

# 9 Linear Programming



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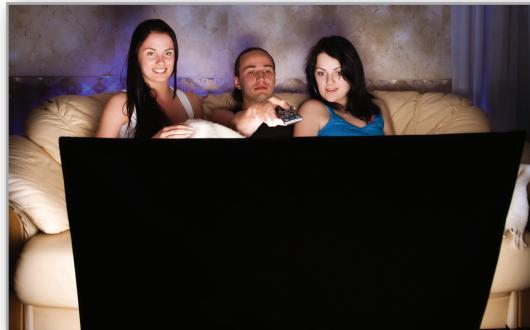
The Simplex Method: Maximization

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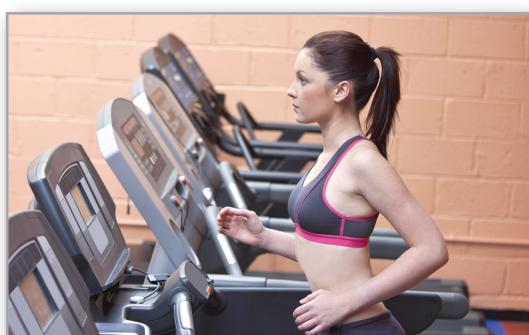
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## 9.1 Systems of Linear Inequalities

- Sketch the graph of a linear inequality.
- Sketch the graph of a system of linear inequalities.

### LINEAR INEQUALITIES AND THEIR GRAPHS

The statements below are inequalities in two variables:

$$3x - 2y < 6 \quad \text{and} \quad x + y \geq 6.$$

An ordered pair  $(a, b)$  is a **solution of an inequality** in  $x$  and  $y$  if the inequality is true when  $a$  and  $b$  are substituted for  $x$  and  $y$ , respectively. For example,  $(1, 1)$  is a solution of the inequality  $3x - 2y < 6$  because

$$3(1) - 2(1) = 1 < 6.$$

The **graph of an inequality** is the collection of all solutions of the inequality. To sketch the graph of an inequality such as

$$3x - 2y < 6$$

begin by sketching the graph of the *corresponding equation*

$$3x - 2y = 6.$$

The graph of the equation separates the plane into two regions. In each region, one of the following two statements listed below must be true.

1. *All* points in the region are solutions of the inequality.
2. *No* point in the region is a solution of the inequality.

So, you can determine whether the points in an entire region satisfy the inequality by simply testing *one* point in the region.

#### REMARK

When possible, use test points that are convenient to substitute into the inequality, such as  $(0, 0)$ .

#### Sketching the Graph of an Inequality in Two Variables

1. Replace the inequality sign with an equal sign, and sketch the graph of the resulting equation. (Use a dashed line for  $<$  or  $>$  and a solid line for  $\leq$  or  $\geq$ .)
2. Test one point in each of the regions formed by the graph in Step 1. If the point satisfies the inequality, then shade the entire region to denote that every point in the region satisfies the inequality.

In this section, you will work with **linear inequalities** of the forms listed below.

$$ax + by < c$$

$$ax + by \leq c$$

$$ax + by > c$$

$$ax + by \geq c$$

The graph of each of these linear inequalities is a half-plane lying on one side of the line  $ax + by = c$ . When the line is dashed, the points on the line *are not* solutions of the inequality; when the line is solid, the points on the line *are* solutions of the inequality. The simplest linear inequalities are those corresponding to horizontal or vertical lines, as shown in Example 1 on the next page.

**EXAMPLE 1****Sketching the Graph of a Linear Inequality**

Sketch the graph of each linear inequality.

- a.  $x > -2$
- b.  $y \leq 3$

**SOLUTION**

- a. The graph of the corresponding equation  $x = -2$  is a vertical line. The points that satisfy the inequality  $x > -2$  are those lying to the right of this line, as shown in Figure 9.1(a).
- b. The graph of the corresponding equation  $y = 3$  is a horizontal line. The points that satisfy the inequality  $y \leq 3$  are those lying below (or on) this line, as shown in Figure 9.1(b).

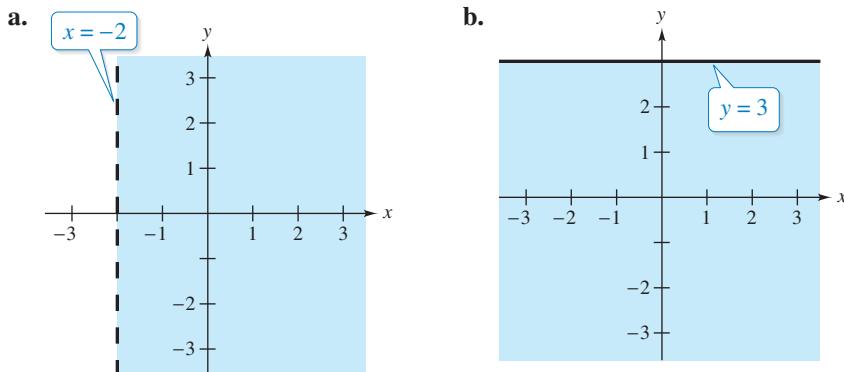


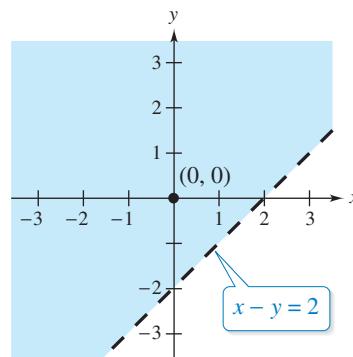
Figure 9.1

**EXAMPLE 2****Sketching the Graph of a Linear Inequality**

Sketch the graph of  $x - y < 2$ .

**SOLUTION**

The graph of the corresponding equation  $x - y = 2$  is a line, as shown below. The origin  $(0, 0)$  satisfies the inequality, so the graph consists of the half-plane lying above the line. (Check a point below the line to see that it does not satisfy the inequality.)



For a linear inequality in two variables, you can sometimes simplify the graphing procedure by writing the inequality in *slope-intercept* form. For example, by writing  $x - y < 2$  in the form

$$y > x - 2$$

you can see that the solution points lie *above* the line  $y = x - 2$ .

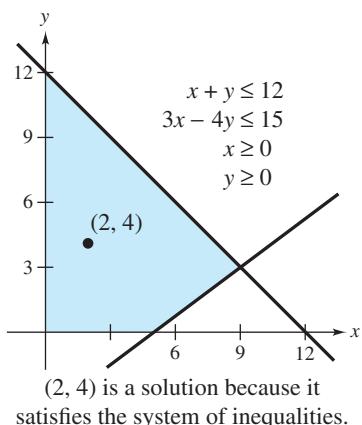


Figure 9.2

## SYSTEMS OF INEQUALITIES

Many practical problems in business, science, and engineering involve systems of linear inequalities. An example of such a system is shown below.

$$\begin{aligned}x + y &\leq 12 \\3x - 4y &\leq 15 \\x &\geq 0 \\y &\geq 0\end{aligned}$$

A **solution** of a system of inequalities in  $x$  and  $y$  is a point  $(x, y)$  that satisfies each inequality in the system. For example,  $(2, 4)$  is a solution of the above system because  $x = 2$  and  $y = 4$  satisfy each of the four inequalities in the system. The **graph** of a system of inequalities in two variables is the collection of all points that are solutions of the system. For example, the graph of the above system is the region shown in Figure 9.2. Note that the point  $(2, 4)$  lies in the shaded region because it is a solution of the system of inequalities.

To sketch the graph of a system of inequalities in two variables, first sketch the graph of each individual inequality (on the same coordinate system) and then find the region that is *common* to every graph in the system. This region represents the **solution set** of the system. For systems of linear inequalities, it is helpful to find the *vertices* of the solution region, as shown in Example 3.

### EXAMPLE 3

#### Solving a System of Inequalities

Sketch the graph (and label the vertices) of the solution set of the system shown below.

$$\begin{aligned}x - y &< 2 \\x &> -2 \\y &\leq 3\end{aligned}$$

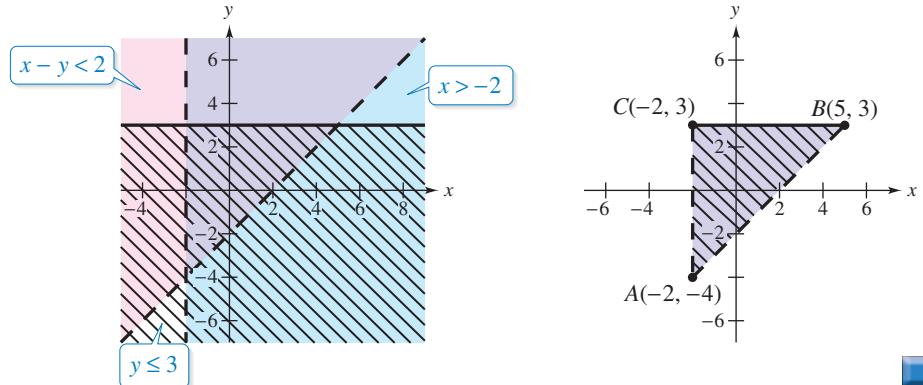
#### SOLUTION

You have already sketched the graph of each of these inequalities in Examples 1 and 2. The region common to all three graphs can be found by superimposing the graphs on the same coordinate plane, as shown below. To find the vertices of the region, find the points of intersection of the boundaries of the region.

Vertex A:  $(-2, -4)$   
Obtained by finding  
the point of  
intersection of  
 $x - y = 2$   
 $x = -2.$

Vertex B:  $(5, 3)$   
Obtained by finding  
the point of  
intersection of  
 $x - y = 2$   
 $y = 3.$

Vertex C:  $(-2, 3)$   
Obtained by finding  
the point of  
intersection of  
 $x = -2$   
 $y = 3.$

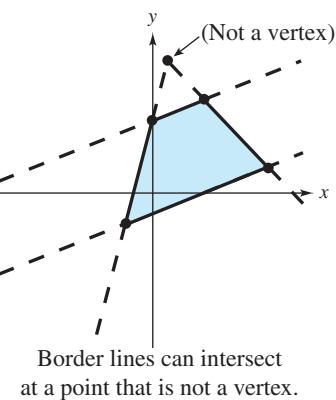
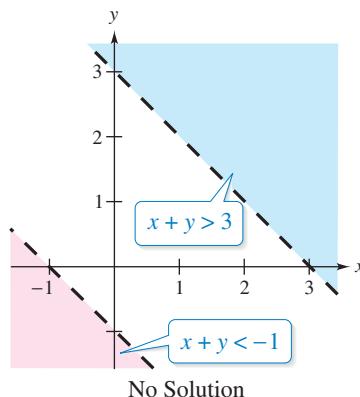


For the triangular region shown in Example 3, each point of intersection of a pair of boundary lines corresponds to a vertex. With more complicated regions, two border lines can sometimes intersect at a point that is not a vertex of the region, as shown at the right. To determine which points of intersection are actually vertices of the region, sketch the region and refer to your sketch as you find each point of intersection.

When solving a system of inequalities, be aware that the system might have no solution. For example, the system

$$\begin{aligned}x + y &> 3 \\x + y &< -1\end{aligned}$$

has no solution points because the quantity  $(x + y)$  cannot be both less than  $-1$  and greater than  $3$ , as shown below.

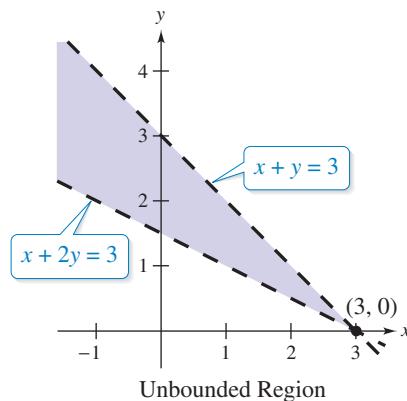


Border lines can intersect at a point that is not a vertex.

Another possibility is that the solution set of a system of inequalities can be unbounded. For example, consider the system below.

$$\begin{aligned}x + y &< 3 \\x + 2y &> 3\end{aligned}$$

The graph of the inequality  $x + y < 3$  is the half-plane that lies below the line  $x + y = 3$ . The graph of the inequality  $x + 2y > 3$  is the half-plane that lies above the line  $x + 2y = 3$ . The intersection of these two half-planes is an *infinite wedge* that has a vertex at  $(3, 0)$ , as shown below. This unbounded region represents the solution set.



Example 4 shows how a system of linear inequalities can arise in an applied problem.

### EXAMPLE 4

### An Application of a System of Inequalities

See [LarsonLinearAlgebra.com](http://LarsonLinearAlgebra.com) for an interactive version of this type of example.

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Set up a system of linear inequalities that describes the minimum daily requirements for calories and vitamins.

#### SOLUTION

Let

- $x$  = number of cups of dietary drink X and  
 $y$  = number of cups of dietary drink Y.

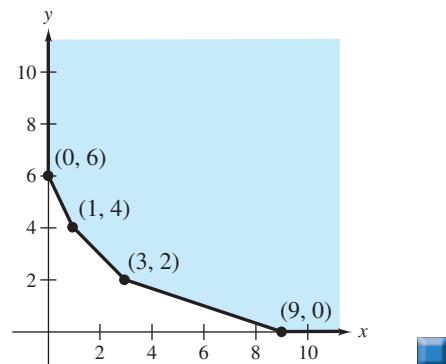
To meet the minimum daily requirements, the inequalities listed below must be satisfied.

#### REMARK

Any point inside the shaded region (or on its boundary) meets the minimum daily requirements for calories and vitamins. For example, 3 cups of dietary drink X and 2 cups of dietary drink Y supply 300 calories, 48 units of vitamin A, and 90 units of vitamin C.

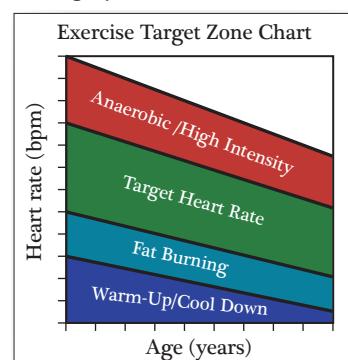
For calories:	$60x + 60y \geq 300$
For vitamin A:	$12x + 6y \geq 36$
For vitamin C:	$10x + 30y \geq 90$
	$x \geq 0$
	$y \geq 0$

The last two inequalities are included because  $x$  and  $y$  cannot be negative. The graph of this system of linear inequalities is shown at the right.



#### LINEAR ALGEBRA APPLIED

A heart rate monitor watch is designed to ensure that a person exercises at a healthy pace. It displays transmissions from a chest strap that monitors the person's pulse electronically. Researchers have identified target heart rate ranges, or zones, for achieving specific results from exercise. Fitness facilities and cardiology treadmill rooms often display wall charts, such as the one shown at the right, to help determine whether a person's exercise heart rate is in a healthy range. In Exercise 60, you are asked to write a system of inequalities for a person's target heart rates during exercise.



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## 9.1 Exercises

**Identifying the Graph of a Linear Inequality** In Exercises 1–6, match the linear inequality with its graph. [The graphs are labeled (a)–(f).]

1.  $x > 3$

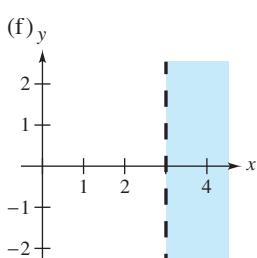
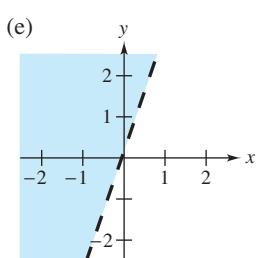
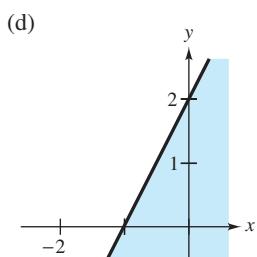
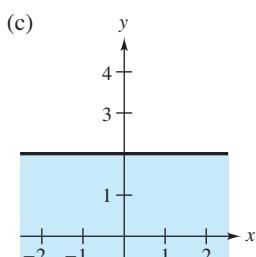
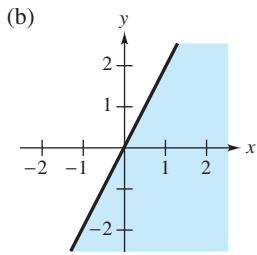
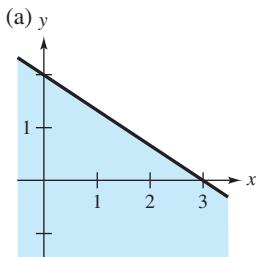
3.  $2x + 3y \leq 6$

5.  $x \geq \frac{y}{2}$

2.  $y \leq 2$

4.  $2x - y \geq -2$

6.  $y > 3x$



**Sketching the Graph of a Linear Inequality** In Exercises 7–28, sketch the graph of the linear inequality.

7.  $x \geq 2$

8.  $x \leq 4$

9.  $y \leq 7$

10.  $y \geq -1$

11.  $y < 2 - x$

12.  $y > 2x - 4$

13.  $2y - x \geq 4$

14.  $5x + 3y \geq -15$

15.  $y \leq x$

16.  $3x > y$

17.  $y \geq 4 - 2x$

18.  $y \leq 3 + x$

19.  $3y + 4 \geq x$

20.  $6 - 2y < x$

21.  $4x - 2y \leq 12$

22.  $y + 3x > 6$

23.  $5x - 2y > 10$

24.  $2x + 7y \leq 28$

25.  $y + \frac{1}{3}x \geq 6$

26.  $y - \frac{1}{2}x < 12$

27.  $\frac{x}{3} + \frac{y}{2} < 1$

28.  $\frac{x}{6} - \frac{y}{5} \geq 1$

**Solving a System of Inequalities** In Exercises 29–32, determine whether each ordered pair is a solution of the system of linear inequalities.

29.  $x \geq -4$

$y > -3$

$y \leq -8x - 3$

(a)  $(0, 0)$

(b)  $(-1, -3)$

(c)  $(-4, 0)$

(d)  $(-3, 11)$

30.  $-2x + 5y \geq 3$

$y < 4$

$-4x - 2y < 7$

(a)  $(0, 2)$

(b)  $(-6, 4)$

(c)  $(-8, -2)$

(d)  $(-3, 2)$

31.  $x \geq 1$

$y \geq 0$

$y \leq 2x + 1$

(a)  $(0, 1)$

(b)  $(1, 3)$

(c)  $(2, 2)$

(d)  $(2, 1)$

32.  $x \geq 0$

$y \geq 0$

$y \leq 4x - 2$

(a)  $(0, -2)$

(b)  $(2, 0)$

(c)  $(3, 1)$

(d)  $(0, -1)$

**Solving a System of Inequalities** In Exercises 33–48, sketch the graph (and label any vertices) of the solution set of the system of linear inequalities.

