# Variational Auto-Encoders (Part 1)

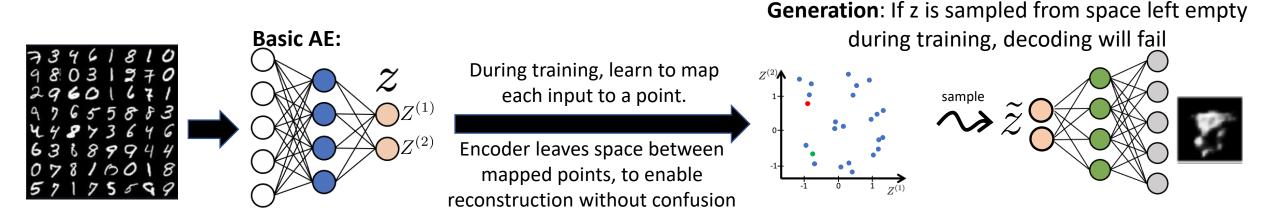
#### What we'll learn about Variational Auto-Encoders:

Part 1 (this video): The "simple" explanation of a VAE, as a regularized AE

- Architecture of a Variational Auto-Encoder (VAE)
- How to train VAEs
- How do different terms of training loss influence what VAE learns
- How does a VAE relate to the basic AE

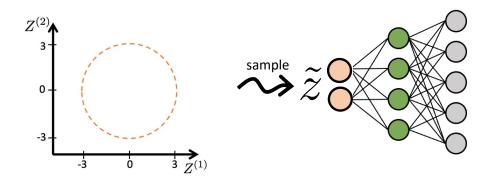
#### Part 2: Applications of VAE for...

- Generation of new data points
- Modifying data via interpolating between 2 inputs
- Modifying specific feature of an input
- Compression and Reconstruction

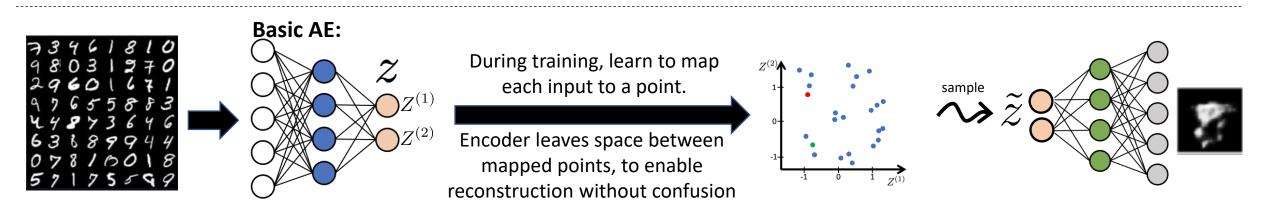


**VAE** (informally):

# **Training**: Enforce codes z to **cover a specified area**



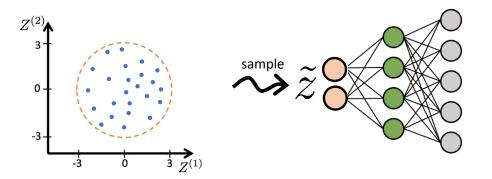
**Generation**: We can then sample from anywhere in the area. Decoding should then work.



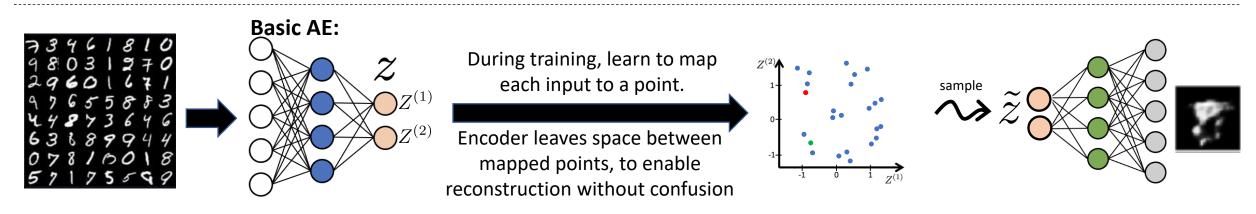
Instead of using Encoder that

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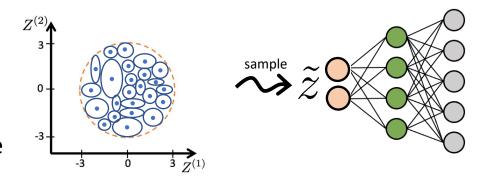


**VAE** (informally):

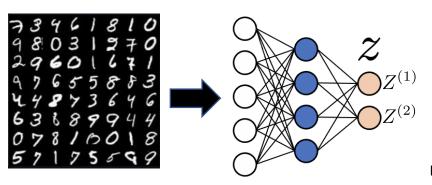
73461810 98031270 29601671 97655863 44873646 63889944 07810018 57175599 Make encoder that maps each input to an 'area'

Forces decoder to learn to decode z from whole covered area.

**Training**: Enforce codes z to **cover a specified area** 

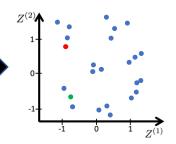


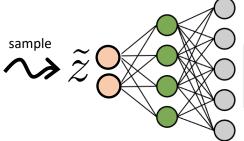
**Generation**: We can then sample from anywhere in the area. Decoding should then work.



During training, learn to map each input to a point.

Encoder leaves space between mapped points, to enable reconstruction without confusion

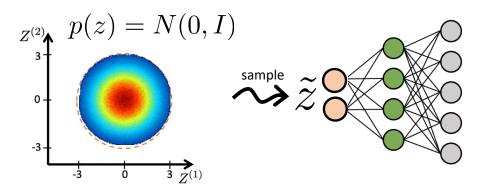




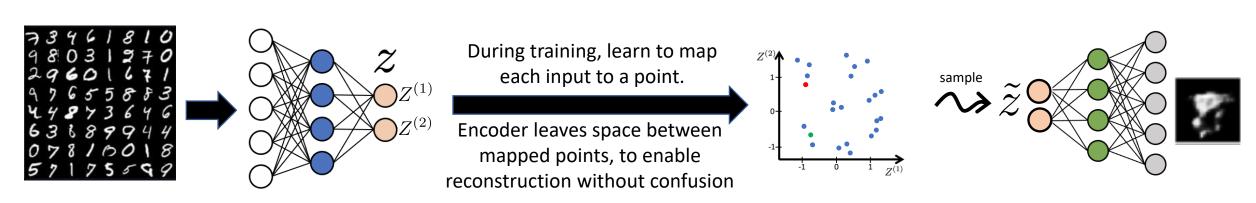


**VAE** (more accurately):

**Training**: Enforce codes z to follow a "prior" distribution p(z)



**Generation**: We can then sample from prior p(z) and decode!

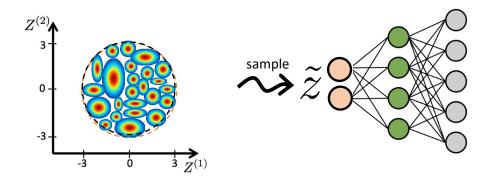


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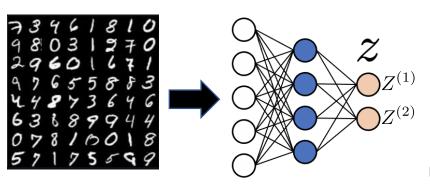
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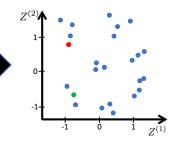


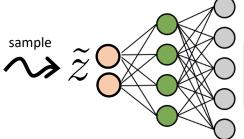
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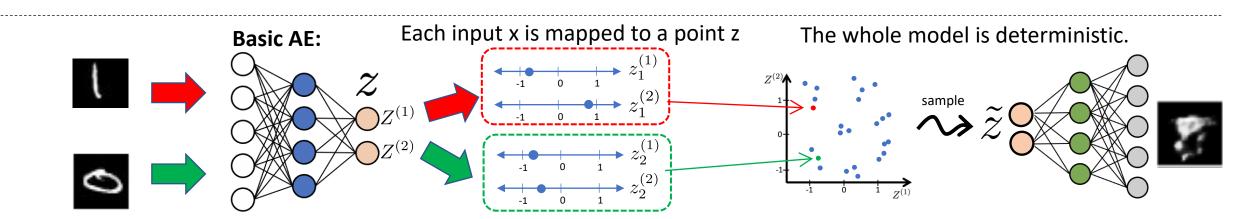
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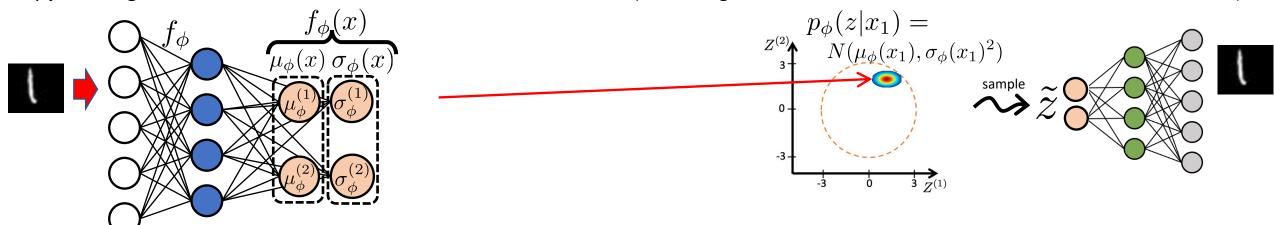


