Relations and Functions

Functions and relations are one the most important topics in Algebra. In most occasions, many people tend to confuse the meaning of these two terms. We are going to define and elaborate on **how you can identify if a relation is a function**. Before we go deeper, let's look at a brief history of functions.

The concept of function was brought to light by mathematicians in 17th century. In 1637, a mathematician and the first modern philosopher, Rene Descartes, talked about many mathematical relationships in his book *Geometry*, but the term "function" was officially first used by German mathematician Gottfried Wilhelm Leibniz after about fifty years. He invented a notation y = x to denote a function, dy/dx to denote the derivative of a function. The notation y = f (x) was introduced by a Swiss mathematician Leonhard Euler in 1734.

Some key concepts used in functions and relations

A set is a collection of distinct or well-defined members or elements**.**

- **What is ordered-pair numbers?**

These are numbers that go hand in hand. Ordered pair numbers are represented within parentheses and separated by a comma. For example, (6, 8) is an ordered-pair number whereby the numbers 6 and 8 are the first and second element respectively.

- **What is a domain?**

A domain is a **set of all input or first values of a function**. Input values are generally 'x' values of a function.

- **What is a range?**

The range of a function is a collection of all output or second values. Output values are 'y' values of a function.

- **What is a function?**

In mathematics, **a function can be defined as rule that relates every element in one set**, called the domain, to exactly one element in another set, called the range. For example, y = x + 3 and y = x^2 – 1 are functions because every x-value produces a different y-value.

- **A relation**

A relation is any set of ordered-pair numbers. In other words, we can define a relation as a bunch ordered pairs.

Types of Relations

Different types of relations are as follows:

- Empty Relations
- Universal Relations
- Identity Relations
- Inverse Relations
- Reflexive Relations
- Symmetric Relations
- Transitive Relations

Empty Relation

When there's no element of set X is related or mapped to any element of X, then the relation R in A is an empty relation, and also called the void relation, i.e R= ∅.

For example, if there are 100 mangoes in the fruit basket. There's no possibility of finding a relation R of getting any apple in the basket. So, R is Void as it has 100 mangoes and no apples.

Universal relation

R is a relation in a set, let's say A is a universal relation because, in this full relation, every element of A is related to every element of A. i.e R = A × A.

It's a full relation as every element of Set A is in Set B.

Identity Relation

If every element of set A is related to itself only, it is called Identity relation.
I={(A, A), ∈ a}.

For Example,

When we throw a dice, the total number of possible outcomes is 36. i..e (1, 1) (1, 2), (1, 3)…..(6, 6). From these, if we consider the relation (1, 1), (2, 2), (3, 3) (4, 4) (5, 5) (6, 6), it is an identity relation.

Inverse Relation

If R is a relation from set A to set B i.e R ∈ A X B. The relation R^{-1} = {(b,a):(a,b) ∈ R}. For Example,

If you throw two dice if R = {(1, 2) (2, 3)}, R^{-1} = {(2, 1) (3, 2)}. Here the domain is the range R^{-1} and vice versa.

Reflexive Relation

A relation is a reflexive relation iIf every element of set A maps to itself, i.e for every a ∈ A, (a, a) ∈ R.

Example − The relation R={(a,a),(b,b) on set X={a,b} is reflexive.

Symmetric Relation

A symmetric relation is a relation R on a set A if (a, b) ∈ R then (b, a) ∈ R, for all a & b ∈ A.

if aRb implies bRa

Example − The relation R={(1,2),(2,1),(3,2),(2,3)} on set A={1,2,3}is symmetric.

Transitive Relation

If (a, b) ∈ R, (b, c) ∈ R, then (a, c) ∈ R, for all a,b,c ∈ A and this relation in set A is transitive.

Example − The relation R={(1,2),(2,3),(1,3)} on set A={1,2,3} is transitive.

Equivalence Relation

If a relation is reflexive, symmetric and transitive, then the relation is called an equivalence relation.

Example − The relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is an equivalence relation since it is reflexive, symmetric, and transitive.

A function is a relation which describes that there should be only one output for each input.

