

MATRICES

EXAM QUESTIONS

(Part One)

Question 1 ()**

The matrices \mathbf{A} , \mathbf{B} and \mathbf{C} are given below in terms of the scalar constants a , b , c and d , by

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix}.$$

Given that $\mathbf{A} + \mathbf{B} = \mathbf{C}$, find the value of a , b , c and d .

$$a = 8, b = 3, c = 2, d = 3$$

$$\begin{aligned} \mathbf{A} + \mathbf{B} = \mathbf{C} &\Rightarrow \begin{pmatrix} -2 & 3 \\ 1 & a \end{pmatrix} + \begin{pmatrix} b & -1 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} -2+b & 2 \\ 1+2 & a-4 \end{pmatrix} = \begin{pmatrix} 1 & c \\ d & 4 \end{pmatrix} \\ \text{So } a-4 &= 4 \quad \cancel{-2+b=1} \quad \cancel{b=3} \quad \cancel{c=-2} \quad \cancel{d=3} \end{aligned}$$

Question 2 ()**

Find, in terms of k , the inverse of the following 2×2 matrix.

$$\mathbf{M} = \begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}.$$

Verify your answer by multiplication.

$$\boxed{\quad}, \quad \mathbf{M}^{-1} = \begin{pmatrix} -k-2 & k+1 \\ k+1 & -k \end{pmatrix}$$

$$\begin{aligned} \bullet \quad \mathbf{M} &= \begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix} \\ \bullet \quad \det(\mathbf{M}) &= k(k+2) - (k+1)(k+1) = k^2 + 2k - (k^2 + 2k + 1) \\ &= k^2 + 2k - k^2 - 2k - 1 = -1 \\ \bullet \quad \mathbf{M}^{-1} &= \frac{1}{-1} \begin{bmatrix} k+2 & -(k+1) \\ -(k+1) & k \end{bmatrix} = - \begin{bmatrix} k+2 & -k-1 \\ -k-1 & k \end{bmatrix} = \begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix} \\ \bullet \quad \text{NOW VERIFYING BY ALGEBRAICALLY} \\ \mathbf{M} \mathbf{M}^{-1} &= \begin{bmatrix} k & k+1 \\ k+1 & k+2 \end{bmatrix} \begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix} \\ &= \begin{bmatrix} k(-k-2) + (k+1)^2 & k(k+1) - k(k+1) \\ (k+1)(-k-2) + (k+2)^2 & (k+1)^2 - k(k+2) \end{bmatrix} \\ &= \begin{bmatrix} -k^2 - 2k + k^2 + 2k + 1 & k^2 + k - k^2 - 2k \\ -k^2 - 2k - k^2 - 2k - 2 & k^2 + 2k + 4 - k^2 - 2k \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \boxed{\mathbf{I}} \end{aligned}$$

VERIFIED THE INVERSE

Question 3 ()**

The 2×2 matrices \mathbf{A} , \mathbf{B} and \mathbf{C} are given below in terms of the scalar constants a , b and c .

$$\mathbf{A} = \begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 4 \\ b & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & c \\ 3 & 2 \end{pmatrix}.$$

Given that

$$2\mathbf{A} - 3\mathbf{B} = 4\mathbf{C},$$

find the value of a , b and c .

$$a = 1, b = -2, c = -2$$

$$\begin{aligned} 2\mathbf{A} - 3\mathbf{B} = 4\mathbf{C} &\Rightarrow 2 \begin{pmatrix} a & 2 \\ 3 & 7 \end{pmatrix} - 3 \begin{pmatrix} 2 & 4 \\ b & 2 \end{pmatrix} = 4 \begin{pmatrix} -1 & c \\ 3 & 2 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} 2a & 4 \\ 6 & 14 \end{pmatrix} - \begin{pmatrix} 6 & 12 \\ 3b & 6 \end{pmatrix} = \begin{pmatrix} -4 & 4c \\ 12 & 8 \end{pmatrix} \\ &\Rightarrow \begin{pmatrix} 2a-6 & -8 \\ 6+3b & 8 \end{pmatrix} = \begin{pmatrix} -4 & 4c \\ 12 & 8 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \cancel{2a-6=-4} & \quad \cancel{6+3b=12} & \cancel{4c=8} \\ \cancel{2a=2} & \quad \cancel{3b=6} & \cancel{c=2} \\ \cancel{a=1} & & \end{aligned}$$

Question 4 ()**

The 2×2 matrix \mathbf{A} represents a rotation by 90° anticlockwise about the origin O .

The 2×2 matrix \mathbf{B} represents a reflection in the straight line with equation $y = -x$.

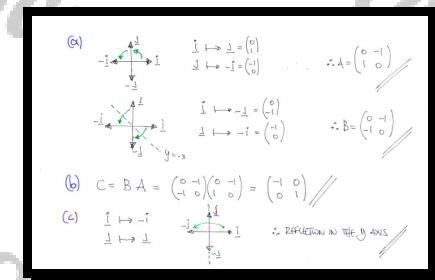
- a) Write down the matrices \mathbf{A} and \mathbf{B} .

The 2×2 matrix \mathbf{C} represents a rotation by 90° anticlockwise about the origin O , followed by a reflection about the straight line with equation $y = -x$.

- b) Find the elements of \mathbf{C} .

- c) Describe geometrically the transformation represented by \mathbf{C} .

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{reflection in the } y \text{ axis}$$



Question 5 ()**

The 2×2 matrix \mathbf{A} , is defined as

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ b & -2 \end{pmatrix}$$

where a and b are constants.

The matrix \mathbf{A} , maps the point $P(2, 5)$ onto the point $Q(-1, 2)$.

- a) Find the value of a and the value of b .

A triangle T_1 with an area of 9 square units is transformed by \mathbf{A} into the triangle T_2 .

- b) Find the area of T_2 .

$$a = -1, \quad b = 6, \quad \text{area} = 18$$

<p>(a) $\begin{pmatrix} 2 & a \\ b & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$</p> $\begin{pmatrix} 4+5a \\ 2b-10 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\begin{cases} 4+5a = -1 \\ 2b-10 = 2 \end{cases}$ $\begin{cases} 5a = -5 \\ 2b = 12 \end{cases}$ $\begin{cases} a = -1 \\ b = 6 \end{cases}$	<p>(b) $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}$</p> $\det \mathbf{A} = -4 + 6 = 2$ $\therefore 9 \times 2 = 18 \text{ units}^2$
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Question 6 ()**

The 2×2 matrices \mathbf{A} , \mathbf{B} and \mathbf{C} are given below in terms of the scalar constants x .

$$\mathbf{A} = \begin{pmatrix} 2 & x \\ 3 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 3x+2 & 7 \\ 7-x & 7 \end{pmatrix}.$$

- a) Find an expression for \mathbf{AB} , in terms of x .
- b) Determine the value of x , given $\mathbf{B}^T \mathbf{A}^T = \mathbf{C}$.

$$\mathbf{AB} = \begin{pmatrix} 4+x & 2+4x \\ 7 & 7 \end{pmatrix}, \quad \boxed{x=1}$$

$$\begin{aligned} \text{(a)} \quad & \mathbf{AB} = \begin{pmatrix} 2 & x \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4+2x & 2+4x \\ 7 & 7 \end{pmatrix} \\ \text{(b)} \quad & \mathbf{B}^T \mathbf{A}^T = (\mathbf{AB})^T = \mathbf{C} \\ & \begin{pmatrix} 4+2x & 7 \\ 2+4x & 7 \end{pmatrix} = \begin{pmatrix} 3x+2 & 7 \\ 7-x & 7 \end{pmatrix} \quad \text{using } a_{ij} : 4+2x = 3x+2 \\ & 2+4x = 7-x \quad \text{and } x = 1 \\ & 2 = 1 \end{aligned}$$

Question 7 ()**

The 2×2 matrices \mathbf{A} and \mathbf{B} are given by

$$\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix}.$$

Find the 2×2 matrix \mathbf{X} that satisfy the equation

$$\mathbf{AX} = \mathbf{B}.$$

$$\mathbf{X} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix} \\ \Rightarrow \mathbf{AX} &= \mathbf{B} \\ \Rightarrow \mathbf{A}^{-1} \mathbf{AX} &= \mathbf{A}^{-1} \mathbf{B} \\ \Rightarrow \mathbf{IX} &= \mathbf{A}^{-1} \mathbf{B} \\ \Rightarrow \mathbf{X} &= \frac{1}{\det(\mathbf{A})} \mathbf{A}^{-1} \mathbf{B} \\ \Rightarrow \mathbf{X} &= \frac{1}{5-2} \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 9 & 12 \\ 4 & 5 \end{pmatrix} \\ \Rightarrow \mathbf{X} &= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \end{aligned}$$

Question 8 (**)

The triangle T_1 is mapped by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix}$$

onto another triangle T_2 , whose vertices have coordinates $A_2(-1, 2)$, $B_2(10, 15)$ and $C_2(-18, -14)$.

Find the coordinates of the vertices of T_1 .

$$A_1(1,1), B_1(4,-3), C_1(-2,8)$$

$$\boxed{\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix} \\ \mathbf{A}^{-1} &= \frac{1}{(1)(-1)-(-2)(3)} \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} \\ \mathbf{A}^{-1} &= \frac{1}{5} \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} \\ \text{Thus } A_1^{-1} \mathbf{A} &= \frac{1}{5} \\ A_1^{-1} \mathbf{A} \mathbf{x} &= A_1^{-1} \mathbf{b} \\ \mathbf{I} \mathbf{x} &= A_1^{-1} \mathbf{b} \\ \mathbf{x} &= \frac{1}{5} \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} -1 & 10 & -18 \\ 2 & 15 & -14 \end{pmatrix} \\ &\quad \uparrow \quad \uparrow \quad \uparrow \\ &\quad A_2 \quad B_2 \quad C_2 \\ \mathbf{x} &= \frac{1}{5} \begin{pmatrix} 5 & 20 & -10 \\ 5 & -15 & 40 \end{pmatrix} \\ \mathbf{x} &= \begin{pmatrix} 1 & 4 & -2 \\ 1 & -3 & 8 \end{pmatrix} \\ \therefore A_1(1,1), B_1(4,-3), C_1(-2,8) \end{aligned}}$$

Question 9 ()**

A plane transformation maps the general point (x, y) onto the general point (X, Y) , by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix},$$

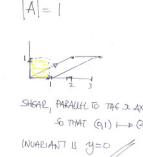
where \mathbf{A} is the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

- Give a geometrical description for the transformation represented by \mathbf{A} , stating the equation of the line of invariant points under this transformation
- Calculate \mathbf{A}^2 and describe geometrically the transformation it represents.

shear parallel to $y = 0$, $(0,1) \mapsto (2,1)$ [line of invariant points $y = 0$],

shear parallel to $y = 0$, $(0,1) \mapsto (4,1)$

a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
 $|\mathbf{A}| = 1$



shear parallel to the x-axis
so that $(0,1) \mapsto (2,1)$
invariant is $y=0$

b) $\mathbf{A}^2 = \mathbf{A}\mathbf{A}$
 $= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$
 $= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

shear parallel to the x-axis
so that $(0,1) \mapsto (4,1)$

Question 10 ()**

The triangle T_1 is mapped by the 2×2 matrix

$$\mathbf{B} = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$$

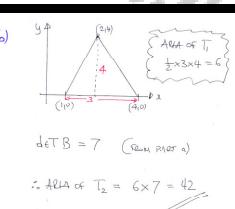
onto a triangle T_2 , whose vertices are the points with coordinates $A_2(4,3)$, $B_2(4,10)$ and $C_2(16,12)$.