33.  $x \geq 0$

$y \geq 0$

$x \leq 2$

$y \leq 4$

34.  $x \geq -1$

$y \geq -1$

$x \leq 1$

$y \leq 2$

35.  $x + y \leq 1$

$-x + y \leq 1$

$y \geq 0$

36.  $3x + 2y < 6$

$x > 0$

$y > 0$

37.  $x + y \leq 5$

$x \geq 2$

$y \geq 0$

38.  $2x + y \geq 2$

$x \leq 2$

$y \leq 1$

39.  $-3x + 2y < 6$

$x + 4y > -2$

$2x + y < 3$

40.  $x - 7y > -36$

$5x + 10y > 20$

$6x - 5y > 6$

41.  $x \geq 1$

$x - 2y \leq 3$

$3x + 2y \geq 9$

$x + y \leq 6$

42.  $x + y < 10$

$2x + y > 10$

$x - y > 2$

43.  $-3x + 2y < 6$

$x - 4y > -2$

$x + 4y < 4$

44.  $-x + 3y \leq 12$

$5x + 2y > 5$

$x - 3y < -3$

45.  $2x + y > 2$

$6x + 3y < 2$

46.  $x - 2y < -6$

$5x - 3y > -9$

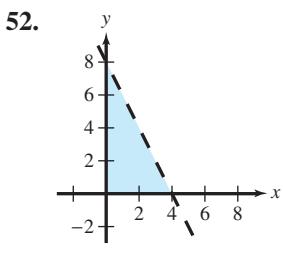
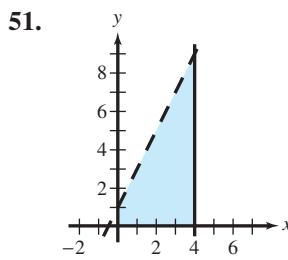
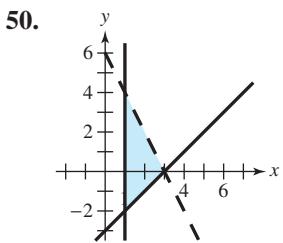
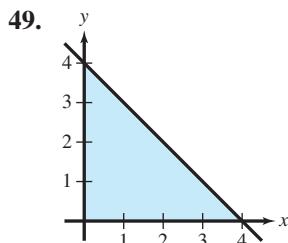
47.  $3x - 6y \leq 5$

$x - 2y \geq -3$

48.  $12x + 15y \geq 60$

$y \leq -\frac{4}{5}x + 4$

**Writing a System of Inequalities** In Exercises 49–52, write a system of inequalities that describes the region.



**Writing a System of Inequalities** In Exercises 53–58, write a system of inequalities that describes the region.

53. Rectangle with vertices at  $(2, 1)$ ,  $(5, 1)$ ,  $(5, 7)$ , and  $(2, 7)$   
 54. Rectangle with vertices at  $(1, 3)$ ,  $(3, 1)$ ,  $(4, 6)$ , and  $(6, 4)$   
 55. Parallelogram with vertices at  $(-6, -2)$ ,  $(1, 8)$ ,  $(6, 5)$ , and  $(-1, -5)$   
 56. Parallelogram with vertices at  $(0, 0)$ ,  $(4, 0)$ ,  $(1, 4)$ , and  $(5, 4)$   
 57. Triangle with vertices at  $(0, 0)$ ,  $(5, 0)$ , and  $(2, 3)$   
 58. Triangle with vertices at  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$

59. **Investment** A person plans to invest no more than \$20,000 in two different interest-bearing accounts. Each account is to contain at least \$5000. Moreover, one account should have at least twice the amount that is in the other account. Write a system of inequalities describing the various amounts that can be deposited in each account, and sketch the graph of the system.

60. **Target Heart Rate** One formula for a person's maximum heart rate is  $220 - x$ , where  $x$  is the person's age in years for  $20 \leq x \leq 70$ . When a person exercises, it is recommended that the person strive for a heart rate that is at least 50% of the maximum and at most 85% of the maximum. (Source: American Heart Association)

- (a) Write a system of inequalities that describes the region corresponding to these heart rate recommendations.  
 (b) Sketch a graph of the region in part (a).  
 (c) Find two solutions of the system and interpret their meanings in the context of the problem.

61. **Furniture Production** A furniture company produces tables and chairs. Each table requires 1 hour in the assembly center and  $1\frac{1}{3}$  hours in the finishing center. Each chair requires  $1\frac{1}{2}$  hours in the assembly center and  $1\frac{1}{2}$  hours in the finishing center. The assembly center is available 12 hours per day, and the finishing center is available 15 hours per day. Let  $x$  and  $y$  be the numbers of tables and chairs produced per day, respectively. Write a system of inequalities describing all possible production levels, and sketch the graph of the system.

62. **Tablet Inventory** A store sells two models of tablet computers. Due to demand levels, it is necessary to stock at least twice as many units of the Pro Series as units of the Deluxe Series. The costs to the store of the two models are \$200 and \$300, respectively. The management does not want more than \$5000 in laptop inventory at any one time, and it wants at least four Pro Series models and two Deluxe Series models in inventory at all times. Write a system of inequalities describing all possible inventory levels, and sketch the graph of the system.

63. **Diet Supplement** A dietitian prescribes a special dietary supplement using two different foods. Each ounce of food X contains 20 units of calcium, 15 units of iron, and 10 units of vitamin B. Each ounce of food Y contains 10 units of calcium, 10 units of iron, and 20 units of vitamin B. The minimum daily requirements for the diet are 300 units of calcium, 150 units of iron, and 200 units of vitamin B. Write a system of inequalities describing the different amounts of food X and food Y that the dietitian can prescribe. Sketch the graph of the system.

64. Rework Exercise 63 using minimum daily requirements of 280 units of calcium, 160 units of iron, and 180 units of vitamin B.

65. **Reasoning** Consider the inequality  $8x - 2y < 5$ . Without graphing, determine whether the solution points lie in the half-plane above or below the boundary line. Explain.

66. **CAPSTONE** Consider the system of inequalities

$$\begin{aligned} ax + by &\leq c \\ x &\geq d \\ y &> e \end{aligned}$$

where  $a > 0$  and  $b > 0$ . Find values of  $a$ ,  $b$ ,  $c$ ,  $d$ , and  $e$  such that (a) the origin is a solution of the system and (b) the origin is not a solution of the system.

67. **Changing the Inequality Symbol** Sketch the graph of  $x + 2y < 6$ . Then describe how the graph of each inequality is different from your graph.

- (a)  $x + 2y \leq 6$  (b)  $x + 2y > 6$

## 9.2 Linear Programming Involving Two Variables

- Find a maximum or minimum of an objective function subject to a system of constraints.
- Find an optimal solution to a real-world linear programming problem.

### SOLVING A LINEAR PROGRAMMING PROBLEM

Many applications in business and economics involve a process called **optimization**, which is used to find such quantities as minimum cost, maximum profit, and minimum use of resources. In this section, you will study an optimization strategy called **linear programming**.

A two-dimensional linear programming problem consists of a linear **objective function** and a system of linear inequalities called **constraints**. The objective function gives the quantity that is to be maximized (or minimized), and the constraints determine the set of **feasible solutions**.

For the function

$$z = ax + by \quad \text{Objective function}$$

subject to the set of constraints that determines the region in Figure 9.3, every point in the region satisfies each constraint. So, it is not clear how to go about finding the point that yields a maximum or minimum value of  $z$ . Fortunately, it can be shown that when there is an optimal solution, it must occur at one of the vertices of the region. This means that *you can find the optimal value by testing  $z$  at each of the vertices*.

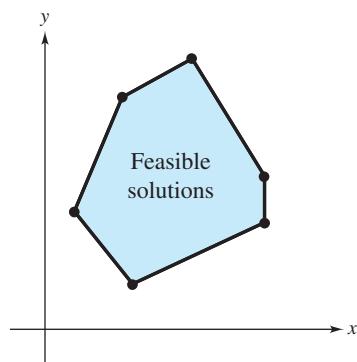


Figure 9.3

#### THEOREM 9.1 Optimal Solution of a Linear Programming Problem

If a linear programming problem has an optimal solution, then it must occur at a vertex of the set of feasible solutions. If the problem has more than one optimal solution, then at least one of them must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is unique.

A linear programming problem can include hundreds, and sometimes even thousands, of variables. However, in this section, you will solve linear programming problems that involve only two variables. The graphical method for solving a linear programming problem in two variables is outlined below.

#### Graphical Method for Solving a Linear Programming Problem

To solve a linear programming problem involving two variables by the graphical method, use the steps listed below.

1. Sketch the region corresponding to the system of constraints. (The points inside or on the boundary of the region are *feasible solutions*.)
2. Find the vertices of the region.
3. Test the objective function at each of the vertices and select the values of the variables that optimize the objective function. For a bounded region, both a minimum and maximum value will exist. (For an unbounded region, if an optimal solution exists, then it will occur at a vertex.)

**EXAMPLE 1****Solving a Linear Programming Problem**

Find the maximum value of

$$z = 3x + 2y \quad \text{Objective function}$$

subject to the constraints listed below.

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + 2y &\leq 4 \\ x - y &\leq 1 \end{aligned} \quad \text{Constraints}$$

**SOLUTION**

The constraints form the region shown in Figure 9.4. At the four vertices of this region, the objective function has the values listed below.

$$\text{At } (0, 0): z = 3(0) + 2(0) = 0$$

$$\text{At } (1, 0): z = 3(1) + 2(0) = 3$$

$$\text{At } (2, 1): z = 3(2) + 2(1) = 8 \quad \text{(Maximum value of } z\text{)}$$

$$\text{At } (0, 2): z = 3(0) + 2(2) = 4$$

So, the maximum value of  $z$  is 8, and this occurs when  $x = 2$  and  $y = 1$ .



In Example 1, test some of the interior points of the region. You will see that the corresponding values of  $z$  are less than 8. Here are some examples.

$$\text{At } (1, 1): z = 3(1) + 2(1) = 5$$

$$\text{At } \left(1, \frac{1}{2}\right): z = 3\left(1\right) + 2\left(\frac{1}{2}\right) = 4$$

$$\text{At } \left(\frac{3}{2}, 1\right): z = 3\left(\frac{3}{2}\right) + 2(1) = \frac{13}{2}$$

To see why the maximum value of the objective function in Example 1 must occur at a vertex, write the objective function in the form

$$y = -\frac{3}{2}x + \frac{z}{2}.$$

This equation represents a family of lines, each of slope  $-3/2$ . Of these infinitely many lines, you want the one that has the largest  $z$ -value, while still intersecting the region determined by the constraints. In other words, of all the lines whose slope is  $-3/2$ , you want the one that has the largest  $y$ -intercept *and* intersects the specified region, as shown below. Such a line will pass through one (or more) of the vertices of the region.

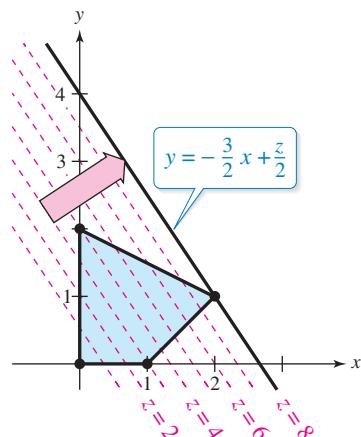


Figure 9.4

The graphical method for solving a linear programming problem works whether the objective function is to be maximized or minimized. The steps used are precisely the same in either case. After you have evaluated the objective function at the vertices of the set of feasible solutions, simply choose the largest value as the maximum and the smallest value as the minimum. For instance, the same test used in Example 1 to find the maximum value of  $z$  can be used to conclude that the minimum value of  $z$  is 0, and that this occurs at the vertex  $(0, 0)$ .

### EXAMPLE 2

### Solving a Linear Programming Problem

Find the maximum value of the objective function

$$z = 4x + 6y \quad \text{Objective function}$$

where  $x \geq 0$  and  $y \geq 0$ , subject to the constraints

$$\left. \begin{array}{l} -x + y \leq 11 \\ x + y \leq 27 \\ 2x + 5y \leq 90 \end{array} \right\} \quad \text{Constraints}$$

### SOLUTION

The region bounded by the constraints is shown in Figure 9.5. By testing the objective function at each vertex, you obtain

$$\text{At } (0, 0): z = 4(0) + 6(0) = 0$$

$$\text{At } (0, 11): z = 4(0) + 6(11) = 66$$

$$\text{At } (5, 16): z = 4(5) + 6(16) = 116 \quad (\text{Maximum value of } z)$$

$$\text{At } (15, 12): z = 4(15) + 6(12) = 132$$

$$\text{At } (27, 0): z = 4(27) + 6(0) = 108.$$

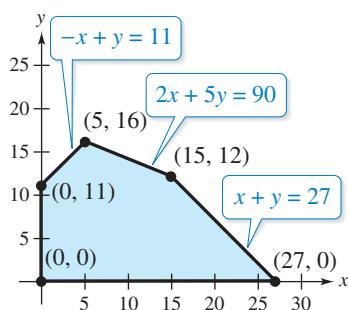


Figure 9.5

So, the maximum value of  $z$  is 132, and this occurs when  $x = 15$  and  $y = 12$ .

### EXAMPLE 3

### Minimizing an Objective Function

Find the minimum value of the objective function

$$z = 5x + 7y \quad \text{Objective function}$$

where  $x \geq 0$  and  $y \geq 0$ , subject to the constraints

$$\left. \begin{array}{l} 2x + 3y \geq 6 \\ 3x - y \leq 15 \\ -x + y \leq 4 \\ 2x + 5y \leq 27 \end{array} \right\} \quad \text{Constraints}$$

### SOLUTION

The region bounded by the constraints is shown in Figure 9.6. By testing the objective function at each vertex, you obtain

$$\text{At } (0, 2): z = 5(0) + 7(2) = 14 \quad (\text{Minimum value of } z)$$

$$\text{At } (0, 4): z = 5(0) + 7(4) = 28$$

$$\text{At } (1, 5): z = 5(1) + 7(5) = 40$$

$$\text{At } (6, 3): z = 5(6) + 7(3) = 51$$

$$\text{At } (5, 0): z = 5(5) + 7(0) = 25$$

$$\text{At } (3, 0): z = 5(3) + 7(0) = 15.$$

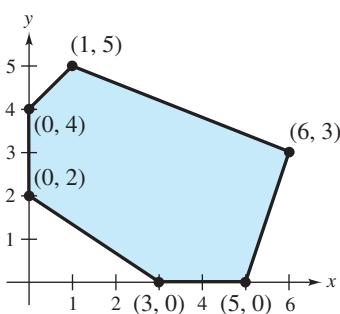


Figure 9.6

So, the minimum value of  $z$  is 14, and this occurs when  $x = 0$  and  $y = 2$ .

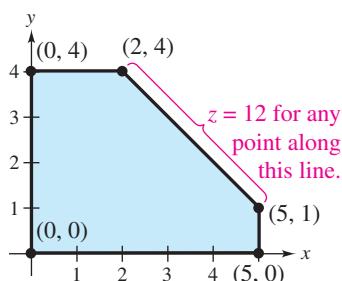


Figure 9.7

When solving a linear programming problem, it is possible that the maximum (or minimum) value occurs at *two* different vertices. For example, at the vertices of the region shown in Figure 9.7, the objective function

$$z = 2x + 2y \quad \text{Objective function}$$

has the values below.

$$\text{At } (0, 0): z = 2(0) + 2(0) = 0$$

$$\text{At } (0, 4): z = 2(0) + 2(4) = 8$$

$$\text{At } (2, 4): z = 2(2) + 2(4) = 12 \quad (\text{Maximum value of } z)$$

$$\text{At } (5, 1): z = 2(5) + 2(1) = 12 \quad (\text{Maximum value of } z)$$

$$\text{At } (5, 0): z = 2(5) + 2(0) = 10$$

In this case, the objective function has a maximum value of 12 not only at the vertices  $(2, 4)$  and  $(5, 1)$ , but at *any point on the line segment connecting these two vertices*.

Some linear programming problems have no optimal solution. This can occur when the region determined by the constraints is *unbounded*. Example 4 illustrates such a problem.

### EXAMPLE 4 An Unbounded Region

See [LarsonLinearAlgebra.com](http://LarsonLinearAlgebra.com) for an interactive version of this type of example.

Find the maximum value of

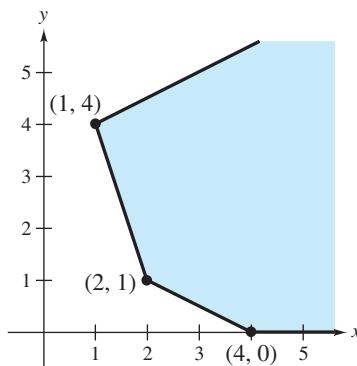
$$z = 4x + 2y \quad \text{Objective function}$$

where  $x \geq 0$  and  $y \geq 0$  subject to the constraints

$$\left. \begin{array}{l} x + 2y \geq 4 \\ 3x + y \geq 7 \\ -x + 2y \leq 7 \end{array} \right\} \quad \text{Constraints}$$

#### SOLUTION

The region determined by the constraints is shown below.



For this unbounded region, there is no maximum value of  $z$ . To see this, note that the point  $(x, 0)$  lies in the region for all values of  $x \geq 4$ . By choosing large values of  $x$ , you can obtain values of

$$\begin{aligned} z &= 4(x) + 2(0) \\ &= 4x \end{aligned}$$

that are large. So, there is no maximum value of  $z$ .

#### REMARK

For this problem, the objective function does have a minimum value,  $z = 10$ , which occurs at the vertex  $(2, 1)$ .



**APPLICATION****EXAMPLE 5****An Application: Optimal Cost**

Example 4 in Section 9.1 set up a system of linear equations for the problem below. The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Now, assume that dietary drink X costs \$0.12 per cup and drink Y costs \$0.15 per cup. How many cups of each drink should be consumed each day to minimize the cost and still meet the daily requirements?

**SOLUTION**

Begin by letting  $x$  be the number of cups of dietary drink X and  $y$  be the number of cups of dietary drink Y. Moreover, to meet the minimum daily requirements, the inequalities listed below must be satisfied.

$$\left. \begin{array}{l} \text{For calories: } 60x + 60y \geq 300 \\ \text{For vitamin A: } 12x + 6y \geq 36 \\ \text{For vitamin C: } 10x + 30y \geq 90 \\ x \geq 0 \\ y \geq 0 \end{array} \right\} \text{Constraints}$$

The cost  $C$  is

$$C = 0.12x + 0.15y. \quad \text{Objective function}$$

The graph of the region corresponding to the constraints is shown in Figure 9.8. To determine the minimum cost, test  $C$  at each vertex of the region, as shown below.

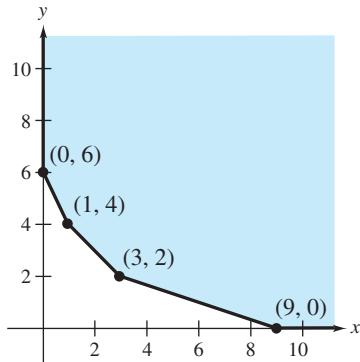
$$\text{At } (0, 6): C = 0.12(0) + 0.15(6) = 0.90$$

$$\text{At } (1, 4): C = 0.12(1) + 0.15(4) = 0.72$$

$$\text{At } (3, 2): C = 0.12(3) + 0.15(2) = 0.66 \quad \text{(Minimum value of } C\text{)}$$

$$\text{At } (9, 0): C = 0.12(9) + 0.15(0) = 1.08$$

So, the minimum cost is \$0.66 per day, and this occurs when three cups of drink X and two cups of drink Y are consumed each day. □



**Figure 9.8**


**LINEAR ALGEBRA APPLIED**

When financial institutions replenish automatic teller machines (ATMs), they need to take into account a large number of variables and constraints to keep the machines stocked appropriately. Demand for cash machines fluctuates with such factors as weather, economic conditions, day of the week, and even road construction. Further complicating the matter in the United States is a penalty for depositing and withdrawing money from the Federal Reserve in the same week. To address this complex problem, a company that specializes in providing financial services technology can create high-end optimization software to set up and solve a linear programming problem with many variables and constraints. The company determines an equation for the objective function to minimize total cash in ATMs, while establishing constraints on travel routes, service vehicles, penalty fees, and so on. The optimal solution generated by the software allows the company to build detailed ATM restocking schedules.

Marquis/Shutterstock.com

## 9.2 Exercises

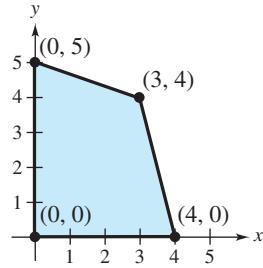
**Solving a Linear Programming Problem** In Exercises 1 and 2, find the minimum and maximum values of each objective function and where they occur, subject to the constraints.

