Nonlinear Optimization

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1 Review

1.1 One-Variable Calculus

1.1.1 Mean Value Theorem

Let $g \in C^1$ on $\mathbb{R}$. We have

$$\frac{g(x+h)-g(x)}{h} = g'(x+\theta h),$$

where $\theta \in (0,1)$ and $\frac{g(x+h)-g(x)}{h}$ is the slope of secant line between $(x, g(x))$ and $(x+h, g(x+h))$. Or we can write $g(x+h) = g(x) + hg'(x+\theta h)$.

1.1.2 First Order Taylor Approximation

Let $g \in C^1$ on $\mathbb{R}$. We have

$$g(x+h) = g(x) + hg'(x) + o(h),$$

where $o(h)$ is the error and we say a function $f(h) = o(h)$ to mean

$$\lim_{h\to 0} \frac{f(h)}{h} = 0.$$

Proof. Want to show $g(x+h) - g(x) - hg'(x) = o(h)$.

We have

$$\begin{aligned}\lim_{h\to 0} \frac{g(x+h)-g(x)-hg'(x)}{h} &= \lim_{h\to 0} \frac{hg'(x+\theta h) - hg'(x)}{h} \\ &= \lim_{h\to 0} g'(x+\theta h) - g'(x) = 0.\end{aligned}$$

$\square$

1.1.3 Second Order MVT

Let $g \in C^2$ on $\mathbb{R}$. We have

$$g(x+h) = g(x) + hg'(x) + \frac{h^2}{2} g''(x+\theta h),$$

where $\theta \in (0,1)$.

1.1.4 Second Order Taylor Approximation

Let $g \in C^2$ on $\mathbb{R}$. We have

$$g(x+h) = g(x) + hg'(x) + \frac{h^2}{2}g''(x) + o(h^2).$$

Proof. W.T.S. $g(x+h) - g(x) - hg'(x) - \frac{h^2}{2}g''(x) = o(h^2)$.

We have

$$\begin{aligned}
\lim_{h\to 0} \frac{g(x+h) - g(x) - hg'(x) - \frac{h^2}{2}g''(x)}{h^2} &= \lim_{h\to 0} \frac{\frac{h^2}{2}g''(x+\theta h) - \frac{h^2}{2}g''(x)}{h^2} \\
&= \lim_{h\to 0} \frac{1}{2}[g''(x+\theta h) - g''(x)] = 0.
\end{aligned}$$

$\square$