- Find the coordinates of the vertices of T_1 .
- Determine the area of T_2 .

$$A_1(1,0), B_1(2,4), C_1(4,0), \text{ area} = 42$$

a) $\bullet \mathbf{B} = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix} \Rightarrow \mathbf{B}^{-1} = \frac{1}{4(1)-3(-1)} \begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}$
 $\Rightarrow \mathbf{B}^{-1} = \frac{1}{7} \begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}$

$\bullet \mathbf{B} \mathbf{x} = \mathbf{b}$
 $\Rightarrow \mathbf{B}^{-1} \mathbf{B} \mathbf{x} = \mathbf{B}^{-1} \mathbf{b}$
 $\Rightarrow \mathbf{x} = \mathbf{B}^{-1} \mathbf{b}$
 $\Rightarrow \mathbf{x} = \frac{1}{7} \begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4 & 4 & 16 \\ -12 & -3 & 12 \end{pmatrix}$
 $\Rightarrow \mathbf{x} = \frac{1}{7} \begin{pmatrix} 7 & 14 & 28 \\ 0 & 12 & 12 \end{pmatrix}$
 $\Rightarrow \mathbf{x} = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 1 \end{pmatrix}$
 $\therefore A_1(1,0), B_1(2,4), C_1(4,0) \parallel$

b) 

$\det \mathbf{B} = 7$ (from part a)

$\therefore \text{Area of } T_2 = 6 \times 7 = 42$

Question 11 ()**

The 2×2 matrix \mathbf{C} is defined, in terms of a scalar constant a , by

$$\mathbf{C} = \begin{pmatrix} 3 & a \\ 5 & 2 \end{pmatrix}.$$

- a) Determine the value of a , given that \mathbf{C} is singular.

The 2×2 matrix \mathbf{D} is given by

$$\mathbf{D} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}.$$

- b) Find the inverse of \mathbf{D} .

The point P is transformed by \mathbf{D} onto the point $Q(6k+1, 14k+1)$, where k is a scalar constant.

- c) Determine, in terms of k , the coordinates of P .

$$a = \frac{6}{5}, \quad \mathbf{D}^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -2 & 1 \end{pmatrix}, \quad P(2k+1, 2k-1)$$

$$\begin{aligned} \text{(a)} \quad \det \mathbf{C} = 0 &\Rightarrow (3)(2) - (5)(a) = 0 \\ &\Rightarrow 6 - 5a = 0 \\ &\Rightarrow 6 = 5a \\ &\Rightarrow a = \frac{6}{5} \quad \boxed{\text{S}} \end{aligned} \quad \begin{aligned} \text{(b)} \quad \mathbf{D}^* &= \frac{1}{(2)(3) - (4)(1)} \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 2 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \mathbf{D} \mathbf{P} &= \mathbf{Q} \\ \Rightarrow \mathbf{D}^{-1} \mathbf{D} \mathbf{P} &= \mathbf{D}^{-1} \mathbf{Q} \\ \Rightarrow \mathbf{I}_2 \mathbf{P} &= \mathbf{D}^{-1} \mathbf{Q} \\ \Rightarrow \mathbf{P} &= \mathbf{D}^{-1} \mathbf{Q} \end{aligned} \quad \left\{ \begin{array}{l} \mathbf{P} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 6k+1 \\ 14k+1 \end{pmatrix} = \begin{pmatrix} \frac{3}{2}(6k+1) - \frac{1}{2}(14k+1) \\ 2(6k+1) + (14k+1) \end{pmatrix} \\ = \begin{pmatrix} 9k + \frac{3}{2} - 7k - \frac{1}{2} \\ 12k + 2 + 14k + 1 \end{pmatrix} = \begin{pmatrix} 2k + 1 \\ 26k + 3 \end{pmatrix} \\ \therefore \mathbf{P}(2k+1, 2k-1) \end{array} \right.$$

Question 12 ()**

A plane transformation maps the general point (x, y) to the general point (X, Y) by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- a) Find the area scale factor of the transformation.

The points on a straight line which passes through the origin remain invariant under this transformation.

- b) Determine the equation of this straight line.

$$\boxed{\text{SF} = 16}, \quad \boxed{y = \frac{3}{4}x}$$

④ Area scale factor $= \begin{vmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{vmatrix} = 67.84 - 51.84 = 16$

⑤ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.4 & -7.2 \\ -7.2 & 10.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6.4x - 7.2y \\ -7.2x + 10.6y \end{pmatrix}$

$$\begin{aligned} 6.4x - 7.2y &= 8x \\ 7.2x + 8y &= 16 \\ 16x + 8y &= 16x \end{aligned}$$

Question 13 ()**

The distinct square matrices \mathbf{A} and \mathbf{B} are non singular.

Simplify the expression, showing all steps in the workings.

$$\mathbf{AB}(\mathbf{A}^{-1}\mathbf{B})^{-1}$$

$$\boxed{\mathbf{A}^2}$$

$$\begin{aligned} \mathbf{AB}(\mathbf{A}^{-1}\mathbf{B})^{-1} &= \mathbf{AB}(\mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1}) = \mathbf{AB}\mathbf{B}^{-1}\mathbf{A} = \mathbf{A}\mathbf{I}\mathbf{A} \\ &= \mathbf{AA} = \mathbf{A}^2 \end{aligned}$$

Question 14 ()**

The distinct square matrices \mathbf{A} and \mathbf{B} have the properties

$$\mathbf{AB} = \mathbf{B}^5 \mathbf{A} \text{ and } \mathbf{B}^6 = \mathbf{I}$$

where \mathbf{I} is the identity matrix.

Prove that

$$\mathbf{BAB} = \mathbf{A}.$$

proof

$$\begin{aligned} \mathbf{AB} &= \mathbf{B}^5 \mathbf{A} \\ \Rightarrow \mathbf{BAB} &= \mathbf{B}^5 \mathbf{A} \\ \Rightarrow \mathbf{BAB} &= \mathbf{B}^5 \mathbf{A} \\ \Rightarrow \mathbf{BAB} &= \mathbf{IA} \\ \Rightarrow \mathbf{BAB} &= \mathbf{A} \end{aligned}$$

Question 15 ()**

The 2×2 matrix \mathbf{A} is given by

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}.$$

The 2×2 matrix \mathbf{B} satisfies

$$\mathbf{BA}^2 = \mathbf{A}.$$

Find the elements of \mathbf{B} .

$$\mathbf{B} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{A} &= \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} & \det \mathbf{A} &= (1 \times 4) - (1 \times 3) = 1 \\ \mathbf{A}^{-1} &= \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} & & \\ \text{Now } \mathbf{BA}^2 &= \mathbf{A} & & \\ \Rightarrow \mathbf{BA} \mathbf{A}^{-1} &= \mathbf{A} \mathbf{A}^{-1} & & \therefore \mathbf{B} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} \\ \Rightarrow \mathbf{BA} &= \mathbf{I} & & \\ \Rightarrow \mathbf{BA} &= \mathbf{I} \mathbf{A}^{-1} & & \\ \Rightarrow \mathbf{BA} &= \mathbf{I} \mathbf{A}^{-1} & & \\ \Rightarrow \mathbf{B} &= \mathbf{A}^{-1} & & \end{aligned}$$

Question 16 ()**

The triangle T_1 is mapped by the 2×2 matrix

$$\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

onto the triangle T_2 , whose vertices have coordinates $A_2(-7, -1)$, $B_2(5, 5)$ and $C_2(7, 16)$.

- Find the coordinates of the vertices of T_1 .
- Determine the area of T_2 .

$[A_1(-4, 1)]$, $[B_1(2, 1)]$, $[C_1(1, 5)]$, $[\text{area} = 60]$

a) Since by finding the inverse of \mathbf{B}

$$\det \mathbf{B} = (2 \cdot 3) - (1 \cdot 1) = 5$$

$$\mathbf{B}^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

Let \mathbf{a} be the coordinates of T_1 & \mathbf{b} be the coordinates of T_2

$$\Rightarrow \mathbf{B}^{-1} \mathbf{a} = \mathbf{b}$$

$$\Rightarrow \mathbf{B}^{-1} \mathbf{B} \mathbf{a} = \mathbf{B}^{-1} \mathbf{b}$$

$$\Rightarrow \mathbf{a} = \mathbf{B}^{-1} \mathbf{b}$$

$$\Rightarrow \mathbf{a} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -7 & 5 & 7 \\ -1 & 5 & 16 \end{pmatrix}$$

$$\Rightarrow \mathbf{a} = \frac{1}{5} \begin{pmatrix} -20 & 10 & 3 \\ 4 & 5 & 25 \end{pmatrix}$$

$$\Rightarrow \mathbf{a} = \begin{pmatrix} -4 & 2 & 3 \\ 1 & 1 & 5 \end{pmatrix}$$

$\therefore A(-4, 1) B(2, 1) C(1, 5)$

b) Since to look the area of T_1 since two of its vertices have the same height

$\text{Area of } T_1 = \frac{1}{2} \times 6 \times 4 = 12$

$$\det \mathbf{B} = 5$$

$\therefore \text{Area of } T_2 = 12 \times 5 = 60 \text{ units}^2$

Question 17 (+)**

The transformation represented by the 2×2 matrix \mathbf{A} maps the point $(3, 4)$ onto the point $(10, 4)$, and the point $(5, -2)$ onto the point $(8, -2)$.

Determine the elements of \mathbf{A} .