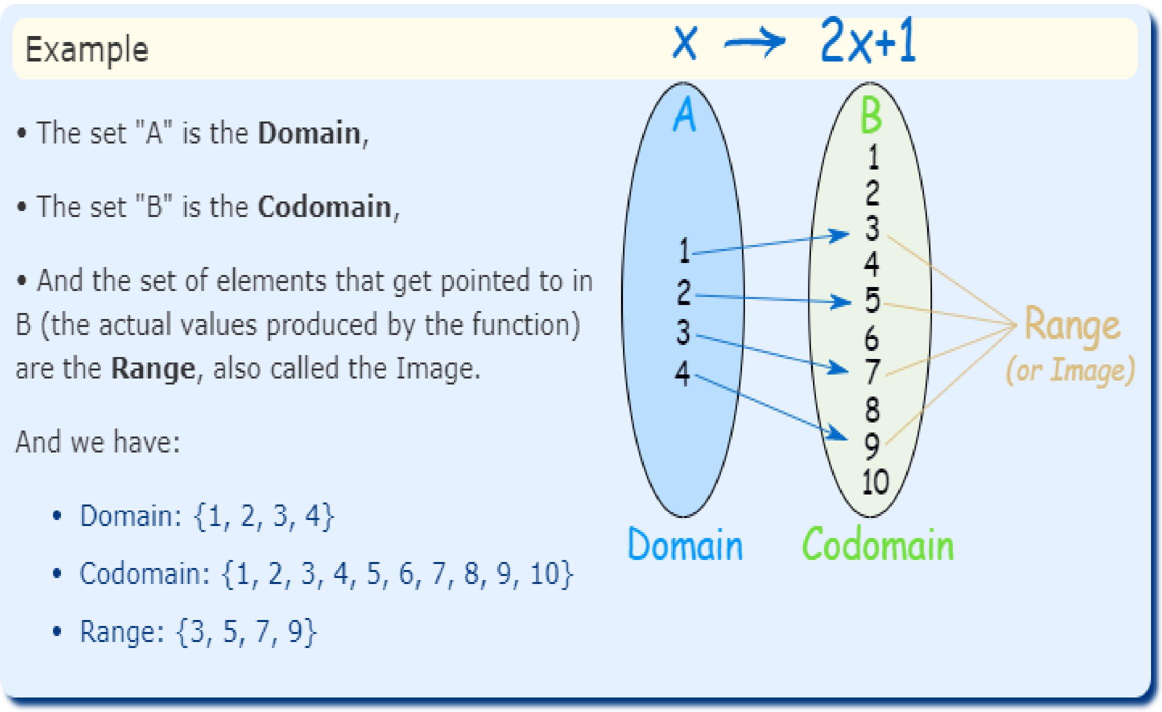
Domain, Codomain and Range

There are special names for **what can go into**, and **what can come out** of a function:

What can go **into** a function is called the **Domain**

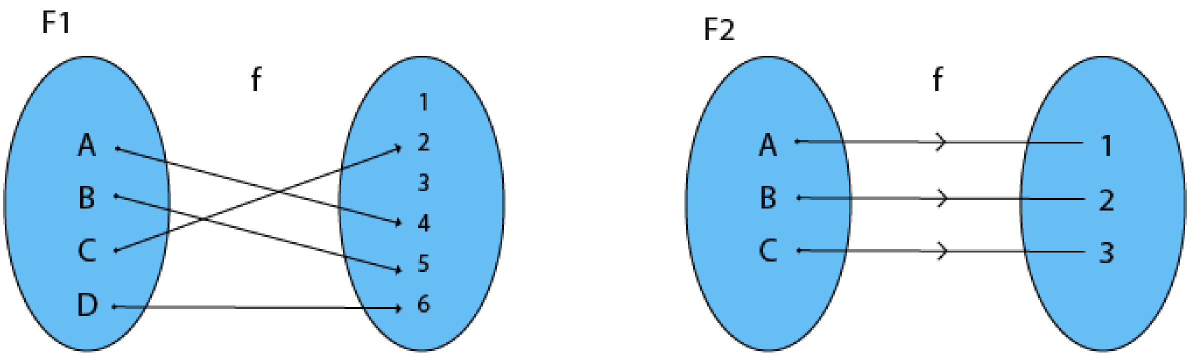
What **may possibly come out** of a function is called the **Codomain**

What **actually comes out** of a function is called the **Range**



Types of Functions

1. Injective (One-to-One) Functions: A function in which one element of Domain Set is connected to one element of Co-Domain Set.

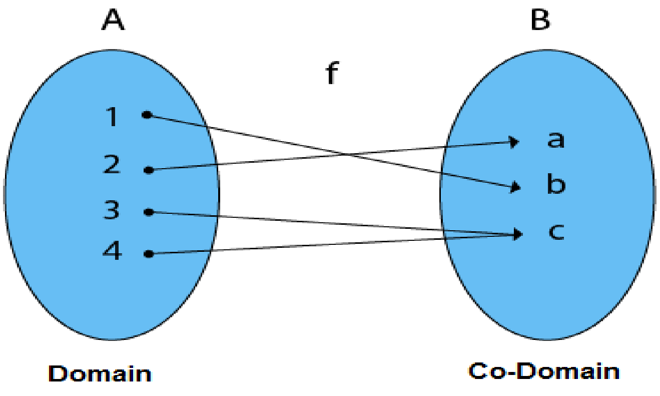


F1 and F2 show one to one Function

2. Surjective (Onto) Functions: A function in which every element of Co-Domain Set has one pre-image.

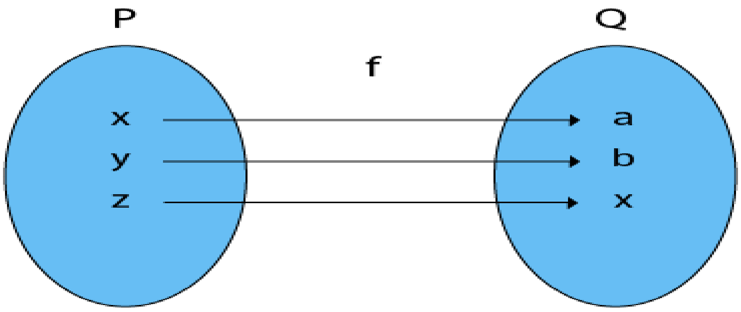
Example: Consider, A = {1, 2, 3, 4}, B = {a, b, c} and f = {(1, b), (2, a), (3, c), (4, c)}.

It is a Surjective Function, as every element of B is the image of some A.



Note: In an Onto Function, Range is equal to Co-Domain.

3. Bijective (One-to-One Onto) Functions: A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.



Example:

Consider P = {x, y, z}

Q = {a, b, c}

and f: P → Q such that

f = {(x, a), (y, b), (z, c)}

The f is a one-to-one function and also it is onto. So it is a bijective function.

4. Into Functions: A function in which there must be an element of co-domain Y does not have a pre-image in domain X.

Example:

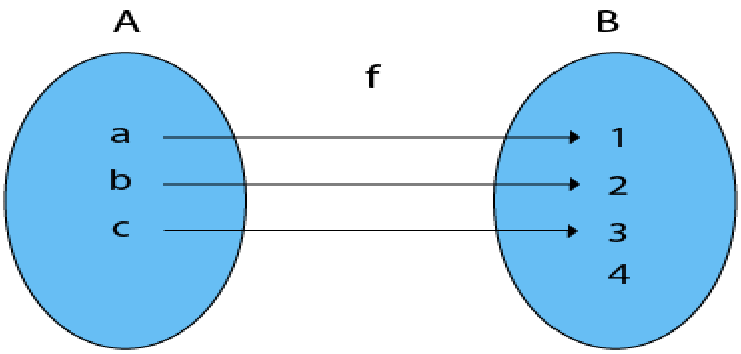
Consider, A = {a, b, c}

B = {1, 2, 3, 4}   and f: A → B such that

f = {(a, 1), (b, 2), (c, 3)}

In the function f, the range i.e., {1, 2, 3} ≠ co-domain of Y i.e., {1, 2, 3, 4}

Therefore, it is an into function



5. One-One Into Functions: Let f: X → Y. The function f is called one-one into function if different elements of X have different unique images of Y.

