INTRODUCTION OF SET THEORY

A **Set** is an unordered collection of objects, known as elements or members of the set. An element 'a' belong to a set A can be written as 'a ∈ A', 'a ∉ A' denotes that a is not an element of the set A.

Examples

N1 = {1,2,3,...} - the set of all natural numbers.

N0 = {0,1,2,3,...} - the set of whole numbers.

Z = { ... -3, -2, -1, 0, 1, 2, 3, ... } - the set of integers.

Q - the set of rational numbers.

P - the set of irrational numbers.

R - the set of real numbers.

The set S that contains no element is called the **empty set** or the **null set** and is denoted by **{ } or ∅.** A set that has only one element is called a **singleton set**.

Representation of a Set

A set can be represented by various methods. 3 common methods used for representing set:

1. Statement form.
2. Roaster form or tabular form method.
3. Set Builder method.

Statement form

In this representation, the well-defined description of the elements of the set is given. Below are some examples of the same.

1. The set of all even number less than 10.
2. The set of the number less than 10 and more than 1.

Roster form

In this representation, elements are listed within the pair of brackets {} and are separated by commas. Below are two examples.

1. Let N is the set of natural numbers less than 5.

N = { 1 , 2 , 3, 4 }.

2. The set of all vowels in the English alphabet.

V = { a , e , i , o , u }.

Set builder form

In Set-builder set is described by a property that its member must satisfy.

1. {x : x is even number divisible by 6 and less than 100}.
2. {x : x is natural number less than 10}.

Equal sets

Two sets are said to be equal if both sets have the **same elements**. For example A = {1, 3, 9, 7} and B = {3, 1, 7, 9} are equal sets.

Equivalent sets

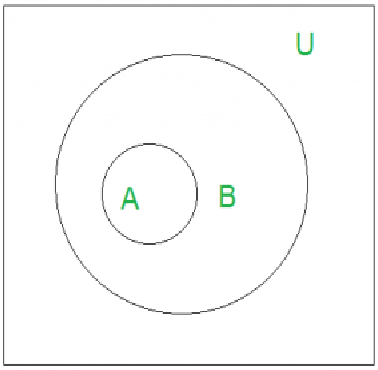
Two sets are said to be equivalent if both sets have the **same size or cardinality**. For example A = {1, 3, 9, 7} and B = {3, 1, 20, 9} are equivalent sets. The reason is that the

Card(A)=4 and the
Card(B)=4
This makes both sets equivalent

NOTE: Order of elements of a set doesn't matter.

Subset

A set A is said to be **subset** of another set B if and only if every element of set A is also a part of other set B.
Denoted by '⊆'.
'A ⊆ B ' denotes A is a subset of B.
To prove A is the subset of B, we need to simply show that if x belongs to A then x also belongs to B.



'U' denotes the universal set.
Above Venn Diagram shows that A is a subset of B.

Size of a Set
Size of a set can be finite or infinite. Size of the set S is known as **Cardinality number**, denoted as |S| or n(S) or Card(S) or ሰ# S
Example: Let A be a set of odd positive integers less than 10.
Solution : A = {1,3,5,7,9}, Cardinality of the set is 5, i.e.,|A| = 5 or n(A) = 5.

Note: Cardinality of a null set is 0.

Power Sets
The power set is the of set all possible subset of the set S. Denoted by P(S).
Example: What is the power set of {0,1,2}?
Solution: All possible subsets
{Ø}, {0}, {1}, {2}, {0,1}, {0,2}, {1,2}, {0,1,2}.
Note: Empty set and set itself is also the member of this set of subsets.

Cardinality of power set is 2^n, where n is the number of elements in a set.
Example

How many subset can be formed from the A={1, 2, 3, 4 }.

Solution

$$2^n = 2^4 = 16$$

Cartesian Products
Let A and B be two sets. Cartesian product of A and B is denoted by A × B, is the set of all ordered pairs (a,b), where a belong to A and b belong to B.
A × B = {(a, b) | a ∈ A ∧ b ∈ B}.