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

LET $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} \Rightarrow \begin{cases} 3a + 4b = 10 \\ 3c + 4d = 4 \end{cases}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} 5a - 2b = 8 \\ 5c - 2d = -2 \end{cases}$$

THUS

$$\begin{cases} 3a + 4b = 10 \\ 5a - 2b = 8 \end{cases} \quad \begin{cases} 3c + 4d = 4 \\ 5c - 2d = -2 \end{cases}$$

$$\begin{cases} 3a + 4b = 10 \\ 10a - 4b = 16 \end{cases} \quad \begin{cases} 3c + 4d = 4 \\ 16c - 4d = -4 \end{cases}$$

$$\begin{cases} 13a = 26 \\ a = 2 \end{cases} \quad \begin{cases} 19c = 0 \\ c = 0 \end{cases}$$

$$\begin{cases} a = 2 \\ b = 1 \end{cases} \quad \begin{cases} d = 1 \\ d = 1 \end{cases}$$

$$\therefore \mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

Question 18 (+)**

The 2×2 matrix \mathbf{A} is given below.

$$\mathbf{A} = \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

Determine the elements of \mathbf{A}^3 and hence describe geometrically the transformation represented by \mathbf{A} .

$$\mathbf{A}^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}, \quad \text{rotation of } 120^\circ \text{, anticlockwise \& enlargement of S.F. 2, both about the origin and in any order.}$$

$$\begin{aligned} \mathbf{A}^3 &= \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} \begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \\ \therefore \mathbf{A}^3 &= \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

THIS IS A ROTATION BY 120° ABOUT THE ORIGIN & UNIFORM ENLARGEMENT ABOUT THE ORIGIN BY SCALE FACTOR 2. (CLOCKWISE)

TO DETERMINE COUNTER ANTICLOCKWISE

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \begin{pmatrix} \cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

ANTIENDEMENT

Question 19 (**+)

It is given that

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & -2 \\ 4 & -1 \end{pmatrix}.$$

- Determine the matrix \mathbf{AB} .
- Find the elements of

$$\mathbf{BA} - 2\mathbf{C}^2.$$

$$\boxed{\mathbf{AB} = \begin{pmatrix} 5 \end{pmatrix}}, \quad \boxed{\mathbf{BA} - 2\mathbf{C}^2 = \begin{pmatrix} 20 & -3 \\ 2 & 13 \end{pmatrix}}$$

$$\begin{aligned} \text{(a)} \quad 4\mathbf{B} &= (2-1)\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \end{pmatrix} \\ \text{(b)} \quad \mathbf{BA} - 2\mathbf{C}^2 &= \begin{pmatrix} 3 \\ 1 \end{pmatrix}(2-1) - 2 \begin{pmatrix} 1-2 \\ 4-1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 4 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -3 \end{pmatrix} - 2 \begin{pmatrix} -7 & 0 \\ 0 & -7 \end{pmatrix} \\ &= \begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} - \begin{pmatrix} -14 & 0 \\ 0 & -14 \end{pmatrix} = \begin{pmatrix} 20 & -3 \\ 2 & 13 \end{pmatrix} \end{aligned}$$

Question 20 (**+)

The 2×2 matrices \mathbf{A} and \mathbf{B} are given below

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}.$$

The matrix \mathbf{C} represents the combined effect of the transformation represented by the \mathbf{B} , followed by the transformation represented by \mathbf{A} .

a) Determine the elements of \mathbf{C} .

b) Describe geometrically the transformation represented by \mathbf{C} .

$$\mathbf{C} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \boxed{\text{enlargement by scale factor 2, reflection in the line } y = x, \text{ in any order}}$$

a) $\mathbf{C} = \mathbf{AB} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

b) Now $\mathbf{C} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

UNIFORM ENLARGEMENT BY SCALE FACTOR 2
(ORDER DOES NOT MATTER.)

REFLECTION IN THE LINE $y = x$

Question 21 (**+)The 2×2 matrix \mathbf{D} is given by

$$\mathbf{D} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}.$$

a) Given that \mathbf{I} is the 2×2 identity matrix, show clearly that ...

i. ... $\mathbf{D}^2 + 5\mathbf{D} = 6\mathbf{I}$.

ii. ... $\mathbf{D}^{-1} = \frac{1}{6}(\mathbf{D} + 5\mathbf{I})$.

The transformation in the x - y plane, which is represented by the matrix \mathbf{D} , maps the point P onto the point Q .

The coordinates of Q are $(7-2k, 9-6k)$, where k is a constant.

b) Determine, in terms of k , the coordinates of P .

$$P(2k+3, 2k+1)$$

(a) $\mathbf{D} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$

- $\mathbf{D}^2 = \mathbf{D}\mathbf{D} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} = \begin{pmatrix} -14 & 25 \\ -30 & 51 \end{pmatrix}$
- $\mathbf{D}^2 + 5\mathbf{D} = \begin{pmatrix} -14 & 25 \\ -30 & 51 \end{pmatrix} + 5 \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} = \begin{pmatrix} -14 & 25 \\ -30 & 51 \end{pmatrix} + \begin{pmatrix} 20 & -25 \\ 30 & -45 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = 6\mathbf{I}$

(b) $\mathbf{D}^2 + 5\mathbf{D} = 6\mathbf{I}$

$$\Rightarrow \mathbf{D}^2 + 5\mathbf{D} = 6\mathbf{D}^{-1}$$

$$\Rightarrow \mathbf{D} + 5\mathbf{I} = \mathbf{D}^{-1}$$

$$\Rightarrow \mathbf{D} = \frac{1}{6}(\mathbf{D} + 5\mathbf{I})$$

At this point, we can multiply both sides by \mathbf{D} to get $\mathbf{D}^2 = \frac{1}{6}(\mathbf{D} + 5\mathbf{I})^2$.

(c) $\mathbf{D}_P = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$

$$\Rightarrow \mathbf{D}_P P = \mathbf{D}_P^2$$

$$\Rightarrow P = \mathbf{D}_P^{-1}$$

$$\Rightarrow P = \mathbf{D}_P^T$$

$$\Rightarrow P = \frac{1}{36} \begin{bmatrix} 35 & -30 \\ -30 & 54 \end{bmatrix}$$

$$\Rightarrow P = \frac{1}{6} \begin{bmatrix} 35 & -30 \\ -30 & 54 \end{bmatrix}$$

$$\Rightarrow P = \frac{1}{6} \begin{bmatrix} 41 & -30 \\ -30 & 54 \end{bmatrix}$$

$$\Rightarrow P = \frac{1}{6} \begin{bmatrix} 11 & -10 \\ -10 & 18 \end{bmatrix}$$

$$\Rightarrow P = \begin{pmatrix} 3 & -5 \\ -5 & 18 \end{pmatrix}$$

$$\Rightarrow P = (2k+3, 2k+1)$$

Question 22 (***)

The 2×2 matrix \mathbf{B} maps the points with coordinates $(-1, 2)$ and $(1, 4)$ onto the points with coordinates $(0, 1)$ and $(6, -1)$, respectively.

- Find the elements of \mathbf{B} .
- Determine whether \mathbf{B} has an invariant line, or a line of invariant points, or both.
- Describe geometrically the transformation represented by \mathbf{B} .

$\boxed{\text{XPC}}$, $\boxed{\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}}$, $\boxed{\text{line of invariant points, } y = -x}$, $\boxed{\text{invariant line } y = -x + c}$,

shear

a) Solve simultaneous equations or use inverses

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ -1 & -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{B} \Delta \mathbf{A}^{-1} = \mathbf{C} \mathbf{A}^{-1}$$

$$\Rightarrow \mathbf{B} \mathbf{I} = \mathbf{C} \mathbf{A}^{-1}$$

$$\Rightarrow \mathbf{B} = \frac{1}{4-2} \begin{pmatrix} 0 & 6 \\ -1 & -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{B} = \frac{1}{2} \begin{pmatrix} 0 & 6 \\ -1 & -1 \end{pmatrix}$$

$$\Rightarrow \mathbf{B} = \begin{pmatrix} 0 & 3 \\ -1 & -0.5 \end{pmatrix}$$

$$\Rightarrow \mathbf{B} = \boxed{\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}}$$

b) Firstly look for line of invariant points

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} 2x+y \\ -x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore y = -x$$

Now look for invariant lines

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x+y \\ -x \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow 2x + y + c = x$$

$$\therefore y = -x + c$$

$$\Rightarrow 2x + y + c = x - c$$

$$\therefore c = -y$$

$$\Rightarrow \frac{2x+y}{1} = \frac{x-c}{-1}$$

$$\therefore y = -\frac{1}{2}x - \frac{1}{2}c$$

Compare with $y = mx + c$

$$m = -\frac{1}{2}$$

$$2m + m^2 = -1$$

$$2(-\frac{1}{2}) + (-\frac{1}{2})^2 = -1$$

$$-\frac{1}{2} + \frac{1}{4} = -1$$

$$\frac{1}{4} = 0$$

$$\therefore m = -1$$

$$\therefore y = -\frac{1}{2}x + \frac{1}{2}c$$

$$\therefore y = -x + c$$

∴ Also invariant lines parallel to $y = -x$

c) Distinguish \mathbf{B}

$\det \mathbf{B} = (2)(0) - (-1)(1) = 1$ (CARA INVARIANT)

POSITION-DETERMINANT \Rightarrow ROTATION OR SHEAR (NO REFLECTION)

INARIANT LINE OR INVARIANT LINE OF POINTS \Rightarrow SHEAR (NO ROTATION)

$\therefore \mathbf{B}$ represents a shear, where $y = -x$ is invariant line of points

Question 23 (*)**

The 2×2 matrices \mathbf{A} and \mathbf{B} are given by

$$\mathbf{A} = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 19 & 36 \\ 8 & 15 \end{pmatrix}.$$

Find the 2×2 matrix \mathbf{X} that satisfy the equation $\mathbf{AX} = \mathbf{B}$

$$\boxed{\mathbf{X} = \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}}$$

$$\begin{aligned} \mathbf{AX} &= \mathbf{B} & \therefore \mathbf{A}^{-1} &= \frac{1}{5 \times 3 - 2 \times 7} \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} \\ \mathbf{A}^{-1} \mathbf{AX} &= \mathbf{A}^{-1} \mathbf{B} & \therefore \mathbf{X} &= \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 19 & 36 \\ 8 & 15 \end{pmatrix} = \begin{pmatrix} 57 - 56 & 108 - 140 \\ -38 + 40 & -72 + 75 \end{pmatrix} \\ \mathbf{X} &= \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} & \therefore \mathbf{X} &= \begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix} \end{aligned}$$

Question 24 (*)**

It is given that \mathbf{A} and \mathbf{B} are 2×2 matrices that satisfy

$$\det(\mathbf{AB}) = 18 \quad \text{and} \quad \det(\mathbf{B}^{-1}) = -3.$$

A square S , of area 6 cm^2 , is transformed by \mathbf{A} to produce an image S' .

Given that S' is also a square, determine its **perimeter**.

72 cm

$$\begin{aligned} \det(\mathbf{AB}) &= 18 & \Rightarrow \det \mathbf{A} \times \det \mathbf{B} &= 18 \\ \det(\mathbf{B}^{-1}) &= -3 & \det \mathbf{A} \times \left(-\frac{1}{3}\right) &= 18 \\ \downarrow & & \det \mathbf{A} &= -54 \\ \det \mathbf{B} &= -\frac{1}{3} & \text{NOW AREA OF IMAGE IS} & \\ & & 6 \times 54 &= 324 \\ & & \text{SIDE LENGTH IS} & \sqrt{324} = 18 \\ & & \therefore \text{PERIMETER OF IMAGE IS} & 18 \times 4 = 72 \text{ cm} \end{aligned}$$

Question 25 (*)**

The 2×2 matrix \mathbf{A} is given by

$$\mathbf{A} = \begin{pmatrix} 2 & a \\ 3 & b \end{pmatrix},$$

where a and b are scalar constants.