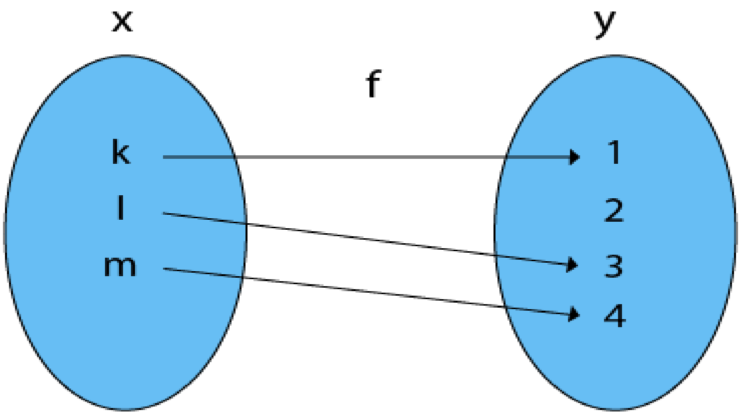
Example:

Consider, X = {k, l, m}

Y = {1, 2, 3, 4} and f: X → Y such that

f = {(k, 1), (l, 3), (m, 4)}

The function f is a one-one into function



6. Many-One Functions: Let f: X → Y. The function f is said to be many-one functions if there exist two or more than two different elements in X having the same image in Y.

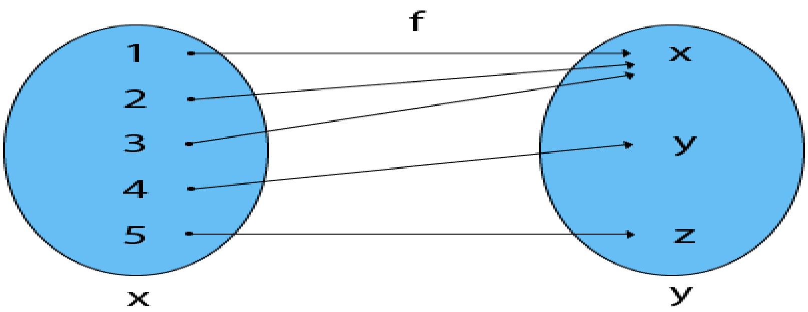
Example:

Consider X = {1, 2, 3, 4, 5}

Y = {x, y, z} and f: X → Y such that

f = {(1, x), (2, x), (3, x), (4, y), (5, z)}

The function f is a many-one function



7. Many-One Into Functions: Let f: X → Y. The function f is called the many-one function if and only if is both many one and into function.

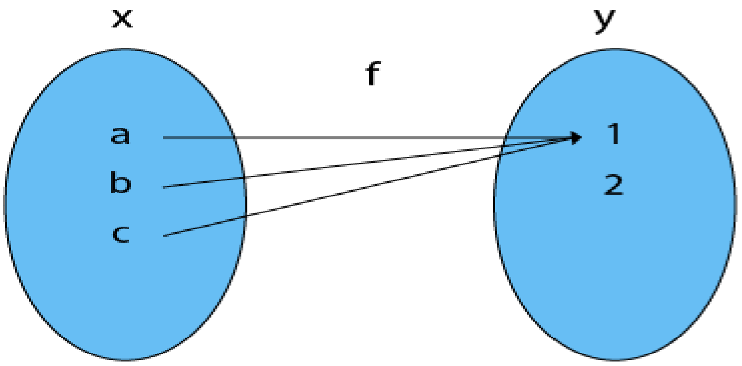
Example:

Consider X = {a, b, c}

Y = {1, 2} and f: X → Y such that

f = {(a, 1), (b, 1), (c, 1)}

As the function f is a many-one and into, so it is a many-one into function.



8. Many-One Onto Functions: Let f: X → Y. The function f is called many-one onto function if and only if is both many one and onto.

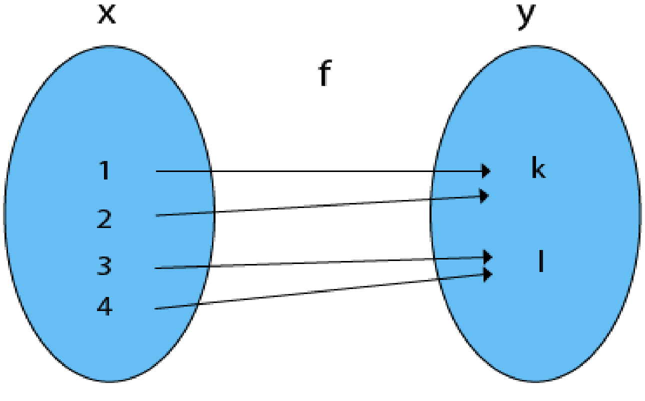
Example:

Consider X = {1, 2, 3, 4}

Y = {k, l} and f: X → Y such that

f = {(1, k), (2, k), (3, l), (4, l)}

The function f is a many-one (as the two elements have the same image in Y) and it is onto (as every element of Y is the image of some element X). So, it is many-one onto function

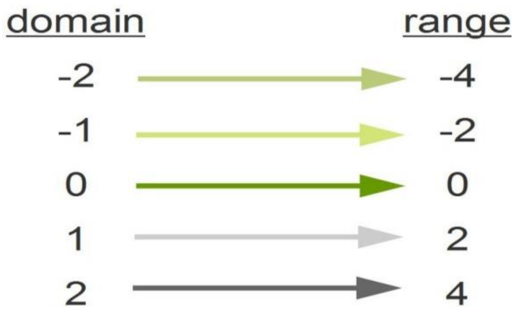


A function is a relation that for each input, there is only one output.