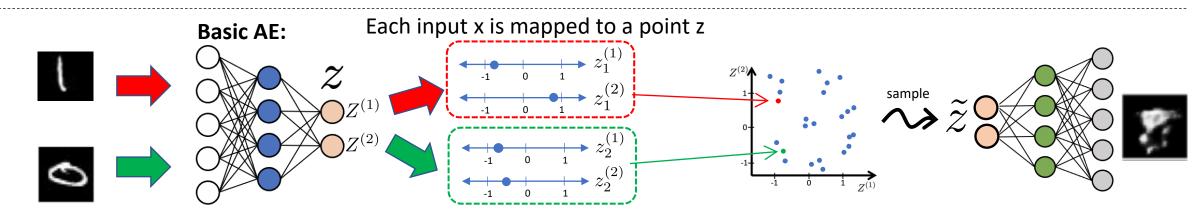
"posterior" distribution of z,

a.k.a. "conditional" distribution of z given x

Each x is mapped to a multi-variate Normal/Gaussian distribution of z,  $p_\phi(z|x) = N(\mu_\phi(x), \sigma_\phi(x)^2)$ 

by predicting a mean and standard deviation for each dimension of z. (Note: diagonal covariance matrix, i.e. each dimension is uncorrelated)



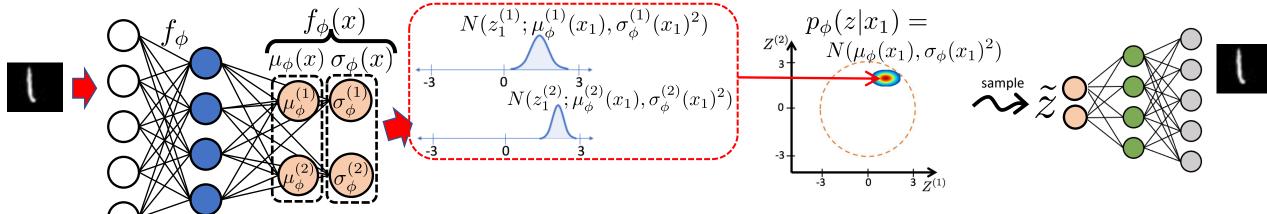


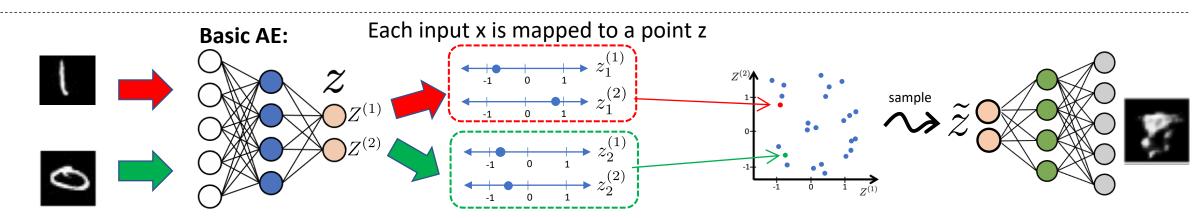
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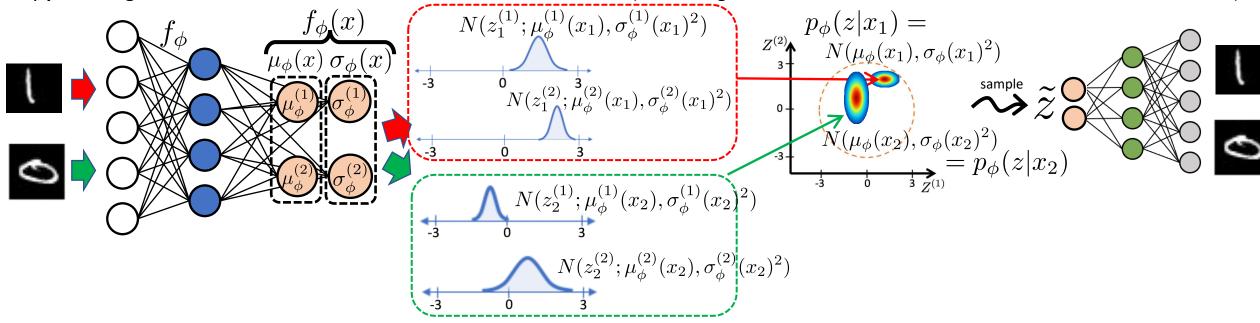
#### "posterior" distribution of z,

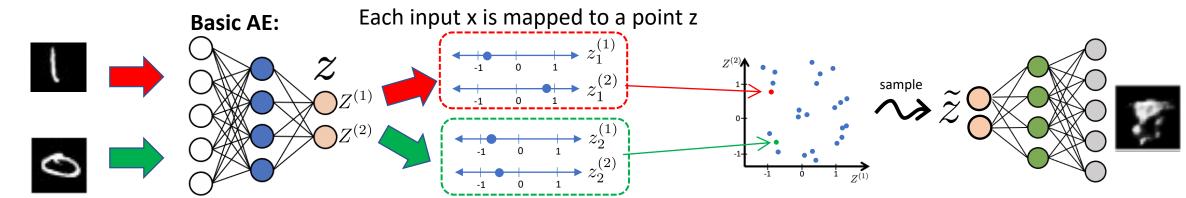
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#### 1. Probabilistic Encoder

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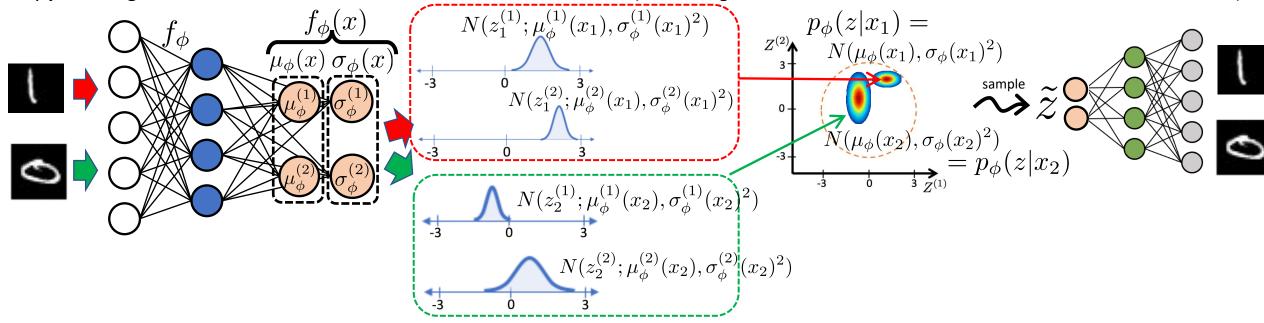


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The final layer of VAE's encoder has 2 times the number of neurons than the encoder of a standard AE.

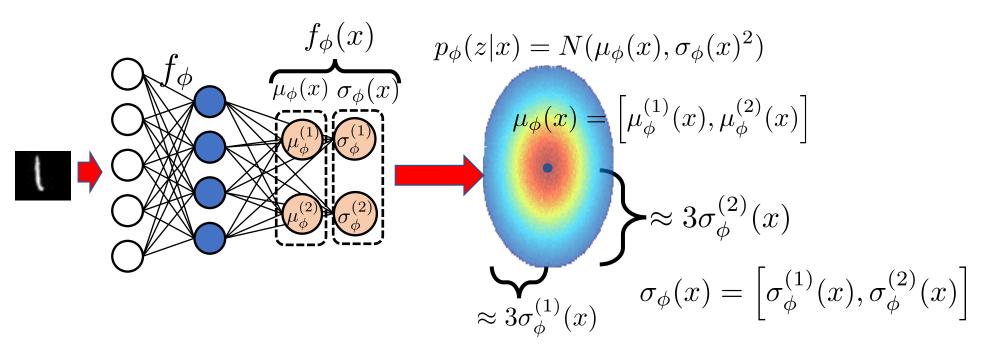
For z of dimension v,  $\,z \in \mathbb{R}^v\,$  , encoder of VAE has  $\,2 imes v\,$  neurons.

For each dimension of z, the **standard AE's** encoder predicts **one value, the code**. The AE's encoder is **deterministic**.

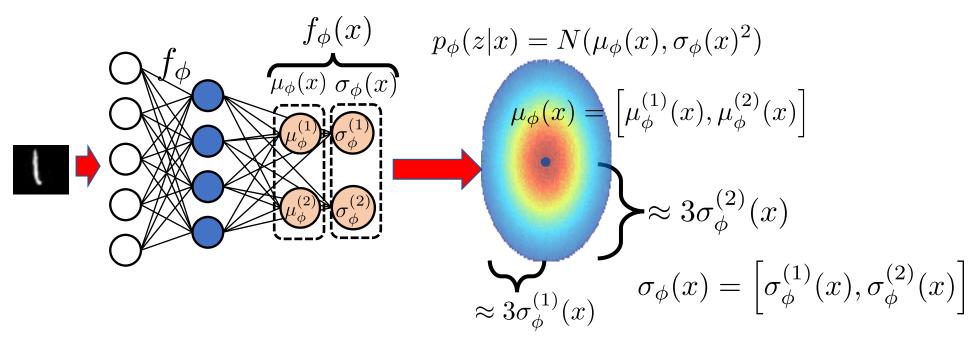
For each dimension of z, the VAE's encoder predicts the **mean** and **standard deviation** of **a Gaussian distribution**.