- If the point with coordinates $(1,1)$ is mapped by \mathbf{A} onto the point with coordinates $(1,3)$, determine the value of a and the value of b .
- Show that

$$\mathbf{A}^2 = 2\mathbf{A} - 3\mathbf{I}.$$

The inverse of \mathbf{A} is denoted by \mathbf{A}^{-1} and \mathbf{I} is the 2×2 identity matrix.

- Use part (b) to show further that ...

i. ... $\mathbf{A}^3 = \mathbf{A} - 6\mathbf{I}$.

ii. ... $\mathbf{A}^{-1} = \frac{1}{3}(2\mathbf{I} - \mathbf{A})$

, $a = -1$, $b = 0$

a) By "Multiplication"

$$\begin{pmatrix} 2 & a \\ 3 & b \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 2x + ax & 2x + 3a \\ 3x + bx & 3x + b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$$

$$\begin{aligned} 2x + ax &= 1 \\ 3x + bx &= 3 \end{aligned} \Rightarrow \begin{aligned} a + 2 &= 1 \\ b + 3 &= 3 \end{aligned} \Rightarrow a = -1 \quad b = 0$$

b) USE BY CALCULATING EACH SIDE SEPARATELY

$$\bullet \mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 2(2) + (-1)(3) & 2(-1) + 0 \\ 3(2) + 0 & 3(0) + 0 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ 6 & 0 \end{pmatrix}$$

$$\therefore \mathbf{A}^2 = \begin{pmatrix} 1 & -2 \\ 6 & 0 \end{pmatrix}$$

$$\bullet 2\mathbf{A} - 3\mathbf{I} = 2 \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 6 & 0 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 6 & 0 \end{pmatrix}$$

$$\therefore \mathbf{A}^2 = 2\mathbf{A} - 3\mathbf{I}$$

c) $\mathbf{A}^2 = 2\mathbf{A} - 3\mathbf{I}$

$$\Rightarrow \mathbf{A}^2\mathbf{A} = 2\mathbf{A}\mathbf{A} - 3\mathbf{I}\mathbf{A}$$

$$\Rightarrow \mathbf{A}^3 = 2\mathbf{A}^2 - 3\mathbf{A}$$

$$\Rightarrow \mathbf{A}^3 = 2(2\mathbf{A} - 3\mathbf{I}) - 3\mathbf{A}$$

$$\Rightarrow \mathbf{A}^3 = \mathbf{A} - 6\mathbf{I}$$

d) $\mathbf{A}^2 = 2\mathbf{A} - 3\mathbf{I}$

$$\Rightarrow \mathbf{A}\mathbf{A}^{-1} = 2\mathbf{A}\mathbf{A}^{-1} - 3\mathbf{I}\mathbf{A}^{-1}$$

$$\Rightarrow \mathbf{A}^{-1} = 2\mathbf{I} - 3\mathbf{A}^{-1}$$

$$\Rightarrow \mathbf{A}^{-1} = 2\mathbf{I} - 3\mathbf{A}^{-1}$$

$$\Rightarrow 3\mathbf{A}^{-1} = 2\mathbf{I} - \mathbf{A}$$

$$\Rightarrow \mathbf{A}^{-1} = \frac{1}{3}(2\mathbf{I} - \mathbf{A})$$

Question 26 (*)**

A transformation in the x - y plane is represented by the 2×2 matrix

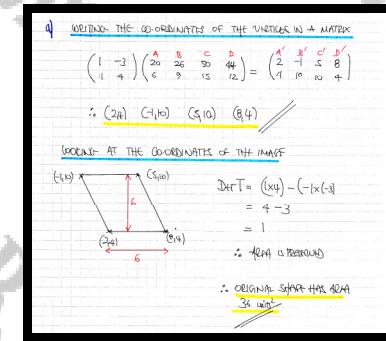
$$\mathbf{T} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}.$$

A quadrilateral Q has vertices at the points with coordinates $(20,6)$, $(26,9)$, $(50,15)$ and $(44,12)$. These coordinates are given in cyclic order.

The vertices of Q are transformed by \mathbf{T} .

- Find the positions of the vertices of the image of Q .
- Determine the area of Q , fully justifying your reasoning.

, $(2,4)$, $(-1,10)$, $(5,10)$, $(8,4)$, area = 36



Question 27 (***)The 2×2 matrix \mathbf{B} is given by

$$\mathbf{B} = \begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix},$$

where a and b are scalar constants.

The point with coordinates $(3,1)$ is mapped by \mathbf{B} onto the point with coordinates $(5,13)$.

- a) Determine the value of a and the value of b .

The inverse of \mathbf{B} is denoted by \mathbf{B}^{-1} and \mathbf{I} is the 2×2 identity matrix.

- b) Show that

$$\mathbf{B}^2 = 5\mathbf{B} + 2\mathbf{I}.$$

- c) Show further that ...

i. ... $\mathbf{B}^3 = 27\mathbf{B} + 10\mathbf{I}$.

ii. ... $\mathbf{B}^{-1} = \frac{1}{2}(\mathbf{B} - 5\mathbf{I})$

$$[a=1], [b=4]$$

$\text{(a)} \quad \begin{pmatrix} a & 2 \\ 3 & b \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \end{pmatrix} \Rightarrow \begin{pmatrix} 3a+2 = 5 \\ 9+b = 13 \end{pmatrix} \Rightarrow \begin{pmatrix} a=1 \\ b=4 \end{pmatrix}$	
$\text{(b)} \quad \mathbf{B}^2 = \begin{pmatrix} a & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ $5\mathbf{B} + 2\mathbf{I} = 5\begin{pmatrix} a & 2 \\ 3 & 4 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5a & 10 \\ 15 & 20 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$ $\therefore \mathbf{B}^2 = 5\mathbf{B} + 2\mathbf{I}$ <small>✓ as required</small>	
$\text{(c)} \quad \mathbf{B}^2 = 5\mathbf{B} + 2\mathbf{I}$ $\Rightarrow \mathbf{B}^2\mathbf{B} = 5\mathbf{B}\mathbf{B} + 2\mathbf{B}\mathbf{I}$ $\Rightarrow \mathbf{B}^3 = 5\mathbf{B}^2 + 2\mathbf{B}$ $\Rightarrow \mathbf{B}^3 = 5(5\mathbf{B} + 2\mathbf{I}) + 2\mathbf{B}$ $\Rightarrow \mathbf{B}^3 = 27\mathbf{B} + 10\mathbf{I}$ <small>✓ as required</small>	$\text{(d)} \quad \mathbf{B}^2 = 5\mathbf{B} + 2\mathbf{I}$ $\Rightarrow \mathbf{B}^{-1}\mathbf{B}^2 = \mathbf{B}^{-1}(5\mathbf{B} + 2\mathbf{I})$ $\Rightarrow \mathbf{B}^2 = 5\mathbf{B} + 2\mathbf{B}$ $\Rightarrow \mathbf{B}^2 = 2\mathbf{B}$ $\Rightarrow \mathbf{B} = \mathbf{B}^{-1} + 2\mathbf{B}^{-1}$ $\Rightarrow \mathbf{B} - \mathbf{B}^{-1} = 2\mathbf{B}^{-1}$ $\Rightarrow \mathbf{B}^{-1} = \frac{1}{2}(\mathbf{B} - \mathbf{B}^{-1})$ <small>✓ as required</small>

Question 28 (***)

A transformation in the x - y plane consists of ...

- ...a reflection about the line with equation $y = x$
- ... followed by an anticlockwise rotation about the origin by 90°
- ... followed by a reflection about the x axis.

Use matrices to describe geometrically the resulting combined transformation.

, rotation about the origin by 180°

• START DRAWING THE THREE MATRICES

REFLECTION ABOUT $y=x$	MAT ROTATION BY 90° ANTICLOCKWISE	FINAL REFLECTION ABOUT x AXIS
---------------------------	---------------------------------------------	------------------------------------

$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

• MULTIPLY IN THE CORRECT ORDER

$$CBA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

q $\begin{pmatrix} 1 \\ 1 \end{pmatrix} \longmapsto \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ WITH POSITIVE DETERMINANT, SO NO REFLECTION

\therefore ROTATION ABOUT 0 BY 180°

Question 29 (***)+

The 2×2 matrices \mathbf{A} and \mathbf{B} are defined by

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -1 \\ -3 & 5 \end{pmatrix}.$$

a) Find \mathbf{A}^{-1} , the inverse of \mathbf{A} .

b) Find a matrix \mathbf{C} , so that

$$(\mathbf{B} + \mathbf{C})^{-1} = \mathbf{A}.$$

c) Describe geometrically the transformation represented by \mathbf{C} .

$$\boxed{\mathbf{A}^{-1}}, \quad \boxed{\mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}}, \quad \boxed{\mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}}, \quad \boxed{\text{rotation about } O, \text{ by } 180^\circ}$$

Q) STANDARD METHOD FOR INVERTING 2×2 MATRICES

$$|\mathbf{A}| = (4 \times 1) - (3 \times 1) = 4 - 3 = 1$$

$$\therefore \mathbf{A}^{-1} = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -3 & 4 \end{pmatrix}$$

4) PROCEED AS FOLLOWS

$$\begin{aligned} &\Rightarrow (\mathbf{B} + \mathbf{C})^{-1} = \mathbf{A} \\ &\Rightarrow (\mathbf{B} + \mathbf{C})(\mathbf{B} + \mathbf{C})^{-1} = (\mathbf{B} + \mathbf{C})\mathbf{A} \\ &\Rightarrow \mathbf{I} = \mathbf{B}\mathbf{A} + \mathbf{C}\mathbf{A} \\ &\Rightarrow \mathbf{C}\mathbf{A} = \mathbf{I} - \mathbf{B}\mathbf{A} \\ &\Rightarrow \mathbf{C}\mathbf{A}\mathbf{A}^{-1} = (\mathbf{I} - \mathbf{B}\mathbf{A})\mathbf{A}^{-1} \\ &\Rightarrow \mathbf{C}\mathbf{I} = \mathbf{I}\mathbf{A}^{-1} - \mathbf{B}\mathbf{A}\mathbf{A}^{-1} \\ &\Rightarrow \mathbf{C} = \mathbf{A}^{-1} - \mathbf{B}\mathbf{I} \\ &\Rightarrow \mathbf{C} = \mathbf{A}^{-1} - \mathbf{B} \\ &\Rightarrow \mathbf{C} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

5) LOOKING AT STATE VECTOR $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{aligned} \mathbf{I} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{i} \\ \mathbf{B} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\mapsto \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \mathbf{b} \\ \mathbf{C} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} &\mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \mathbf{c} \end{aligned}$$

∴ rotation about O , by 180°

Question 30 (***)

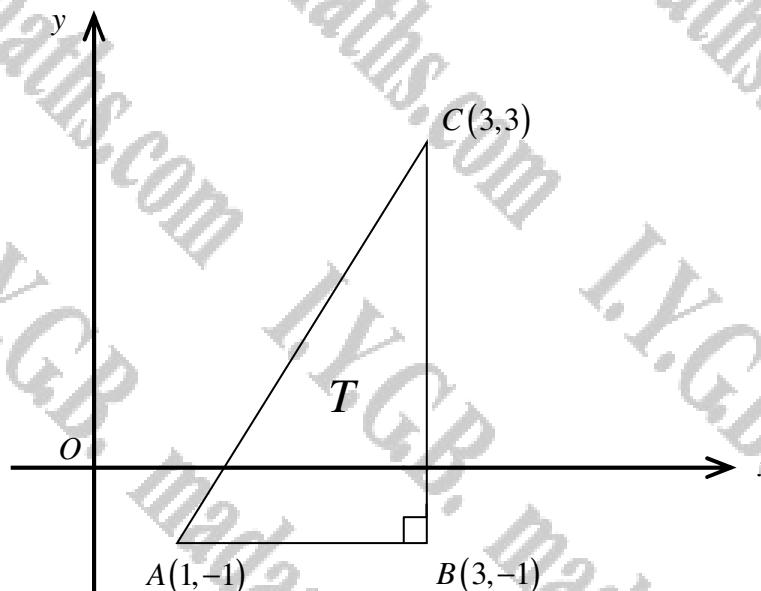
The 2×2 matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

represent linear transformations in the x - y plane.