1. Objective function:

- (a)  $z = 3x + 8y$   
(b)  $z = 5x + 0.5y$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\x + 3y &\leq 15 \\4x + y &\leq 16\end{aligned}$$

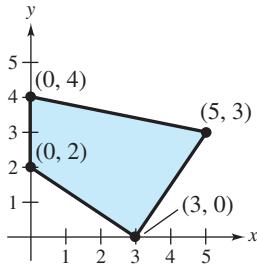


2. Objective function:

- (a)  $z = 4x + 3y$   
(b)  $z = x + 6y$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\2x + 3y &\geq 6 \\3x - 2y &\leq 9 \\x + 5y &\leq 20\end{aligned}$$



**Solving a Linear Programming Problem** In Exercises 3–8, sketch the region determined by the constraints. Then find the minimum and maximum values of each objective function (if possible) and where they occur, subject to the constraints.

3. Objective function:

- (a)  $z = 10x + 7y$   
(b)  $z = 25x + 30y$

Constraints:

$$\begin{aligned}x &\geq 0 \\x &\leq 60 \\y &\geq 0 \\y &\leq 45 \\5x + 6y &\leq 420\end{aligned}$$

5. Objective function:

- (a)  $z = 4x + 5y$   
(b)  $z = 2x - y$   
(c)  $z = -5x + y$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\2x + 2y &\leq 10 \\x + 2y &\leq 6 \\3x + 5y &\geq 30\end{aligned}$$

4. Objective function:

- (a)  $z = 50x + 35y$   
(b)  $z = 16x + 18y$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\8x + 9y &\leq 7200 \\8x + 9y &\geq 5400\end{aligned}$$

6. Objective function:

- (a)  $z = 4x + 5y$   
(b)  $z = 2x + 7y$   
(c)  $z = -x - 3y$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\4x + 3y &\geq 27 \\x + y &\geq 8 \\3x + 5y &\geq 30\end{aligned}$$

7. Objective function:

- (a)  $z = x$   
(b)  $z = y$   
(c)  $z = x + y$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\2x + 3y &\leq 60 \\2x + y &\leq 28 \\4x + y &\leq 48\end{aligned}$$

8. Objective function:

- (a)  $z = 4x + y$   
(b)  $z = x + 4y$   
(c)  $z = 4x + 4y$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\x + 2y &\leq 40 \\x + y &\geq 30 \\2x + 3y &\geq 72\end{aligned}$$

**Finding Minimum and Maximum Values** In Exercises 9–14, find the minimum and maximum values of the objective function and the points  $(x_1, x_2)$  where they occur, subject to the constraints  $3x_1 + x_2 \leq 15$  and  $4x_1 + 3x_2 \leq 30$ , where  $x_1, x_2 \geq 0$ .

9.  $z = 2x_1 + x_2$   
10.  $z = 5x_1 + x_2$   
11.  $z = x_1 + x_2$   
12.  $z = 3x_1 + x_2$   
13.  $z = x_1 + 5x_2$   
14.  $z = 4x_1 + 5x_2$

**Describing an Unusual Characteristic** In Exercises 15–20, the linear programming problem has an unusual characteristic. Sketch a graph of the solution region for the problem and describe the unusual characteristic. (In each problem, the objective function is to be maximized.)

15. Objective function:

$$z = 2.5x + y$$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\3x + 5y &\leq 15 \\5x + 2y &\leq 10\end{aligned}$$

16. Objective function:

$$z = x + y$$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\-x + y &\leq 1 \\-x + 2y &\leq 4\end{aligned}$$

17. Objective function:

$$z = -x + 2y$$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\x &\leq 10 \\x + y &\leq 7\end{aligned}$$

18. Objective function:

$$z = x + y$$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\-x + y &\leq 0 \\-3x + y &\geq 3\end{aligned}$$

19. Objective function:

$$z = 3x + 4y$$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\x + y &\leq 1 \\2x + y &\geq 4\end{aligned}$$

20. Objective function:

$$z = x + 2y$$

Constraints:

$$\begin{aligned}x &\geq 0 \\y &\geq 0 \\x + 2y &\leq 4 \\2x + y &\geq 4\end{aligned}$$

- 21. Optimal Profit** The costs to a store for two models of Global Positioning System (GPS) receivers are \$80 and \$100. The \$80 model yields a profit of \$25 and the \$100 model yields a profit of \$30. Market tests and available resources determined the constraints below.

- The merchant estimates that the total monthly demand will not exceed 200 units.
- The merchant does not want to invest more than \$18,000 in GPS receiver inventory.

What is the optimal inventory level for each model? What is the optimal profit?

- 22. Optimal Profit** A fruit grower has 150 acres of land for raising crops A and B. The profit is \$185 per acre for crop A and \$245 per acre for crop B. Research and available resources determined the constraints below.

- It takes 1 day to trim an acre of crop A and 2 days to trim an acre of crop B, and there are 240 days per year available for trimming.
- It takes 0.3 day to pick an acre of crop A and 0.1 day to pick an acre of crop B, and there are 30 days per year available for picking.

What is the optimal acreage for each fruit? What is the optimal profit?

- 23. Optimal Cost** A farming cooperative mixes two brands of cattle feed. Brand X costs \$30 per bag, and brand Y costs \$25 per bag. Research and available resources determined the constraints below.

- Brand X contains two units of nutritional element A, two units of element B, and two units of element C.
- Brand Y contains one unit of nutritional element A, nine units of element B, and three units of element C.
- The minimum requirements for nutrients A, B, and C are 12 units, 36 units, and 24 units, respectively.

What is the optimal number of bags of each brand that should be mixed? What is the optimal cost?

- 24. Optimal Cost** Rework Exercise 23 assuming that brand Y now costs \$32 per bag, and it now contains one unit of nutritional element A, twelve units of element B, and four units of element C.

- 25. Optimal Revenue** An accounting firm charges \$2500 for an audit and \$350 for a tax return. The table shows the times (in hours) required for staffing and reviewing.

Component	Audit	Tax Return
Staffing	75	12.5
Reviewing	10	2.5

The firm has 900 hours of staff time and 155 hours of review time available each week. What numbers of audits and tax returns will bring in an optimal revenue?

- 26. Optimal Revenue** The accounting firm in Exercise 25 lowers its charge for an audit to \$2000. What numbers of audits and tax returns will bring in an optimal revenue?

- 27. Determine**, for each vertex, values of  $t$  such that the objective function has a maximum value at the vertex (if possible).

Objective function:

$$z = 3x + ty$$

Constraints:

$$x \geq 0$$

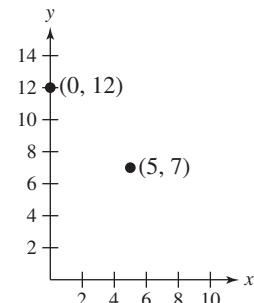
$$y \geq 0$$

$$7x + 14y \leq 84$$

$$5x + 4y \leq 30$$

- (0, 6)
- (2, 5)
- (6, 0)
- (0, 0)

- 28. CAPSTONE** A company determining an optimal profit finds that the objective function has a maximum value at the vertices shown in the graph.



- Can you conclude that it also has a maximum value at the point (3, 9)? Explain.
- Can you conclude that it also has a maximum value at the point (6, 6)? Explain.
- Find two additional points that maximize the objective function.

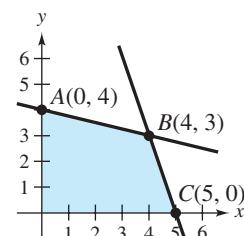
**Finding an Objective Function** In Exercises 29–32, find an objective function that has a maximum or minimum value at the specified vertex of the constraint region shown below. (There are many correct answers.)

29. Maximum at vertex A

30. Maximum at vertex B

31. Maximum at vertex C

32. Minimum at vertex C



## 9.3 The Simplex Method: Maximization

- Write the simplex tableau for a linear programming problem.
- Use pivoting to find an improved solution.
- Use the simplex method to solve a linear programming problem that maximizes an objective function.
- Use the simplex method to find an optimal solution to a real-world application.

### THE SIMPLEX TABLEAU

For linear programming problems involving two variables, the graphical solution method introduced in Section 9.2 is convenient. For problems involving more than two variables or large numbers of constraints, it is better to use methods that are adaptable to technology. One such method is the **simplex method**, developed by George Dantzig in 1946. It provides a systematic way of examining the vertices of the feasible region to determine the optimal value of the objective function.

Say you want to find the maximum value of  $z = 4x_1 + 6x_2$ , where the **decision variables**  $x_1$  and  $x_2$  are nonnegative, subject to the constraints

$$\begin{aligned} -x_1 + x_2 &\leq 11 \\ x_1 + x_2 &\leq 27 \\ 2x_1 + 5x_2 &\leq 90. \end{aligned}$$

The left-hand side of each *inequality* is less than or equal to the right-hand side, so there must exist nonnegative numbers  $s_1$ ,  $s_2$ , and  $s_3$  that can be added to the left side of each equation to produce the system of linear *equations*

$$\begin{aligned} -x_1 + x_2 + s_1 &= 11 \\ x_1 + x_2 + s_2 &= 27 \\ 2x_1 + 5x_2 + s_3 &= 90. \end{aligned}$$

The numbers  $s_1$ ,  $s_2$ , and  $s_3$  are called **slack variables** because they represent the “slack” in each inequality.

#### REMARK

Note that for a linear programming problem in standard form, the objective function is to be maximized, not minimized. (Minimization problems are discussed in Sections 9.4 and 9.5.)

### Standard Form of a Linear Programming Problem

A linear programming problem is in **standard form** when it seeks to *maximize* the objective function  $z = c_1x_1 + c_2x_2 + \dots + c_nx_n$  subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \end{aligned}$$

where  $x_i \geq 0$  and  $b_i \geq 0$ . After adding slack variables, the corresponding system of **constraint equations** is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + s_1 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + s_2 &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n + s_m &= b_m \end{aligned}$$

where  $s_i \geq 0$ .

A **basic solution** of a linear programming problem in standard form is a solution  $(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m)$  of the constraint equations in which *at most*  $m$  variables are nonzero, and the variables that are nonzero are called **basic variables**. A basic solution for which all variables are nonnegative is a **basic feasible solution**.

The simplex method is carried out by performing elementary row operations on a matrix called the **simplex tableau**. This tableau consists of the augmented matrix corresponding to the constraint equations together with the coefficients of the objective function written in the form

$$-c_1x_1 - c_2x_2 - \dots - c_nx_n + (0)s_1 + (0)s_2 + \dots + (0)s_m + z = 0.$$

In the tableau, it is customary to omit the coefficient of  $z$ . For example, the simplex tableau for the linear programming problem

$$\begin{array}{rcl} z = 4x_1 + 6x_2 & & \text{Objective function} \\ -x_1 + x_2 + s_1 & = 11 \\ x_1 + x_2 + s_2 & = 27 \\ 2x_1 + 5x_2 + s_3 & = 90 \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Constraints}$$

is shown below.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	1	1	0	0	11	$s_1$
1	1	0	1	0	27	$s_2$
2	5	0	0	1	90	$s_3$
-4	-6	0	0	0	0	↑ Current $z$ -value

For this **initial simplex tableau**, the **basic variables** are  $s_1, s_2$ , and  $s_3$ , and the **nonbasic variables** are  $x_1$  and  $x_2$ . Note that the basic variables are labeled to the right of the simplex tableau next to the appropriate rows. This technique is important as you proceed through the simplex method. It helps keep track of the changing basic variables, as shown in Example 1.

$x_1$  and  $x_2$  are the nonbasic variables in this initial tableau, so they have an initial value of zero, yielding a current  $z$ -value of zero. From the columns that are farthest to the right, the basic variables have initial values of  $s_1 = 11$ ,  $s_2 = 27$ , and  $s_3 = 90$ . So the current solution is

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ s_1 &= 11 \\ s_2 &= 27 \\ s_3 &= 90. \end{aligned}$$

This solution is a basic feasible solution and is often written as

$$(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90).$$

The entry in the lower right corner of the simplex tableau is the current value of  $z$ . Note that the bottom-row entries under  $x_1$  and  $x_2$  are the negatives of the coefficients of  $x_1$  and  $x_2$  in the objective function

$$z = 4x_1 + 6x_2.$$

To perform an **optimality check** for a solution represented by a simplex tableau, look at the entries in the bottom row of the tableau. If any of these entries are negative (as above), then the current solution is *not* optimal.

## PIVOTING

After you have set up the initial simplex tableau for a linear programming problem, the simplex method consists of checking for optimality and then, when the current solution is not optimal, improving the current solution. (An improved solution is one that has a larger  $z$ -value than the current solution.) To improve the current solution, bring a new basic variable into the solution, the **entering variable**. This implies that one of the current basic variables (the **departing variable**) must leave, otherwise you would have too many variables for a basic solution. Choose the entering and departing variables as listed below.

1. The **entering variable** corresponds to the least (the most negative) entry in the bottom row of the tableau, excluding the “ $b$ -column.”
2. The **departing variable** corresponds to the least nonnegative ratio  $b_i/a_{ij}$  in the column determined by the entering variable, when  $a_{ij} > 0$ .
3. The entry in the simplex tableau in the entering variable’s column and the departing variable’s row is the **pivot**.

Finally, to form the improved solution, apply Gauss-Jordan elimination to the column that contains the pivot, as illustrated in Example 1. (This process is called **pivotting**.)

### EXAMPLE 1

#### Pivotting to Find an Improved Solution

Use the simplex method to find an improved solution for the linear programming problem represented by the tableau shown below.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	1	1	0	0	11	$s_1$
1	1	0	1	0	27	$s_2$
2	5	0	0	1	90	$s_3$
-4	-6	0	0	0	0	

### SOLUTION

The objective function for this problem is

$$z = 4x_1 + 6x_2.$$

Note that the current solution

$$(x_1 = 0, x_2 = 0, s_1 = 11, s_2 = 27, s_3 = 90)$$

corresponds to a  $z$ -value of 0. To improve this solution, choose  $x_2$  as the entering variable, because  $-6$  is the least entry in the bottom row.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	1	1	0	0	11	$s_1$
1	1	0	1	0	27	$s_2$
2	5	0	0	1	90	$s_3$
-4	-6	0	0	0	0	

↑  
Entering

To see *why* you choose  $x_2$  as the entering variable, remember that  $z = 4x_1 + 6x_2$ . So, it appears that a unit change in  $x_2$  produces a change of 6 in  $z$ , whereas a unit change in  $x_1$  produces a change of only 4 in  $z$ .

**REMARK**

In the event of a tie when choosing entering and/or departing variables, any of the tied variables may be chosen.

To find the departing variable, locate the  $b_i$ 's that have corresponding positive elements in the entering variable's column and form the ratios

$$\frac{11}{1} = 11, \quad \frac{27}{1} = 27, \quad \text{and} \quad \frac{90}{5} = 18.$$

Here the least nonnegative ratio is 11, so choose  $s_1$  as the departing variable.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	(1)	1	0	0	11	$s_1 \leftarrow$ Departing
1	1	0	1	0	27	$s_2$
2	5	0	0	1	90	$s_3$
-4	-6	0	0	0	0	
		↑ Entering				

Note that the pivot is the entry in the first row and second column. Now, use Gauss-Jordan elimination to obtain the improved solution shown below.

$$\begin{array}{c} \text{Before Pivoting} \\ \left[ \begin{array}{cccccc} -1 & 1 & 1 & 0 & 0 & 11 \\ 1 & 1 & 0 & 1 & 0 & 27 \\ 2 & 5 & 0 & 0 & 1 & 90 \\ -4 & -6 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{array}{c} \text{After Pivoting} \\ \left[ \begin{array}{cccccc} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{array} \right] \end{array} \end{array} \begin{array}{l} -R_1 + R_2 \\ -5R_1 + R_3 \\ 6R_1 + R_4 \end{array}$$

The new tableau is shown below.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	1	1	0	0	11	$x_2$
2	0	-1	1	0	16	$s_2$
7	0	-5	0	1	35	$s_3$
-10	0	6	0	0	66	

Note that  $x_2$  has replaced  $s_1$  in the basic variables column and the improved solution

$$(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$$

has a  $z$ -value of

$$z = 4x_1 + 6x_2 = 4(0) + 6(11) = 66.$$



In Example 1, the improved solution is not optimal because the bottom row has a negative entry. So, apply another iteration of the simplex method to improve the solution further. Choose  $x_1$  as the entering variable. Moreover, the lesser of the ratios  $16/2 = 8$  and  $35/7 = 5$  is 5, so  $s_3$  is the departing variable. Gauss-Jordan elimination produces the matrices shown below.

$$\begin{array}{c} \left[ \begin{array}{cccccc} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 7 & 0 & -5 & 0 & 1 & 35 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{array} \right] \rightarrow \left[ \begin{array}{cccccc} -1 & 1 & 1 & 0 & 0 & 11 \\ 2 & 0 & -1 & 1 & 0 & 16 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ -10 & 0 & 6 & 0 & 0 & 66 \end{array} \right] \begin{array}{l} \frac{1}{7}R_3 \\ R_1 + R_3 \\ R_2 - 2R_3 \\ R_4 + 10R_3 \end{array} \\ \rightarrow \left[ \begin{array}{cccccc} 0 & 1 & \frac{2}{7} & 0 & \frac{1}{7} & 16 \\ 0 & 0 & \frac{3}{7} & 1 & \frac{2}{7} & 6 \\ 1 & 0 & -\frac{5}{7} & 0 & \frac{1}{7} & 5 \\ 0 & 0 & -\frac{8}{7} & 0 & \frac{10}{7} & 116 \end{array} \right] \end{array}$$

So, the new simplex tableau is as shown below.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	$x_2$
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	$s_2$
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	$x_1$
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

In this tableau, there is still a negative entry in the bottom row. So, choose  $s_1$  as the entering variable and  $s_2$  as the departing variable, as shown in the next tableau.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	$x_2$
0	0	$(\frac{3}{7})$	1	$-\frac{2}{7}$	6	$s_2$ ← Departing
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	$x_1$
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

↑  
Entering

One more iteration of the simplex method gives the tableau below. (Check this.)

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	$x_2$
0	0	1	$\frac{7}{3}$	$-\frac{2}{3}$	14	$s_1$
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	$x_1$
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	← Maximum $z$ -value

In this tableau, there are no negative elements in the bottom row. So, the optimal solution is

$$(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$$

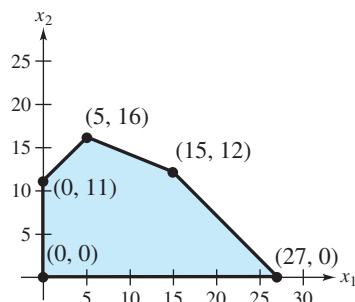
with

$$z = 4x_1 + 6x_2 = 4(15) + 6(12) = 132.$$

The linear programming problem in Example 1 involved only two decision variables,  $x_1$  and  $x_2$ , so you could have used a graphical solution technique, as in Section 9.2, Example 2. Notice in the figure below that each iteration in the simplex method corresponds to moving from one vertex to an adjacent vertex with an improved  $z$ -value.

$$(0, 0) \rightarrow (0, 11) \rightarrow (5, 16) \rightarrow (15, 12) \rightarrow (15, 12)$$

$z = 0 \quad z = 66 \quad z = 116 \quad z = 132$



## THE SIMPLEX METHOD

Here is a summary of the steps involved in the simplex method.

### The Simplex Method (Standard Form)

To solve a linear programming problem in standard form, use the steps below.

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Locate the most negative entry in the bottom row, excluding the “ $b$ -column.” This entry is called the entering variable, and its column is the **entering column**. (If ties occur, then any of the tied entries can be used to determine the entering column.)
4. Form the ratios of the entries in the “ $b$ -column” with their corresponding positive entries in the entering column. (If all entries in the entering column are 0 or negative, then there is no maximum solution.) The **departing row** corresponds to the least nonnegative ratio  $b_i/a_{ij}$ . (For ties, choose any corresponding row.) The entry in the departing row and the entering column is called the **pivot**.
5. Use elementary row operations to change the pivot to 1 and all other entries in the entering column to 0. This process is called **pivoting**.
6. When all entries in the bottom row are zero or positive, this is the final tableau. Otherwise, go back to Step 3.
7. If you obtain a final tableau, then the linear programming problem has a maximum solution. The maximum value of the objective function is the entry in the lower right corner of the tableau.