Example 1. What is Cartesian product of A = {1,2} and B = {p, q, r}.
Solution : A × B ={(1, p), (1, q), (1, r), (2, p), (2, q), (2, r) };

The cardinality of A × B is N*M, where N is the Cardinality of A and M is the cardinality of B.
Note: A × B is not the same as B × A.

OPERATIONS ON SETS

Definition: The **intersection of two sets S and T** is the collection of all objects that are in both sets. It is written

S ∩ T . Using curly brace notation

S ∩ T = {x : (x ∈ S) and (x ∈ T)}

Example Intersections of sets

Suppose S = {1, 2, 3, 5},

T = {1, 3, 4, 5}, and U = {2, 3, 4, 5}. Then:

S ∩ T = {1, 3, 5},

S ∩ U = {2, 3, 5}, and

T ∩ U = {3, 4, 5}

Definition: If A and B are sets and A ∩ B = ∅ then we say that A and B are **disjoint**, or **disjoint sets**.

Definition: The **union of two sets S and T** is the collection of all objects that are in either set. It is written S ∪ T . Using curly brace notion

S ∪ T = {x : (x ∈ S) or (x ∈ T)}

Example Unions of sets.

$$Suppose\ \ S = \{1,2,3\},\ \ T = \{1,3,5\},\ \ and\ \ U = \{2,3,4,5\}.$$

Then:

$S \cup T = \{1,2,3,5\}$,

$S \cup U = \{1, 2, 3, 4, 5\}, \quad and\, T \cup U = \{1, 2, 3, 4, 5\}$

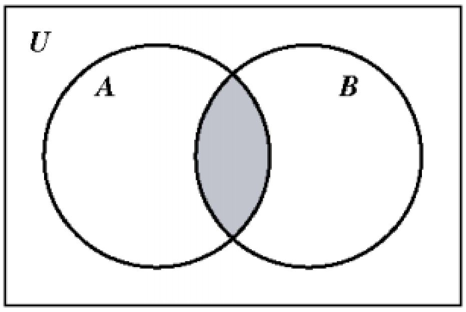
When performing set theoretic computations, you should declare the domain in which you are working. In set theory this is done by declaring a universal set.

Definition: The **universal set,** at least for a given collection of set theoretic computations, is the set of all possible objects. The universal set is commonly written U.

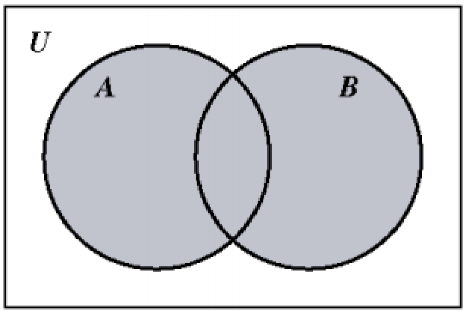
Venn Diagrams

A Venn diagram is a way of depicting the relationship between sets. Each set is shown as a circle and circles overlap if the sets intersect.

Example. The following are Venn diagrams for the intersection and union of two sets. The shaded parts of the diagrams are the intersections and unions respectively.



$A \cap B$



$A \cup B$

Notice that the rectangle containing the diagram is labelled with a U representing the universal set.

Definition The **compliment of a set** S is the collection of objects in the universal set that are not in S. The compliment is written S^c. In curly brace notation

$S^c = \{x : (x \in U) \wedge (x \notin S)\}$ or more compactly as

$S^c = \{x : x \notin S\}$

however it should be apparent that the compliment of a set always depends on which universal set is chosen.

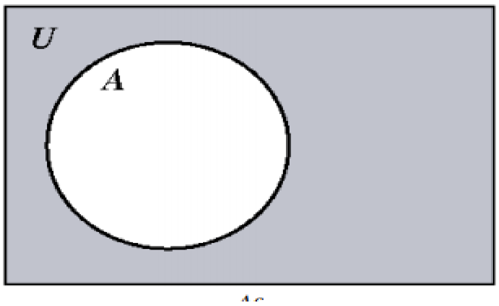
Example Set Compliments

(i) Let the universal set be the integers. Then the compliment of the even integers is the odd integers.