- a) Give full geometrical descriptions for each of the transformations represented by \mathbf{A} and \mathbf{B} .

The figure below shows a right angled triangle T , with vertices at the points $A(1, -1)$, $B(3, -1)$ and $C(3, 3)$.



The triangle T is first transformed by \mathbf{A} and then by \mathbf{B} , producing the triangle T' .

- b) Find the single matrix that represents this composite transformation.
- c) Determine the coordinates of the vertices of T' .
- d) Calculate the area of T' .

[continues overleaf]

[continued from overleaf]

The triangle T' is then reflected in the straight line with equation $y = -x$ to give the triangle T'' .

- e) Find the single matrix that maps T'' back onto T

, rotation about O , 90° , clockwise, enlargement, in x only, scale factor 2

$$\boxed{\mathbf{BA} = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}}, \boxed{A'(-2, -1), B'(-2, -3), C'(6, -3)}, \boxed{\text{area} = 8}, \boxed{\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}}$$

a) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$\det A = 1$ (possibly rotation)

$\det B = 2$.

$\begin{array}{c} \text{A} \\ \downarrow \\ \begin{array}{c} \text{B} \\ \downarrow \\ \text{C} \end{array} \end{array}$

$j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$

$\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1$

$i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1$

$j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 1$

$\therefore \text{EQUATIONS PARALLEL TO THE } x \text{-axis BY SIGN-POSS OF 2}$

= ROTATION AROUND O BY 90°
ORIGINALLY

b) TRANSFORMATIONS, A FOLLOWED BY B, is BA

$BA = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$

c) WRITING THE 3 SETS OF COORDINATES AS A SINGLE MATRIX

$BA \Delta = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -1 & 2 \\ 2 & 1 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 3 & 0 \end{pmatrix}$

$\therefore A'(-2, 2), B'(3, 0), C'(1, 3)$

d) AREA OF T is $\frac{1}{2} \times 2 \times 4 = 4 \text{ units}^2$

$\det(AB) = \det A \times \det B = 1 \times 2 = 2$

AREA OF T' is $\frac{1}{2} \times 2 \times 5 = 5 \text{ units}^2$

e) FIND THE MATRIX WHICH REPRESENTS REFLECTION ABOUT y = -x

$\begin{array}{c} \text{A} \\ \downarrow \\ \begin{array}{c} \text{B} \\ \downarrow \\ \text{C} \end{array} \end{array}$

$\begin{array}{c} \text{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \text{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \text{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \text{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$

$\therefore \text{REQUIRED MATRIX IS } \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

THE MATRIX WHICH DOES THE 3 TRANSFORMATIONS IN THE CORRECT ORDER IS

$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

↑
FOLLOWING
 $\text{B}'(-2, 0) \rightarrow \text{A}'(0, 2)$

FINDING THE DETERMINANT OF THE MATRIX

$\det \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = -2$

$\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$

Question 31 (*)+**

The 2×2 matrix A given below, represents a transformation in the x - y plane.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- a) Describe geometrically the transformation represented by A .

The transformation described by \mathbf{A} is equivalent to a reflection about the straight line with equation $y = -x$, followed by another transformation described by the matrix \mathbf{C} .

- b) Find the matrix \mathbf{C} , and describe it geometrically

[] , rotation about O , by 90° , anticlockwise , $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, [] reflection about the x axis

a) LOOKING AT THE BASIC IDEAS

$$\begin{aligned} \mathbf{i} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{j} \\ \mathbf{j} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{i} \end{aligned}$$

\Rightarrow ROTATION ABOUT \mathbf{j} BY 90° ANTICLOCKWISE

b) SETTING UP A MATRIX EQUATION - LET THE REQUIRED MATRIX BE C

$$\Rightarrow \mathbf{A} = \mathbf{C} \mathbf{B}$$

DEFINITION ROTATE \mathbf{i} BY -90°

$$\begin{aligned} \mathbf{i} &\mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\ \mathbf{j} &\mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{aligned}$$

$$\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

AND ITS INVERSE IS ALSO

R (DEFINITION IS A SELF-INVERSE PROCESS)

$$\begin{aligned} \mathbf{A}^{-1} &= \mathbf{C} \mathbf{B}^{-1} \\ \mathbf{A} \mathbf{B}^{-1} &= \mathbf{C} \mathbf{I} \\ \mathbf{C} &= \mathbf{A} \mathbf{B}^{-1} \\ \mathbf{C} &= \mathbf{A} \mathbf{B} \\ \mathbf{C} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ \mathbf{C} &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

DEFINITION $\det(\mathbf{C}) = -1$ WHILE $\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

\Rightarrow REFLECTION ALONG THE \mathbf{j} AXIS

Question 32 (***)

The 2×2 matrix \mathbf{M} is defined by

$$\mathbf{M} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}.$$

Find, by calculation, the equations of the two lines which pass through the origin, that remain invariant under the transformation represented by \mathbf{M} .

, $y = \pm x$

METHOD A

LET A LINE THROUGH THE ORIGIN HAVING EQUATION $y = mx$, WHERE m IS THEN MAPPED TO $y = Mx$

$$\begin{bmatrix} Y \\ Y \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} X \\ Mx \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ mx \end{bmatrix} = \begin{bmatrix} 3mx \\ 3x \end{bmatrix}$$

∴ Hence we obtain the equations

$$\begin{cases} X = 3mx \\ Mx = 3x \end{cases} \Rightarrow \begin{array}{l} \text{DIVIDING THE EQUATIONS WE OBTAIN} \\ \frac{1}{M} = 1 \\ M^2 = 1 \\ M = \pm 1 \end{array}$$

∴ THE REQUIRED LINES ARE $y = x$ & $y = -x$

METHOD B (BY EIGENVECTORS)

FIND THE CHARACTERISTIC EQUATION OF \mathbf{M}

$$\begin{vmatrix} 0-\lambda & 3 \\ 3 & 0-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda)^2 - 9 = 0$$

$$\Rightarrow \lambda^2 - 9 = 0$$

$$\Rightarrow (\lambda - 3)(\lambda + 3) = 0$$

$$\Rightarrow \lambda = 3 \quad \lambda = -3$$

FINDING THE EIGENVALUES AND HENCE THE LINES

IF $\lambda = 3$

$$\begin{array}{l} 3y = 3x \\ 3x = 3y \end{array}$$

$$\therefore y = x$$

IF $\lambda = -3$

$$\begin{array}{l} 3y = -3x \\ 3x = -3y \end{array}$$

$$\therefore y = -x$$

Question 33 (***)

Find the image of the straight line with equation

$$2x + 3y = 10,$$

under the transformation represented by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}.$$

$$\boxed{\quad}, \quad 11x + y = 70$$

METHOD A

BY INSPECTION $A(3,0)$ & $B(2,2)$ LIE ON THE LINE.
MAP THESE POINTS onto their new positions

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 15 & 4 \end{pmatrix}$$

$\begin{matrix} A & B \\ A' & B' \end{matrix}$

FIND EQUATION OF $A(5,6)$ & $B(15,4)$ $\Rightarrow m = \frac{6-4}{5-15} = \frac{-2}{-10} = \frac{1}{5}$

$$\begin{aligned} \therefore y - y_0 &= m(x - x_0) \\ y - 4 &= 1(x - 15) \\ y - 4 &= -15 + 6 \\ y &= -11x + 70 \end{aligned}$$

METHOD B

$$\begin{aligned} 2x + 3y &= 10 \\ 3y &= 2x + 10 \\ y &= -\frac{2}{3}x + \frac{10}{3} \end{aligned}$$

LET A POINT C ON ABOVE LINE HAVE COORDINATES $(t, -\frac{2}{3}t + \frac{10}{3})$

THUS

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} t \\ -\frac{2}{3}t + \frac{10}{3} \end{pmatrix} = \begin{pmatrix} t - \frac{4}{3}t + \frac{20}{3} \\ 3t + \frac{2}{3}t - \frac{10}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}t + \frac{20}{3} \\ \frac{11}{3}t - \frac{10}{3} \end{pmatrix}$$

$\begin{matrix} \text{NEW } x \\ \text{NEW } y \end{matrix}$

$$\begin{aligned} x &= -\frac{1}{3}t + \frac{20}{3} \\ y &= \frac{11}{3}t - \frac{10}{3} \end{aligned} \Rightarrow \begin{aligned} 3x &= -t + 20 \\ 3y &= 11t - 10 \end{aligned} \Rightarrow \begin{aligned} 3x &= 11t + 20 \\ 3y &= 11t - 10 \end{aligned}$$

ADD EQUATIONS $\begin{aligned} 33x + 3y &= 210 \\ 11x + y &= 70 \end{aligned}$ $\cancel{\text{as before}}$

Question 34 (***)

The 2×2 matrix $\mathbf{M} = \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix}$ is given.

Under the transformation represented by \mathbf{M} a straight line passing through the origin remains invariant.

Determine the equation of this line.

, $y = 3x$

WORKING AS FOLLOWS

- "OBJECT LINE" $y = mx$
- "IMAGE LINE" $Y = mX$

$$\rightarrow \begin{pmatrix} -2 & 1 \\ -9 & 4 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} -2x + m \\ -9x + 4mx \end{pmatrix} = \begin{pmatrix} X \\ mX \end{pmatrix}$$

DIVIDING THE EQUATIONS

$$\rightarrow \frac{-2x + m}{-9x + 4mx} = \frac{1}{m}$$
$$\rightarrow -2x + m^2 = 4m - 9$$
$$\rightarrow m^2 - 6m + 9 = 0$$
$$\rightarrow (m - 3)^2 = 0$$
$$\rightarrow m = 3$$

\therefore Required line $y = 3x$

Question 35 (****)

The 2×2 matrix $A = \begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix}$ is given.

Under the transformation represented by A , a straight line passing through the origin is reflected about the y axis.

Determine the possible equations of this line.