Note that the basic feasible solution of an initial simplex tableau is

$$(x_1, x_2, \dots, x_n, s_1, s_2, \dots, s_m) = (0, 0, \dots, 0, b_1, b_2, \dots, b_m).$$

This solution is basic because at most  $m$  variables are nonzero (namely, the slack variables). It is feasible because each variable is nonnegative.

The next two examples illustrate the use of the simplex method to solve a problem involving three decision variables.



### LINEAR ALGEBRA APPLIED

There are many commercially available optimization software packages to aid operations researchers in such areas as allocating resources to maximize profits and minimize costs. Many of these packages use the simplex method as their foundation. Also, free online tools are available that enable you to check your work in this chapter. Many of these are user-friendly interfaces that use the simplex method as well as the graphical method to solve linear programming problems. These freeware programs show the steps in the solution, which can be very helpful in the learning process. Use online optimization freeware to check the solution of Example 1 in this section and to check the solution of Example 2 in Section 9.2. As you will see in the next section, the simplex method can also be applied to minimization problems. In addition to optimizing the objective function, commercially available software packages often contain built-in sensitivity reporting to analyze how changes or errors in the data will affect the outcome.

**EXAMPLE 2****The Simplex Method with Three Decision Variables**

See [LarsonLinearAlgebra.com](http://LarsonLinearAlgebra.com) for an interactive version of this type of example.

Use the simplex method to find the maximum value of

$$z = 2x_1 - x_2 + 2x_3 \quad \text{Objective function}$$

subject to the constraints

$$2x_1 + x_2 \leq 10$$

$$x_1 + 2x_2 - 2x_3 \leq 20$$

$$x_2 + 2x_3 \leq 5$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ .

**SOLUTION**

Using the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 10, 20, 5)$$

the initial and subsequent simplex tableaus for this problem are shown below. (Check the computations, and note the “tie” that occurs when choosing the first entering variable.)

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
2	1	0	1	0	0	10	$s_1$
1	2	-2	0	1	0	20	$s_2$
0	1	(2)	0	0	1	5	$s_3$ ← Departing
-2	1	-2	0	0	0	0	
↑ Entering							

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
(2)	1	0	1	0	0	10	$s_1$ ← Departing
1	3	0	0	1	1	25	$s_2$
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	$x_3$
-2	2	0	0	0	1	5	
↑ Entering							

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	5	$x_1$
0	$\frac{5}{2}$	0	$-\frac{1}{2}$	1	1	20	$s_2$
0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$\frac{5}{2}$	$x_3$
0	3	0	1	0	1	15	

This implies that the optimal solution is

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (5, 0, \frac{5}{2}, 0, 20, 0)$$

and the maximum value of  $z$  is 15.

Note that  $s_2 = 20$ . The optimal solution yields a maximum value of  $z = 15$  provided that  $x_1 = 5$ ,  $x_2 = 0$ , and  $x_3 = \frac{5}{2}$ . Check that these values satisfy the constraints giving equality in the first and third constraints, yet the second constraint has a slack of 20.



Occasionally, the constraints in a linear programming problem will include an equation. In such cases, add a “slack variable” called an **artificial variable** to form the initial simplex tableau. Technically, this new variable is not a slack variable (because there is no slack to be taken). Once you have determined an optimal solution in such a problem, check that any equations in the original constraints are satisfied. Example 3 illustrates such a case.

### EXAMPLE 3

### The Simplex Method with Three Decision Variables

Use the simplex method to find the maximum value of

$$z = 3x_1 + 2x_2 + x_3 \quad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 4x_1 + x_2 + x_3 &= 30 \\ 2x_1 + 3x_2 + x_3 &\leq 60 \\ x_1 + 2x_2 + 3x_3 &\leq 40 \end{aligned}$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ .

#### SOLUTION

Using the basic feasible solution  $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 30, 60, 40)$ , the initial and subsequent simplex tableaus for this problem are shown below. (Note that  $s_1$  is an artificial variable, rather than a slack variable.)

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
4	1	1	1	0	0	30	$s_1 \leftarrow$ Departing
2	3	1	0	1	0	60	$s_2$
1	2	3	0	0	1	40	$s_3$
-3	-2	-1	0	0	0	0	

↑  
Entering

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	0	0	$\frac{15}{2}$	$x_1$
0	$\frac{5}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	45	$s_2 \leftarrow$ Departing
0	$\frac{7}{4}$	$\frac{11}{4}$	$-\frac{1}{4}$	0	1	$\frac{65}{2}$	$s_3$
0	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$	0	0	$\frac{45}{2}$	

↑  
Entering

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	0	$\frac{1}{5}$	$\frac{3}{10}$	$-\frac{1}{10}$	0	3	$x_1$
0	1	$\frac{1}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	0	18	$x_2$
0	0	$\frac{12}{5}$	$\frac{1}{10}$	$-\frac{7}{10}$	1	1	$s_3$
0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	45	

So the optimal solution is  $(x_1, x_2, x_3, s_1, s_2, s_3) = (3, 18, 0, 0, 0, 1)$  and the maximum value of  $z$  is 45. This solution satisfies the equation provided in the constraints, because  $4(3) + 1(18) + 1(0) = 30$ .



## APPLICATIONS

Example 4 shows how to use the simplex method to maximize profits in a business application.

### EXAMPLE 4

### A Business Application: Maximum Profit

A manufacturer produces three types of plastic fixtures. The table below shows the times required for molding, trimming, and packaging. (Times are in hours per dozen fixtures, and profits are in dollars per dozen fixtures.)

Process	Type A	Type B	Type C
<i>Molding</i>	1	2	$\frac{3}{2}$
<i>Trimming</i>	$\frac{2}{3}$	$\frac{2}{3}$	1
<i>Packaging</i>	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$
<i>Profit</i>	\$11	\$16	\$15

The maximum amounts of production time that the manufacturer can allocate to each component are listed below.

Molding: 12,000 hours

Trimming: 4600 hours

Packaging: 2400 hours

How many dozen units of each type of fixture should the manufacturer produce to obtain a maximum profit?

### SOLUTION

Let  $x_1$ ,  $x_2$ , and  $x_3$  represent the numbers of dozens of types A, B, and C fixtures, respectively. The objective function to be maximized is

$$\text{Profit} = 11x_1 + 16x_2 + 15x_3$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ . Moreover, using the information in the table, you can write the constraints below.

$$x_1 + 2x_2 + \frac{3}{2}x_3 \leq 12,000$$

$$\frac{2}{3}x_1 + \frac{2}{3}x_2 + x_3 \leq 4600$$

$$\frac{1}{2}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3 \leq 2400$$

So, the initial simplex tableau with the basic feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 12,000, 4600, 2400)$$

is as shown below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	(2)	$\frac{3}{2}$	1	0	0	12,000	$s_1 \leftarrow$ Departing
$\frac{2}{3}$	$\frac{2}{3}$	1	0	1	0	4600	$s_2$
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	0	0	1	2400	$s_3$
-11	-16	-15	0	0	0	0	
↑ Entering							

Now, to this initial tableau, apply the simplex method as shown on the next page.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$	0	0	6000	$x_2$
$\frac{1}{3}$	0	$\frac{1}{2}$	$-\frac{1}{3}$	1	0	600	$s_2$
$(\frac{1}{3})$	0	$\frac{1}{4}$	$-\frac{1}{6}$	0	1	400	$s_3$ ← Departing
-3	0	-3	8	0	0	96,000	

↑  
Entering

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
0	1	$\frac{3}{8}$	$\frac{3}{4}$	0	$-\frac{3}{2}$	5400	$x_2$
0	0	$(\frac{1}{4})$	$-\frac{1}{6}$	1	-1	200	$s_2$ ← Departing
1	0	$\frac{3}{4}$	$-\frac{1}{2}$	0	3	1200	$x_1$
0	0	$-\frac{3}{4}$	$\frac{13}{2}$	0	9	99,600	

↑  
Entering

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
0	1	0	1	$-\frac{3}{2}$	0	5100	$x_2$
0	0	1	$-\frac{2}{3}$	4	-4	800	$x_3$
1	0	0	0	-3	6	600	$x_1$
0	0	0	6	3	6	100,200	

So the maximum profit is \$100,200, obtained by producing 600 dozen units of Type A, 5100 dozen units of Type B, and 800 dozen units of Type C. ■

In Example 4, note the “tie” that occurs when choosing the second entering variable. Verify that choosing  $x_3$  instead of  $x_1$  as the second entering variable results in the two intermediate simplex tableaus below, and the same final tableau (and optimal solution) shown in Example 4.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
$\frac{1}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$	0	0	6000	$x_2$
$\frac{1}{3}$	0	$(\frac{1}{4})$	$-\frac{1}{3}$	1	0	600	$s_2$ ← Departing
$\frac{1}{3}$	0	$\frac{1}{4}$	$-\frac{1}{6}$	0	1	400	$s_3$
-3	0	-3	8	0	0	96,000	

↑  
Entering

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
0	1	0	1	$-\frac{3}{2}$	0	5100	$x_2$
$\frac{2}{3}$	0	1	$-\frac{2}{3}$	2	0	1200	$x_3$
$(\frac{1}{6})$	0	0	0	$-\frac{1}{2}$	1	100	$s_3$ ← Departing
-1	0	0	6	6	0	99,600	

↑  
Entering

Example 5 shows how to use the simplex method to maximize the audience in an advertising campaign.

### EXAMPLE 5

### A Business Application: Media Selection

Advertising alternatives for a company include television, radio, and newspaper. The table below shows the costs and estimates of audience coverage for each type of media.

	<i>Cost per advertisement</i>	<i>Audience per advertisement</i>
<i>Television</i>	\$2000	100,000
<i>Newspaper</i>	\$600	40,000
<i>Radio</i>	\$300	18,000

The newspaper limits the number of weekly advertisements from a single company to ten. Moreover, to balance the advertising among the three types of media, no more than half of the total number of advertisements should occur on the radio, and at least 10% should occur on television. The weekly advertising budget is \$18,200. How many advertisements should run in each of the three types of media to maximize the total audience?

#### SOLUTION

Let  $x_1$ ,  $x_2$ , and  $x_3$  represent the numbers of advertisements on television, in the newspaper, and on the radio, respectively. The objective function to be maximized is

$$z = 100,000x_1 + 40,000x_2 + 18,000x_3 \quad \text{Objective function}$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ . The constraints for this problem are shown below.

$$\begin{aligned} 2000x_1 + 600x_2 + 300x_3 &\leq 18,200 \\ x_2 &\leq 10 \\ x_3 &\leq 0.5(x_1 + x_2 + x_3) \\ x_1 &\geq 0.1(x_1 + x_2 + x_3) \end{aligned}$$

Here is a more manageable form of this system of constraints.

$$\left. \begin{aligned} 20x_1 + 6x_2 + 3x_3 &\leq 182 \\ x_2 &\leq 10 \\ -x_1 - x_2 + x_3 &\leq 0 \\ -9x_1 + x_2 + x_3 &\leq 0 \end{aligned} \right\} \quad \text{Constraints}$$

So, the initial simplex tableau is below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
$\frac{20}{6}$	6	3	1	0	0	0	182	$s_1 \leftarrow \text{Departing}$
0	1	0	0	1	0	0	10	$s_2$
-1	-1	1	0	0	1	0	0	$s_3$
-9	1	1	0	0	0	1	0	$s_4$
-100,000	-40,000	-18,000	0	0	0	0	0	

↑  
Entering

Now, to this initial tableau, apply the simplex method, as shown below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
1	$\frac{3}{10}$	$\frac{3}{20}$	$\frac{1}{20}$	0	0	0	$\frac{91}{10}$	$x_1$
0	$\frac{1}{10}$	0	0	1	0	0	10	$s_2 \leftarrow$ Departing
0	$-\frac{7}{10}$	$\frac{23}{20}$	$\frac{1}{20}$	0	1	0	$\frac{91}{10}$	$s_3$
0	$\frac{37}{10}$	$\frac{47}{20}$	$\frac{9}{20}$	0	0	1	$\frac{819}{10}$	$s_4$
0	-10,000	-3000	5000	0	0	0	910,000	
↑ Entering								

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
1	0	$\frac{3}{20}$	$\frac{1}{20}$	$-\frac{3}{10}$	0	0	$\frac{61}{10}$	$x_1$
0	1	0	0	1	0	0	10	$x_2$
0	0	$\frac{23}{20}$	$\frac{1}{20}$	$\frac{7}{10}$	1	0	$\frac{161}{10}$	$s_3 \leftarrow$ Departing
0	0	$\frac{47}{20}$	$\frac{9}{20}$	$-\frac{37}{10}$	0	1	$\frac{449}{10}$	$s_4$
0	0	-3000	5000	10,000	0	0	1,010,000	
↑ Entering								

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
1	0	0	$\frac{1}{23}$	$-\frac{9}{23}$	$-\frac{3}{23}$	0	4	$x_1$
0	1	0	0	1	0	0	10	$x_2$
0	0	1	$\frac{1}{23}$	$\frac{14}{23}$	$\frac{20}{23}$	0	14	$x_3$
0	0	0	$\frac{8}{23}$	$-\frac{118}{23}$	$-\frac{47}{23}$	1	12	$s_4$
0	0	0	$\frac{118,000}{23}$	$\frac{272,000}{23}$	$\frac{60,000}{23}$	0	1,052,000	

From this final simplex tableau, the maximum weekly audience for an advertising budget of \$18,200 is

$$z = 1,052,000 \quad \text{Maximum weekly audience}$$

which occurs when

$$x_1 = 4$$

$$x_2 = 10$$

$$x_3 = 14.$$

The table below summarizes the results.

Media	Number of advertisements	Cost	Audience
Television	4	\$8000	400,000
Newspaper	10	\$6000	400,000
Radio	14	\$4200	252,000
<b>Total</b>	28	\$18,200	1,052,000



## 9.3 Exercises

**Standard Form** In Exercises 1–4, explain why the linear programming problem is not in standard form.

1. (Minimize) Objective function:

$$z = x_1 + x_2$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

2. (Maximize) Objective function:

$$z = x_1 + x_2$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\leq 6 \\ 2x_1 - x_2 &\leq -1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. (Maximize) Objective function:

$$z = x_1 + x_2$$

Constraints:

$$\begin{aligned} x_1 + x_2 + 3x_3 &\leq 5 \\ 2x_1 - 2x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

4. (Maximize) Objective function:

$$z = x_1 + x_2$$

Constraints:

$$\begin{aligned} x_1 + x_2 &\geq 4 \\ 2x_1 + x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

**Writing a Simplex Tableau** In Exercises 5–8, write the simplex tableau for the linear programming problem. You do not need to solve the problem.

5. (Maximize) Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 8 \\ x_1 + x_2 &\leq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

6. (Maximize) Objective function:

$$z = x_1 + 3x_2$$

Constraints:

$$\begin{aligned} x_1 + x_2 &\leq 4 \\ x_1 - x_2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

7. (Maximize) Objective function:

$$z = 2x_1 + 3x_2 + 4x_3$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\leq 12 \\ x_1 + x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

8. (Maximize) Objective function:

$$z = 6x_1 - 9x_2$$

Constraints:

$$\begin{aligned} 2x_1 - 3x_2 &\leq 6 \\ x_1 + x_2 &\leq 20 \\ x_1, x_2 &\geq 0 \end{aligned}$$

**Pivoting to Find an Improved Solution** In Exercises 9 and 10, use one iteration of pivoting to find an improved solution for the linear programming problem represented by the tableau.

9.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-10	3	1	0	0	32	$s_1$
2	-2	0	1	0	16	$s_2$
3	12	0	0	1	28	$s_3$
-5	-2	0	0	0	0	

10.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
9	15	1	0	0	750	$s_1$
15	-10	0	1	0	60	$s_2$
-12	50	0	0	1	3000	$s_3$
-45	-78	0	0	0	0	

**Using the Simplex Method** In Exercises 11–24, use the simplex method to maximize the objective function, subject to the constraints.

11. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$\begin{aligned} x_1 + 4x_2 &\leq 8 \\ x_1 + x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

12. Objective function:

$$z = x_1 + x_2$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\leq 6 \\ 3x_1 + 2x_1 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

13. Objective function:

$$z = 3x_1 + 2x_2$$

Constraints:

$$\begin{aligned} x_1 + 3x_2 &\leq 15 \\ 4x_1 + x_2 &\leq 16 \\ x_1, x_2 &\geq 0 \end{aligned}$$

14. Objective function:

$$z = 4x_1 + 5x_2$$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 &\leq 10 \\ x_1 + 2x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

15. Objective function:

$$z = 10x_1 + 7x_2$$

Constraints:

$$\begin{aligned} x_1 &\leq 60 \\ x_2 &\leq 45 \\ 5x_1 + 6x_2 &\leq 420 \\ x_1, x_2 &\geq 0 \end{aligned}$$

16. Objective function:

$$z = 25x_1 + 35x_2$$

Constraints:

$$\begin{aligned} 18x_1 + 9x_2 &\leq 7200 \\ 8x_1 + 9x_2 &\leq 3600 \\ x_1, x_2 &\geq 0 \end{aligned}$$

17. Objective function:

$$z = 4x_1 + 5x_2$$

Constraints:

$$\begin{aligned} x_1 + x_2 &\leq 10 \\ 3x_1 + 7x_2 &\leq 42 \\ -3x_1 + 7x_2 &\leq 28 \\ x_1, x_2 &\geq 0 \end{aligned}$$

18. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$\begin{aligned} x_1 + 3x_2 &\leq 15 \\ 2x_1 - x_2 &\leq 12 \\ -4x_1 + 3x_2 &\leq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

19. Objective function:

$$z = 5x_1 + 2x_2 + 8x_3$$

Constraints:

$$\begin{aligned} 2x_1 - 4x_2 + x_3 &\leq 42 \\ 2x_1 + 3x_2 - x_3 &\leq 42 \\ 6x_1 - x_2 + 3x_3 &\leq 42 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

20. Objective function:

$$z = x_1 - x_2 + 2x_3$$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 &\leq 8 \\ x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

21. Objective function:

$$z = x_1 - x_2 + x_3$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 - 3x_3 &\leq 40 \\ x_1 + x_3 &\leq 25 \\ 2x_2 + 3x_3 &\leq 32 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

22. Objective function:

$$z = 3x_1 + 4x_2 + x_3 + 7x_4$$

Constraints:

$$\begin{aligned} 8x_1 + 3x_2 + 4x_3 + x_4 &\leq 7 \\ 2x_1 + 6x_2 + x_3 + 5x_4 &\leq 3 \\ x_1 + 4x_2 + 5x_3 + 2x_4 &\leq 8 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

23. Objective function:

$$z = x_1 + 2x_2 - x_4$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\leq 24 \\ 3x_2 + 7x_3 + x_4 &\leq 42 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

24. Objective function:

$$z = x_1 + 2x_2 + x_3 - x_4$$

Constraints:

$$\begin{aligned} x_1 + x_2 + 3x_3 + 4x_4 &\leq 60 \\ x_2 + 2x_3 + 5x_4 &\leq 50 \\ 2x_1 + 3x_2 + 6x_4 &\leq 72 \\ 4x_1 - x_3 + 3x_4 &\leq 48 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

**Using Artificial Variables** In Exercises 25 and 26, use an artificial variable to solve the linear programming problem, and check your solution.

25. Objective function:

$$z = 10x_1 + 5x_2 + 12x_3$$

Constraints:

$$\begin{aligned} -2x_1 + x_2 + 2x_3 &\leq 80 \\ x_1 - x_3 &= 35 \\ -x_2 + 3x_3 &\leq 60 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

26. Objective function:

$$z = 9x_1 + x_2 + 2x_3$$

Constraints:

$$\begin{aligned} 2x_1 + 6x_2 &= 8 \\ 4x_1 + 2x_3 &\leq 21 \\ -2x_1 - 3x_2 + x_3 &\leq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

27. **Optimal Revenue** An accounting firm charges \$2000 for an audit and \$300 for a tax return. The table shows the times (in hours) required for staffing and reviewing.