(ii) Let the universal set be {1, 2, 3, 4, 5}, then the compliment of S = {1, 2, 3} is S^c = {4, 5} while the compliment of T = {1, 3, 5} is T^c = {2, 4}.

(iii) Let the universal set be the letters {a, e, i, o, u, y}. Then $\{y\}^c$ = {a, e, i, o, u}.

The Venn diagram for A^c is



Examples:

1. Let A = {1, 2, 4, 18} and B = {x : x is an integer, $0 < x \leq 5$}. Then,

$A \cup B$ = {1, 2, 3, 4, 5, 18} and $A \cap B$ = {1, 2, 4}.

2. Let $S = \{x \in R : 0 \leq x \leq 1\}$ and $T = \{x \in R : .5 \leq x < 7\}$. Then,

$S \cup T = \{x \in R : 0 \leq x < 7\}$ and $S \cap T = \{x \in R : .5 \leq x \leq 1\}$.

3. Let X = {{b, c}, {{b}, {c}}, b} and Y = {a, b, c}. Then

$X \cap Y$ = {b} and $X \cup Y$ = {a, b, c, {b, c}, {{b}, {c}} }.

Definition: The difference of two sets S and T is the collection of objects in S that are not in T . The difference is written S − T . In curly brace notation

$S - T = \{x : x \in (S \cap (T^c))\}$,

Laws of Set theory

Some rules governing set algebra. Binary operations of union ∪ and intersection ∩ are **commutative and associative**. In mathematical notation these rules are given as:

• **Commutative Law** - $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

• **Associative Law** - $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.

The Distributive Laws. We now consider the two distributive laws known as 'union over intersection' and 'intersection over union'. I will provide proofs as these laws are very important.

• Union Over Intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

• Intersection Over Union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

DeMorgan's Laws.

• De Morgan's First Law - the complement of the union equals the intersection of the complements. That is, $(A \cup B)^c = A^c \cap B^c$.

• De Morgan's Second Law - the complement of the intersection equals the union of the complements. That is, $(A \cap B)^c = A^c \cup B^c$.

Identity Laws

Let X be the universal set and let $A \subset X$ be an arbitrary subset of X. The following identity rules hold: • $A \cup \emptyset = A$.

• $A \cap X = A$.

• $A \cup A = A$.

• $A \cap A = A$.

Complement Laws.

- $X^c = \emptyset$.
- $\emptyset^c = X$.
- $A \cup A^c = X$.
- $A \cap A^c = \emptyset$.

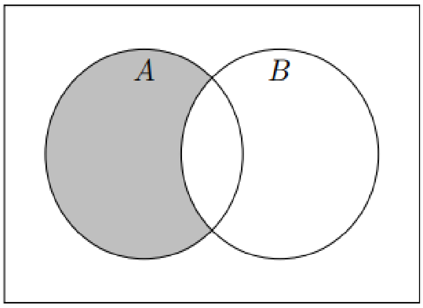
Difference of Sets.

The difference of two sets A and B is the collection of objects in A that are not in B. The difference is written $A - B$. In curly brace notation

$A - B = \{x : x \in (A \cap (B^c))\}$,

or alternately

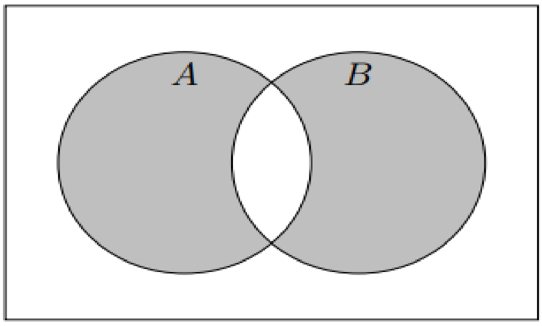
$A - B = \{x : (x \in A) \wedge (x \notin B)\}$



The set $A - B = A \cap B^c$. Looking at the above diagram, it is apparent $A - B$ must also equate to $A \cap (A \cap B)^c$.