, ,

LET THE REQUIRED LINE HAVE EQUATION $y = mx$

LOOKING AT THE DIAGRAM, THE REFLECTED LINE WILL HAVE EQUATION $y = -mx$

$$\Rightarrow \begin{pmatrix} 2 & 1 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} X \\ -mX \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2x + mx \\ -6x + 3mx \end{pmatrix} = \begin{pmatrix} X \\ -mX \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 2x + mx = X \\ -6x + 3mx = -mX \end{array} \quad \text{DEVIDING}$$

$$\begin{array}{l} \frac{2+m}{-6+3m} = \frac{1}{-m} \\ -6+3m = -2m - m^2 \\ m^2 + 5m - 6 = 0 \\ (m+6)(m-1) = 0 \\ \therefore m = -6 \quad \text{OR} \\ \therefore m = 1 \end{array}$$

$$\therefore y = 2 \quad \text{OR} \quad y = -6x$$

Question 36 (****)

Find the image of the circle with equation

$$x^2 + y^2 = 4,$$

under the transformation represented by the 2×2 matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$.

$$\boxed{\quad}, \quad 20x^2 - 32xy + 13y^2 = 16$$

LET THE 'BEFORE' COORDINATES BE (x,y) AND THE 'AFTER' COORDINATES BE DENOTED BY (X,Y)

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{if } \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 2x + 4y \end{pmatrix}$$

Here we now have

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4x - 3y \\ -2x + 2y \end{pmatrix} = \begin{pmatrix} 2x - \frac{3}{2}y \\ -x + y \end{pmatrix}$$

SUBSTITUTE INTO THE CIRCLE & Tidy

$$\begin{aligned} \Rightarrow x^2 + y^2 &= 4 \\ \Rightarrow (2x - \frac{3}{2}y)^2 + (-x + y)^2 &= 4 \\ \Rightarrow 4x^2 - 6xy + \frac{9}{4}y^2 + x^2 - 2xy + y^2 &= 4 \\ \Rightarrow 5x^2 - 8xy + \frac{13}{4}y^2 &= 4 \\ \Rightarrow 20x^2 - 32xy + 13y^2 &= 16 \end{aligned}$$

It $20x^2 - 32xy + 13y^2 = 16$

Question 37 (****)

The 2×2 matrix \mathbf{R} represents a reflection where the point $(2,1)$ gets mapped onto the point $(6,-5)$, and the line with equation $y = -\frac{1}{2}x$ is a line of invariant points.

- a) Determine the elements of \mathbf{R}

The 2×2 matrix M represents the combined transformation of the reflection represented by R , followed by another transformation T .

$$\mathbf{M} = \begin{pmatrix} 0 & -0.4 \\ 2.5 & 2.8 \end{pmatrix}$$

- b) Given that T is also a reflection determine, in exact simplified form, the equation of the line of reflection of T .

$$[\quad], \quad \mathbf{R} = \begin{pmatrix} 2 & 2 \\ -\frac{3}{2} & -2 \end{pmatrix}, \quad \frac{1}{2}x = -\frac{1}{7}y = \frac{1}{2}z$$

a) START BY CREATING THE MATRIX FOR THE REFLECTION

$(2,1) \longmapsto (6,-5)$

"Line $y = -\frac{1}{2}x$ is line of invariant points"

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}$

$2a+b=6$

$2c+d=-5$

$at - \frac{1}{2}bt = t$

$ct - \frac{1}{2}dt = -5$

$a - \frac{1}{2}b = 1$

$c - \frac{1}{2}d = -5$

SOLVING SIMULTANEOUSLY

$2a+b=6$

$a - \frac{1}{2}b = 1$

9

$2c+d=-5$

$c - \frac{1}{2}d = -5$

$2a+b=6$

$2a - b = 2$

Adding $4a = 8$

$a = 2$

Adding $4c = -2$

$c = -\frac{1}{2}$

4

$4 - b = 2$

$b = -2$

$4 - d = -5$

$d = -1$

\therefore THE REFLECTION MATRIX IS $\begin{bmatrix} 2 & -2 \\ -\frac{1}{2} & -1 \end{bmatrix}$

4) $M = \begin{pmatrix} 0 & -6 & 11 \\ 2 & 5 & 20 \end{pmatrix} = \text{REF}(M) \text{ followed by } \text{row reduction}$

$\Rightarrow M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

$\Rightarrow M^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

FIND THE INVERSE OF R

$\Rightarrow R = \begin{pmatrix} 2 & 2 \\ \frac{1}{2} & -2 \end{pmatrix} \quad \det R = -4 + 3 = -1$

$R^{-1} = \frac{1}{-1} \begin{pmatrix} 2 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -\frac{1}{2} & 2 \end{pmatrix}$

$[R \text{ is self inverse if it is a reflection/rotating matrix}]$

$\Rightarrow T = \begin{pmatrix} 0 & -6 \\ 2 & 20 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ \frac{1}{2} & 2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix} \quad \det = -1$

COMPARING WITH THE REFLECTION (SYMMETRIC) MATRIX

$\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \quad \text{REFLECTION ABOUT } y = (\tan \theta)z$

$\sin \theta = 0.8 \quad \cos \theta = 0.6$



$\tan \theta = \frac{4}{3}$

$$\begin{aligned}
 & \Rightarrow \tan(2B) = \frac{4}{3} \\
 & \Rightarrow \frac{2 \sin B}{1 - \sin^2 B} = \frac{4}{3} \\
 & \Rightarrow \frac{\sin B}{1 - \sin^2 B} = \frac{2}{3} \\
 & \Rightarrow 3 \sin B = 2 - 2 \sin^2 B \\
 & \Rightarrow 2 \sin^2 B + 3 \sin B - 2 = 0 \\
 & \Rightarrow (2 \sin B - 1)(\sin B + 2) = 0 \\
 & \Rightarrow \sin B = \frac{1}{2} \quad (\sin B \neq -2) \\
 & \Rightarrow \tan B = \frac{\frac{1}{2}}{\sqrt{1 - \frac{1}{4}}} \\
 & \Rightarrow \tan B = \pm \frac{1}{\sqrt{3}} \\
 & \Rightarrow B = 60^\circ \quad (\text{as } B \in A(0^\circ, 90^\circ))
 \end{aligned}$$

Question 38 (****)

Under the transformation represented by the 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & -7 \end{pmatrix},$$

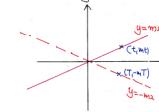
the straight line with equation $y = mx$ is reflected about the x axis.

Find the possible values of m .

, $m = 1, m = 2$

WORKING AT A DIMENSION

under this transformation $(t, mt) \mapsto (T, -mT)$



hence we obtain

$$\begin{pmatrix} 1 & 2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} t \\ mt \end{pmatrix} = \begin{pmatrix} T \\ -mT \end{pmatrix} \Rightarrow \begin{pmatrix} t+2mt \\ 4t-mt \end{pmatrix} = \begin{pmatrix} T \\ -mT \end{pmatrix}$$

$$\Rightarrow \begin{cases} t+2mt = T \\ 4t-mt = -mT \end{cases}$$

$$\Rightarrow \begin{cases} t+2mt = T \\ 4t-mt = -mT \end{cases}$$

$$\Rightarrow \frac{t+2mt}{t-mt} = \frac{T}{-mT}$$

$$\Rightarrow \frac{1+2m}{4-m} = \frac{1}{-m}$$

$$\Rightarrow 1+2m = \frac{1}{-m}$$

$$\Rightarrow -m - 2m^2 = 4 - 7m$$

$$\Rightarrow 0 = 2m^2 - 6m + 4$$

$$\Rightarrow m^2 - 3m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\Rightarrow m = 1, m = 2$$

Question 39 (****)

A transformation in two dimensional space maps a general point with coordinates (x, y) onto the point with coordinates (X, Y) according to the equation

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \mathbf{B} \begin{pmatrix} x \\ y \end{pmatrix},$$

where \mathbf{B} is the 2×2 matrix $\begin{pmatrix} 3 & -2 \\ 4 & -6 \end{pmatrix}$.

Investigate whether this transformation has any lines of invariant points or any invariant lines, giving any relevant equations of such lines if they exist.

You may not use eigenvalue/eigenvector methods in this question

, no lines of invariant points , invariant lines : $y = \frac{1}{2}x \cup y = 4x$

WORKING FOR LINES OF INVARIANT POINTS FIRST SO $(2, 4) \mapsto (2x, 4y)$

$$\begin{aligned} x &= 3x - 2y \\ y &= 4x - 6y \end{aligned} \Rightarrow \begin{aligned} 2x &= 3x - 2y \Rightarrow 2x = 2y \\ 4x &= 4x - 6y \Rightarrow 0 = -6y \end{aligned}$$

\therefore NO LINES OF INVARIANT POINTS

WORKING FOR INVARIANT LINES OF THE FORM $y = mx + c$, FOR WHICH

WE EXPECT $c = 0$

$$\begin{aligned} x &= 3x - 2y \\ y &= 4x - 6y \end{aligned}$$

SUBSTITUTE INTO $y = mx + c$

$$\begin{aligned} \Rightarrow 4x - 6(mx + c) &= m[3x - 2(mx + c)] \\ \Rightarrow 4x - 6mx - 6c &= 3mx - 2mx - 2mc \\ \Rightarrow 2x^2 - 9mx + 6c &= 7c - 2mc \\ \Rightarrow (2m^2 - 9m + 6)x &= c(1 - 2m) \\ \Rightarrow (2m - 1)(m - 4) &= c(2m - 1) = 0 \end{aligned}$$

THIS RELATIONSHIP CAN ONLY BE SATISFIED IF $m = \frac{1}{2}$, $c = 0$ OR $m = 4$, $c = 0$

\therefore INVARIANT LINES ARE $y = \frac{1}{2}x$ \cup $y = 4x$

Question 40 (****)

The 2×2 matrix \mathbf{A} is given below.

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}.$$

- a) Find scalar constants, k and h , so that

$$\mathbf{A}^2 + k\mathbf{I} = h\mathbf{A}.$$

- b) Use part (a) to determine \mathbf{A}^{-1} , the inverse of \mathbf{A} .

No credit will be given for finding \mathbf{A}^{-1} by a direct method.

$$\boxed{[]}, \boxed{k=1}, \boxed{h=8}, \boxed{\mathbf{A}^{-1} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}}$$

a) BY COMPARING ELEMENTS IN THE MATRIX EQUATION

$$\begin{aligned} \mathbf{A}^2 + k\mathbf{I} &= h\mathbf{A} \\ \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} + k \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} &= h \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} \\ (23, 16) + (k, 0) &= (3h, 2h) \\ (23, 16) + (k, 0) &= (7h, 5h) \end{aligned}$$

$$\begin{aligned} \text{looking at } a_{11} & \quad \text{looking at } a_{11} \\ 16 + 0 &= 2h \quad 23 + k = 3h \\ h = 8 & \quad \quad \quad k = 1 \end{aligned}$$

b) DIVIDE THE EQUATION BY \mathbf{A}

$$\begin{aligned} \mathbf{A}^2 + \mathbf{I} &= h\mathbf{A} \\ \mathbf{A}^2 \mathbf{A}^{-1} + \mathbf{I} \mathbf{A}^{-1} &= h\mathbf{A} \mathbf{A}^{-1} \\ \mathbf{A} + \mathbf{A}^{-1} &= h\mathbf{I} \\ \mathbf{A}^{-1} &= \mathbf{A} - \mathbf{A} \\ \mathbf{A}^{-1} &= \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix} - \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} \\ \mathbf{A}^{-1} &= \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix} \end{aligned}$$

$$\boxed{\mathbf{A}^{-1} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}}$$

Question 41 (**)**

The 2×2 matrix \mathbf{A} given below represents a transformation in the x - y plane.