Component	Audit	Tax Return
Staffing	100	12.5
Reviewing	10	2.5

The firm has 900 hours of staff time and 100 hours of review time available each week. Use the simplex method to find the numbers of audits and tax returns that will bring in a maximum revenue.

28. **Optimal Revenue** The accounting firm in Exercise 27 raises its charge for an audit to \$2500. Use the simplex method to find the numbers of audits and tax returns that will bring in a maximum revenue.

29. **Optimal Profit** A fruit juice company makes two special drinks by blending apple and pineapple juices. The profit per liter is \$0.60 for the first drink and \$0.50 for the second drink. The table shows the portions of apple and pineapple juice in each drink.

Ingredient	First drink	Second drink
Apple juice	30%	60%
Pineapple juice	70%	40%

There are 1000 liters of apple juice and 1500 liters of pineapple juice available. Use the simplex method to find the numbers of liters of each drink that should be produced in order to maximize the profit.

30. **Optimal Profit** Rework Exercise 29 assuming that the second drink was changed to contain 80% apple juice and 20% pineapple juice.

31. **Optimal Profit** A manufacturer produces three models of bicycles. The table shows the times (in hours) required for assembling, painting, and packaging each model.

	Model A	Model B	Model C
Assembling	2	2.5	3
Painting	1.5	2	1
Packaging	1	0.75	1.25

The total times available for assembling, painting, and packaging are 4006 hours, 2495 hours, and 1500 hours, respectively. The profits per unit are \$45 for model A, \$50 for model B, and \$55 for model C. What is the optimal production level for each model? What is the optimal profit?

**32. Optimal Profit** Rework Exercise 31 assuming that the total times available for assembling, painting, and packaging are 4000 hours, 2500 hours, and 1500 hours, respectively, and that the profits per unit are \$48 for model A, \$50 for model B, and \$52 for model C.

**33. Optimal Profit** A grower raises crops X, Y, and Z. The profit is \$60 per acre for crop X, \$20 per acre for crop Y, and \$30 per acre for crop Z. Research and available resources determined the constraints below.

- The grower has 50 acres of land available.
- The costs per acre of producing crops X, Y, and Z are \$200, \$80, and \$140, respectively.
- The grower's total cost cannot exceed \$10,000.

Use the simplex method to find the optimal number of acres of land for each crop. What is the maximum profit?

**34. Investment** An investor has up to \$450,000 to invest in three types of investments. Type A investments pay 6% annually and have a risk factor of 0. Type B investments pay 10% annually and have a risk factor of 0.06. Type C investments pay 12% annually and have a risk factor of 0.08. To have a well-balanced portfolio, the investor imposes some conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-half of the total portfolio is to be allocated to type A investments and at least one-fourth of the portfolio is to be allocated to type B investments. How much should the investor allocate to each type of investment to obtain a maximum return?

**35.** Use the simplex method to find the value(s) of  $t$  such that the objective function  $z = 2x_1 - tx_2 + x_3$  has a maximum value at  $(x_1, x_2, x_3) = (8, 0, 6)$  using the constraints

$$\begin{aligned} 2x_1 - 3x_2 &\leq 16 \\ x_2 + 2x_3 &\leq 12 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

**36. CAPSTONE** Consider the initial simplex tableau below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	1	3	1	0	0	15	$s_1$
1	1	-2	0	1	0	5	$s_2$
9	3	12	0	0	1	48	$s_3$
-1	0	-2	0	0	0	0	

- List the objective function and constraints corresponding to the tableau.
- What solution does the tableau represent? Is it optimal? Explain.
- To find an improved solution using an iteration of the simplex method, which entering and departing variables would you select? Explain.

**Unbounded Solutions** In the simplex method, it may happen that in selecting the departing variable, all the calculated ratios are negative. This signifies an *unbounded solution*. Demonstrate this in Exercises 37 and 38.

$$\begin{array}{ll} \text{37. (Maximize)} & \text{38. (Maximize)} \\ \text{Objective function:} & \text{Objective function:} \\ z = x_1 + 2x_2 & z = x_1 + 3x_2 \\ \text{Constraints:} & \text{Constraints:} \\ x_1 - 3x_2 \leq 1 & -x_1 + x_2 \leq 20 \\ -x_1 + 2x_2 \leq 4 & -2x_1 + x_2 \leq 50 \\ x_1, x_2 \geq 0 & x_1, x_2 \geq 0 \end{array}$$

**Other Optimal Solutions** If the simplex method terminates and one or more decision variables are nonbasic and have bottom-row entries of zero, then bringing these variables into the solution will determine other optimal solutions. Demonstrate this in Exercises 39 and 40.

$$\begin{array}{ll} \text{39. (Maximize)} & \text{40. (Maximize)} \\ \text{Objective function:} & \text{Objective function:} \\ z = 2.5x_1 + x_2 & z = x_1 + \frac{1}{2}x_2 \\ \text{Constraints:} & \text{Constraints:} \\ 3x_1 + 5x_2 \leq 15 & 2x_1 + x_2 \leq 20 \\ 5x_1 + 2x_2 \leq 10 & x_1 + 3x_2 \leq 35 \\ x_1, x_2 \geq 0 & x_1, x_2 \geq 0 \end{array}$$

 **Maximizing a Function** In Exercises 41–44, use a software program or a graphing utility to maximize the objective function subject to the constraints

$$\begin{aligned} x_1 + x_2 + 0.83x_3 + 0.5x_4 &\leq 65 \\ 1.2x_1 + x_2 + x_3 + 1.2x_4 &\leq 96 \\ 0.5x_1 + 0.7x_2 + 1.2x_3 + 0.4x_4 &\leq 80 \end{aligned}$$

where  $x_1, x_2, x_3, x_4 \geq 0$ .

- $z = 2x_1 + 7x_2 + 6x_3 + 4x_4$
- $z = 1.2x_1 + x_2 + x_3 + x_4$
- $z = 3x_1 - 0.1x_2 + 5x_3 - 0.9x_4$
- $z = 2.8x_1 + 2.7x_2 + 2.2x_3 + 2.3x_4$

**True or False?** In Exercises 45 and 46, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- After creating the initial simplex tableau, the entering column is chosen by locating the most positive entry in the bottom row.
- If all entries in the entering column are 0 or negative, then there is no maximum solution.

## 9.4 The Simplex Method: Minimization

- Determine the dual of a linear programming problem that minimizes an objective function.
- Use the simplex method to solve a linear programming problem that minimizes an objective function.

### THE DUAL OF A MINIMIZATION PROBLEM

In Section 9.3, you applied the simplex method to linear programming problems in standard form where the objective function was to be maximized. In this section, you will extend this procedure to linear programming problems in which the objective function is to be minimized.

A minimization problem is in **standard form** when the objective function

$$w = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is to be minimized, subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\geq b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

where  $x_i \geq 0$  and  $b_i \geq 0$ . The basic procedure used to solve such a problem is to convert it to a *maximization problem* in standard form, and then apply the simplex method as discussed in Section 9.3.

Consider Example 5 in Section 9.2, where you used geometric methods to solve the minimization problem shown below.

*Minimization Problem:* Find the minimum value of

$$w = 0.12x_1 + 0.15x_2 \quad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 60x_1 + 60x_2 &\geq 300 \\ 12x_1 + 6x_2 &\geq 36 \\ 10x_1 + 30x_2 &\geq 90 \end{aligned} \quad \text{Constraints}$$

where  $x_1 \geq 0$  and  $x_2 \geq 0$ .

To solve this problem using the simplex method, you must first convert it to a maximization problem. The first step is to form the augmented matrix for this system of inequalities. To this augmented matrix, add a last row that represents the coefficients of the objective function, as shown below.

$$\left[ \begin{array}{ccc|c} 60 & 60 & 300 \\ 12 & 6 & 36 \\ 10 & 30 & 90 \\ 0.12 & 0.15 & 0 \end{array} \right]$$

Next, form the transpose of this matrix.

$$\left[ \begin{array}{cccc} 60 & 12 & 10 & 0.12 \\ 60 & 6 & 30 & 0.15 \\ 300 & 36 & 90 & 0 \end{array} \right]$$

Now, interpret this transposed matrix as a *maximization problem*, as shown on the next page.

To interpret the transposed matrix as a maximization problem, introduce new variables,  $y_1$ ,  $y_2$ , and  $y_3$ . This corresponding maximization problem is called the **dual** of the original minimization problem.

*Dual Maximization Problem:* Find the maximum value of

$$z = 300y_1 + 36y_2 + 90y_3 \quad \text{Dual objective function}$$

subject to the constraints

$$\begin{aligned} 60y_1 + 12y_2 + 10y_3 &\leq 0.12 \\ 60y_1 + 6y_2 + 30y_3 &\leq 0.15 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Constraints}$$

where  $y_1 \geq 0$ ,  $y_2 \geq 0$ , and  $y_3 \geq 0$ .

The solution of the original minimization problem can be found by applying the simplex method to the new dual problem, as shown below.

$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$b$	Basic Variables
60	12	10	1	0	0.12	$s_1 \leftarrow$ Departing
60	6	30	0	1	0.15	$s_2$
-300	-36	-90	0	0	0	

↑  
Entering

$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$b$	Basic Variables
1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{60}$	0	$\frac{1}{500}$	$y_1$
0	-6	20	-1	1	$\frac{3}{100}$	$s_2 \leftarrow$ Departing
0	24	-40	5	0	$\frac{3}{5}$	

↑  
Entering

$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$b$	Basic Variables
1	$\frac{1}{4}$	0	$\frac{1}{40}$	$-\frac{1}{120}$	$\frac{7}{4000}$	$y_1$
0	$-\frac{3}{10}$	1	$-\frac{1}{20}$	$\frac{1}{20}$	$\frac{3}{2000}$	$y_3$
0	12	0	3	2	$\frac{33}{50}$	

↑  
↑  
 $x_1$        $x_2$

So, the solution of the dual maximization problem is  $z = \frac{33}{50} = 0.66$ . This is the same value obtained in the minimization problem in Example 5 in Section 9.2. The  $x$ -values corresponding to this optimal solution are the entries in the bottom row corresponding to slack variable columns. In other words, the optimal solution occurs when  $x_1 = 3$  and  $x_2 = 2$ .

The fact that a dual maximization problem has the same solution as its original minimization problem is stated formally below without proof in the **von Neumann Duality Principle**, named after American mathematician John von Neumann (1903–1957).

### THEOREM 9.2 The von Neumann Duality Principle

The objective value  $w$  of a minimization problem in standard form has a minimum value if and only if the objective value  $z$  of the dual maximization problem has a maximum value. Moreover, the minimum value of  $w$  is equal to the maximum value of  $z$ .

## SOLVING A MINIMIZATION PROBLEM

A summary of the steps used to solve a minimization problem is below.

### Solving a Minimization Problem

A minimization problem is in standard form when the objective function

$$w = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

is to be minimized, subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq b_2$$

⋮

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq b_m$$

where

$$x_i \geq 0 \quad \text{and} \quad b_i \geq 0.$$

To solve this problem, use the steps below.

1. Form the **augmented matrix** for the system of inequalities, and add a bottom row consisting of the coefficients of the objective function.

$$\left[ \begin{array}{ccccc} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \\ c_1 & c_2 & \dots & c_n & 0 \end{array} \right]$$

2. Form the transpose of this matrix.

$$\left[ \begin{array}{ccccc} a_{11} & a_{21} & \dots & a_{m1} & c_1 \\ a_{12} & a_{22} & \dots & a_{m2} & c_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} & c_n \\ b_1 & b_2 & \dots & b_m & 0 \end{array} \right]$$

3. Form the **dual maximization problem** corresponding to this transposed matrix. That is, find the maximum of the objective function

$$z = b_1y_1 + b_2y_2 + \dots + b_my_m$$

subject to the constraints

$$a_{11}y_1 + a_{21}y_2 + \dots + a_{m1}y_m \leq c_1$$

$$a_{12}y_1 + a_{22}y_2 + \dots + a_{m2}y_m \leq c_2$$

⋮

$$a_{1n}y_1 + a_{2n}y_2 + \dots + a_{mn}y_m \leq c_n$$

where

$$y_1 \geq 0, \quad y_2 \geq 0, \quad \dots, \quad \text{and} \quad y_m \geq 0.$$

4. Apply the **simplex method** to the dual maximization problem. The maximum value of  $z$  will be the minimum value of  $w$ . Moreover, the values of  $x_1, x_2, \dots, x_n$  will occur in the bottom row of the final simplex tableau, in the columns corresponding to the slack variables.

Examples 1 and 2 illustrate the steps used to solve a minimization problem.

**EXAMPLE 1****Solving a Minimization Problem**

Find the minimum value of

$$w = 3x_1 + 2x_2 \quad \text{Objective function}$$

subject to the constraints

$$\begin{cases} 2x_1 + x_2 \geq 6 \\ x_1 + x_2 \geq 4 \end{cases} \quad \text{Constraints}$$

where  $x_1 \geq 0$  and  $x_2 \geq 0$ .

**SOLUTION**

The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 2 & 1 & 6 \\ 1 & 1 & 4 \\ 3 & 2 & 0 \end{bmatrix}$$

Minimization Problem

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}$$

Dual Maximization Problem

*Dual Maximization Problem:* Find the maximum value of

$$z = 6y_1 + 4y_2 \quad \text{Dual objective function}$$

subject to the constraints

$$\begin{cases} 2y_1 + y_2 \leq 3 \\ y_1 + y_2 \leq 2 \end{cases} \quad \text{Dual constraints}$$

where  $y_1 \geq 0$  and  $y_2 \geq 0$ . Now apply the simplex method to the dual problem, as shown below.

$y_1$	$y_2$	$s_1$	$s_2$	$b$	Basic Variables
(2)	1	1	0	3	$s_1 \leftarrow$ Departing
1	1	0	1	2	$s_2$
-6	-4	0	0	0	

↑  
Entering

$y_1$	$y_2$	$s_1$	$s_2$	$b$	Basic Variables
1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	$y_1$
0	(2)	- $\frac{1}{2}$	1	$\frac{1}{2}$	$s_2 \leftarrow$ Departing
0	-1	3	0	9	

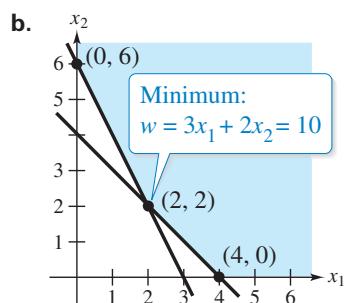
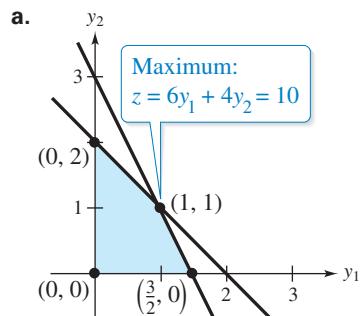
↑  
Entering

$y_1$	$y_2$	$s_1$	$s_2$	$b$	Basic Variables
1	0	1	-1	1	$y_1$
0	1	-1	2	1	$y_2$
0	0	2	2	10	

↑  
↑  
 $x_1$      $x_2$

**REMARK**

Both the minimization and maximization linear programming problems in Example 1 could have been solved with a graphical method, as shown below. Note that the maximum value of  $z = 6y_1 + 4y_2$  in part (a) is the same as the minimum value of  $w = 3x_1 + 2x_2$  in part (b).



From this final simplex tableau, the maximum value of  $z$  is 10. So, the minimum value of  $w$  is 10, and this occurs when  $x_1 = 2$  and  $x_2 = 2$ .

## EXAMPLE 2

## Solving a Minimization Problem

See [LarsonLinearAlgebra.com](http://LarsonLinearAlgebra.com) for an interactive version of this type of example.

Find the minimum value of  $w = 2x_1 + 10x_2 + 8x_3$  subject to the constraints

$$\begin{array}{l} x_1 + x_2 + x_3 \geq 6 \\ \qquad x_2 + 2x_3 \geq 8 \\ -x_1 + 2x_2 + 2x_3 \geq 4 \end{array} \left. \right\} \text{Constraints}$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ .

## SOLUTION

The augmented matrices corresponding to this problem are shown below.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ -1 & 2 & 2 & 4 \\ 2 & 10 & 8 & 0 \end{bmatrix}$$

## Minimization Problem

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 1 & 2 & 10 \\ 1 & 2 & 2 & 8 \\ 6 & 8 & 4 & 0 \end{bmatrix}$$

## Dual Maximization Problem

*Dual Maximization Problem:* Find the maximum value of

$$z = 6y_1 + 8y_2 + 4y_3 \quad \text{Dual objective function}$$

subject to the constraints

$$\left. \begin{array}{l} y_1 - y_3 \leq 2 \\ y_1 + y_2 + 2y_3 \leq 10 \\ y_1 + 2y_2 + 2y_3 \leq 8 \end{array} \right\} \quad \text{Dual constraints}$$

where  $y_1 \geq 0$ ,  $y_2 \geq 0$ , and  $y_3 \geq 0$ . Now apply the simplex method to the dual problem as shown below.

$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	0	-1	1	0	0	2	$s_1$
1	1	2	0	1	0	10	$s_2$
1	2	2	0	0	1	8	$s_3$
-6	-8	-4	0	0	0	0	← Departing

$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	0	-1	1	0	0	2	$y_1$
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	5	$s_2$
0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	3	$y_2$
0	0	2	2	0	4	36	
			↑	↑	↑		
			$x_1$	$x_2$	$x_3$		

From this final simplex tableau, the maximum value of  $z$  is 36. So, the minimum value of  $w$  is 36, and this occurs when  $x_1 = 2$ ,  $x_2 = 0$ , and  $x_3 = 4$ . ■

### EXAMPLE 3

### A Business Application: Minimum Cost

A petroleum company owns two refineries. Refinery 1 costs \$20,000 per day to operate, and refinery 2 costs \$25,000 per day to operate. The table shows the numbers of barrels of each grade of oil the refineries can produce each day.

Grade	Refinery 1	Refinery 2
High-grade	400	300
Medium-grade	300	400
Low-grade	200	500

The company has orders totaling 25,000 barrels of high-grade oil, 27,000 barrels of medium-grade oil, and 30,000 barrels of low-grade oil. How many days should it run each refinery to minimize its costs and still refine enough oil to meet its orders?

#### SOLUTION

Let  $x_1$  and  $x_2$  represent the numbers of days the two refineries operate. Then the total cost is

$$C = 20,000x_1 + 25,000x_2 \quad \text{Objective function}$$

The constraints are

$$\begin{aligned} (\text{High-grade}) \quad 400x_1 + 300x_2 &\geq 25,000 \\ (\text{Medium-grade}) \quad 300x_1 + 400x_2 &\geq 27,000 \\ (\text{Low-grade}) \quad 200x_1 + 500x_2 &\geq 30,000 \end{aligned} \quad \text{Constraints}$$

where  $x_1 \geq 0$  and  $x_2 \geq 0$ . The augmented matrices corresponding to this problem are shown below.

$$\left[ \begin{array}{ccc|c} 400 & 300 & 25,000 \\ 300 & 400 & 27,000 \\ 200 & 500 & 30,000 \\ 20,000 & 25,000 & 0 \end{array} \right]$$

#### Minimization Problem

$$\left[ \begin{array}{cccc|c} 400 & 300 & 200 & 20,000 \\ 300 & 400 & 500 & 25,000 \\ 25,000 & 27,000 & 30,000 & 0 \end{array} \right]$$

#### Dual Maximization Problem

Now apply the simplex method to the dual problem.