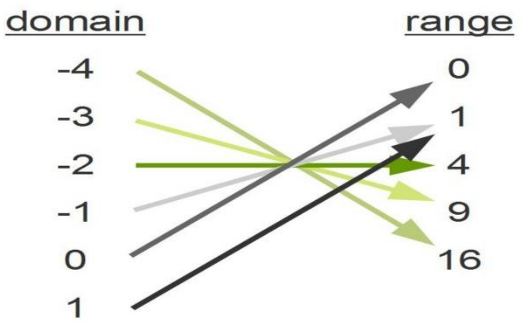
How to Determine if a Relation is a Function?

- Examine the x or input values.
- Examine also the y or output values.
- If all the input values are different, then the relation becomes a function, and if the values are repeated, the relation is not a function

Here are mappings of functions. The **domain** is the input or the **x-value**, and the **range** is the output, or the **y-value**.

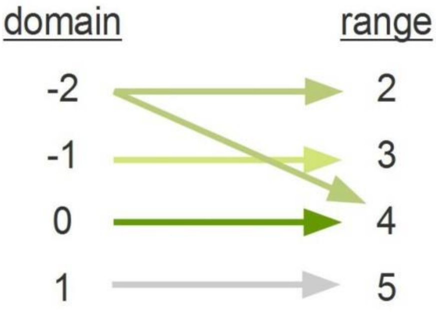


Each x-value is related to only one y-value.



Although the inputs equal to -1 and 1 have the same output, this relation is still a function because each input has just one output.

This mapping is not a function. The input for -2 has more than one output.



Special Functions in Algebra

Some of the important functions are as follow:

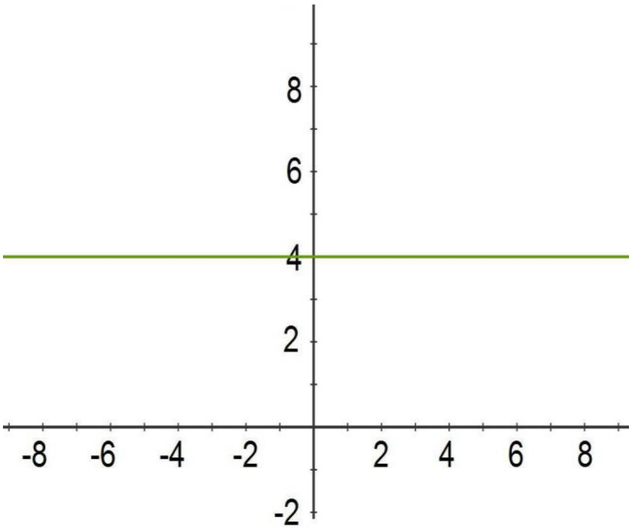
- Constant Function
- Identity Function
- Linear Function
- Absolute Value Function
- Inverse Functions

Special Functions

Special functions and their equations have recognizable characteristics.

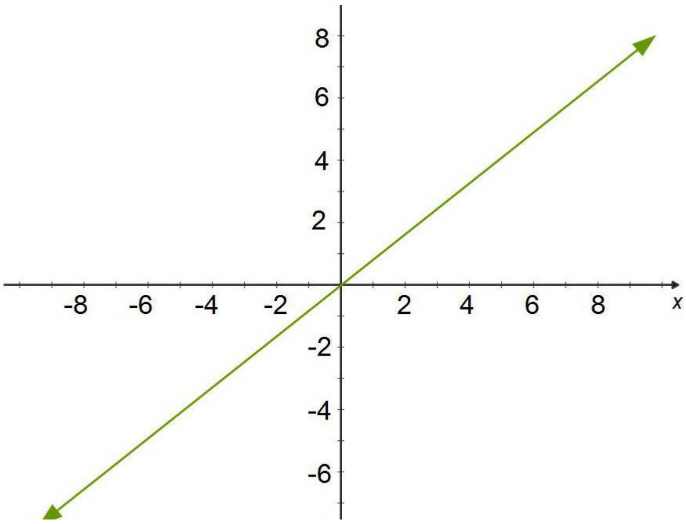
Constant Function
f(x) = c

The c-value can be any number, so the graph of a constant function is a horizontal line. Here is the graph of f(x) = 4



Identity Function
f(x) = x

For the **identity function**, the x-value is the same as the y-value. The graph is a diagonal line going through the origin.

