Numerical Methods

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Contents

1 Scientific Computing

1.1 Absolute Error and Relative Error

Let A be approximate value, T be true value. Absolute error and relative error are defined as follows:

$$\text{Absolute Error} = A - T,$$
$$\text{Relative Error} = \frac{A - T}{T} \text{ assuming } T \neq 0.$$

If numbers written in scientific notation agree to p significant digits, then the magnitude of the relative error is about 10^{-p} (within a factor of 10).

Example 1.1. $A = 5.46729 \times 10^{-12}, T = 5.46417 \times 10^{-12}$. Thus, $A - T = 0.00312 \times 10^{-12}$ and $\frac{A-T}{T} = \frac{3.12 \times 10^{-3}}{5.46417}$.

Example 1.2. $A = 1.00596 \times 10^{-10}, T = 0.99452 \times 10^{-10}$. Thus, $A - T = 0.01144 \times 10^{-10}$ and $\frac{A-T}{T} = \frac{1.144 \times 10^{-2}}{0.99452}$. A and T agree to 2 significant digits.

1.2 Data Error and Computational Error

The difference between exact function values due to error in the input and thus can be viewed as data error.

The difference between the exact and approximate functions for the same input and thus can be considered computational error.

1.3 Truncation Error

Truncation error is the difference between the true result (for the actual input) and the result that would be produced by a given algorithm using exact arithmetic. It is due to approximations such as truncating an infinite series, replacing derivatives by finite differences, or terminating an iterative sequence before convergence.

Example 1.3. $f(x) = \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \widehat{f}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$. The part $-\frac{x^7}{7!} + \cdots$ is the truncation error.

1.4 Forward Error and Backward Error

Suppose we want to compute the value of a function, $y = f(x)$, but we obtain instead an approximate value $\widehat{y}$. The discrepancy between the computed and true values, $\Delta y = \widehat{y} - y$, is called the forward error.

The quantity $\Delta x = \widehat{x} - x$, where $f(\widehat{x}) = \widehat{y}$, is called the backward error.

Example 1.4. As an approximation to $y = \sqrt{2}$, the value $\widehat{y} = 1.4$ has a forward error

$$\Delta y = \widehat{y} - y = 1.4 - \sqrt{2} \approx -0.0142$$

or a relative forward error -1.004×10^{-2}. We find that $\sqrt{1.96} = 1.4$, so the backward error is

$$\Delta x = \widehat{x} - x = 1.96 - 2 = -0.04$$

or the relative backward error -2×10^{-2}.

1.5 Conditioning

A problem is well-conditioned if a small change to the input produces a small change to the output; ill conditioned if there are some examples for which a small change in the input produces a large change in output.

Define the condition number of a problem to be the ratio of the relative change in the solution to the relative change in the input, i.e.,

$$\text{Condition number} = \frac{|(f(\widehat{x}) - f(x))/f(x)|}{|(\widehat{x} - x)/x|} = \frac{|(\widehat{y} - y)/y|}{|(\widehat{x} - x)/x|} = \frac{|\Delta y/y|}{|\Delta x/x|}.$$

A problem is ill-conditioned if its condition number is much larger than 1. We can rephrase the relationship

$$|\text{Relative forward error}| = \text{condition number} \times |\text{relative backward error}|.$$

Since

$$\text{Absolute forward error} = f(x + \Delta x) - f(x) \approx f'(x)\Delta x,$$

so

$$\text{Condition number} \approx \left| \frac{f'(x)\Delta x/y}{\Delta x/x} \right| = \left| \frac{x f'(x)}{f(x)} \right|.$$