to $\cdot$, and the union operation corresponds to $+$.

On the Venn diagram, sets are shown as closed contours, i.e., circles, square, ellipses, and the like.

The Venn diagram is a powerful tool for visualizing not only the postulates but also the important theorems of Boolean algebra.

Venn diagrams for the sets a, b, $a \cdot b$, and $a + b$ are shown in the Figure 1 below:

Explaining Postulates with Venn Diagrams

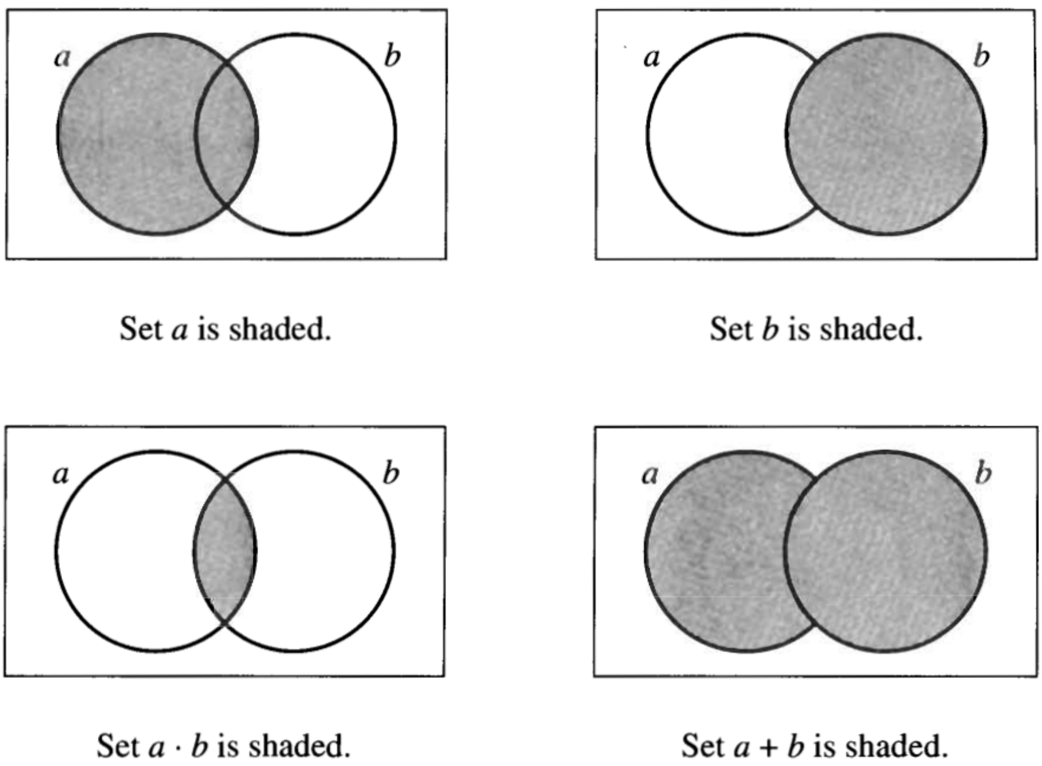

Figure 1

Facets for Postulate 6

Let's examine some facets of Postulate 6 from Venn diagrams from Figure 2 below.

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Facets for Postulate 6

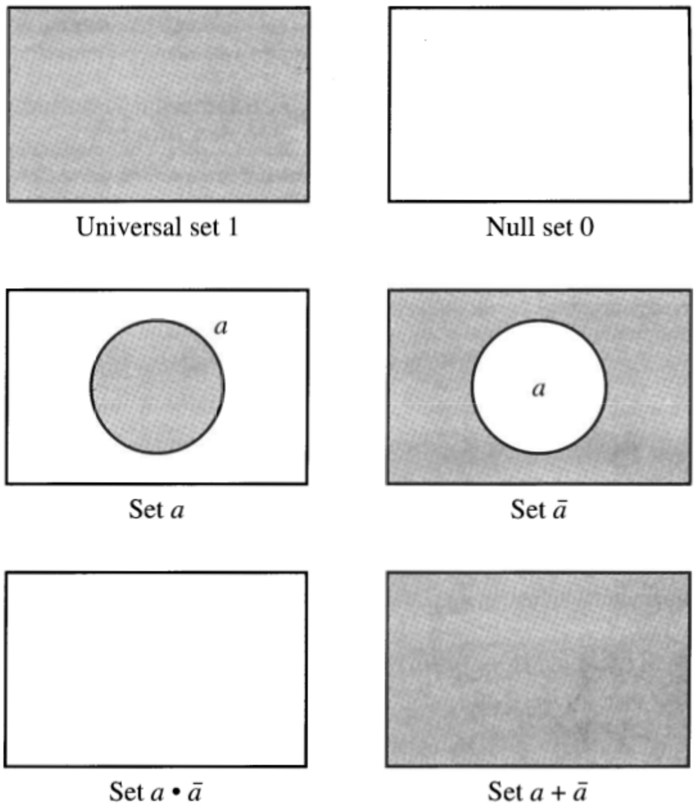


Figure 2

Facets for Postulate 6

If a is the shaded set, the complement of a, $\overline{a}$, is the area outside a in the universal set.