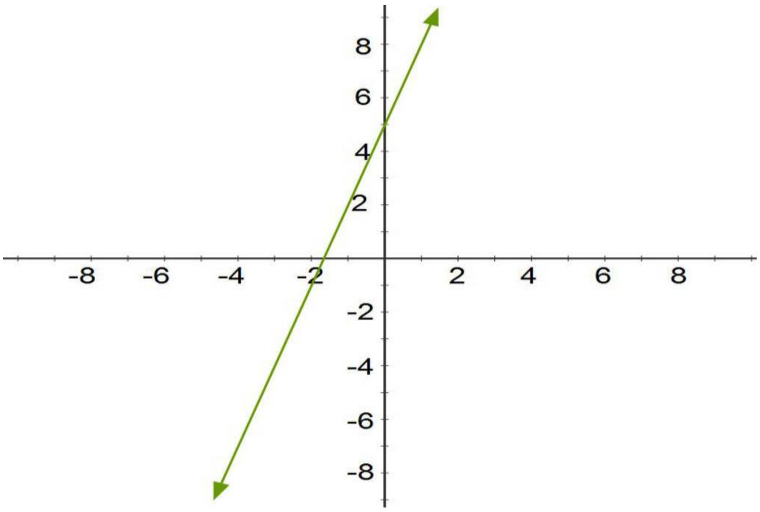


Linear Function

f(x) = mx + b

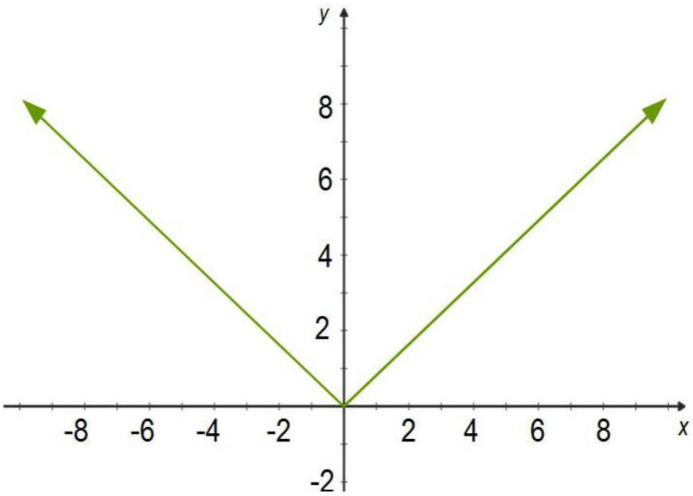
An equation written in the **slope-intercept form** is the equation of a **linear function**, and the graph of the function is a straight line.

Here is the graph of f(x) = 3x +4



Absolute Value Function
f(x) = |x|

The **absolute value function** is easy to recognize with its V-shaped graph. The graph is in two pieces and is one of the piecewise functions.



Example 1

Identify the range and domain of the relation below:

{(-2, 3), {4, 5), (6, -5), (-2, 3)}

<u>Solution</u>

Since the x values are the domain, the answer is therefore,

$\Longrightarrow$ {-2, 4, 6}

The range is {-5, 3, 5}.

Example 2

Check whether the following relation is a function:

B = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)}

<u>Solution</u>

B = {(1, 5), (1, 5), (3, -8), (3, -8), (3, -8)}

Though a relation is not classified as a function if there is repetition of x – values, this problem is a bit tricky because x values are repeated with their corresponding y-values.

Example 3

Determine the domain and range of the following function: Z = {(1, 120), (2, 100), (3, 150), (4, 130)}.

<u>Solution</u>

Domain of z = {1, 2, 3, 4} and the range is {120, 100, 150, 130}

Example 4

Check if the following ordered pairs are functions:

1. W= {(1, 2), (2, 3), (3, 4), (4, 5)
2. Y = {(1, 6), (2, 5), (1, 9), (4, 3)}

<u>Solution</u>

1. All the first values in W = {(1, 2), (2, 3), (3, 4), (4, 5)} are not repeated, therefore, this is a function.
2. Y = {(1, 6), (2, 5), (1, 9), (4, 3)} is not a function because, the first value 1 has been repeated twice.

Example 5

Determine whether the following ordered pairs of numbers is a function.

R = (1,1); (2,2); (3,1); (4,2); (5,1); (6,7)

<u>Solution</u>

There is no repetition of x values in the given set of ordered pair of numbers.

Therefore, R = (1,1); (2,2); (3,1); (4,2); (5,1); (6,7) is a function.

Function Notation

Function notation is a way in which a function can be represented using symbols and signs. Function notation is a simpler method of describing a function without a lengthy written explanation.

Consider a linear function y = 3x + 7. To write such function in function notation, we simply replace the variable y with the phrase f(x) to get;

f(x) = 3x + 7. This function f(x) = 3x + 7 is read as the value of f at x or as f of x.

How to Evaluate Functions?

Function evaluation is the process of determining output values of a function. This is done by substituting the input values in the given function notation.

Example 1

Write $y = x^2 + 4x + 1$ using function notation and evaluate the function at $x = 3$.

<u>Solution</u>

Given, $y = x^2 + 4x + 1$

By applying function notation, we get

$f(x) = x^2 + 4x + 1$

Evaluation:

Substitute x with 3

$f(3) = 3^2 + 4 \times 3 + 1 = 9 + 12 + 1 = 22$

Example 2

Evaluate the function $f(x) = 3(2x+1)$ when $x = 4$.

<u>Solution</u>

Plug $x = 4$ in the function $f(x)$.

$f(4) = 3[2(4) + 1]$

$f(4) = 3[8 + 1]$

$f(4) = 3 \times 9$

$f(4) = 27$

Example 3

Write the function $y = 2x^2 + 4x – 3$ in function notation and find $f (2a + 3)$.

Solution

$y = 2x^2 + 4x – 3 \Longrightarrow f (x) = 2x^2 + 4x – 3$

Substitute x with $(2a + 3)$.

$f (2a + 3) = 2(2a + 3)^2 + 4(2a + 3) – 3$

$= 2(4a^2 + 12a + 9) + 8a + 12 – 3$
$= 8a^2 + 24a + 18 + 8a + 12 – 3$
$= 8a^2 + 32a + 27$

Example 4

Represent $y = x^3 – 4x$ using function notation and solve for y at $x = 2$.

Solution

Given the function $y = x^3 – 4x$, replace y with f(x) to get;

$f(x) = x^3 – 4x$

Now evaluate f(x) when x = 2

$\Longrightarrow f (2) = 2^3 – 4 \times 2 = 8 -8 = 0$

Therefore, the value of y at x=2 is 0

Real life examples of function notation

Function notation can be applied in real life to evaluate mathematical problems as shown in the following examples:

Example 1

To manufacture a certain product, a company spends x dollars on raw materials and y dollars on the labor. If the production cost is described by the function f (x, y) = 36000 + 40x + 30y + xy/100. Calculate cost of production when the firm spends 10,000 and1,000 on raw materials and labor respectively.