This distribution describes "which values of z are likely the correct code" for input x. VAE's encoder is probabilistic/stochastic.

#### A closer look



# How to interpret that for each input x, the encoder predicts a <u>probability distribution</u> for z?

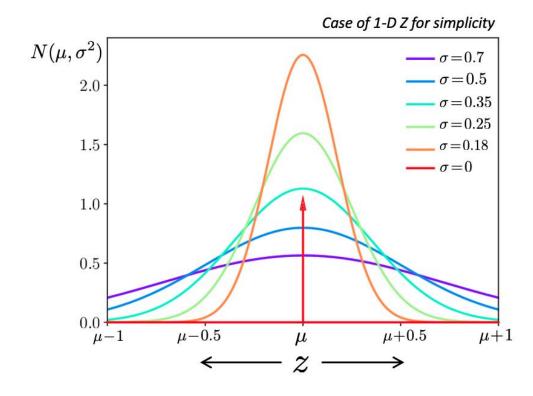


VAE's encoder expresses "uncertainty":

For each sample x, the Gaussian distribution of z predicted by the encoder,  $p_{\phi}(z|x) = N(\mu_{\phi}(x), \sigma_{\phi}(x)^2)$  expresses how "likely" is that a specific value of z is the correct code for x.

In probabilistic terms, it expresses the *conditional probability of z given x*,  $p_{\phi}(z|x)$ , according to the model (encoder) with parameters  $\phi$ .

# Uncertainty (low probability) in encoding

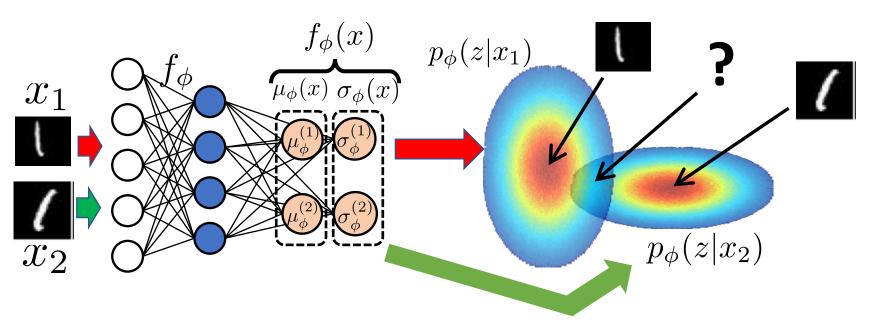


Q: If predicted std. deviation  $\sigma_{\phi}(x) = 0$  ?

A: Totally **certain (prob=1)** that **z is the predicted mean!**This is the **basic Auto-Encoder's behavior!** 

Larger standard deviation => larger range of values z may be the correct code of x => less certainty (probability) each of these z values is the "ideal" code for x.

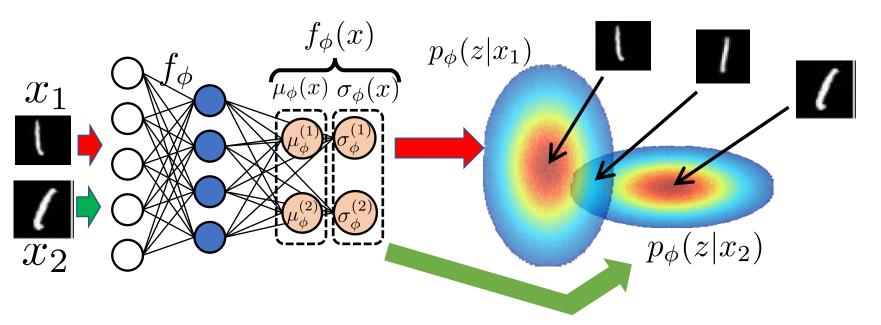
# How to interpret "overlaps" of predicted distributions for z?



This formulation of the encoder leads to existence of "overlaps" in the encoding of different samples. (in fact, Gaussians extend to infinity, even for very small values, so in fact, there are overlaps everywhere)

Q: What would be the appearance of an image that would be mapped with a mean in the overlapping area of  $x_1$  and  $x_2$ ?

# How to interpret "overlaps" of predicted distributions for z?

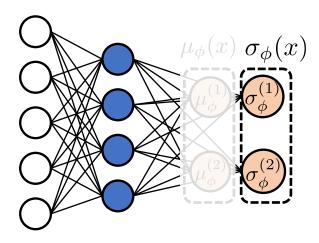


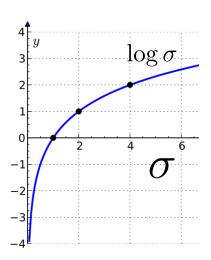
This formulation of the encoder leads to existence of "overlaps" in the encoding of different samples. (in fact, Gaussians extend to infinity, even for very small values, so in fact, there are overlaps everywhere)

Q: What would be the appearance of an image that would be mapped with a mean in the overlapping area of  $x_1$  and  $x_2$ ?

A: Appearance that would be in-between those of  $x_1$  and  $x_2$ 

# Implementation trick: predict log(std.dev.)

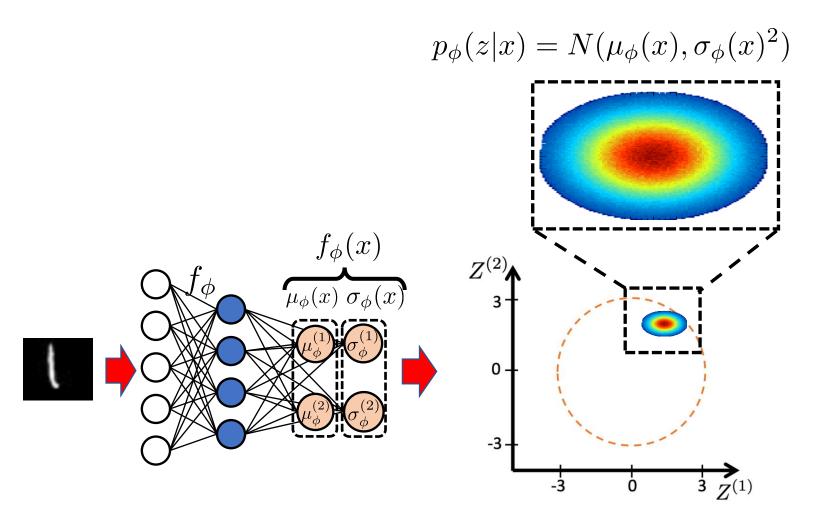


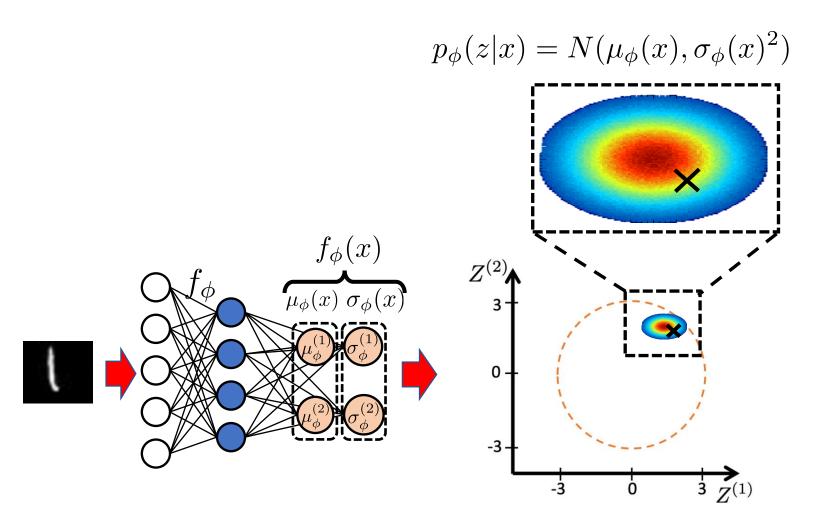


- Standard deviation \*must\* be positive.
   => Cant use activation functions that allow negatives.
- 2. Standard deviations for each x tend to be small, close to zero=> Cant use ReLU. Not good gradients around 0,but std.devs can and will take such low values.