In other words, a and $\overline{a}$ are mutually exclusive and lie inside the universal set.

Since they are mutually exclusive, they contain no area in common and hence their intersection is the null set:

$a \cdot \overline{a} = 0$.

The union of a and $\overline{a}$ is by definition the universal set:

$a + \overline{a} = 1$

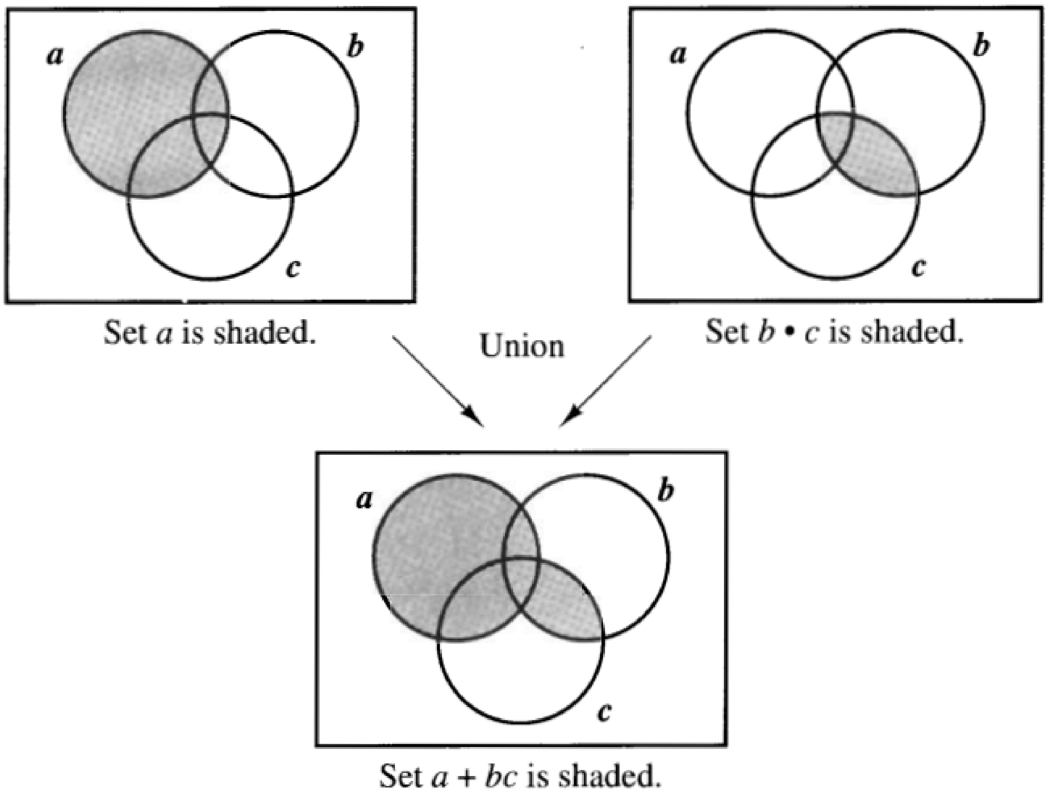
Furthermore, since the universal set, 1, contains all other sets, it's complement must be the null set, 0.