$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$b$	Basic Variables
400	300	200	1	0	20,000	$s_1$
300	400	500	0	1	25,000	$s_2 \leftarrow$ Departing
-25,000	-27,000	-30,000	0	0	0	
↑ Entering						
$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$b$	Basic Variables
280	140	0	1	$-\frac{2}{5}$	10,000	$s_1 \leftarrow$ Departing
$-\frac{3}{5}$	$\frac{4}{5}$	1	0	$\frac{1}{500}$	50	$y_3$
-7000	-3000	0	0	60	1,500,000	
↑ Entering						
$y_1$	$y_2$	$y_3$	$s_1$	$s_2$	$b$	Basic Variables
1	$\frac{1}{2}$	0	$\frac{1}{280}$	$-\frac{1}{700}$	$\frac{250}{7}$	$y_1$
0	$\frac{1}{2}$	1	$-\frac{3}{1400}$	$\frac{1}{350}$	$\frac{200}{7}$	$y_3$
0	500	0	25	50	1,750,000	
↑ ↑ $x_1$ $x_2$						

From this final simplex tableau, the minimum cost is

$$C = \$1,750,000 \quad \text{Minimum cost}$$

and this occurs when

$$x_1 = 25 \quad \text{and} \quad x_2 = 50.$$

So, the two refineries should be operated for the numbers of days shown below.

Refinery 1: 25 days

Refinery 2: 50 days

Note that by operating the two refineries for these numbers of days, the company produces the amounts of oil listed below.

High-grade oil:  $400(25) + 300(50) = 25,000$  barrels

Medium-grade oil:  $300(25) + 400(50) = 27,500$  barrels

Low-grade oil:  $200(25) + 500(50) = 30,000$  barrels

So, the company refines enough of each grade of oil to meet its orders (with a surplus of 500 barrels of medium-grade oil). 



### LINEAR ALGEBRA APPLIED

Minimization has a wide variety of real-life applications. For example, consider a situation in which steel coils must be custom cut to a precise measurement. This can cause a sizable amount of waste metal. To minimize waste, a manufacturer determines an objective function and establishes detailed constraints on costs, production time, order specifications, and available materials. The manufacturer then uses specialized optimization software to develop a production plan for each order, and can adapt the constraints for different orders, changing conditions, or varying customers' needs.

## 9.4 Exercises

**Finding the Dual** In Exercises 1–6, determine the dual of the minimization problem.

1. Objective function:

$$w = 2x_1 + 2x_2$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 &\geq 6 \\ x_1 + 2x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

3. Objective function:

$$w = 9x_1 + 6x_2$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 5 \\ 2x_1 + 2x_2 &\geq 8 \\ 2x_1 + x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

5. Objective function:

$$w = 14x_1 + 20x_2 + 24x_3$$

Constraints:

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 7 \\ x_1 + 2x_2 + x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

2. Objective function:

$$w = 2x_1 + x_2$$

Constraints:

$$\begin{aligned} 5x_1 + x_2 &\geq 9 \\ 2x_1 + 2x_2 &\geq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

4. Objective function:

$$w = 4x_1 + x_2 + x_3$$

Constraints:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\geq 23 \\ x_1 + x_3 &\geq 10 \\ 8x_1 + x_2 + 2x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

6. Objective function:

$$w = 9x_1 + 4x_2 + 10x_3$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\geq 6 \\ 6x_1 + x_2 + x_3 &\geq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

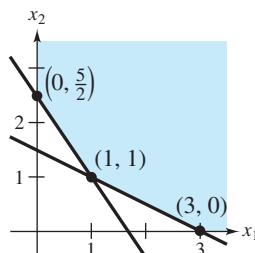
**Solving a Minimization Problem** In Exercises 7–14, (a) solve the minimization problem by the graphical method, (b) formulate the dual problem, and (c) solve the dual problem by the graphical method.

7. Objective function:

$$w = 3x_1 + 3x_2$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ 3x_1 + 2x_2 &\geq 5 \\ x_1, x_2 &\geq 0 \end{aligned}$$

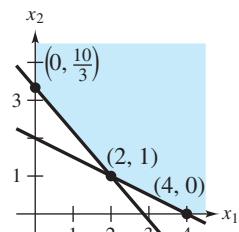


8. Objective function:

$$w = 14x_1 + 20x_2$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 4 \\ 7x_1 + 6x_2 &\geq 20 \\ x_1, x_2 &\geq 0 \end{aligned}$$



9. Objective function:

$$w = 5x_1 + 8x_2$$

Constraints:

(See Exercise 7.)

10. Objective function:

$$w = 7x_1 + 6x_2$$

Constraints:

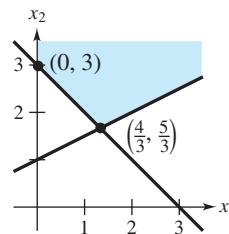
(See Exercise 8.)

11. Objective function:

$$w = x_1 + 4x_2$$

Constraints:

$$\begin{aligned} x_1 + x_2 &\geq 3 \\ -x_1 + 2x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

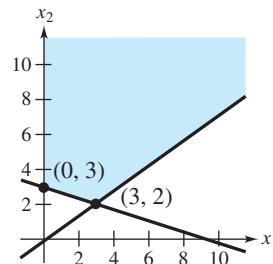


12. Objective function:

$$w = 2x_1 + 6x_2$$

Constraints:

$$\begin{aligned} -2x_1 + 3x_2 &\geq 0 \\ x_1 + 3x_2 &\geq 9 \\ x_1, x_2 &\geq 0 \end{aligned}$$

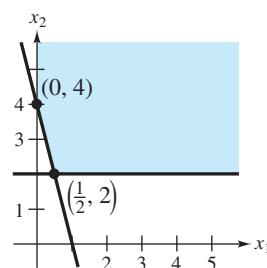


13. Objective function:

$$w = 6x_1 + 3x_2$$

Constraints:

$$\begin{aligned} 4x_1 + x_2 &\geq 4 \\ x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

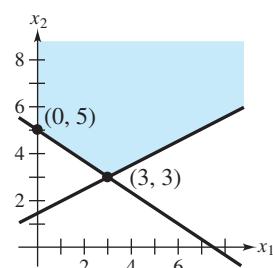


14. Objective function:

$$w = x_1 + 6x_2$$

Constraints:

$$\begin{aligned} 2x_1 + 3x_2 &\geq 15 \\ -x_1 + 2x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$



**Solving a Minimization Problem** In Exercises 15–24, solve the minimization problem by solving the dual maximization problem with the simplex method.

15. Objective function:

$$w = x_2$$

Constraints:

$$\begin{aligned} x_1 + 5x_2 &\geq 10 \\ -6x_1 + 5x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

16. Objective function:

$$w = 3x_1 + 8x_2$$

Constraints:

$$\begin{aligned} 2x_1 + 7x_2 &\geq 9 \\ x_1 + 2x_2 &\geq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

17. Objective function:

$$w = 4x_1 + x_2$$

Constraints:

$$\begin{aligned} x_1 + x_2 &\geq 8 \\ 3x_1 + 5x_2 &\geq 30 \\ x_1, x_2 &\geq 0 \end{aligned}$$

18. Objective function:

$$w = x_1 + x_2$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 &\geq 40 \\ 2x_1 + 3x_2 &\geq 72 \\ x_1, x_2 &\geq 0 \end{aligned}$$

19. Objective function:

$$w = x_1 + 2x_2$$

Constraints:

$$3x_1 + 5x_2 \geq 15$$

$$5x_1 + 2x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

21. Objective function:

$$w = 8x_1 + 4x_2 + 6x_3$$

Constraints:

$$3x_1 + 2x_2 + x_3 \geq 6$$

$$4x_1 + x_2 + 3x_3 \geq 7$$

$$2x_1 + x_2 + 4x_3 \geq 8$$

$$x_1, x_2, x_3 \geq 0$$

23. Objective function:

$$w = 6x_1 + 2x_2 + 3x_3$$

Constraints:

$$3x_1 + 2x_2 + x_3 \geq 28$$

$$6x_1 + x_2 + x_3 \geq 24$$

$$3x_1 + x_2 + 2x_3 \geq 40$$

$$x_1, x_2, x_3 \geq 0$$

25. **Manufacturing** An electronics manufacturing company has three production plants, and each produces three different models of a smart phone. The table shows the daily capacities (in thousands of units) of the three plants.

	Basic Model	Gold Model	Platinum Model
Plant 1	8	4	8
Plant 2	6	6	3
Plant 3	12	4	8

The total demands are 300,000 units of the Basic model, 172,000 units of the Gold model, and 249,500 units of the Platinum model. The daily operating costs are \$55,000 for plant 1, \$60,000 for plant 2, and \$60,000 for plant 3. How many days should each plant operate to fill the total demand while keeping the operating cost at a minimum?

26. **Manufacturing** The company in Exercise 25 lowers the daily operating cost for plant 3 to \$50,000. How many days should each plant operate in order to fill the total demand while keeping the operating cost at a minimum?

27. **Standard Form** Use a geometric argument to explain why, for a minimization problem in standard form, the left-hand side of each constraint is greater than or equal to the right-hand side.

20. Objective function:

$$w = 2x_1 + 5x_2$$

Constraints:

$$x_1 + 3x_2 \geq 9$$

$$2x_1 + x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

22. Objective function:

$$w = 8x_1 + 16x_2 + 18x_3$$

Constraints:

$$2x_1 + 2x_2 - 2x_3 \geq 4$$

$$-4x_1 + 3x_2 - x_3 \geq 1$$

$$x_1 - x_2 + 3x_3 \geq 8$$

$$x_1, x_2, x_3 \geq 0$$

24. Objective function:

$$w = 42x_1 + 5x_2 + 17x_3$$

Constraints:

$$3x_1 - x_2 + 7x_3 \geq 5$$

$$-3x_1 - x_2 + 3x_3 \geq 8$$

$$6x_1 + x_2 + x_3 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

28. **CAPSTONE** Consider the two related optimization problems below.

**Problem 1**

Objective function:

$$M = 8y_1 + 50y_2 + 10y_3$$

Constraints:

$$2y_1 + y_2 + 2y_3 \leq 5$$

$$y_1 + y_2 \leq 10$$

$$y_3 \geq 6y_1 - 65$$

$$y_1, y_2, y_3 \geq 0$$

**Problem 2**

Objective function:

$$N = 5x_1 + 10x_2 + 65x_3$$

Constraints:

$$2x_1 + x_2 + 6x_3 \geq 8$$

$$x_1 + x_2 \geq 50$$

$$x_3 \leq 2x_1 - 10$$

$$x_1, x_2, x_3 \geq 0$$

- (a) Which is the minimization problem? Explain.  
 (b) Which is the dual maximization problem? Explain.  
 (c) Write the corresponding augmented matrices and explain how they are related.

**Optimal Cost** In Exercises 29–32, two sports drinks supply protein and carbohydrates. Drink A provides 1 unit of protein and 3 units of carbohydrates in each liter. Drink B supplies 2 units of protein and 2 units of carbohydrates in each liter. An athlete requires 3 units of protein and 5 units of carbohydrates. Find the amount of each drink the athlete should consume to minimize the cost and still meet the minimum dietary requirements.

29. Drink A costs \$2 per liter and drink B costs \$3 per liter.  
 30. Drink A costs \$4 per liter and drink B costs \$2 per liter.  
 31. Drink A costs \$1 per liter and drink B costs \$3 per liter.  
 32. Drink A costs \$1 per liter and drink B costs \$2 per liter.



**Minimizing a Function** In Exercises 33 and 34, use a software program or a graphing utility to minimize the objective function subject to the constraints

$$1.5x_1 + x_2 + 2x_4 \geq 35$$

$$2x_2 + 6x_3 + 4x_4 \geq 120$$

$$x_1 + x_2 + x_3 + x_4 \geq 50$$

$$0.5x_1 + 2.5x_3 + 1.5x_4 \geq 75$$

where  $x_1, x_2, x_3, x_4 \geq 0$ .

$$33. w = x_1 + 0.5x_2 + 2.5x_3 + 3x_4$$

$$34. w = 1.5x_1 + x_2 + 0.5x_3 + 2x_4$$

## 9.5 The Simplex Method: Mixed Constraints

- Find the maximum of an objective function subject to mixed constraints.
- Find the minimum of an objective function subject to mixed constraints.

### MIXED CONSTRAINTS AND MAXIMIZATION

In Sections 9.3 and 9.4, you studied linear programming problems in *standard form*. The constraints for the maximization problems all involved  $\leq$  inequalities, and the constraints for the minimization problems all involved  $\geq$  inequalities.

As mentioned earlier, linear programming problems for which the constraints involve *both* types of inequalities are called **mixed-constraint** problems. For example, consider the linear programming problem below.

*Mixed-Constraint Problem:* Find the maximum value of

$$z = x_1 + x_2 + 2x_3 \quad \text{Objective function}$$

subject to the constraints

$$\left. \begin{array}{l} 2x_1 + x_2 + x_3 \leq 50 \\ 2x_1 + x_2 \geq 36 \\ x_1 + x_3 \geq 10 \end{array} \right\} \quad \text{Constraints}$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ . This is a maximization problem, so you would expect each of the inequalities in the set of constraints to involve  $\leq$ . The first inequality does involve  $\leq$ , so add a slack variable to form the equation

$$2x_1 + x_2 + x_3 + s_1 = 50.$$

For the other two inequalities, a new type of variable, a **surplus variable**, is introduced, as shown below.

$$\begin{array}{rcl} 2x_1 + x_2 - s_2 & = 36 \\ x_1 + x_3 - s_3 & = 10 \end{array}$$

Notice that surplus variables are *subtracted from* (not added to) the left side of each equation. They are called *surplus* variables because they represent the amounts by which the left sides of the inequalities exceed the right sides. Surplus variables must be nonnegative.

Now, to solve the problem, form an initial simplex tableau, as shown below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
2	1	1	1	0	0	50	$s_1$
2	1	0	0	-1	0	36	$s_2$
1	0	1	0	0	-1	10	$s_3$
-1	-1	-2	0	0	0	0	

#### REMARK

In fact, the values  $x_1 = x_2 = x_3 = 0$  do not even satisfy the constraint equations.

Solving mixed-constraint problems can be difficult. One reason for this is that there is no convenient feasible solution to begin the simplex method. Note that the solution represented by the initial tableau above

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 0, 50, -36, -10)$$

is not a feasible solution because the values of the two surplus variables are negative.

To eliminate the surplus variables from the current solution, use “trial and error.” That is, in an effort to find a feasible solution, arbitrarily choose new entering variables. For example, it seems reasonable to select  $x_3$  as the entering variable, because its column has the most negative entry in the bottom row.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
2	1	1	1	0	0	50	$s_1$
2	1	0	0	-1	0	36	$s_2$
1	0	(1)	0	0	-1	10	$s_3 \leftarrow$ Departing
-1	-1	-2	0	0	0	0	
↑ Entering							

After pivoting, the new simplex tableau is as shown below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	1	0	1	0	1	40	$s_1$
2	(1)	0	0	-1	0	36	$s_2 \leftarrow$ Departing
1	0	1	0	0	-1	10	$x_3$
1	-1	0	0	0	-2	20	
↑ Entering							

The current solution  $(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 0, 10, 40, -36, 0)$  is still not feasible, so choose  $x_2$  as the entering variable and pivot to obtain the simplex tableau below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	0	0	1	1	(1)	4	$s_1 \leftarrow$ Departing
2	1	0	0	-1	0	36	$x_2$
1	0	1	0	0	-1	10	$x_3$
3	0	0	0	-1	-2	56	
↑ Entering							

At this point, you obtain a feasible solution

$$(x_1, x_2, x_3, s_1, s_2, s_3) = (0, 36, 10, 4, 0, 0).$$

From here, continue by applying the simplex method as usual. Note that the next entering variable is  $s_3$ . After pivoting one more time, you obtain the final simplex tableau shown below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	0	0	1	1	1	4	$s_3$
2	1	0	0	-1	0	36	$x_2$
0	0	1	1	1	0	14	$x_3$
1	0	0	2	1	0	64	

Note that this tableau is final because it represents a feasible solution and there are no negative entries in the bottom row. So, the maximum value of the objective function is  $z = 64$  and this occurs when  $x_1 = 0$ ,  $x_2 = 36$ , and  $x_3 = 14$ .

**EXAMPLE 1****A Maximization Problem with Mixed Constraints**

Find the maximum value of

$$z = 3x_1 + 2x_2 + 4x_3 \quad \text{Objective function}$$

subject to the constraints

$$\left. \begin{array}{l} 3x_1 + 2x_2 + 5x_3 \leq 18 \\ 4x_1 + 2x_2 + 3x_3 \leq 16 \\ 2x_1 + x_2 + x_3 \geq 4 \end{array} \right\} \quad \text{Constraints}$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ .

**SOLUTION**

To begin, add a slack variable to each of the first two inequalities and subtract a surplus variable from the third inequality to produce the system of linear equations below.

$$\begin{array}{rcl} 3x_1 + 2x_2 + 5x_3 + s_1 & = 18 \\ 4x_1 + 2x_2 + 3x_3 + s_2 & = 16 \\ 2x_1 + x_2 + x_3 - s_3 & = 4 \end{array}$$

Next form the initial simplex tableau.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
3	2	5	1	0	0	18	$s_1$
4	2	3	0	1	0	16	$s_2$
2	1	1	0	0	-1	4	$s_3 \leftarrow$ Departing
-3	-2	-4	0	0	0	0	

This tableau does not represent a feasible solution because the value of  $s_3$  is negative. So,  $s_3$  should be the departing variable. There are no real guidelines as to which variable should enter the solution, and in fact, any choice will work. However, some entering variables will require more tedious computations than others. For example, choosing  $x_1$  as the entering variable produces the tableau below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
0	$\frac{1}{2}$	$\frac{7}{2}$	1	0	$\frac{3}{2}$	12	$s_1$
0	0	1	0	1	2	8	$s_2$
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	2	$x_1$
0	$-\frac{1}{2}$	$-\frac{5}{2}$	0	0	$-\frac{3}{2}$	6	

Choosing  $x_2$  as the entering variable on the initial tableau instead produces the tableau shown below, which contains “nicer” numbers.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	0	3	1	0	2	10	$s_1$
0	0	1	0	1	2	8	$s_2$
2	1	1	0	0	-1	4	$x_2$
1	0	-2	0	0	-2	8	

Choosing  $x_3$  as the entering variable on the initial tableau will also produce a tableau that does not contain fractions. (Verify this.)

Notice that both of the tableaus shown represent feasible solutions. Any choice of entering variables will lead to a feasible solution, so use trial and error to find an entering variable that yields “nice” numbers. Once you have reached a feasible solution, follow the standard pivoting procedure to choose entering variables and eventually reach an optimal solution.

The rest of this solution uses the tableau produced by choosing  $x_2$  as the entering variable. This tableau does represent a feasible solution, so proceed as usual, choosing the most negative entry in the bottom row to be the entering variable. (In this case, there is a tie, so arbitrarily choose  $x_3$  to be the entering variable.)

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-1	0	( $\hat{3}$ )	1	0	2	10	$s_1 \leftarrow$ Departing
0	0	1	0	1	2	8	$s_2$
2	1	1	0	0	-1	4	$x_2$
1	0	-2	0	0	-2	8	
↑ Entering							

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
$-\frac{1}{3}$	0	1	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{10}{3}$	$x_3$
$\frac{1}{3}$	0	0	$-\frac{1}{3}$	1	( $\frac{4}{3}$ )	$\frac{14}{3}$	$s_2 \leftarrow$ Departing
$\frac{7}{3}$	1	0	$-\frac{1}{3}$	0	$-\frac{5}{3}$	$\frac{2}{3}$	$x_2$
$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	$-\frac{2}{3}$	$\frac{44}{3}$	
↑ Entering							

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
$-\frac{1}{2}$	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1	$x_3$
$\frac{1}{4}$	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{7}{2}$	$s_3$
$\frac{11}{4}$	1	0	$-\frac{3}{4}$	$\frac{5}{4}$	0	$\frac{13}{2}$	$x_2$
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	17	

### REMARK

Verify that choosing  $x_1$  or  $x_3$  as the entering variable on the initial simplex tableau yields the same solution.