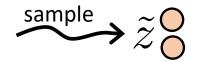
#### Common solution in implementations:

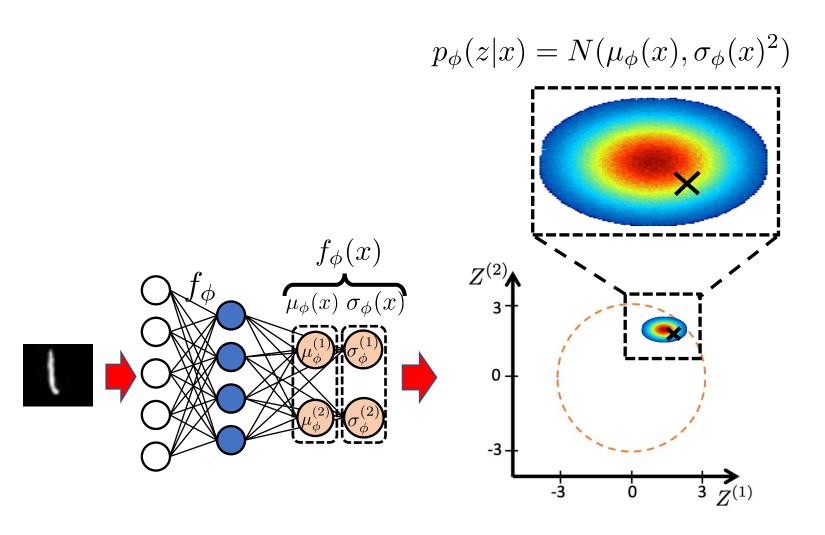
- Treat Encoder as if it predicts  $\log(\sigma_{\phi}(x))$   $\log_{\phi}(x) = \max_{x \in \mathcal{X}} \min_{x \in \mathcal{$
- Compute sigma as:  $\sigma_{\phi}(x) = \exp(\log(\sigma_{\phi}(x)))$ stddev = exp(log\_stddev)



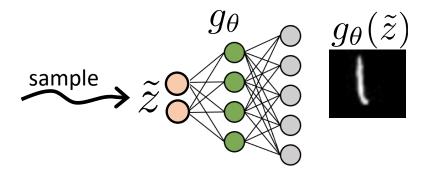


**Step 1.** Sample a code  $\widetilde{\mathcal{Z}}$  from the predicted Gaussian  $p_{\phi}(z|x)$ 





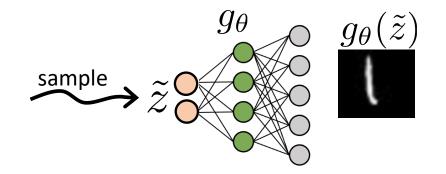
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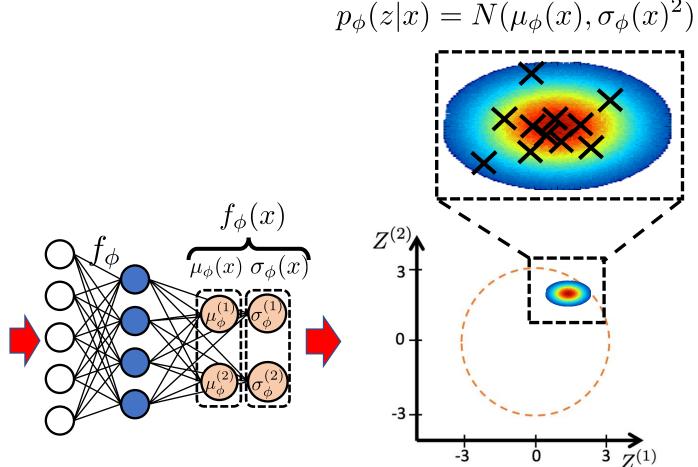


Step 2.  $\sim$  Decode  $\stackrel{\sim}{\mathcal{Z}}$  with a standard decoder, similar to a basic AE.

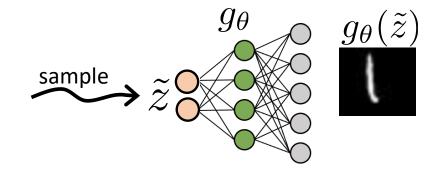
 $p_{\phi}(z|x) = N(\mu_{\phi}(x), \sigma_{\phi}(x)^2)$  $f_{\phi}(x)$  $\mu_{\phi}(x) \ \overline{\sigma_{\phi}(x)}$ 0 -3 Z<sup>(1)</sup>

At <u>each training iteration</u> of SGD, for <u>each input x</u> in the batch, we <u>draw 1 sample</u>.





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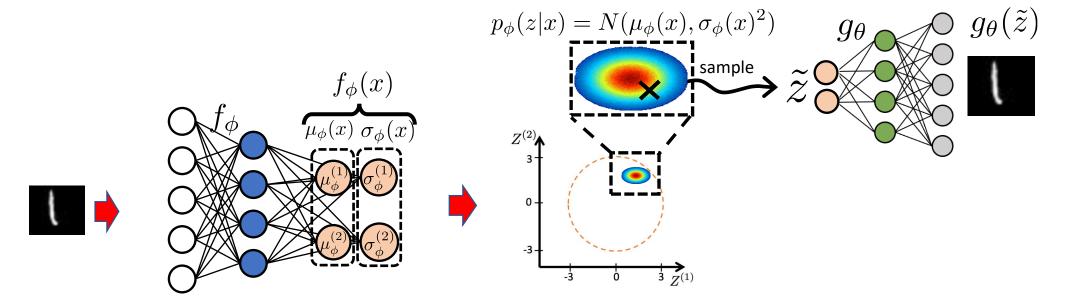


However, at <u>different SGD iterations</u>, <u>different samples</u> may be drawn <u>for the same x</u>.

For any input x and predicted  $p_{\phi}(z|x)$  it is **more likely** that we will sample value  $\widetilde{z}$  that is close to the predicted mean  $\mu_{\phi}(x)$ , rather than further.

At different SGD iterations, the decoder should try to reconstruct the input for **any drawn sample**, even the unlikely ones.

#### 3. Training Objective



What do we want to train this model to do?

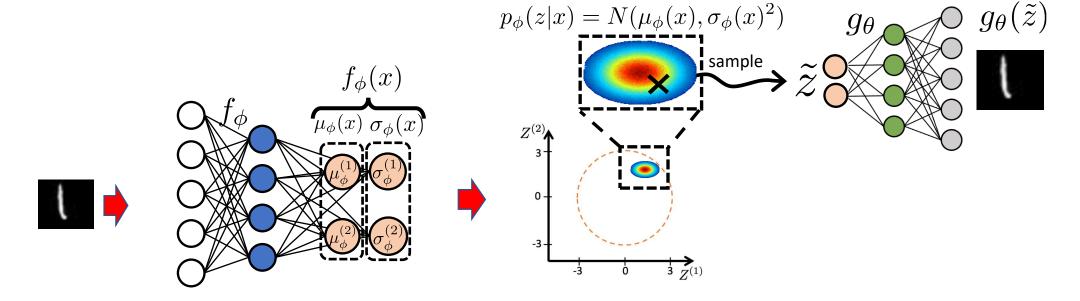


Create <u>realistic images</u> with the decoder



Predicted (distributions of) codes to **cover the prior p(z)** 

#### 3. Training Objective



#### **Reconstruction loss**

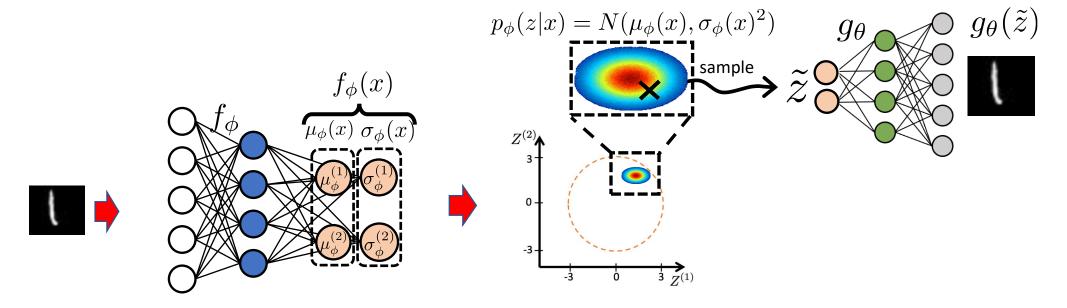
#### Regularizer

$$\mathcal{L}_{VAE} = \mathcal{L}_{rec} + \lambda \mathcal{L}_{reg}$$

#### **Strength of Regularizer**

(chosen by us through experimentation)

#### 3. Training Objective



**Reconstruction loss** 

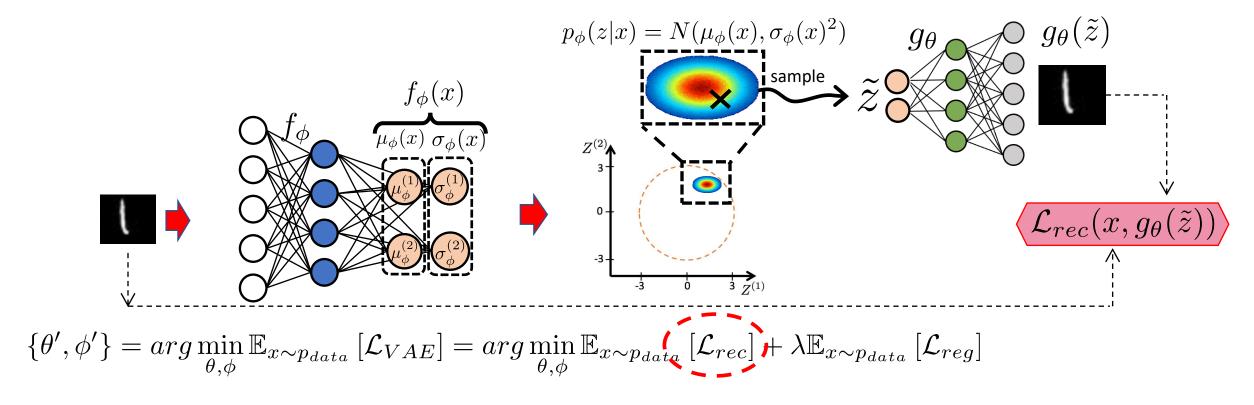
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$$\{\theta', \phi'\} = \arg\min_{\theta, \phi} \mathbb{E}_{x \sim p_{data}} \left[\mathcal{L}_{VAE}\right] = \arg\min_{\theta, \phi} \mathbb{E}_{x \sim p_{data}} \left[\mathcal{L}_{rec}\right] + \lambda \mathbb{E}_{x \sim p_{data}} \left[\mathcal{L}_{reg}\right]$$