Therefore, $\overline{1} = 0$ and $\overline{0} = 1$

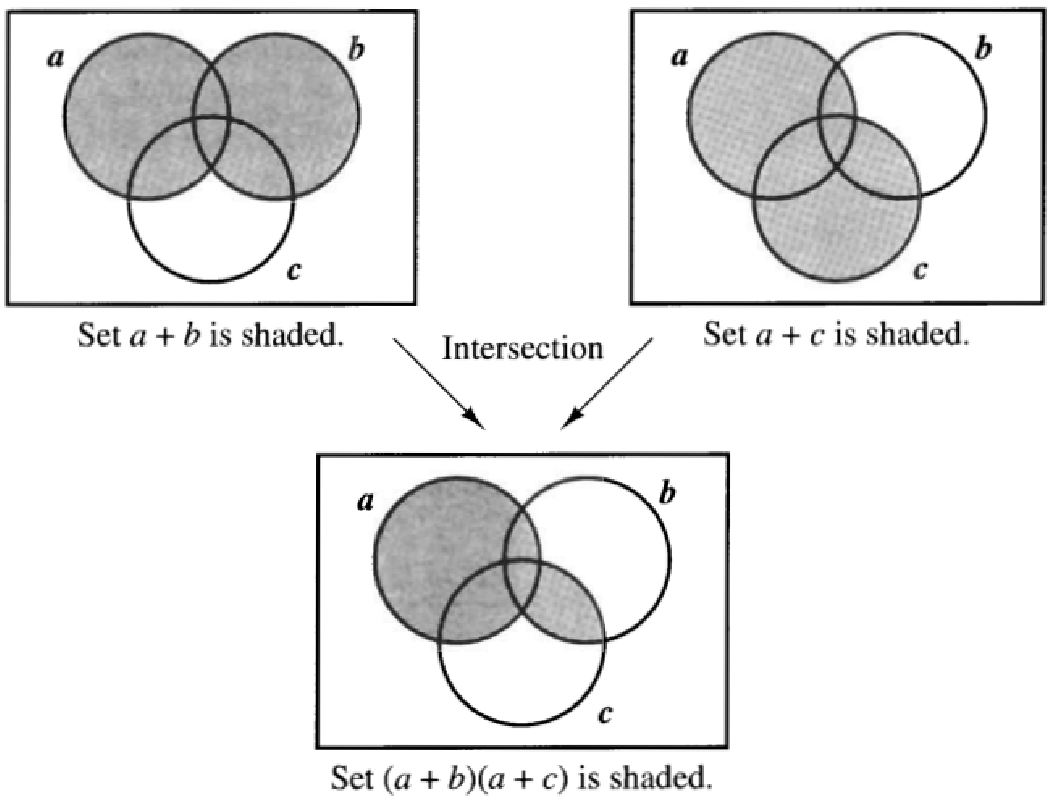
Example 1

Use Venn diagrams to prove that $a + bc = (a + b)(a + c)$.

Example 1



Example 1



The Duality Principle

The principle of *duality* states that if an expression is valid in Boolean algebra, the dual of the expression is also valid.

The dual expression is found by replacing all $+$ operators with $\cdot$, all $\cdot$ with $+$, all ones with zeros and all zeros with ones.

Example 1

Find the dual expression $a + (bc) = (a + b)(a + c)$

Example 2: Solution

Changing all $+$ to $\cdot$ and vice versa, the dual expression of $a + (bc) = (a + b)(a + c)$ is $a(b + c) = ab + ac$

Notes on Duality

When obtaining a dual, care must be taken not to alter the location of the parentheses, if they are present.

Note also that the two expressions above are parts (a) and (b) of Postulate 5.

Theorem 1: Idempotency

(a) $a + a = a$

(b) $a \cdot a = a$

Proof. We may prove either part (a) or (b) of this theorem. Suppose we prove part (a):

$$\begin{aligned}
a + a &= (a + a)1 && [P2(b)] \ldots \textit{slide } 4 \\
&= (a + a)(a + \bar{a}) && [P6(a)] \ldots \textit{slide } 8 \\
&= a + a\bar{a} && [P5(a)] \ldots \textit{slide } 7 \\
&= a + 0 && [P6(b)] \ldots \textit{slide } 8 \\
&= a && [P2(a)] \ldots \textit{slide } 4
\end{aligned}$$

An important point to remember is that symbols on opposite sides of the equal sign may be used interchangeably. E.g. Theorem 1 tells us that we may exchange a for $a \cdot a$, and vice versa.

Exercise

Prove part (b) of the example from the previous slide.

Theorem 2: Null elements for $+$ and $\cdot$ operators

(a) $a + 1 = 1$

(b) $a \cdot 0 = 0$

Proof. Let's prove part (a).

$$
\begin{aligned}
a + 1 &= (a+1)1 && [P2(b)] \ldots \textit{slide } 4 \\
&= a \cdot (a+1) && [P3(b)] \ldots \textit{slide } 5 \\
&= (a+\bar{a})(a+1) && [P6(a)] \ldots \textit{slide } 8 \\
&= a + \bar{a} \cdot 1 && [P5(a)] \ldots \textit{slide } 7 \\
&= a + \bar{a} && [P2(b)] \ldots \textit{slide } 4 \\
&= 1 && [P6(a)] \ldots \textit{slide } 8
\end{aligned}
$$

Since part (a) of this theorem is valid, it follows from the principle of duality that part (b) is valid also.

Exercise

Prove part (b) of the example from the previous slide.