So, the maximum value of the objective function is  $z = 17$ , and this occurs when

$$x_1 = 0, x_2 = \frac{13}{2}, \text{ and } x_3 = 1.$$



### LINEAR ALGEBRA APPLIED

As with standard form maximization and minimization, mixed constraint optimization has a wide variety of real-life applications. For example, in Exercise 55 you are asked to determine how to allocate a company’s advertising budget to maximize the total audience. This type of mixed constraint problem is typical in advertising because market research establishes minimums on the number of each type of ad, while budgets put upper limits on expenses. To maximize an audience, researchers must first determine the ideal demographic and then estimate the time and place at which those people will most likely see the ad, considering traditional venues as well as digital formats and social media. Other common real-life applications of constrained optimization problems involve minimizing costs.

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## MIXED CONSTRAINTS AND MINIMIZATION

Section 9.4 discusses the solution of minimization problems in *standard form*. Minimization problems that are not in standard form are more difficult to solve. One technique is to change a mixed-constraint minimization problem to a mixed-constraint maximization problem by multiplying each coefficient in the objective function by  $-1$ . The next example demonstrates this technique.

### EXAMPLE 2

### A Minimization Problem with Mixed Constraints

See [LarsonLinearAlgebra.com](http://LarsonLinearAlgebra.com) for an interactive version of this type of example.

Find the minimum value of

$$w = 4x_1 + 2x_2 + x_3 \quad \text{Objective function}$$

subject to the constraints

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 14 \\ 3x_1 + x_2 + 5x_3 &\geq 4 \\ x_1 + 4x_2 + 3x_3 &\geq 6 \end{aligned} \quad \text{Constraints}$$

where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$ .

### SOLUTION

First, rewrite the objective function by multiplying each of its coefficients by  $-1$ , as shown below.

$$z = -4x_1 - 2x_2 - x_3 \quad \text{Revised objective function}$$

Maximizing this revised objective function is equivalent to minimizing the original objective function. Next, add a slack variable to the first inequality and subtract surplus variables from the second and third inequalities to produce the initial simplex tableau below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
2	3	4	1	0	0	14	$s_1$
3	1	5	0	-1	0	4	$s_2$
1	4	3	0	0	-1	6	$s_3$
4	2	1	0	0	0	0	

Note that the bottom row contains the negatives of the coefficients of the revised objective function. Another way of looking at this is that when using this technique to solve minimization problems in nonstandard form, the bottom row of the initial simplex tableau consists of the coefficients of the original objective function.

As with maximization problems with mixed constraints, this initial simplex tableau does not represent a feasible solution. By trial and error, choose  $x_2$  as the entering variable and  $s_2$  as the departing variable.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
2	3	4	1	0	0	14	$s_1$
3	( $\frac{1}{2}$ )	5	0	-1	0	4	$s_2 \leftarrow$ Departing
1	4	3	0	0	-1	6	$s_3$
4	2	1	0	0	0	0	
				↑			Entering

After pivoting, you obtain the tableau shown below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-7	0	-11	1	3	0	2	$s_1$
3	1	5	0	-1	0	4	$x_2$
-11	0	-17	0	4	-1	-10	$s_3$
-2	0	-9	0	2	0	-8	

To transform the tableau into one that represents a feasible solution, multiply the third row by -1.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-7	0	-11	1	3	0	2	$s_1$
3	1	5	0	-1	0	4	$x_2$
11	0	17	0	-4	1	10	$s_3$
-2	0	-9	0	2	0	-8	

Now that you have obtained a simplex tableau that represents a feasible solution, continue with pivoting operations until you obtain an optimal solution.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-7	0	-11	1	3	0	2	$s_1$
3	1	5	0	-1	0	4	$x_2$
11	0	17	0	-4	1	10	$s_3$ ← Departing
-2	0	-9	0	2	0	-8	

↑  
Entering

### REMARK

Recall from page 458 that your choices for the entering variable exclude the entry in the “ $b$ -column.” So, although  $-\frac{46}{17}$  is the least entry in the bottom row here, it does not represent a variable that can enter the solution.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
$\frac{2}{17}$	0	0	1	$\frac{7}{17}$	$\frac{11}{17}$	$\frac{144}{17}$	$s_1$
$-\frac{4}{17}$	1	0	0	$-\frac{3}{17}$	$-\frac{5}{17}$	$\frac{18}{17}$	$x_2$ ← Departing
$\frac{11}{17}$	0	1	0	$-\frac{4}{17}$	$\frac{1}{17}$	$\frac{10}{17}$	$x_3$
$\frac{65}{17}$	0	0	0	$-\frac{2}{17}$	$\frac{9}{17}$	$-\frac{46}{17}$	

↑  
Entering

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
$\frac{2}{3}$	$-\frac{7}{3}$	0	1	0	$\frac{4}{3}$	6	$s_1$
$-\frac{4}{3}$	$\frac{17}{3}$	0	0	1	$-\frac{5}{3}$	6	$s_2$
$\frac{1}{3}$	$\frac{4}{3}$	1	0	0	$-\frac{1}{3}$	2	$x_3$
$\frac{11}{3}$	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	-2	

The maximum value of the revised objective function is  $z = -2$ , and so the minimum value of the original objective function is

$$w = 2$$

(the negative of the entry in the lower-right corner). This occurs when

$$x_1 = 0, x_2 = 0, \text{ and } x_3 = 2.$$



**EXAMPLE 3****A Business Application:  
Minimum Shipment Costs**

An automobile company has two factories. One factory has 400 cars of one model in stock and the other factory has 300 cars (of the same model) in stock. Two customers order this car model. The first customer needs 200 cars, and the second customer needs 300 cars. The table shows the costs of shipping cars from the two factories to the customers.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	\$30	\$25
<i>Factory 2</i>	\$36	\$30

How should the company ship the cars to minimize the shipping costs? What is the minimum cost?

**SOLUTION**

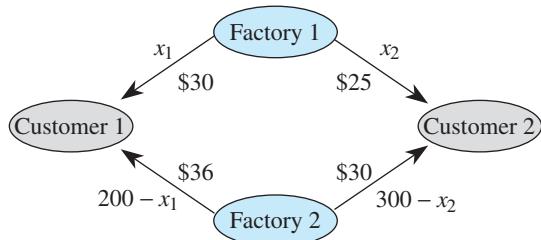
To begin, let  $x_1$  and  $x_2$  represent the numbers of cars shipped from factory 1 to the first and second customers, respectively. This means that the number of cars shipped from factory 2 to the first customer is

$$200 - x_1$$

and that the number of cars shipped from factory 2 to the second customer is

$$300 - x_2.$$

(See figure.)



The total cost of shipping is

$$C = 30x_1 + 25x_2 + 36(200 - x_1) + 30(300 - x_2) = 16,200 - 6x_1 - 5x_2.$$

The constraints for this minimization problem are as listed below.

$$\begin{aligned}
 x_1 + x_2 &\leq 400 \\
 (200 - x_1) + (300 - x_2) &\leq 300 \quad \rightarrow \quad x_1 + x_2 \geq 200 \\
 x_1 &\leq 200 \\
 x_2 &\leq 300
 \end{aligned}$$

The corresponding maximization problem is to maximize  $z = 6x_1 + 5x_2 - 16,200$ . The initial simplex tableau is shown below.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
1	1	1	0	0	0	400	$s_1$
1	1	0	-1	0	0	200	$s_2$
1	0	0	0	1	0	200	$s_3$
0	1	0	0	0	1	300	$s_4$
-6	-5	0	0	0	0	-16,200	

Note that the current  $z$ -value is  $-16,200$  because

$$z = 6x_1 + 5x_2 - 16,200 = 6(0) + 5(0) - 16,200 = -16,200.$$

Now, to the initial tableau, apply the simplex method, as shown below.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
1	1	1	0	0	0	400	$s_1$
( $\hat{1}$ )	1	0	-1	0	0	200	$s_2 \leftarrow$ Departing
1	0	0	0	1	0	200	$s_3$
0	1	0	0	0	1	300	$s_4$
-6	-5	0	0	0	0	-16,200	

↑  
Entering

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
0	0	1	1	0	0	200	$s_1$
1	1	0	-1	0	0	200	$x_1$
0	-1	0	( $\hat{1}$ )	1	0	0	$s_3 \leftarrow$ Departing
0	1	0	0	0	1	300	$s_4$
0	1	0	-6	0	0	-15,000	

↑  
Entering

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
0	( $\hat{1}$ )	1	0	-1	0	200	$s_1 \leftarrow$ Departing
1	1	0	0	1	0	200	$x_1$
0	-1	0	1	1	0	0	$s_2$
0	1	0	0	0	1	300	$s_4$
0	-5	0	0	6	0	-15,000	

↑  
Entering

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$s_4$	$b$	Basic Variables
0	1	1	0	-1	0	200	$x_2$
1	0	0	0	1	0	200	$x_1$
0	0	1	1	0	0	200	$s_2$
0	0	-1	0	1	1	100	$s_4$
0	0	5	0	1	0	-14,000	

From this final simplex tableau, the minimum shipping cost is \$14,000.  $x_1 = 200$  and  $x_2 = 200$ , so you can conclude that the numbers of cars that should ship from the two factories are as shown in the table.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	200 cars	200 cars
<i>Factory 2</i>	0	100 cars



## 9.5 Exercises

**Slack and Surplus Variables** In Exercises 1–8, add the appropriate slack and surplus variables to the system and form the initial simplex tableau.

1. (Maximize)  
Objective function:

$$z = 10x_1 + 4x_2$$

Constraints:

$$2x_1 + x_2 \geq 4$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

3. (Minimize)  
Objective function:

$$w = x_1 + x_2$$

Constraints:

$$2x_1 + x_2 \leq 4$$

$$x_1 + 3x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

5. (Maximize)  
Objective function:

$$z = x_1 + x_3$$

Constraints:

$$4x_1 + x_2 \geq 10$$

$$x_1 + x_2 + 3x_3 \leq 30$$

$$2x_1 + x_2 + 4x_3 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

7. (Minimize)  
Objective function:

$$w = 4x_1 + 2x_2 + x_3$$

Constraints:

$$5x_1 + 4x_2 + 5x_3 \geq 12$$

$$x_1 + 6x_2 \leq 5$$

$$x_1, x_2, x_3 \geq 0$$

2. (Maximize)  
Objective function:

$$z = x_1 + 5x_2$$

Constraints:

$$x_1 + x_2 \leq 16$$

$$3x_1 + 2x_2 \geq 19$$

$$x_1, x_2 \geq 0$$

4. (Minimize)  
Objective function:

$$w = 2x_1 + 3x_2$$

Constraints:

$$3x_1 + x_2 \geq 4$$

$$4x_1 + 2x_2 \leq 3$$

$$x_1, x_2 \geq 0$$

6. (Minimize)  
Objective function:

$$w = 3x_1 + x_2 + x_3$$

Constraints:

$$x_1 + 2x_2 + x_3 \leq 10$$

$$x_2 + 5x_3 \geq 6$$

$$4x_1 - x_2 + x_3 \geq 16$$

$$x_1, x_2, x_3 \geq 0$$

8. (Maximize)  
Objective function:

$$z = 4x_1 + x_2 + x_3$$

Constraints:

$$2x_1 + x_2 + 4x_3 \leq 60$$

$$x_2 + x_3 \geq 40$$

$$x_1, x_2, x_3 \geq 0$$

**Solving a Mixed-Constraint Problem** In Exercises 9–14, use the specified entering and departing variables to solve the mixed-constraint problem.

9. (Maximize)  
Objective function:

$$z = -x_1 + 2x_2$$

Constraints:

$$x_1 + x_2 \geq 3$$

$$x_1 + x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Entering  $x_2$ , departing  $s_1$

10. (Maximize)  
Objective function:

$$z = 2x_1 + x_2$$

Constraints:

$$x_1 + x_2 \geq 4$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Entering  $x_1$ , departing  $s_1$

11. (Minimize)  
Objective function:

$$w = x_1 + 2x_2$$

Constraints:

$$2x_1 + 3x_2 \leq 25$$

$$x_1 + 2x_2 \geq 16$$

$$x_1, x_2 \geq 0$$

Entering  $x_2$ , departing  $s_2$

13. (Maximize)  
Objective function:

$$z = x_1 + x_2$$

Constraints:

$$-4x_1 + 3x_2 + x_3 \leq 40$$

$$-2x_1 + x_2 + x_3 \geq 10$$

$$x_2 + x_3 \leq 20$$

$$x_1, x_2, x_3 \geq 0$$

Entering  $x_2$ , departing  $s_2$

12. (Minimize)  
Objective function:

$$w = 3x_1 + 2x_2$$

Constraints:

$$x_1 + x_2 \geq 20$$

$$3x_1 + 4x_2 \leq 70$$

$$x_1, x_2 \geq 0$$

Entering  $x_1$ , departing  $s_1$

14. (Maximize)  
Objective function:

$$z = x_1 + 2x_2 + 2x_3$$

Constraints:

$$x_1 + x_2 \geq 50$$

$$2x_1 + x_2 + x_3 \leq 70$$

$$x_2 + 3x_3 \geq 40$$

$$x_1, x_2, x_3 \geq 0$$

Entering  $x_2$ , departing  $s_1$

**Solving a Mixed-Constraint Problem** In Exercises 15–20, rework the stated exercise using the specified entering and departing variables.

15. Exercise 9; entering  $x_1$ , departing  $s_1$

16. Exercise 10; entering  $x_2$ , departing  $s_1$

17. Exercise 11; entering  $x_1$ , departing  $s_2$

18. Exercise 12; entering  $x_2$ , departing  $s_1$

19. Exercise 13; entering  $x_1$ , departing  $s_2$

20. Exercise 14; entering  $x_1$ , departing  $s_1$

**Solving a Mixed-Constraint Problem** In Exercises 21–30, use the simplex method to solve the problem.

21. (Maximize)  
Objective function:

$$z = 2x_1 + 5x_2$$

Constraints:

$$x_1 + 2x_2 \geq 4$$

$$x_1 + x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

22. (Maximize)  
Objective function:

$$z = -x_1 + 3x_2$$

Constraints:

$$2x_1 + x_2 \leq 4$$

$$x_1 + 5x_2 \geq 5$$

$$x_1, x_2 \geq 0$$

23. (Maximize)  
Objective function:

$$z = x_1 + 3x_2$$

Constraints:

$$2x_1 + x_2 \leq 36$$

$$x_1 + x_2 \geq 18$$

$$x_1, x_2 \geq 0$$

24. (Minimize)  
Objective function:

$$w = 5x_1 + 3x_2$$

Constraints:

$$3x_1 - 4x_2 \geq 28$$

$$x_1 + \frac{1}{2}x_2 \leq 64$$

$$x_1, x_2 \geq 0$$

25. (Minimize)  
Objective function:  
 $w = x_1 + x_2$   
Constraints:  
 $x_1 + 2x_2 \geq 25$   
 $2x_1 + 5x_2 \leq 60$   
 $x_1, x_2 \geq 0$

27. (Maximize)  
Objective function:  
 $z = 2x_1 + x_2 + 3x_3$   
Constraints:  
 $x_1 + 4x_2 + 2x_3 \leq 85$   
 $x_2 - 5x_3 \geq 20$   
 $3x_1 + 2x_2 + 11x_3 \geq 49$   
 $x_1, x_2, x_3 \geq 0$

29. (Minimize)  
Objective function:  
 $w = -2x_1 + 4x_2 - x_3$   
Constraints:  
 $3x_1 - 6x_2 + 4x_3 \leq 30$   
 $2x_1 - 8x_2 + 10x_3 \geq 18$   
 $x_1, x_2, x_3 \geq 0$

26. (Minimize)  
Objective function:  
 $w = 2x_1 + 3x_2$   
Constraints:  
 $3x_1 + 2x_2 \leq 22$   
 $x_1 + x_2 \geq 10$   
 $x_1, x_2 \geq 0$

28. (Maximize)  
Objective function:  
 $z = 3x_1 + 5x_2 + 2x_3$   
Constraints:  
 $9x_1 + 4x_2 + x_3 \geq 70$   
 $5x_1 + 2x_2 + x_3 \leq 40$   
 $4x_1 + x_2 \geq 16$   
 $x_1, x_2, x_3 \geq 0$

30. (Minimize)  
Objective function:  
 $w = x_1 + x_2 + x_3$   
Constraints:  
 $x_1 + 2x_2 + x_3 \geq 30$   
 $6x_2 + x_3 \leq 54$   
 $x_1 + x_2 + 3x_3 \geq 20$   
 $x_1, x_2, x_3 \geq 0$

**Maximizing with Mixed Constraints** In Exercises 31–38, maximize the objective function subject to the constraints listed below.

$$\begin{aligned}x_1 + x_2 &\leq 5 \\-x_1 + x_2 &\leq 3 \\x_2 &\geq 1 \\x_1, x_2 &\geq 0\end{aligned}$$

31.  $z = 2x_1 + x_2$

33.  $z = x_2$

35.  $z = -3x_1 + 2x_2$

37.  $z = 4x_1 + 5x_2$

32.  $z = x_1 + 2x_2$

34.  $z = -x_1 - x_2$

36.  $z = 3x_1$

38.  $z = -4x_1 + 2x_2$

**Maximizing with Mixed Constraints** In Exercises 39–46, maximize the objective function subject to the constraints listed below.

$$\begin{aligned}3x_1 + 2x_2 &\geq 6 \\x_1 - x_2 &\leq 2 \\-x_1 + 2x_2 &\leq 6 \\x_1 &\leq 4 \\x_1, x_2 &\geq 0\end{aligned}$$

39.  $z = -3x_2$

41.  $z = x_1 + x_2$

43.  $z = -4x_1 + x_2$

45.  $z = -2x_1 - 2x_2$

40.  $z = 4x_1$

42.  $z = x_1 - 2x_2$

44.  $z = 4x_1 - x_2$

46.  $z = -6x_1 + 2x_2$

**Minimizing Cost** In Exercises 47–50, a tire company has two suppliers,  $S_1$  and  $S_2$ .  $S_1$  has 900 tires on hand and  $S_2$  has 800 tires on hand. Customer  $C_1$  needs 500 tires and customer  $C_2$  needs 600 tires. Minimize the cost of filling the orders subject to the data in the table (shipping costs per tire).

	$C_1$	$C_2$
$S_1$	0.60	1.20
$S_2$	1.00	1.80

	$C_1$	$C_2$
$S_1$	1.20	1.00
$S_2$	1.00	1.20

	$C_1$	$C_2$
$S_1$	0.80	1.00
$S_2$	1.00	0.80

**Minimizing Cost** In Exercises 51 and 52, a manufacturer has two assembly plants, plant A and plant B, and two distribution outlets, outlet I and outlet II. Plant A can assemble 5000 units of a product in a year and plant B can assemble 4000 units of the same product in a year. Outlet I must have 3000 units per year and outlet II must have 5000 units per year. The table shows the costs of transportation from each plant to each outlet. Find the shipping schedule that will produce the minimum cost. What is the minimum cost?

	<i>Outlet I</i>	<i>Outlet II</i>
<i>Plant A</i>	\$4	\$5
<i>Plant B</i>	\$5	\$6

	<i>Outlet I</i>	<i>Outlet II</i>
<i>Plant A</i>	\$4	\$5
<i>Plant B</i>	\$6	\$4

**53. Minimizing Cost** An automobile company has two factories. One factory has 400 cars of one model in stock and the other factory has 300 cars (of the same model) in stock. Two customers order this car model. The first customer needs 200 cars, and the second customer needs 300 cars. The table shows the costs of shipping cars from the two factories to the two customers.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	\$36	\$30
<i>Factory 2</i>	\$30	\$25

(a) How should the company ship the cars to minimize the shipping costs?

(b) What is the minimum cost?

- 54. Minimizing Cost** Rework Exercise 53 assuming that the shipping costs for the two factories are as shown in the table below.

	<i>Customer 1</i>	<i>Customer 2</i>
<i>Factory 1</i>	\$25	\$30
<i>Factory 2</i>	\$35	\$30

- 55. Advertising** A company is determining how to advertise a product nationally on television and in a newspaper. Each television ad is expected to be seen by 15 million viewers, and each newspaper ad is expected to be seen by 3 million readers. The company has the constraints below.