"Minimize the **Expected Value** of the VAE loss, when x is sampled from the data distribution" In other words: "Minimize the VAE loss **on average** over the training data x"

#### 3. Training Objective: Reconstruction loss

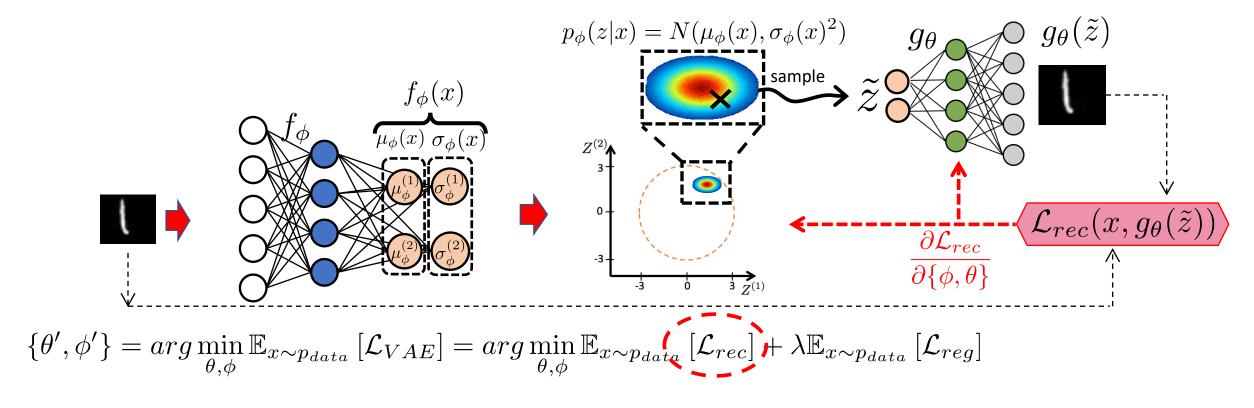


#### **Reconstruction loss:**

$$\mathcal{L}_{rec} = \underbrace{1}_{j=1}^{a} (x^{(j)} - g_{\theta}(\tilde{z})^{(j)})^2 \quad \text{for } \tilde{z} \sim p_{\phi}(z|x) \quad \text{Reconstruct input x, for any } \tilde{z} \quad \text{sampled from } p_{\phi}(z|x).$$

Dimensionality of x

#### 3. Training Objective: Reconstruction loss



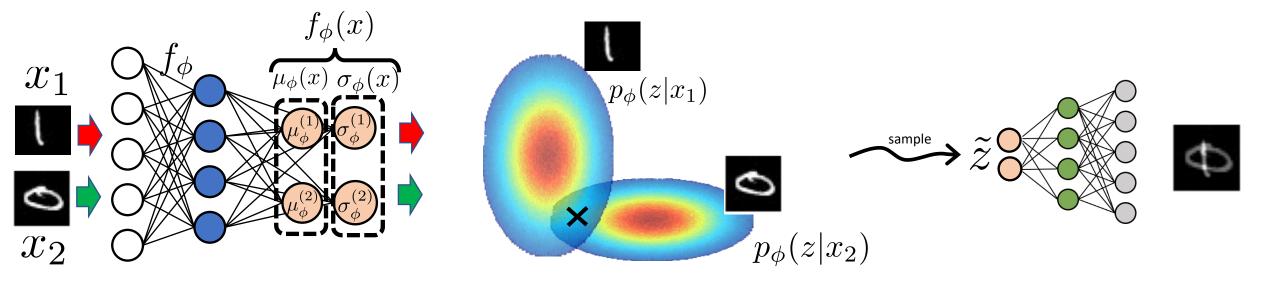
#### **Reconstruction loss:**

$$\mathcal{L}_{rec} = \frac{1}{d} \sum_{j=1}^{d} (x^{(j)} - g_{\theta}(\tilde{z})^{(j)})^{2}$$
Function of  $\theta$  (encoder params) (decoder params)

The Reconstruction loss is a **function of all parameters**.

Therefore it trains BOTH the encoder AND the decoder.

#### Effect of Reconstruction loss:

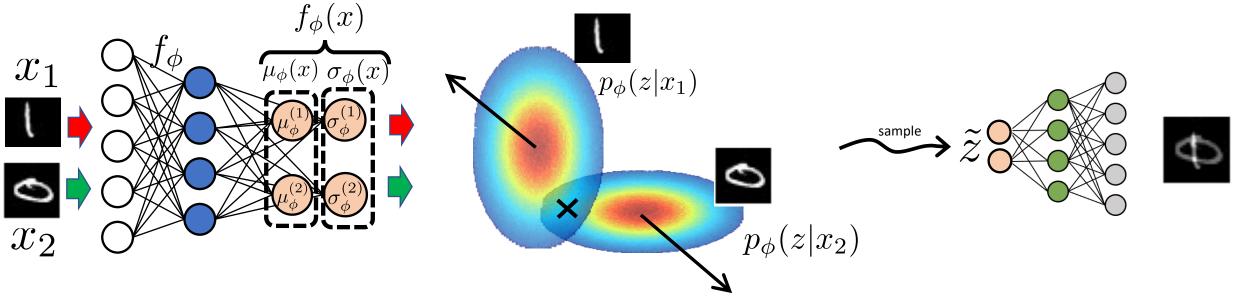


Assume that an encoder learns to encode these two digits as shown.

Q: For what values of z do you expect *high reconstruction error*?

A: Overlapping area

#### Effect of Reconstruction loss: on the predicted means

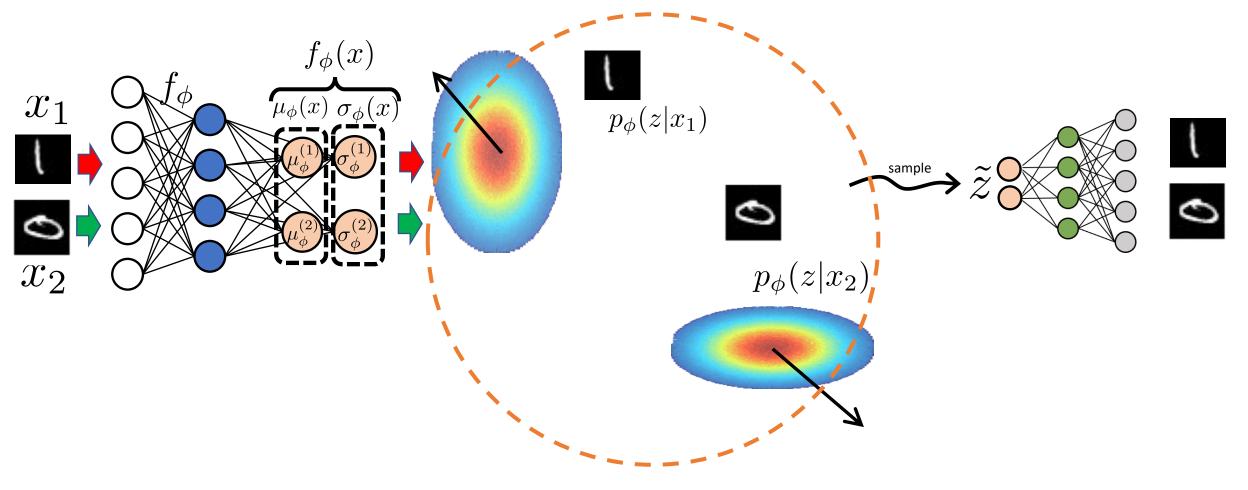


Grads of Recon loss encourages learning means of different samples away from each other in latent space  ${\mathcal Z}$ 

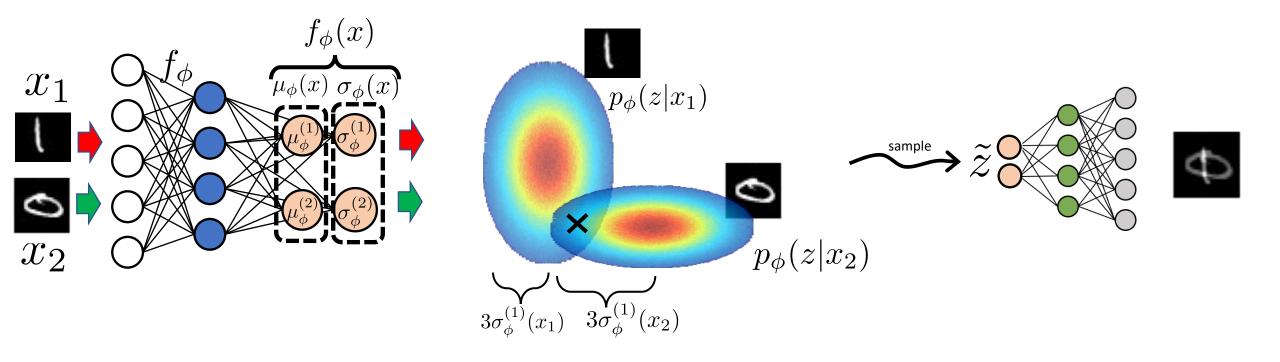
Q: How can the encoder reduce overlapping areas by changing the <u>predicted means</u>?