Theorem 3: Involution

$$\overline{\overline{a}} = a$$

Proof: From Postulate 5, $a \cdot \overline{a} = 0$ and $a + \overline{a} = 1$. Therefore, $\overline{a}$ is the complement of a, and also a is the complement of $\overline{a}$. Since the complement of $\overline{a}$ is *unique*, it follows that $\overline{\overline{a}} = a$

Theorem 4: Absorption

(a) $a + ab = a$

(b) $a(a + b) = a$

Proof. Let's prove part (a).

$$\begin{aligned}
a + ab &= a \cdot 1 + ab && [P2(b)] \ldots \textit{slide } 4 \\
&= a(1 + b) && [P5(b)] \ldots \textit{slide } 7 \\
&= a(b + 1) && [P3(b)] \ldots \textit{slide } 5 \\
&= a \cdot 1 && [T2(a)] \ldots \textit{slide } 23 \\
&= a && [P2(b)] \ldots \textit{slide } 4
\end{aligned}$$

Exercise

Prove part (b) of the example from the previous slide.

Theorem 4: Absorption Examples

Theorem 4 can be easily visualized using a Venn diagram.

The following examples illustrate the use of this theorem.

$(X+Y)+(X+Y)Z = X+Y$ [T4(a)]…slide 26

$A\overline{B}(A\overline{B}+\overline{B}C) = A\overline{B}$ [T4(b)]…slide 26

$A\overline{B}C+\overline{B} = \overline{B}$ [T4(a)]…slide 26

The following Theorems 5, 6 & 7 are similar to absorption in that they can be employed to eliminate extra elements from a Boolean expression.

Theorem 5

(a) $a + \overline{a}b = a + b$

(b) $a(\overline{a} + b) = ab$

Proof. Let's prove part (a).

$$\begin{aligned} a + \overline{a}b &= (a + \overline{a})(a + b) && [P5(a)] \ldots \textit{slide } 7 \\ &= 1 \cdot (a + b) && [P6(a)] \ldots \textit{slide } 8 \\ &= (a + b) \cdot 1 && [P3(b)] \ldots \textit{slide } 5 \\ &= (a + b) && [P2(b)] \ldots \textit{slide } 4 \end{aligned}$$

Exercise

Prove part (b) of the example from the previous slide.

Theorem 5 Examples

The following examples illustrate the use of Theorem 5 in simplifying Boolean expressions.

$B + A\overline{BC}D = B + A\overline{C}D$ $[T5(a)]$...slide 29

$\overline{Y}(X + Y + Z) = \overline{Y}(X + Z)$ $[T5(b)]$...slide 29

$(X + Y)((\overline{X + Y}) + Z) = (X + Y)Z$ $[T5(b)]$...slide 29

Theorem 6

(a) $ab + a\overline{b} = a$

(b) $(a + b)(a + \overline{b}) = a$

Proof. Let's prove part (a).

$$\begin{aligned} ab + a\overline{b} &= a(b + \overline{b}) && [P5(b)] \ldots \textit{slide } 7 \\ &= a \cdot 1 && [P6(a)] \ldots \textit{slide } 8 \\ &= a && [P2(b)] \ldots \textit{slide } 4 \end{aligned}$$

Exercise

Prove part (b) of the example from the previous slide.

Theorem 6 Examples

The following examples illustrate the use of Theorem 6 in simplifying Boolean expressions.

$ABC + A\overline{B}C = AC$ $\quad [T6(a)]$...slide 32

$(AD + B + C)(AD + (\overline{B + C})) = AD$ $\quad [T6(b)]$ ...slide 32

Theorem 7

(a) $ab + a\overline{b}c = ab + ac$

(b) $(a+b)(a+\overline{b}+c) = (a+b)(a+c)$

Proof. Let's prove part (a) as follows:

$$\begin{aligned} ab + a\overline{b}c &= a(b + \overline{b}c) && [P5(b)] \ldots \textit{slide } 7 \\ &= a(b+c) && [T5(a)] \ldots \textit{slide } 29 \\ &= ab + ac && [P5(b)] \ldots \textit{slide } 7 \end{aligned}$$

Exercise

Prove part (b) of the example from the previous slide.

Theorem 7 Examples

$$xy + x\overline{y}(\overline{w} + \overline{z}) = xy + x(\overline{w} + \overline{z}) \qquad [T7(a)]\ldots\text{slide } 35$$

$$(\overline{x}\,\overline{y} + z)(w + \overline{x}\,\overline{y} + \overline{z}) = (\overline{x}\,\overline{y} + z)(w + \overline{x}\,\overline{y}) \qquad [T7(b)]\ldots\text{slide } 35$$

Assignment

Complete and submit via vcampus the proofs mentioned in the slides.