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

The straight line L with equation

$$y = 2x + 1$$

is transformed by \mathbf{A} into the straight line L' .

- a) Find a Cartesian equation of L' .

The straight line M is transformed by \mathbf{A} into the straight line M' with equation

$$11x + 6y = 4.$$

- b) Find a Cartesian equation of M .

$$L': y = 1 - x, \quad M: y = 4 - 3x$$

a) GIVE TO PARAMETERIZE L
 $y = 2x + 1$ has a fixed point $(\frac{1}{2}, \frac{3}{2})$

THUS

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{3}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \cdot \frac{1}{2} - \frac{3}{2} \\ -5 \cdot \frac{1}{2} + 2 \cdot \frac{3}{2} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{3}{2} - \frac{3}{2} \\ -\frac{5}{2} + 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$\begin{cases} x = \frac{1}{2} - t \\ y = \frac{1}{2} + t \end{cases}$$

ADD THE PARAMETRIC

$$x + y = 1$$

$$\therefore y = 1 - x$$

b) $\therefore A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\rightarrow A^T A \begin{pmatrix} x \\ y \end{pmatrix} = A^T \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightarrow I \begin{pmatrix} x \\ y \end{pmatrix} = A^T \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = A^T \begin{pmatrix} x \\ y \end{pmatrix}$$

Now $A^T = \frac{1}{3x+2y-4} \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

WE NEED TO PARAMETERIZE BUT TO TRANSFORM BACK
 TWO POINT WHICH EASILY USE ON M'

SAY $(2, 3)$ AND $(8, 14)$ ARE ON M'

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 14 \\ 5 & 14 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

SO $(1, 1)$ & $(2, 2)$ ARE ON M

$$\text{SLOPES} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 1}{2 - 1} = -3$$

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -3(x - 1)$$

$$y - 1 = -3x + 3$$

$$y = 4 - 3x$$

Question 42 (****)

Describe fully the transformation given by the following 2×2 matrix

$$\begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

The description must be supported by mathematical calculations.

reflection in $y = 2x$

$$\det \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} = -\frac{3}{5} \times \frac{3}{5} - \frac{4}{5} \times \frac{4}{5} = -\frac{9}{25} - \frac{16}{25} = -1$$

COMPARING THE MATRIX WITH THE STANDARD REFLECTION MATRIX

$$\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$$

$$\cos \theta = -\frac{3}{5}$$

$$2\theta = 126.87^\circ \pm 360^\circ$$

$$2\theta = 233.13^\circ \pm 360^\circ$$

$$\theta = 63.43^\circ \pm 180^\circ$$

$$\theta = 116.57^\circ \pm 180^\circ$$

$\theta = 63.43^\circ$ PRODUCES ALL THE REQUIRED CORRECTNESS

∴ MATRIX REPRESENTS REFLECTION ACROSS THE LINE $y = (\pm \sqrt{2})x$

$\therefore y = \tan(63.43^\circ)x$

$y = 2x$

∴ $\cos \theta = -\frac{3}{5}$ $\tan \theta = 2$

∴ $\cos \theta = +\frac{1}{\sqrt{5}}$ $(\theta \text{ is acute } 0 < \theta < 90^\circ)$

$\cos^2 \theta + \sin^2 \theta = 1$

$2\cos^2 \theta - 1 = -\frac{3}{5}$

$2\cos^2 \theta = \frac{3}{5}$

$\cos^2 \theta = \frac{3}{10}$

$\cos \theta = \pm \frac{\sqrt{3}}{\sqrt{10}}$

Question 43 (**)**

A composite transformation in the x - y plane consists of ...

- ... a uniform enlargement about the origin of scale factor k , $k > 0$, denoted by the matrix \mathbf{E} .
- ... a shear parallel to the straight line L , denoted by the matrix \mathbf{S} .

It is given that $\mathbf{ES} = \mathbf{SE} = \begin{pmatrix} 12 & 16 \\ -9 & 36 \end{pmatrix}$

- Show clearly that $k = 24$.
- Find a Cartesian equation of L .

$$y = \frac{3}{4}x$$

a) Firstly $\det \begin{pmatrix} 12 & 16 \\ -9 & 36 \end{pmatrix} = (2x3 - (-7))x16 = 42x16 = 516 \rightarrow \text{RIGHT SCALE FACTOR}$
 AS THE ENLARGEMENT IS UNKNOWN, THE SCALE FACTOR MUST BE $\sqrt{516} = 24 \therefore k = 24$

b) To find L , we need to find an invariant line through 0
 This $\begin{pmatrix} 12 & 16 \\ -9 & 36 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 24 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 12x + 16y = 24x \\ -9x + 36y = 24y \end{cases} \Rightarrow \text{DIVIDE EQUATIONS}$
 $\Rightarrow \frac{12x + 16y}{-9x + 36y} = \frac{24}{24}$
 $\Rightarrow \frac{12 + 16y}{-9 + 36y} = \frac{1}{1}$
 $\Rightarrow 12y + 16y^2 = -9 + 36y$
 $\Rightarrow 16y^2 - 24y + 9 = 0$
 $\Rightarrow (4y - 3)^2 = 0$
 $\therefore y = \frac{3}{4}$
 $\therefore y = \frac{3}{4}x$

Question 44 (**)**

A plane transformation maps the general point (x, y) onto the general point (X, Y) , by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Find the area scale factor of the transformation.
- Determine the equation of the straight line of invariant points under this transformation.
- Show that all the straight lines with equation of the form

$$x + y = c,$$

where c is a constant, are invariant lines under this transformation.

- Hence describe the transformation geometrically.

$[SF = 3]$, $[y = x]$, stretch perpendicular to the line $y = x$, by area scale factor 3

a) $\det \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 2 \cdot 2 - (-1)(-1) = 3$
 \therefore AREA SCALE FACTOR IS 3

b) IF A POINT IS INvariant THEN $(x, y) \mapsto (x, y)$
 $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
 $\begin{cases} 2x - y = x \\ -x + 2y = y \end{cases} \Rightarrow y = x$

c) INvariant LINE (POINTS NOT INvariant)
 $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
 $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} = \begin{pmatrix} 3x - y \\ -x + 2y \end{pmatrix}$
 $\begin{cases} X = 3x - y \\ Y = -x + 2y \end{cases}$ Adding gives $X + Y = 2x$
 Adding gives $X + Y = 2x$

d) SINCE POINTS ON LINE $y = x$ ARE INvariant AND DIRECTION $y = -x$ IS INVARIANT THE MATRIX REPRESENTS A STRETCH perpendicular TO $y = x$, BY AREA SCALE FACTOR 3

Question 45 (****)

A transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is represented by the following 2×2 matrix .

$$\mathbf{A} = \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}.$$

- Find the determinant of \mathbf{A} and explain its significance with reference to its sign and its magnitude.
- Find the equation of the straight line of the invariant points under the transformation represented by \mathbf{A} .
- Determine the entries of the 2×2 matrix \mathbf{B} which represents a reflection about the straight line found in part (b), giving all its entries as simple fractions.

The transformation represented by \mathbf{A} , consists of a shear represented by the matrix \mathbf{C} , followed by a reflection represented by the matrix \mathbf{B} .

- Determine the matrix \mathbf{C} and describe the shear.

$$\boxed{\det \mathbf{A} = -1}, \quad \boxed{y = \frac{1}{2}x}, \quad \boxed{\mathbf{B} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}}, \quad \boxed{\mathbf{C} = \begin{pmatrix} -\frac{13}{5} & \frac{36}{5} \\ -\frac{9}{5} & \frac{23}{5} \end{pmatrix}}$$

(a) $\det \mathbf{B} = \begin{vmatrix} -2 & 8 \\ -1 & 3 \end{vmatrix} = -9 - (-8) = -1$ • AREA OF PARALLELOGRAM
HENCE \mathbf{B} IS DIRECTED (MIRROR) INVERSE

(b) $\begin{pmatrix} -2 & 8 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \Rightarrow \begin{pmatrix} -2x + 8y \\ -x + 3y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \Rightarrow 4x = 8y \Rightarrow x = 2y \Rightarrow y = \frac{1}{2}x$

(c) ROTATION REFLECTION ABOUT $y = \tan 60^\circ x = 0$

$$\begin{pmatrix} \cos 60^\circ & \sin 60^\circ \\ \sin 60^\circ & -\cos 60^\circ \end{pmatrix} \quad \begin{array}{c} \text{7/4} \\ 1 \\ 0 \\ 3/4 \end{array} \quad \begin{array}{c} \tan 60^\circ = \frac{1}{\sqrt{3}} \\ \sin 60^\circ = \frac{\sqrt{3}}{2} \\ \cos 60^\circ = \frac{1}{2} \end{array}$$

$$\therefore \sin 60^\circ = 2 \sin 60^\circ \cos 60^\circ = 2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \cos^2 60^\circ - \sin^2 60^\circ = (\frac{1}{2})^2 - (\frac{\sqrt{3}}{2})^2 = -\frac{2}{4} = -\frac{1}{2}$$

$$\therefore \mathbf{B} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

(d) $\mathbf{A} = \mathbf{BC}$
 $\Rightarrow \mathbf{B}^T \mathbf{A} = \mathbf{B}^T \mathbf{B} \mathbf{C}$
 $\Rightarrow \mathbf{C} = \mathbf{B}^T \mathbf{A}$
 $\Rightarrow \mathbf{C} = \frac{1}{5} \begin{pmatrix} 7 & -3 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$
 $\Rightarrow \mathbf{C} = \frac{1}{5} \begin{pmatrix} 2 & 4 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} -3 & 8 \\ -1 & 3 \end{pmatrix}$
 $\Rightarrow \mathbf{C} = \frac{1}{5} \begin{pmatrix} -13 & 36 \\ -9 & 23 \end{pmatrix}$

SHARP PRODUCT TO $\mathbf{C} = \frac{1}{5} \mathbf{A}$
 \Rightarrow THAT $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Question 46 (****+)

A transformation in two dimensional space maps a general point with coordinates (x, y) onto the point with coordinates (X, Y) according to the equation

$$\begin{pmatrix} X-4 \\ Y+4 \end{pmatrix} = \mathbf{A} \begin{pmatrix} x \\ y \end{pmatrix},$$

where \mathbf{A} is the 2×2 matrix $\begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix}$.

Investigate whether this transformation has any lines of invariant points or any invariant lines, giving any relevant equations of such lines if they exist.