A: Learn parameters that predict means that are <u>away from each other</u>.

#### Effect of Reconstruction loss: on the predicted means

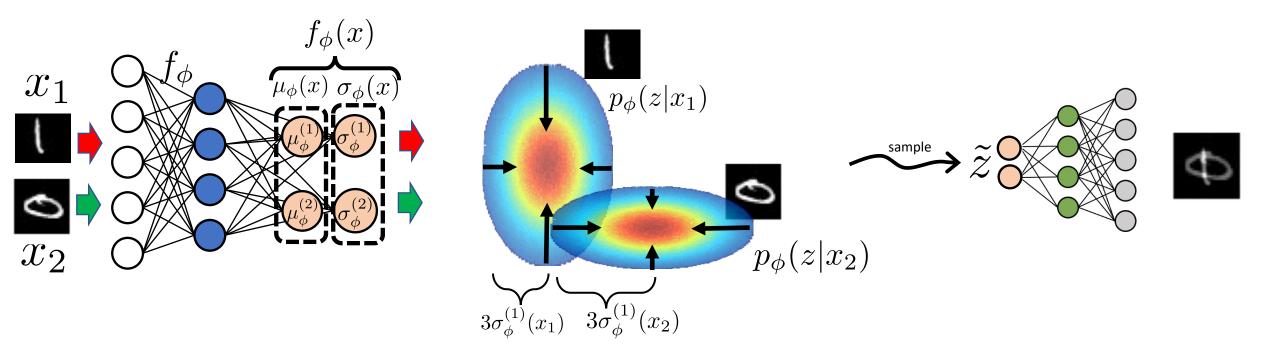


If we would minimize ONLY the Reconstruction loss, encoder can learn to predict means of p(z|x) that are **arbitrarily far**. They can take values outside the area covered by the "prior" p(z).



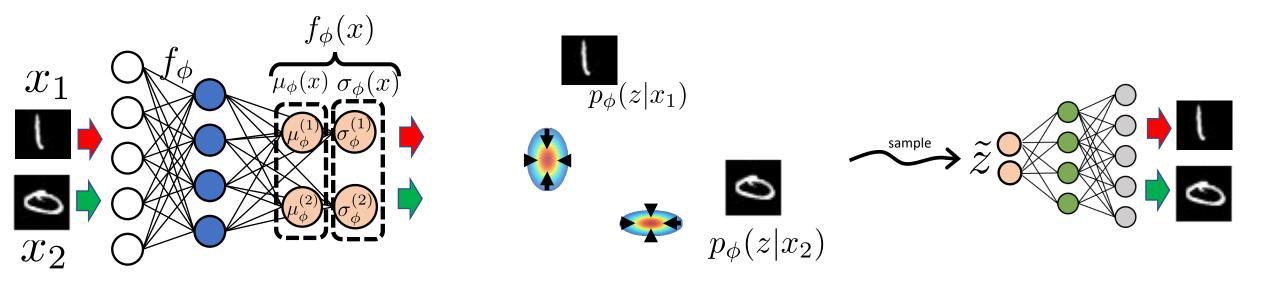
Q: How can the encoder reduce overlapping areas *by changing the predicted <u>standard deviations</u>?

A: Learn parameters that predict <u>small</u> <u>standard deviations</u>.* 



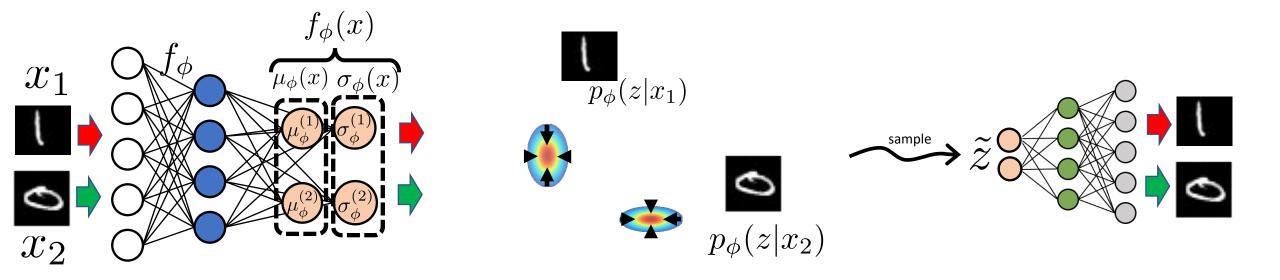
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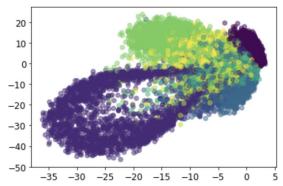


If we would minimize ONLY the Reconstruction loss, "ideal" standard deviations for reconstruction loss are <u>zero</u> for every x!

### Influence of Reconstruction loss:

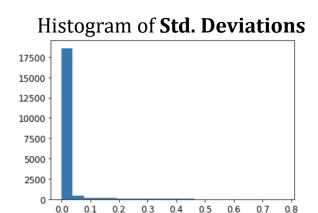
If we would train the VAE to **only** minimize the Reconstruction loss:

**Means** in 2D space of Z:



For each x, predicted **means**:

- Dissimilar x to different means of z.
   More similar samples can be closer.
- Means are pushed to arbitrarily high values



For each x, predicted **standard deviations**:

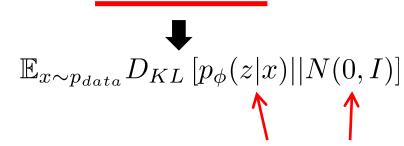
- Tend to be small, ideally 0.
- Distributions p(z|x) tend to collapse to their mean. Therefore **cant cover prior p(z)**.
- The encoder tends to become deterministic.

If we would train a VAE only with the Reconstruction loss, its learned parameters would make it behave exactly like the basic deterministic AE!

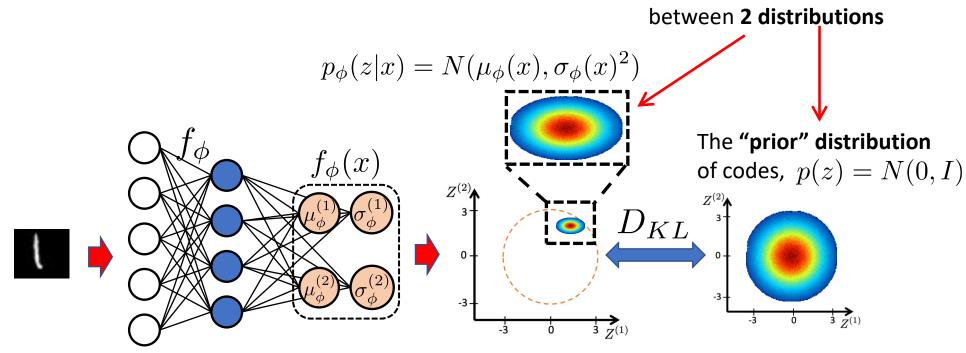
So, the "probabilistic encoder" architecture is not sufficient! We need the Regularizer!

$$\{\theta', \phi'\} = arg \min_{\theta, \phi} \mathbb{E}_{x \sim p_{data}} \left[ \mathcal{L}_{VAE} \right] = arg \min_{\theta, \phi} \mathbb{E}_{x \sim p_{data}} \left[ \mathcal{L}_{rec} \right] + \lambda \mathbb{E}_{x \sim p_{data}} \left[ \mathcal{L}_{reg} \right]$$

"Expected value (average over training data) of the **Kullback-Leibler (KL) Divergence**" between two distributions



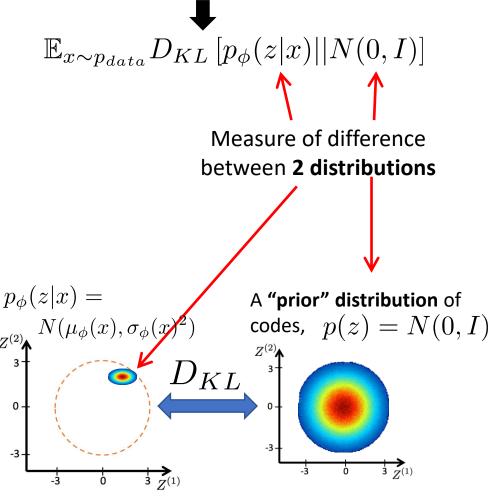
Measure of difference



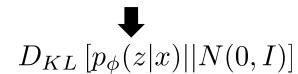
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#### What this regularizer does:

"Learn encoder parameters  $\phi$  such that on average over the training data, we minimize the "Kullback Leibler (KL) Divergence" between the predicted posterior distribution of z codes,  $p_{\phi}(z|x)$ , and the Gaussian distribution N(0,I) "