This is easily verified using the distributive laws: $A \cap (A \cap B)^c = A \cap (A^c \cup B^c) = (A \cap A^c) \cup (A \cap B^c) = \emptyset \cup (A \cap B^c) = A \cap B^c = A - B$

The Symmetric Difference. We may conceptually extend the difference to form another set, the symmetric difference of sets. Denoted by $A \Delta B$, the symmetric difference (also known as the disjunctive union) is simply the union of $A - B$ and $B - A$ viz, $A \Delta B := (A - B) \cup (B - A) = (A \cap B^c) \cup (B \cap A^c)$. We illustrate the symmetric difference in the Venn diagram below.



We note the symmetric difference may also be expressed as

$$A \,\Delta\, B = (A \cup B) - (A \cap B)$$

Some Rules Regarding the Null Set.

As mentioned earlier, the null or empty set is that set containing no elements. It is denoted by Ø which actually is a Danish letter and not the Greek letter φ. **The null set is a subset of every set** and thus is a subset of itself: $\emptyset \subseteq \emptyset$. Now let us consider the following questions.

Power Sets

For a set X, the power set of X is the set of all subsets of X. We denote the power set of X by 2^X. Example.

0) The set Ø has 0 elements. Its power set is $2^{\emptyset} = \{\emptyset\}$, which has $1 = 2^0$ elements.

(i) The set [1] = {1} has 1 element. Its power set is $2^{[1]} = \{\emptyset, \{1\}\}$, which has $2 = 2^1$ elements.

(ii) The set [2] = {1, 2} has 2 elements. Its power set is $\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, which has $4 = 2^2$ elements.

Proposition. Let S be a finite set of cardinality n. Then the power set 2^S is finite of cardinality 2^n.

Partitions

Let X be a nonempty set. A partition of X is, roughly, an exhaustive division of X into nonoverlapping nonempty pieces. More precisely, a partition of X is a set P of subsets of X satisfying all of the following properties:

(P1) $\bigcup_{S \in P} S = X$.

(P2) For distinct elements $S \neq T$ in P, we have $S \cap T = \emptyset$.

(P3) If $S \in P$ then $S \neq \emptyset$.

Example.

Let X = [5] = {1, 2, 3, 4, 5}. Then:

a) The set P_1 = {{1, 3}, {2}, {4, 5}} is a partition of X.

b) The set P_2 = {{1, 2, 3}, {4}} is not a partition of X: $5 \in X$, but 5 is not an element of any element of P_2, so (P_1) fails. However, (P_2) and (P_3) both hold.

c) The set P_3 = {{1, 2, 3}, {3, 4, 5}} is not a partition of X because {1, 2, 3} and {3, 4, 5} are not disjoint sets. However, (P1) and (P3) both hold.

d) The set P_4 = {{1, 2, 3, 4, 5}, ∅} is not a partition of X because it contains the empty set, so (P3) fails. However, (P1) and (P2) both hold.

Example.

a) Let X = [1] = {1}. There is exactly one partition, P = {X}. b) Let X = [2] = {1, 2}. There are two partitions on X, P1 = {{1, 2}}, P2 = {{1}, {2}}

Computer Representation of Sets

Example Let U = {0, 1, 2, 3, 4, 5, 6, 7} and let A = {0, 1, 6, 7} Then the bit vector representing A is

1100 0011

Set operations become almost trivial when sets are represented by bit vectors. In particular, the bit-wise Or corresponds to the union operation. The bit-wise And corresponds to the intersection operation.

Let U = {0, 1, 2, 3, 4, 5, 6, 7}, A = {0, 1, 6, 7} and B = {0, 4, 5}

The bit vector for A is 1100 0011 and

B is 1000 1100

The union, $A \cup B$ can be computed by $1100\ 0011 \vee 1000\ 1100 = 1100\ 1111$

The intersection, $A \cap B$ can be computed by $1100\ 0011 \wedge 1000\ 1100 = 1000\ 0000$