- The company has budgeted a maximum of \$600,000 to advertise the product.
- Each minute of television time costs \$60,000 and each one-page newspaper ad costs \$15,000.
- The company's market research department recommends using at least 6 television ads and at least 4 newspaper ads.

How should the company allocate its advertising budget to maximize the total audience? What is the maximum audience?

- 56. CAPSTONE** Consider the initial simplex tableau below.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
3	-1	8	$a$	0	0	48	$s_1$
0	12	4	0	$b$	0	96	$s_2$
-2	2	0	0	0	$c$	5	$s_3$
-9	-5	-2	0	0	0	0	

- Does this tableau represent a maximization problem or a minimization problem? Explain.
- Find values of  $a$ ,  $b$ , and  $c$  such that the solution is feasible.
- Use your answer to part (b) to write an objective function and a set of constraints that the simplex tableau could be used to represent.
- Find values of  $a$ ,  $b$ , and  $c$  such that the solution is not feasible.
- Let  $a = 1$ ,  $b = 1$ , and  $c = 1$ . Which variable should you choose as the departing variable in the first iteration of pivoting? Explain.
- Say you perform the simplex method on the tableau above and obtain a final simplex tableau. Does the entry in the lower-right corner represent the optimal value of the original objective function, or its opposite? Explain.

**Identifying Feasible Solutions** In Exercises 57–62, determine whether the simplex tableau represents a feasible solution. Explain.

57.

$x_1$	$x_2$	$s_1$	$s_2$	$b$	Basic Variables
25	-19	1	0	12	$s_1$
-19	25	0	1	12	$s_2$
6	-6	0	0	0	

58.

$x_1$	$x_2$	$s_1$	$s_2$	$b$	Basic Variables
5	15	-1	0	60	$s_1$
2	-2	0	1	8	$s_2$
25	20	0	0	0	

59.

$x_1$	$x_2$	$s_1$	$s_2$	$b$	Basic Variables
3	9	-1	0	-28	$s_1$
8	4	0	-1	-36	$s_2$
-16	72	0	0	0	

60.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-14	13	1	0	0	4	$s_1$
-10	2	0	-1	0	3	$s_2$
-8	6	0	0	1	5	$s_3$
-2	-2	0	0	0	0	

61.

$x_1$	$x_2$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
1	2	1	0	0	32	$s_1$
1	-5	0	1	0	10	$s_2$
0	2	0	0	-1	-50	$s_3$
-4	-8	0	0	0	0	

62.

$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	$b$	Basic Variables
-10	0	1	-1	0	0	75	$s_1$
0	0	5	0	-1	0	30	$s_2$
3	1	12	0	0	1	26	$s_3$
7	-24	16	0	0	0	0	

**True or False?** In Exercises 63 and 64, determine whether each statement is true or false. If a statement is true, give a reason or cite an appropriate statement from the text. If a statement is false, provide an example that shows the statement is not true in all cases or cite an appropriate statement from the text.

- One technique that can be used to change a mixed-constraint minimization problem to a mixed-constraint maximization problem is to multiply each coefficient of the objective function by  $-1$ .
- Surplus variables, like slack variables, are always positive because they represent the amount by which the left side of the inequality is less than the right side.

## 9 Review Exercises

**Solving a System of Inequalities** In Exercises 1–6, sketch the graph (and label any vertices) of the solution set of the system of inequalities.

1.  $x + 2y \leq 160$   
 $3x + y \leq 180$   
 $x \geq 0$   
 $y \geq 0$

2.  $2x + 3y \leq 24$   
 $2x + y \leq 16$   
 $x \geq 0$   
 $y \geq 0$

3.  $3x + 2y \geq 24$   
 $x + 2y \geq 12$   
 $2 \leq x \leq 15$   
 $y \leq 15$

4.  $2x + y \geq 16$   
 $x + 3y \geq 18$   
 $0 \leq x \leq 25$   
 $0 \leq y \leq 15$

5.  $2x - 3y \geq 0$   
 $2x - y \leq 8$   
 $0 \leq y < 1$

6.  $x - y \leq 10$   
 $2x + 3y < 30$   
 $x \geq 0$

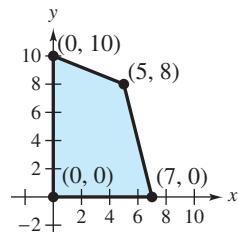
**Solving a Linear Programming Problem** In Exercises 7–18, find the minimum and maximum values of the objective function (if possible), and where they occur, by the graphical method.

7. Objective function:

$$z = 3x + 4y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + 5y &\leq 50 \\ 4x + y &\leq 28 \end{aligned}$$

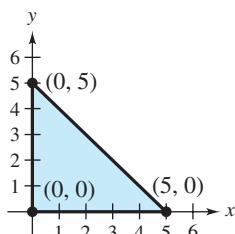


9. Objective function:

$$z = 4x + 3y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\leq 5 \end{aligned}$$

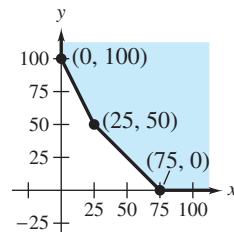


8. Objective function:

$$z = 10x + 7y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x + y &\geq 100 \\ x + y &\geq 75 \end{aligned}$$

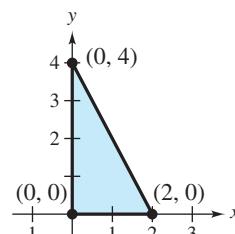


10. Objective function:

$$z = 2x + 8y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 2x - y &\leq 4 \end{aligned}$$

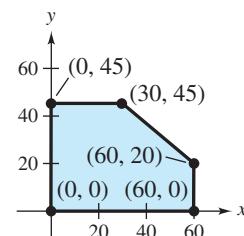


11. Objective function:

$$z = 25x + 30y$$

Constraints:

$$\begin{aligned} 0 &\leq x \leq 60 \\ 0 &\leq y \leq 45 \\ 5x + 6y &\leq 420 \end{aligned}$$

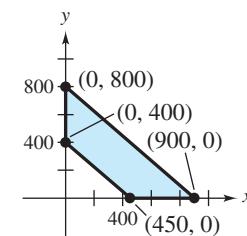


12. Objective function:

$$z = 15x + 20y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 8x + 9y &\leq 7200 \\ 8x + 9y &\geq 3600 \end{aligned}$$

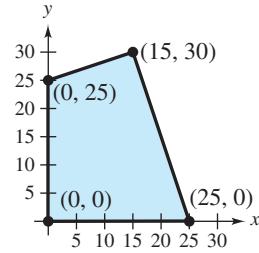


13. Objective function:

$$z = 5x + 0.5y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ -x + 3y &\leq 75 \\ 3x + y &\leq 75 \end{aligned}$$

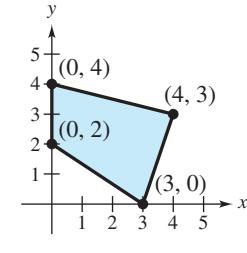


14. Objective function:

$$z = 2x + y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ 2x + 3y &\geq 6 \\ 3x - y &\leq 9 \\ x + 4y &\leq 16 \end{aligned}$$



15. Objective function:

$$z = x + 3y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\geq 3 \\ x - y &\leq 3 \\ x + 5y &\leq 15 \end{aligned}$$

17. Objective function:

$$z = 3x - y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x &\leq 3y \\ -x + 2y &\leq 12 \\ 4x + 3y &\leq 40 \end{aligned}$$

16. Objective function:

$$z = 4x - y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ x + y &\geq 2 \\ x &\geq y \\ 3x - y &\leq 12 \end{aligned}$$

18. Objective function:

$$z = x - 2y$$

Constraints:

$$\begin{aligned} x &\geq 0 \\ y &\geq 0 \\ 5x + y &\leq 36 \\ -x + 2y &\leq 4 \\ 2x + 5y &\geq 19 \end{aligned}$$

**Using the Simplex Method** In Exercises 19–26, use the simplex method to maximize the objective function, subject to the constraints.

19. Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 &\leq 31 \\ x_1 + 4x_2 &\leq 40 \\ x_1, x_2 &\geq 0 \end{aligned}$$

20. Objective function:

$$z = 5x_1 + 4x_2$$

Constraints:

$$\begin{aligned} x_1 - x_2 &\leq 22 \\ x_1 + 2x_2 &\leq 43 \\ x_1, x_2 &\geq 0 \end{aligned}$$

21. Objective function:

$$z = x_1 + 2x_2 + x_3$$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 + x_3 &\leq 20 \\ x_1 + x_2 - 2x_3 &\leq 23 \\ -2x_1 + x_2 - 2x_3 &\leq 8 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

22. Objective function:

$$z = 4x_1 + 5x_2 + 6x_3$$

Constraints:

$$\begin{aligned} 4x_1 + 2x_2 + x_3 &\leq 30 \\ x_1 + 3x_2 + 2x_3 &\leq 54 \\ x_1 + x_2 + 2x_3 &\leq 24 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

23. Objective function:

$$z = x_1 + x_2$$

Constraints:

$$\begin{aligned} 3x_1 + x_2 &\leq 432 \\ x_1 + 4x_2 &\leq 628 \\ x_1, x_2 &\geq 0 \end{aligned}$$

24. Objective function:

$$z = 6x_1 + 8x_2$$

Constraints:

$$\begin{aligned} 20x_1 + 40x_2 &\leq 200 \\ 30x_1 + 42x_2 &\leq 228 \\ x_1, x_2 &\geq 0 \end{aligned}$$

25. Objective function:

$$z = 3x_1 + 5x_2 + 4x_3$$

Constraints:

$$\begin{aligned} 6x_1 - 2x_2 + 3x_3 &\leq 24 \\ 3x_1 - 3x_2 + 9x_3 &\leq 33 \\ -2x_1 + x_2 - 2x_3 &\leq 25 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

26. Objective function:

$$z = 2x_1 + 5x_2 - x_3$$

Constraints:

$$\begin{aligned} -x_1 + 3x_2 + 2x_3 &\leq 92 \\ -2x_1 + 2x_2 + 12x_3 &\leq 76 \\ 3x_1 + x_2 - 6x_3 &\leq 24 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

**Finding the Dual** In Exercises 27 and 28, determine the dual of the minimization problem.

27. Objective function:

$$w = 7x_1 + 3x_2 + x_3$$

Constraints:

$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 30 \\ 3x_1 + 6x_2 + 4x_3 &\geq 75 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

28. Objective function:

$$w = 2x_1 + 3x_2 + 4x_3$$

Constraints:

$$\begin{aligned} x_1 + 5x_2 + 3x_3 &\geq 90 \\ x_1 + 2x_2 + x_3 &\geq 75 \\ 3x_1 + 2x_2 + x_3 &\geq 56 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

**Solving a Minimization Problem** In Exercises 29–34, solve the minimization problem by solving the dual maximization problem by the simplex method.

29. Objective function:

$$w = 9x_1 + 15x_2$$

Constraints:

$$\begin{aligned} x_1 + 5x_2 &\geq 15 \\ 4x_1 - 10x_2 &\geq 0 \\ x_1, x_2 &\geq 0 \end{aligned}$$

30. Objective function:

$$w = 16x_1 + 18x_2$$

Constraints:

$$\begin{aligned} 2x_1 - 3x_2 &\geq 14 \\ -4x_1 + 9x_2 &\geq 8 \\ x_1, x_2 &\geq 0 \end{aligned}$$

31. Objective function: 32. Objective function:

$$w = 24x_1 + 22x_2 + 18x_3$$

Constraints:

$$\begin{aligned} 2x_1 + 2x_2 - 3x_3 &\geq 24 \\ 6x_1 - 2x_3 &\geq 21 \\ -8x_1 - 4x_2 + 8x_3 &\geq 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$w = 32x_1 + 36x_2 + 4x_3$$

Constraints:

$$\begin{aligned} 4x_1 + 3x_2 - x_3 &\geq 8 \\ -8x_1 + 6x_2 - 6x_3 &\geq 0 \\ -4x_1 + 9x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

33. Objective function: 34. Objective function:

$$w = 16x_1 + 54x_2 + 48x_3$$

Constraints:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\geq 2 \\ 2x_1 + 7x_2 + 4x_3 &\geq 5 \\ x_1 + 3x_2 + 4x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$w = 22x_1 + 27x_2 + 18x_3$$

Constraints:

$$\begin{aligned} -2x_1 + 7x_2 + 3x_3 &\geq 4 \\ 2x_1 + x_2 - 3x_3 &\geq 12 \\ x_1 - 5x_2 + 2x_3 &\geq 16 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

**Solving a Mixed-Constraint Problem** In Exercises 35–40, use the simplex method to solve the mixed-constraint problem.

35. (Maximize)

Objective function:

$$z = x_1 + 2x_2$$

Constraints:

$$\begin{aligned} -4x_1 + 2x_2 &\leq 26 \\ -3x_1 + x_2 &\geq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

36. (Maximize)

Objective function:

$$z = 2x_1 + 3x_2$$

Constraints:

$$\begin{aligned} -x_1 + x_2 &\geq 40 \\ x_2 &\leq 61 \\ x_1, x_2 &\geq 0 \end{aligned}$$

37. (Maximize)

Objective function:

$$z = 2x_1 + x_2 + x_3$$

Constraints:

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 60 \\ -4x_1 + 2x_2 + x_3 &\geq 52 \\ 2x_1 + x_3 &\geq 40 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

38. (Maximize)

Objective function:

$$z = 3x_1 + 2x_2 + x_3$$

Constraints:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &\leq 52 \\ x_1 + x_2 + 2x_3 &\geq 24 \\ 2x_2 + x_3 &\leq 52 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

39. (Minimize)

Objective function:

$$w = 9x_1 + 4x_2 + 10x_3$$

Constraints:

$$\begin{aligned} 32x_1 + 16x_2 + 8x_3 &\leq 344 \\ 20x_1 - 40x_2 + 20x_3 &\geq 200 \\ -45x_1 + 15x_2 + 30x_3 &\leq 525 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

40. (Minimize)

Objective function:

$$w = 4x_1 - 2x_2 - x_3$$

Constraints:

$$\begin{aligned} 2x_1 - x_2 - x_3 &\leq 41 \\ x_1 - 2x_2 - x_3 &\geq 10 \\ -x_1 - 7x_2 + 5x_3 &\leq 11 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

**Graphing a System of Inequalities** In Exercises 41 and 42, write a system of inequalities that models the description, and sketch a graph of the solution of the system.

41. A Pennsylvania fruit grower has 1500 bushels of apples and divides them between markets in Harrisburg and Philadelphia. These two markets need at least 400 bushels and 600 bushels, respectively.
42. A warehouse operator has 24,000 square meters of floor space in which to store two products. Each unit of product I requires 20 square meters of floor space and costs \$12 per day to store. Each unit of product II requires 30 square meters of floor space and costs \$8 per day to store. The total storage cost per day cannot exceed \$12,400.
43. **Optimal Revenue** A tailor has 12 square feet of cotton, 21 square feet of silk, and 11 square feet of wool. A vest requires 1 square foot of cotton, 2 square feet of silk, and 3 square feet of wool. A purse requires 2 square feet of cotton, 1 square foot of silk, and 1 square foot of wool. The purse sells for \$80 and the vest sells for \$50.
- (a) How many purses and vests should the tailor make to maximize revenue?
- (b) What is the maximum revenue?

44. **Optimal Income** A wood carpentry workshop has 400 board-feet of plywood, 487 board-feet of birch, and 795 board-feet of pine. A bar stool requires 1 board-foot of plywood, 2 board-feet of birch, and 1 board-foot of pine. A step stool requires 1 board-foot of plywood, 1 board-foot of birch, and 3 board-feet of pine. An ottoman requires 2 board-feet of plywood, 1 board-foot of birch, and 1 board-foot of pine. The bar stool sells for \$22, the step stool sells for \$42, and the ottoman sells for \$29. What combination of products yields the maximum income?

**Optimal Cost** In Exercises 45–48, an athlete uses two dietary supplement drinks that provide the nutritional elements shown in the table.

Drink	Protein	Carbohydrates	Vitamin D
I	4	2	1
II	1	5	1

Find the combination of drinks of minimum cost that will meet the minimum requirements of 4 units of protein, 10 units of carbohydrates, and 3 units of vitamin D.

45. Drink I costs \$5 per liter and drink II costs \$8 per liter.
46. Drink I costs \$7 per liter and drink II costs \$4 per liter.
47. Drink I costs \$1 per liter and drink II costs \$5 per liter.
48. Drink I costs \$8 per liter and drink II costs \$1 per liter.

49. **Optimal Cost** A farming cooperative mixes two brands of cattle feed. Brand X costs \$35 per bag, and brand Y costs \$40 per bag. Research and available resources determined the constraints below.

- (a) Brand X contains one unit of nutritional element A, one unit of element B, and one unit of element C.
- (b) Brand Y contains three units of nutritional element A, one unit of element B, and six units of element C.
- (c) The minimum requirements for nutrients A, B, and C are 21 units, 9 units, and 30 units, respectively.

What is the optimal number of bags of each brand that should be mixed? What is the optimal cost?

50. **Investment** An investor has up to \$250,000 to invest in three types of investments. Type A investments pay 8% annually and have a risk factor of 0. Type B investments pay 10% annually and have a risk factor of 0.06. Type C investments pay 14% annually and have a risk factor of 0.10. To have a well-balanced portfolio, the investor imposes some conditions. The average risk factor should be no greater than 0.05. Moreover, at least one-fourth of the total portfolio is to be allocated to type A investments and at least one-fourth of the portfolio is to be allocated to type B investments. How much should the investor allocate to each type of investment to obtain a maximum return?

51. **Optimal Cost** A company owns three mines that have the daily production levels (in metric tons) shown in the table.

Mine	Grade of Ore		
	High	Medium	Low
A	1	2	3
B	1	2	2
C	2	1	1

The company needs 60 metric tons of high-grade ore, 48 metric tons of medium-grade ore, and 55 metric tons of low-grade ore. How many days should each mine operate to minimize the cost of meeting these requirements when the daily operating costs are \$200 for mine A, \$200 for mine B, and \$100 for mine C, and what is the minimum total cost?

52. **Optimal Cost** Rework Exercise 51 using the daily production schedule shown in the table.

Mine	Grade of Ore		
	High	Medium	Low
A	2	1	2
B	1	1	1
C	4	2	1

# 9 Projects



## 1 Beach Sand Replenishment (I)

A lakeside state park is replenishing the sand on its beaches that was lost due to erosion. The table shows the distances from the sand source sites to the beaches where the sand will be used, the maximum numbers of truckloads of sand per day that can be obtained from each site, and the minimum numbers of truckloads of sand required per day at each beach.

Sand source	Distance to beach (miles)			Maximum truckloads per day
	Beach 1	Beach 2	Beach 3	
A	8	15	50	20
B	10	17	20	30
C	30	26	15	45
Demand (truckloads per day)	30	35	30	

The average round-trip cost to operate the trucks is \$2 per mile, regardless of whether the trucks are loaded or empty. The **northwest corner method** can be used to allocate truckloads to meet the demands.

1. Rewrite the table to show the daily round-trip costs from each source to each beach.
2. In the top entry in the first column, corresponding to source A and beach 1, allocate either all the demand for the row or all the supply for the column, whichever is lesser. Repeat for each entry in the first column (if needed) until the demand for that column is depleted.
3. Repeat part 2 for the second column, taking into consideration the amounts already allocated in the first column.
4. Repeat part 2 for the third column, taking into consideration the amounts already allocated in the first and second columns.
5. Find the daily transportation cost corresponding to the allocations you made in parts 2–4.
6. Use the Internet or some other reference source to research the northwest corner method. Does the method give an optimal solution? Explain.

## 2 Beach Sand Replenishment (II)

Rework Project 1 using the simplex method.

1. Write an objective function to represent the situation.
2. Write the corresponding set of constraints to represent the situation.
3. Use a software program or a graphing utility to determine the daily numbers of truckloads of sand that should be hauled from each source to each beach to minimize the total daily transportation cost. Then find the minimum daily transportation cost.
4. Compare your solution to that of Project 1.