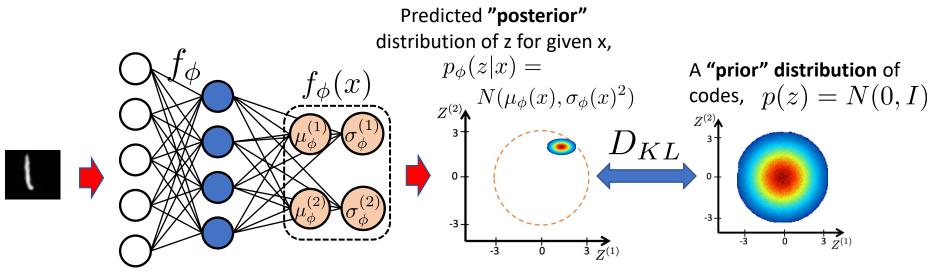


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In the specific case of KL divergence between two Gaussian distributions, it is equal to:

$$= \frac{1}{2} \sum_{j=1}^{v} \left[ (\mu_{\phi}^{(j)}(x))^2 + (\sigma_{\phi}^{(j)}(x))^2 - 2\log \sigma_{\phi}^{(j)}(x) - 1 \right]$$

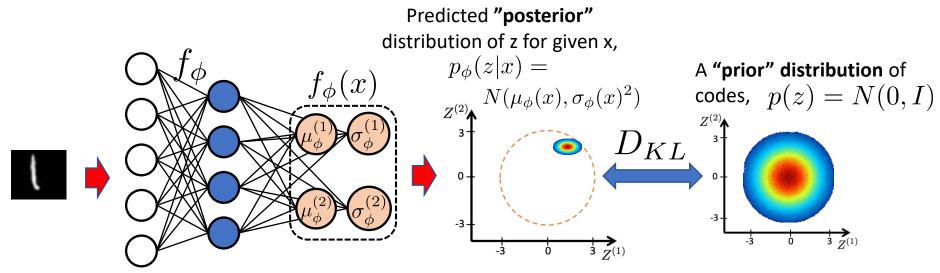


$$\{\theta', \phi'\} = arg \min_{\theta, \phi} \mathbb{E}_{x \sim p_{data}} \left[ \mathcal{L}_{VAE} \right] = arg \min_{\theta, \phi} \mathbb{E}_{x \sim p_{data}} \left[ \mathcal{L}_{rec} \right] + \lambda \mathbb{E}_{x \sim p_{data}} \left[ \mathcal{L}_{reg} \right]$$

The regularizer is **ONLY** a function of  $\phi$ , the parameters of the encoder!

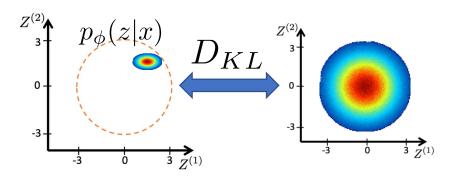
It does NOT train the decoder!

$$= \frac{1}{2} \sum_{j=1}^{v} \left[ (\mu_{\phi}^{(j)}(x))^{2} + (\sigma_{\phi}^{(j)}(x))^{2} - 2\log \sigma_{\phi}^{(j)}(x) - 1 \right]$$



## Effect of Regularizer on means and standard deviations:

$$\begin{split} \phi' &= arg \min_{\phi} \mathcal{L}_{reg}(x) \Rightarrow \\ &= arg \min_{\phi} D_{KL} \left[ p_{\phi}(z|x) || N(0,I) \right] \Rightarrow \\ &= arg \min_{\phi} \frac{1}{2} \sum_{j=1}^{v} \left[ (\mu_{\phi}^{(j)}(x))^2 + (\sigma_{\phi}^{(j)}(x))^2 - 2\log \sigma_{\phi}^{(j)}(x) - 1 \right] \end{split}$$
 To minimize the regularizer....



...for each of  $\,v$  dimensions of z...

...mean<sup>2</sup> must be minimized, which means ideally the predicted means must be....????

...must be 0!

## Effect of Regularizer on means and standard deviations:

$$\phi' = \arg\min_{\phi} \mathcal{L}_{reg}(x) \Rightarrow$$

$$= \arg\min_{\phi} D_{KL} \left[ p_{\phi}(z|x) || N(0,I) \right] \Rightarrow$$

$$= \arg\min_{\phi} \frac{1}{2} \sum_{\sigma} \left[ (u^{(j)}(\sigma))^2 + (\sigma^{(j)}(\sigma))^2 \right]$$

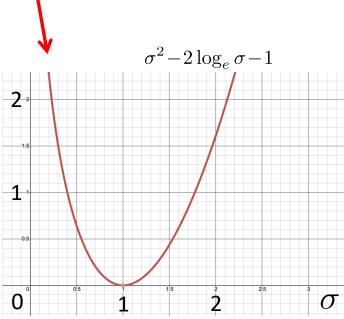
 $= \arg\min_{\phi} \frac{1}{2} \sum_{j=1}^{v} \left[ (\mu_{\phi}^{(j)}(x))^{2} + (\sigma_{\phi}^{(j)}(x))^{2} - 2\log\sigma_{\phi}^{(j)}(x) - 1 \right]$ 

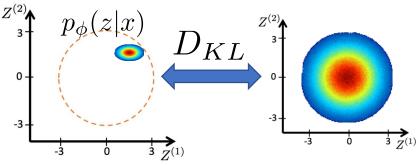
To <u>minimize</u> the regularizer....

...for each of  $\,v$  dimensions of z...

...mean<sup>2</sup> must be minimized, which means ideally the predicted means must be....????

...must be 0!





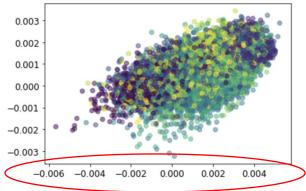
...the predicted standard deviation must **ideally be** ????....

... must be =1!

### Effect of Regularizer:

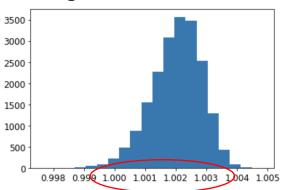
If we would train the VAE to **only** minimize the Regularizer:





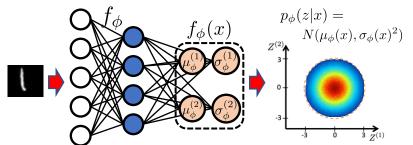
For each x, predicted **mean tends to 0.** 

#### Histogram of **Std. Deviations**



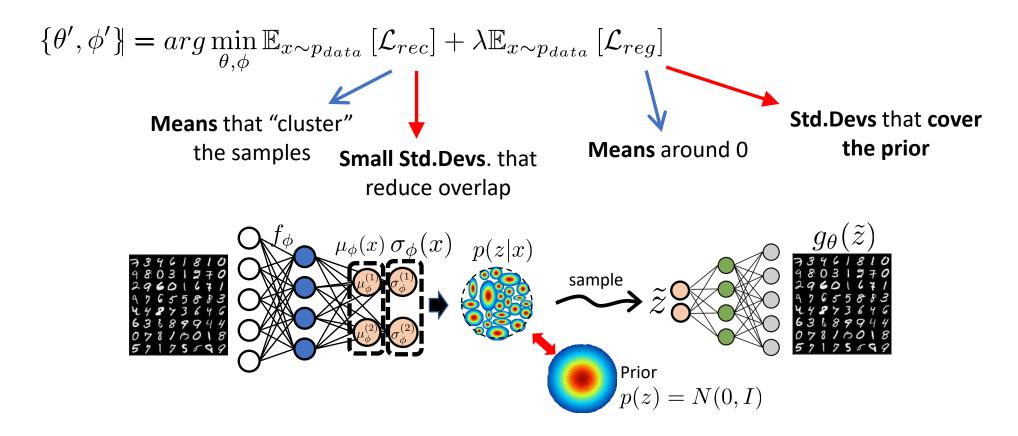
For each x, predicted std. deviation tends to 1.

For every sample, the predicted posterior p(z|x) takes the shape of the prior p(z)

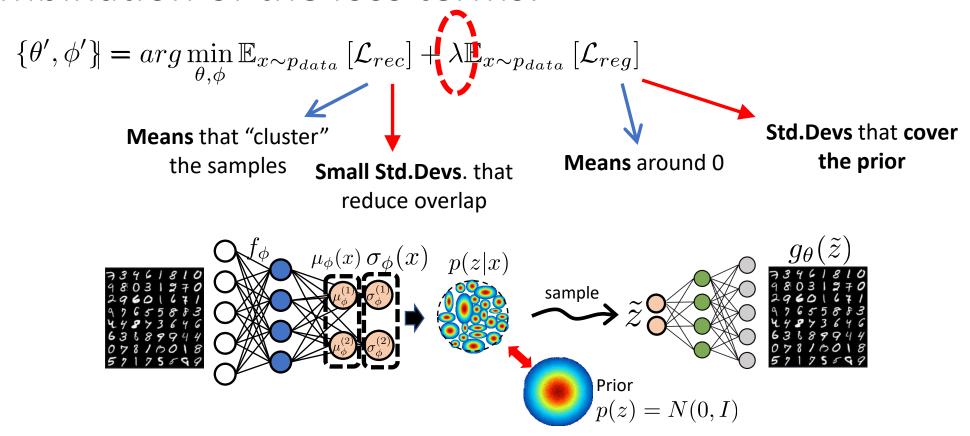


Useless by itself!
But useful if combined with Reconstruction..!