Solution

Given x = 10,000andy=1,000

Substitute the values of x and y in the production cost function

$\Longrightarrow$f (10000, 1000) = 36000 + 40(10000) + 30(1000) + (10000) (1000)/100.

$\Longrightarrow$ f (10000, 1000) = 36000 + 400000 + 30000 + 100000

$\Longrightarrow$ \$566,000.

Example 2

Mary saves 100 weekly for her upcoming birthday party. If she already has 1000, how much will she have after 22 weeks.

Solution

Let x = number of weeks, and f(x) = total amount. We can write this problem in function notation as;

f(x)=100x + 1000
Now evaluate the function when x =22
f (22) =100(22) +1000
f (22) =3200

Therefore, the total amount is \$3200.

Example 3

A certain number is such that when its added to 142, the result is 64 more than thrice the original number. Find the number.

<u>Solution</u>

Let x = the original number and f(x) be the resultant number after adding 142.

f(x) = 142 + x = 3x + 64

2x = 78

x = 39

Example 4

If the product of two consecutive positive integers is 1122, find the two integers.

<u>Solution</u>

Let x be the first integer;

second integer = x + 1

Now form the function as;

f(x) = x (x + 1)

find the value of x if f(x) = 1122

Replace the function f(x) by 1122

1122 = x (x + 1)

1122 = x^2 + x

$$x^2 + x - 1122 = 0$$

$$x^2 + 34x - 33x - 1122$$

$$X(x+34)-33(x+34)$$

$$(x+34)(x-33)$$

$$x-33=0 \qquad x+34=0$$

$$x=33 \qquad x=-34$$

We will choose the positive integer 33, and 33+1 = 34

Arithmetic Operations on Functions

Functions can also be added, subtracted, multiplied and divided by following the same rules and steps. Although function notation will look different at first, but you will still arrive at the correct answer.

Addition of Functions

To add functions, we collect the like terms and add them together. Variables are added by taking the sum of their coefficients.

Example 1

Add f(x) = x + 2 and g(x) = 5x – 6

Solution

(f + g) (x) = f(x) + g(x)
= (x + 2) + (5x – 6)
= 6x – 4

Example 2

Add the following functions: f(x) = $3x^2$ – 4x + 8 and g(x) = 5x + 6

Solution

$\Longrightarrow$ (f + g) (x) = $(3x^2 - 4x + 8) + (5x + 6)$

Collect the like terms

$= 3x^2 + (-4x + 5x) + (8 + 6)$

$= 3x^2 + x + 14$

Example 3

Add the following functions: $f(x) = 5x^2 + 7x - 6$, $g(x) = 3x^2 + 4x$ and $h(x) = 9x^2 - 9x + 2$

Solution

$5x^2 + 7x - 6$

$+ 3x^2 + 4x$

$\underline{+ 9x^2 - 9x + 2}$

$\underline{16x^2 + 2x - 4}$

Therefore, $(f + g + h)(x) = 16x^2 + 2x - 4$

Subtract of Functions

Example 4

Subtract the function $g(x) = 5x - 6$ from $f(x) = x + 2$

Solution

$(f - g)(x) = f(x) - g(x)$

Place the second function in parentheses.

$= x + 2 - (5x - 6)$

Remove the parentheses by changing the sign within the parentheses.

$= x + 2 – 5x + 6$

Combine like terms

$= x – 5x + 2 + 6$

$= –4x + 8$

Example 5

Subtract $f(x) = 3x^2 – 6x – 4$ from $g(x) = – 2x^2 + x + 5$

<u>Solution</u>

$(g - f) (x) = g(x) - f(x) = – 2x^2 + x + 5 – (3x^2 – 6x – 4)$

Remove the parentheses and change the operators

$= – 2x^2 + x + 5 – 3x^2 + 6x + 4$

Collect like terms

$= -2x^2 – 3x^2 + x + 6x + 5 + 4$

$= -5x^2 + 7x + 9$

Multiplication of Functions

To multiply variables between two or more functions, multiply their coefficients and then add the exponents of the variables.

Example 6

Multiply $f(x) = 2x + 1$ by $g(x)= 3x^2 − x + 4$

Solution

Apply the distributive property

$\Longrightarrow$ (f *g) (x) = f(x) * g(x) = $2x (3x^2 - x + 4) + 1(3x^2 – x + 4)$
$\Longrightarrow (6x^3 - 2x^2 + 8x) + (3x^2 – x + 4)$

Combine and add like terms.

$\Longrightarrow 6x^3 + (-2x^2 + 3x^2) + (8x - x) + 4$

$= 6x^3 + x^2 + 7x + 4$

Example 7

multiply f(x) = x + 2 and g(x) = 5x – 6

Solution

$\Longrightarrow$ (f * g) (x) = f(x) * g(x)
= (x + 2) (5x – 6)
$= 5x^2 + 4x – 12$

Example 8

Find the product of f(x) = x – 3 and g(x) = 2x – 9

Solution

Apply FOIL method

(f * g) (x) = f(x) * g(x) = (x – 3) (2x – 9)

Product of first terms.

$= (x) * (2x) = 2x^2$

Product of outermost terms.

= (x) *(–9) = –9x

Product of the inner terms.

= (–3) * (2x) = –6x

Product of last terms

= (–3) * (–9) = 27

Sum the partial products

= $2x^2 – 9x – 6x + 27$

= $2x^2 – 15x +27$

How to Divide Functions?

Just like polynomials, functions can be also divided using synthetic or long division method.