EDD, line of invariant points : $3x - 2y = 4$, invariant line : $y = -x + C$

SEARCH FOR LINES OF INVARIANT POINTS : IF $(2, 4) \mapsto (2, 4)$

$$\begin{pmatrix} 2-4 \\ 4+4 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 2-4 \\ 4+4 \end{pmatrix} = \begin{pmatrix} -2+2 \cdot 2 \\ 3-1 \cdot 4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-4 \\ 4+4 \end{pmatrix} = \begin{pmatrix} -2+2y \\ 3-4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2-4 = -2+2y \\ 4+4 = 3-4 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3x-2y = 4 \\ 4 = 3x-2y \end{pmatrix}$$

$\therefore 3x-2y = 4$ IS A LINE OF INVARIANT POINTS

NEXT INVESTIGATE INVARIANT LINE, SAY $y = mx + c$

$$\begin{cases} X-4 = -2+2y \\ Y+4 = 3x-2y \end{cases} \Rightarrow \begin{cases} X = 2+2y \\ Y = 4+3x-2y \end{cases}$$

SUBSTITUTE INTO $Y = mx + c$

$$\begin{aligned} &\Rightarrow -4+3x-2(mx+c) = [4+2+2(2mx+c)]m+c \\ &\Rightarrow -4+3x-2m-2c = [6+2m+2c]m+c \\ &\Rightarrow -4+3x-2m-2c = 14m+2mx+2m^2+2mc+c \\ &\Rightarrow -4-2c-14m-2mc = -2m^2+2m^2+2mc-3x \\ &\Rightarrow -[2mc+14m+2c+4] = [2m^2-2m-4]x \\ &\Rightarrow -[2m(c+2)+2(c+2)] = (2m-2)(m+1)x \\ &\Rightarrow -2m(c+2)(c+2) = (2m-2)(m+1)x \\ &\Rightarrow -2(m+1)(c+2) = (2m-2)(m+1)x \\ &\Rightarrow [2m-2](m+1)x + 2(m+1)(c+2) = 0 \end{aligned}$$

NOW LOOKING AT THIS EQUATION

- IF $m = -1$ IT IS AUTOMATICALLY SATISFIED FOR ALL C
 - $y = -x + C$ IS AN INVARIANT LINE
- IF $m = \frac{3}{2}$ AND $C = 2$ IT IS ALSO SATISFIED

$$\begin{aligned} y &= \frac{3}{2}x - 2 \\ 2y &= 3x - 4 \\ 0 &= 3x - 2y - 4 \\ 3x - 2y &= 4 \end{aligned}$$

which was found to be a line of invariant points

Question 47 (****+)

A transformation T , maps the general point (x, y) onto the general point (X, Y) , by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

- Find the area scale factor of the transformation.
- Determine the equation of the line of invariant points under this transformation.
- Show that all the straight lines of the form

$$y = x + c,$$

where c is a constant, are invariant lines under T .

- Hence state the name of T .
- Show that the acute angle formed by the straight line with equation $y = -x$ and its image under T is

$$\frac{3\pi}{4} - \arctan\left(\frac{5}{3}\right).$$

[SF = 1], $y = x$, shear

a) $\det \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} = -1 \cdot 3 - (-2) \cdot 2 = -3 + 4 = 1$

b) INvariant POINTS $(x, y) \mapsto (x, y)$
 $\begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$
 $\begin{cases} -x + 2y = x \\ -2x + 3y = y \end{cases} \Rightarrow y = x$

c) INVARIANT DIRECTION
 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x + 2y \\ -2x + 3y \end{pmatrix}$
 $\begin{cases} x = -x + 2y \\ y = -2x + 3y \end{cases}$ subtract $x - y = -c$
 $y = x + c$

d) IT IS A shear, parallel to the line $y = x$

e) PARAMETERIZE THE UNIT $y = -x$
 $\begin{cases} x = t \\ y = -t \end{cases}$
 $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} -t - 2t \\ -2t + 3t \end{pmatrix} = \begin{pmatrix} -3t \\ t \end{pmatrix}$
 $\begin{cases} x = -3t \\ y = t \end{cases} \Rightarrow \text{divide } \begin{cases} x = \frac{3}{3}t \\ y = \frac{1}{3}t \end{cases}$
 $\therefore y = \frac{1}{3}x$

THE REQUIRED ANGLE IS $T\frac{\pi}{2} + \frac{\pi}{4} - \arctan\frac{5}{3}$
 $= \frac{3\pi}{4} - \arctan\frac{5}{3}$

Question 48 (****+)

A curve has equation

$$5x^2 - 16xy + 13y^2 = 25.$$

This curve is to be mapped onto another curve C , under the transformation defined by the 2×2 matrix A , given below.

$$A = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

Show that the equation of C is the circle with equation

$$x^2 + y^2 = 25.$$

, proof

DETERMINE THE TRANSFORMATION EQUATIONS

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \iff \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{-3+4} \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

From we have

- $x = 3x - 2y$
- $y = 2x - y$

SUBSTITUTE INTO THE EQUATION $5x^2 - 16xy + 13y^2 = 25$

$$\begin{aligned} &\Rightarrow 5(3x - 2y)^2 - 16(3x - 2y)(2x - y) + 13(2x - y)^2 = 25 \\ &\Rightarrow 5(9x^2 - 12xy + 4y^2) - 16(6x^2 - 7xy + 2y^2) + 13(4x^2 - 16xy + y^2) = 25 \\ &\Rightarrow \left\{ \begin{array}{l} 45x^2 - 60xy + 20y^2 \\ -96x^2 + 112xy - 32y^2 \\ 52x^2 - 62xy + 13y^2 \end{array} \right\} = 25 \\ &\Rightarrow x^2 + y^2 = 25 \\ &\text{or} \\ &x^2 + y^2 = 25 \end{aligned}$$

Question 49 (***)+

The 2×2 matrix \mathbf{P} is given below.

$$\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

The points on the x - y plane which lie on the curve with equation

$$13x^2 - 16xy + 5y^2 + 8x - 6y = 4,$$

are transformed by \mathbf{P} onto the points which lie on another curve C .

Determine an equation for C and hence describe it geometrically.

$$\boxed{\quad}, \boxed{(x-1)^2 + (y-2)^2 = 9}$$

START BY OBTAINING THE INVERSE OF \mathbf{P}

$$\mathbf{P}^{-1} = \frac{1}{2a-3b} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

THIS CAN BE INVERTED BY THE TRANSFORMATION EQUATIONS

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

1.6 $\begin{cases} x = 2x' + y' \\ y = 3x' + 2y' \end{cases}$

SUBSTITUTING INTO THE EQUATION WE OBTAIN

$$\begin{aligned} & \Rightarrow 13x^2 - 16xy + 5y^2 + 8x - 6y = 4 \\ & \Rightarrow 13(2x'+y')^2 - 16(2x'+y')(3x'+2y') + 5(3x'+2y')^2 + 8(2x'+y') - 6(3x'+2y') = 4 \\ & \Rightarrow 13(4x'^2 + 4xy' + y'^2) - 16(6x'^2 + 10xy' + 2y'^2) + 5(9x'^2 + 12xy' + 4y'^2) - 2x - 4y = 4 \\ & \Rightarrow 52x'^2 + 52xy' + 13y'^2 - 96x'^2 - 112xy' - 32y'^2 + 45x'^2 + 60xy' + 20y'^2 - 2x - 4y = 4 \\ & \Rightarrow x^2 + y^2 - 2x - 4y = 4 \\ & \Rightarrow (x-1)^2 + (y-2)^2 = 4 \\ & \Rightarrow (x-1)^2 + (y-2)^2 = 9 \\ & \therefore \text{A CIRCLE OF RADIUS 3, CENTRE AT (1,2)} \end{aligned}$$

Question 50 (****+)

The points $P(7,5)$ and $Q(4,-3)$ are given.

The point Q is rotated by 90° anticlockwise about the point P .

V , \boxed{P} , $R(15,2)$

THE STANDARD ROTATION FOR 90° ABOUT O IS GIVEN BY

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} -y \\ x \end{pmatrix} \text{ i.e. } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

TRANSLATE THE CENTRE OF ROTATION ON THE ORIGIN

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -a \\ -b \end{pmatrix}$$

DO THE SAME FOR THE OTHER POINT

$$\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \begin{pmatrix} -7 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 4 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

ROTATE ABOUT THE CENTRE

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

FINALLY INVERSE THE TRANSLATION

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$$

$\therefore (15,2)$

Question 51 (*****)

A shear is defined by the 2×2 matrix

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & 4 \end{pmatrix},$$

where a , b and c are scalar constants.

Under this transformation the point with coordinates $(1, 2)$ is mapped onto the point with coordinates $(-8, 11)$.

The shear defined by \mathbf{M} has an invariant line L , which passes through the point with coordinates $(0, 1)$.

Determine an equation of L .

$$\boxed{\quad}, \boxed{L : y = 1 - x}$$

AS THIS IS A SHEAR THE DETERMINANT MUST BE 1

$$\Rightarrow 4a - bc = 1$$

USING THE MAPPING $(1, 2) \rightarrow (-8, 11)$

$$\begin{pmatrix} a & b \\ c & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ 11 \end{pmatrix} \Rightarrow \begin{cases} a + 2b = -8 \\ c + 8 = 11 \end{cases}$$

SEEING TO GET

- $c = 3$
- $4a - 3b = 1 \quad \left\{ \begin{array}{l} 8a - 6b = 2 \\ 3a + 6b = -24 \end{array} \right. \Rightarrow 11a = -22 \Rightarrow a = -2$
- $a + 2b = -8$
 $-2 + 2b = -8$
 $2b = -6$
 $b = -3$

$$\therefore \mathbf{M} = \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix}$$

NOOW WE LOOK FOR INVARIANT LINES OF THE FORM $y = mx + 1$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} \Rightarrow \begin{cases} x = -2 - 3m \\ y = 3 + 4m \end{cases}$$

$$\Rightarrow \begin{cases} x = -2 - 3(1 - m) \\ y = 3 + 4(1 - m) \end{cases}$$

$$\Rightarrow \begin{cases} x = -2 - 3m + 3 \\ y = 3 + 4m + 4 \end{cases}$$

$$\Rightarrow \begin{cases} x = 1 - 3m \\ y = 7 + 4m \end{cases}$$

NOW SUB INTO $y = mx + 1$

$$\Rightarrow 3x + 4m + 4 = m(-2x - 3m + 3) + 1$$

$$\Rightarrow 3x + 4m + 4 = -2mx - 3m^2 - 3m + 1$$

$$\Rightarrow 3m^2 + 6mx + 3x + 3m + 3 = 0$$

$$\Rightarrow 3x(m^2 + 2m + 1) + 3(m + 1) = 0$$

$$\Rightarrow 3x(m+1)^2 + 3(m+1) = 0$$

$$\Rightarrow 3(m+1)^2 + 3(m+1) = 0$$

EQUATION IS SATISFIED FOR $m = -1$

\therefore INVARIANT LINE IS $y = 1 - x$