### Combination of the loss terms:



### Combination of the loss terms:



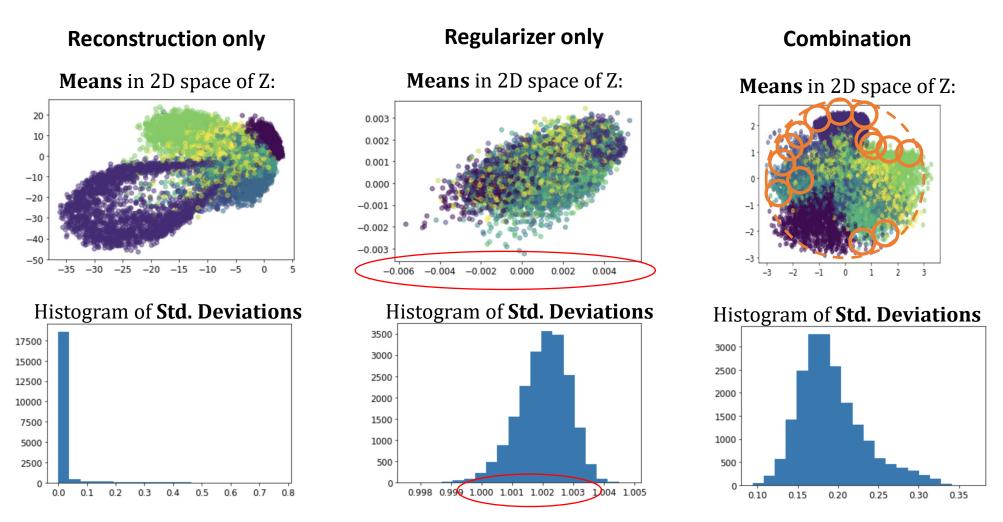
Under \*ideal\* optimization: and appropriate choice of  $\lambda$  (via cross-val observing rec loss, means and stds of z) This combination leads that all together (not each individually) the *posteriors* p(z|x) of the whole training database *cover the "prior"* N(0,I).\*

Simultaneously, encoder arranges posteriors of codes such that it facilitates reconstruction.

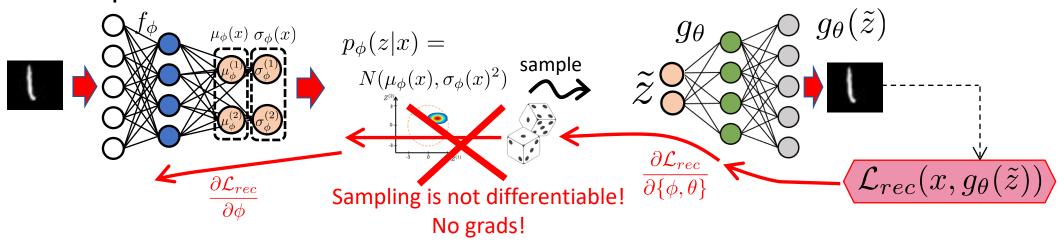
<sup>\*</sup> The explanation why this happens is in the probabilistic derivation of the VAE (further reading/optional).

### Combination of the loss terms:

Example **in practice**, with stochastic optimization and non-optimal training parameters: (from our Tutorial - VAEs on MNIST)



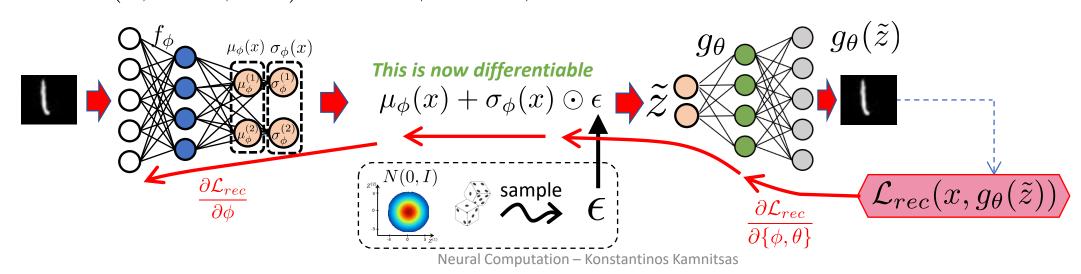
### 4. Re-parameterization trick



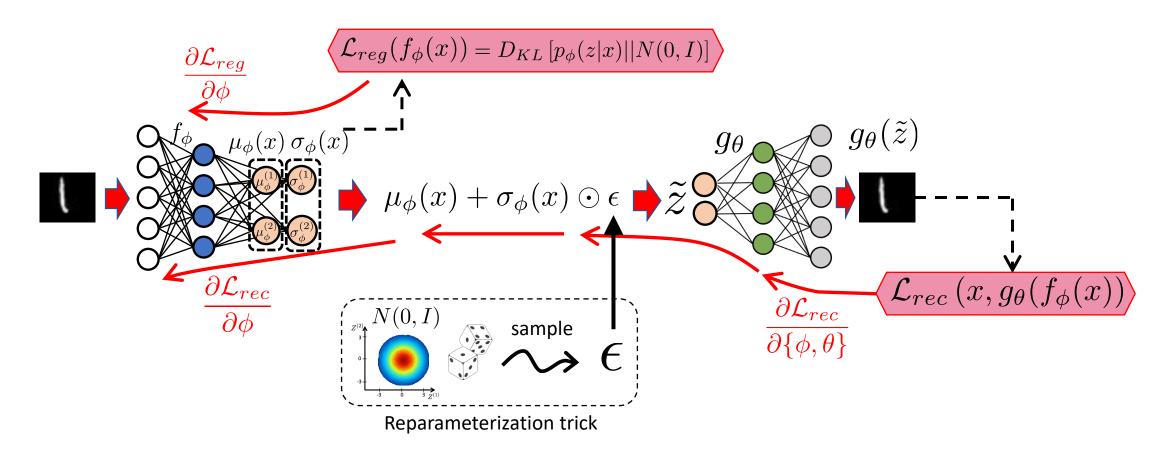
#### **Solution: The Re-parameterization trick**

For  $p_{\phi}(z|x)$  a Gaussian (as in VAEs):

$$\tilde{z} \sim N\left(\mu_{\phi}(x), \sigma_{\phi}(x)^2\right) \Leftrightarrow \tilde{z} = \mu_{\phi}(x) + \sigma_{\phi}(x) \odot \epsilon$$
, with  $\epsilon \sim N(0, I)$ 

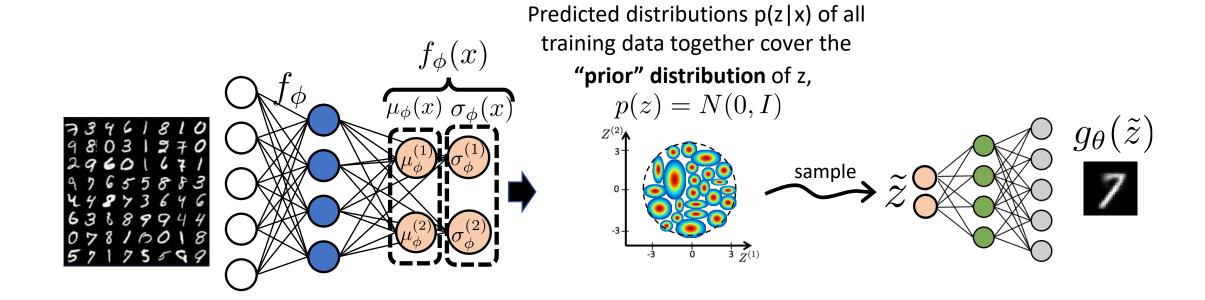


## Bringing everything together: Training



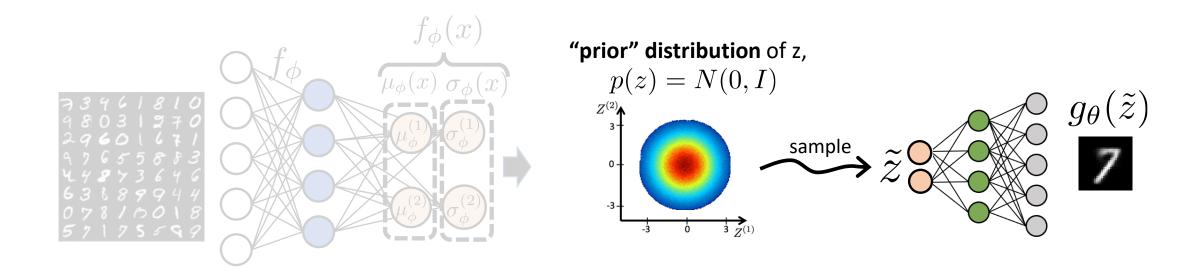
Kingma & Welling, Auto-encoding variational bayes, ICLR 2014
Rezende et al, Stochastic Backpropagation and Approximate Inference in Deep Generative Models, ICML 2014

### Bringing everything together: Training



Kingma & Welling, Auto-encoding variational bayes, ICLR 2014
Rezende et al, Stochastic Backpropagation and Approximate Inference in Deep Generative Models, ICML 2014

### Bringing everything together: Generating new data



Kingma & Welling, Auto-encoding variational bayes, ICLR 2014
Rezende et al, Stochastic Backpropagation and Approximate Inference in Deep Generative Models, ICML 2014

### In this video:

- What is the architecture of the VAE
- What is the training loss: Reconstruction + Regularizer
- What is the effect of minimizing each loss term
- The Re-Parameterization trick for the sampling

#### Next video:

Application of VAE for...

- Generation of new data points
- Modifying data via interpolating between 2 inputs
- Modifying specific feature of an input
- Compression and Reconstruction

# Thank you very much