Example		
Problem	$f(x) = 12x^3 + 15x^2 - 6x$ $g(x) = 3x$ **Find** $\left(\frac{f}{g}\right)(x)$.	
	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ $= \frac{12x^3 + 15x^2 - 6x}{3x}, \quad x \neq 0$ $= \frac{3x(4x^2 + 5x - 2)}{3x}$ $= 1 \cdot (4x^2 + 5x - 2)$ $= 4x^2 + 5x - 2, \qquad x \neq 0$	To find the quotient, divide *f* by *g*. Substitute the polynomials in for *f*(*x*) and *g*(*x*) and divide. We add $x \neq 0$ because *x* = 0 would make the denominator *g*(*x*) =0 and $\frac{f(x)}{g(x)}$ undefined. Remember to rename $\frac{3x}{3x}$ as 1.
Answer	$\left(\frac{f}{g}\right)(x) = 4x^2 + 5x - 2$	

Example	
Problem	$f(x) = 8x^3 - 3x^2$ $g(x) = 4x^3 + 9x^2$ $h(x) = 3x^2$ **Find** $\left(\frac{f+g}{h}\right)(x)$.
$\left(\frac{f+g}{h}\right)(x) = \frac{f(x)+g(x)}{h(x)}$ $= \frac{(8x^3 - 3x^2)+(4x^3+9x^2)}{3x^2}, \quad x \neq 0$	Replace *f*(*x*), *g*(*x*), and *h*(*x*) with the equivalent polynomials. We add $x \neq 0$ because that would make the denominator *h*(*x*) of $\frac{f(x)+g(x)}{h(x)}$ zero and the fraction undefined.
$\left(\frac{f+g}{h}\right)(x) = \frac{8x^3 + 4x^3 + 9x^2 - 3x^2}{3x^2}$ $= \frac{12x^3 + 6x^2}{3x^2}$	Add *f*(*x*) and *g*(*x*).
$\left(\frac{f+g}{h}\right)(x) = \frac{3x^2(4x+2)}{3x^2}$ $= 1 \bullet (4x+2)$ $= 4x+2 \quad x \neq 0$	Divide by *h*(*x*). Pull out a factor of $3x^2$ from the numerator, and then simplify the expression, using $\frac{3x^2}{3x^2} = 1$.
Answer	$\left(\frac{f+g}{h}\right)(x) = 4x+2 \quad x \neq 0$

Example

Divide the functions $f(x) = 6x^5 + 18x^4 – 3x^2$ by $g(x) = 3x^2$

<u>Solution</u>

$\Longrightarrow (f \div g)(x) = f(x) \div g(x) = (6x^5 + 18x^4 – 3x^2) \div (3x^2)$

$\Longrightarrow 6x^5/ 3x^2 + 18x^4/3x^2 – 3x^2/3x^2$

$= 2x^3 + 6x^2 – 1.$

Composite Functions

A composite function is generally a function that is written inside another function. Composition of a function is done by substituting one function into another function.

Solving a composite function means, finding the composition of two functions. We use a small circle (∘) for the composition of a function. Here are the steps on how to solve a composite function:

- Rewrite the composition in a different form.

For example

$(f \circ g)(x) = f[g(x)]$

$(f \circ g)(x^2) = f[g(x^2)]$

- Substitute the variable x that is in the outside function with the inside function.
- Simplify the function.

Note: The order in the composition of a function is important because $(f \circ g)(x)$ is NOT the same as $(g \circ f)(x)$.

Let's look at the following problems:

Example 1

Given the functions $f(x) = x^2 + 6$ and $g(x) = 2x - 1$, find $(f \circ g)(x)$.

Solution

Substitute x with $2x - 1$ in the function $f(x) = x^2 + 6$.

$(f \circ g)(x) = (2x - 1)^2 + 6 = (2x - 1)(2x - 1) + 6$

Apply FOIL

$= 4x^2 - 4x + 1 + 6$

$= 4x^2 - 4x + 7$

Example 2

Given the functions $g(x) = 2x - 1$ and $f(x) = x^2 + 6$, find $(g \circ f)(x)$.

Solution

Substitute x with $x^2 + 6$ in the function $g(x) = 2x - 1$

$(g \circ f)(x) = 2(x^2 + 6) - 1$

Use the distributive property to remove the parentheses.

$= 2x^2 + 12 - 1$

$= 2x^2 + 11$

Example 3

Given $f(x) = 2x + 3$, find $(f \circ f)(x)$.

Solution

$(f \circ f)(x) = f[f(x)]$

$= 2(2x + 3) + 3$

$= 4x + 9$

Example 4

Find $(g \circ f)(x)$ given that, $f(x) = 2x + 3$ and $g(x) = -x^2 + 5$

$\Longrightarrow (g \circ f)(x) = g[f(x)]$

Replace x in $g(x) = -x^2 + 5$ with $2x + 3$
$= -(2x + 3)^2 + 5$
$= -(4x^2 + 12x + 9) + 5$
$= -4x^2 - 12x - 9 + 5$
$= -4x^2 - 12x - 4$

Example 5

Evaluate $f[g(6)]$ given that, $f(x) = 5x + 4$ and $g(x) = x - 3$

Solution

First, find the value of $f(g(x))$.

$\Longrightarrow f(g(x)) = 5(x - 3) + 4$

$= 5x - 15 + 4$

$= 5x - 11$

Now substitute x in $f(g(x))$ with 6

$\Longrightarrow 5(6) - 11$

$\Longrightarrow 30 - 11$

$= 19$

Therefore, $f[g(6)] = 19$

Example 6

Given $g(x) = 2x + 8$ and $f(x) = 8x^2$, Find $(f \circ g)(x)$

Solution

$(f \circ g)(x) = f[g(x)]$

Replace x in $f(x) = 8x^2$ with $(2x + 8)$

$\Longrightarrow (f \circ g)(x) = f[g(x)] = 8(2x + 8)^2$

$\Longrightarrow 8[4x^2 + 8^2 + 2(2x)(8)]$

$\Longrightarrow 8[4x^2 + 64 + 32x]$

$\Longrightarrow 32x^2 + 512 + 256x$

$\Longrightarrow 32x^2 + 256x + 512$

Example 7

Find $(g \circ f)(x)$ if, $f(x) = 6x^2$ and $g(x) = 14x + 4$

Solution

$\Longrightarrow (g \circ f)(x) = g[f(x)]$

Substitute x in $g(x) = 14x + 4$ with $6x^2$

$\Longrightarrow g[f(x)] = 14(6x^2) + 4$

$= 84x^2 + 4$

Example 8

Calculate $(f \circ g)(x)$ using $f(x) = 2x + 3$ and $g(x) = -x^2 + 1$,

Solution

$(f \circ g)(x) = f(g(x))$
$= 2(g(x)) + 3$
$= 2(-x^2 + 1) + 3$
$= -2x^2 + 5$

Inverse of a Function

In mathematics, an inverse function is a function that undoes the action of another function.

The inverse of a function can be viewed as the reflection of the original function over the line y = x. In simple words, the inverse function is obtained by swapping the (x, y) of the original function to (y, x).

We use the symbol f^{-1} to denote an inverse function. For example, if f (x) and g (x) are inverses of each other, then we can symbolically represent this statement as:

$g(x) = f^{-1}(x)$ or $f(x) = g^{-1}(x)$

One thing to note about inverse function is that, the inverse of a function is not the same as its reciprocal i.e. $f^{-1}(x) \neq 1/f(x)$.

Since not all functions have an inverse, it is therefore important to check whether or not a function has an inverse before embarking on the process of determining its inverse.

We check whether or not a function has an inverse in order to avoid wasting time trying to find something that does not exist.

How to Find the Inverse of a Function?

Finding the inverse of a function is a straight forward process, though there are a couple of steps that we really need to be careful with. In this article, we are going to assume that all functions we are going to deal with are one to one.

Here is the procedure of finding of the inverse of a function f(x):

- Replace the function notation f(x) with y.
- Swap x with y and vice versa.
- From step 2, solve the equation for y. Be careful with this step.
- Finally, change y to $f^{-1}(x)$. This is the inverse of the function.
- You can verify your answer by checking if the following two statements are true:

$\Longrightarrow (f \circ f^{-1})(x) = x$

$\Longrightarrow (f^{-1} \circ f)(x) = x$

Let’s work a couple of examples.

Example 1

Given the function $f(x) = 3x - 2$, find its inverse.

<u>Solution</u>

$f(x) = 3x - 2$

Replace $f(x)$ with y.

$\Longrightarrow y = 3x - 2$

Swap x with y

$\Longrightarrow x = 3y - 2$

Solve for y

$x + 2 = 3y$

Divide through by 3 to get;

$1/3(x + 2) = y$

$x/3 + 2/3 = y$

Finally, replace y with $f^{-1}(x)$.

$f^{-1}(x) = x/3 + 2/3$

Verify $(f \circ f^{-1})(x) = x$

$(f \circ f^{-1})(x) = f[f^{-1}(x)]$

$= f(x/3 + 2/3)$

$\Longrightarrow 3(x/3 + 2/3) – 2$

$\Longrightarrow x + 2 – 2$

$= x$

Hence, $f^{-1}(x) = x/3 + 2/3$ is the correct answer.

Example 2

Given $f(x) = 2x + 3$, find $f^{-1}(x)$.

<u>Solution</u>

$f(x) = y = 2x + 3$

$2x + 3 = y$

Swap x and y

$\Longrightarrow 2y + 3 = x$

Now solve for y

$\Longrightarrow 2y = x – 3$

$\Longrightarrow y = x/2 – 3/2$

Finally substitute y with $f^{-1}(x)$

$\Longrightarrow f^{-1}(x) = (x– 3)/2$

Example 3

Find the inverse of the following function $g(x) = (x + 4)/ (2x -5)$

<u>Solution</u>

$g(x) = (x + 4)/ (2x -5) \Longrightarrow y = (x + 4)/ (2x -5)$

Interchange y with x and vice versa

$y = (x + 4)/ (2x -5) \Rightarrow x = (y + 4)/ (2y -5)$

$\Rightarrow x(2y-5) = y + 4$

$\Rightarrow 2xy - 5x = y + 4$

$\Rightarrow 2xy - y = 4 + 5x$

$\Rightarrow (2x - 1) y = 4 + 5x$

Divide both side of the equation by $(2x - 1)$.

$\Rightarrow y = (4 + 5x)/ (2x - 1)$

Replace y with $g^{-1}(x)$

$= g^{-1}(x) = (4 + 5x)/ (2x - 1)$

Example 5

Determine the inverse of the following function $f(x) = 2x - 5$

Solution

Replace f(x) with y.

$f(x) = 2x - 5 \Rightarrow y = 2x - 5$

Switch x and y to get;

$\Rightarrow x = 2y - 5$

Isolate the variable y.

$2y = x + 5$

$\Longrightarrow y = x/2 + 5/2$

Change y back to $f^{-1}(x)$.

$\Longrightarrow f^{-1}(x) = (x + 5)/2$

Example 6

Find the inverse of $h(x) = (4x + 3)/(2x + 5)$

<u>Solution</u>

Replace h (x) with y.

$h(x) = (4x+3)/(2x+5) \Longrightarrow y = (4x + 3)/(2x + 5)$

Swap x and y.

$\Longrightarrow x = (4y + 3)/(2y + 5)$.

Solve for y in the above equation as follows:

$\Longrightarrow x = (4y + 3)/(2y + 5)$

Multiply both sides by (2y + 5)

$\Longrightarrow x(2y + 5) = 4y + 3$

Distribute the x

$\Longrightarrow 2xy + 5x = 4y + 3$

Isolate y.

$\Longrightarrow 2xy – 4y = 3 – 5x$

$\Longrightarrow y(2x – 4) = 3 – 5x$

Divide through by 2x – 4 to get;

$\Longrightarrow y = (3 - 5x)/ (2x - 4)$

Finally replace y with $h^{-1}(x)$.

$\Longrightarrow h^{-1}(x) = (3 - 5x)/ (2x - 4)$