Chaos, Fractals, and Dynamics

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Contents

1 Introduction

1.1 Dynamical systems

Definition 1.1. A repeated movement is called a ***dynamical system***. It is described in two parts:
(1) Space you are moving around in the state space.
(2) How to move, i.e., the dynamical map.

Example 1.1. (Standard) Quadratic maps

- State space: $\mathbb{R}$.
- Dynamical map: $Q_c(x) = x^2 - c$.

Example 1.2. Rotation maps

- State space: The unit circle $\mathbb{T}$.
- Dynamical map: Rotate the circle α radians counterclockwise. $R_\alpha(\theta) \equiv \theta + \alpha$.

Example 1.3. Doubling maps

- State space: $\mathbb{T}$.
- Dynamical map: $D(\theta) \equiv 2\theta$.

Example 1.4. Shift maps

- State space: The set of sequences of 0 and 1, $2^{\mathbb{N}}$.
- Dynamical map: Erase the first digit.

There are two ways to look at repetition. Say we have a dynamical system with dynamical map F and state space $Y, F : Y \to Y$.

- Follow individual points. For $y \in Y$, look at the sequence of points $y, F(y), F(F(y)), ..,$ called the ***orbit*** of y.
- Look at the whole state space at once, i.e., look at the sequence of functions $F, F \circ F, F \circ F \circ F, ...$ $\underbrace{F \circ F \circ \cdots \circ F}_{n \text{ times}}$ is called the nth ***iterate*** of F, F^n.
    - Note: $F^2(y) = F \circ F(y) = F(F(y)), F(y)^2 = F(y) \cdot F(y)$.

1.2 Fixed points

Definition 1.2. x is a ***fixed point*** of F iff it satisfies the equation $F(x) = x$. Fixed points often make good landmarks in the state space of a dynamical system.

Example 1.5. $F(x) = x^2 - 0.3 = x \Rightarrow p_{\pm} = \frac{1}{2}(1 \pm \sqrt{1 + 4 \times 0.3})$.

Example 1.6. Rotation map, R_α with $\alpha \not\equiv 0$: There are no fixed points.

Example 1.7. Doubling map, the point θ is fixed iff $D(\theta) \equiv \theta \Rightarrow 2\theta \equiv \theta \Rightarrow \theta \equiv 0$.

Example 1.8. Shift map: Only fixed points are $\overline{0}$ and $\overline{1}$.

1.3 Eventually fixed points

Definition 1.3. An ***eventually fixed point*** of $F : Y \to Y$ is a point whose orbit eventually reacts a fixed point, i.e., $F^{n+1}(y) = F^n(y)$, for some $n \in \{0, 1, 2, ...\}$.

Example 1.9. The orbit of $\frac{2\pi}{8}$ under the doubling map.

Example 1.10. Eventually fixed points of the shift map on $2^{\mathbb{N}}$ are $0011\overline{0}, 0100\overline{1}$, and so on.

1.4 Periodic points

Definition 1.4. A ***periodic point*** of $F : Y \to Y$ is a point where orbit eventually returns to its starting point, that is y periodic if $F^n(y) = y$ for some $n \in \mathbb{N}$.

Example 1.11. $G(x) = x^2 - 1$ on $\mathbb{R}$. The orbit of 0(-1) eventually returns to 0(-1).

Definition 1.5. An ***n-periodic point*** of $F : Y \to Y$ is a point y with $F^n(y) = y$. If y is n-periodic, it is also $2n/3n/4n/cdots$-periodic. The smallest period of a periodic point is called its ***minimum/prime period***.

2 Graphical Analysis of Dynamics

[Try to draw some graphs to analyze the dynamical systems.]

2.1 Attracting and repelling fixed points

Definition 2.1. Let $F : \mathbb{R} \to \mathbb{R}, p \in \mathbb{R}$ s.t. $F(p) = p$. A ***basin of attraction*** for p is an open interval $p \in I \subset \mathbb{R}$ s.t. every $x \in I$ is mapped to $F(x) \in I$, i.e., every orbit starting in I stays in I forever, or $F(x) \in I, \forall x \in I \Leftrightarrow F(I) \subset I$, and every orbit in I limits to p.

Definition 2.2. A ***region of repulsion*** for p is an open interval $p \in I$ s.t. every $x \in I$ eventually leaves I (allowed to com back) unless $x = p$.

If p has a b.o.a., p is attracting. If p has a r.o.r., p is repelling. For some cases, the orbit never leaves the open ball but does not limit to p, then p is ***neither attracting nor repelling***.

2.2 Fixed points of linear and approximating linear functions

For $F(x) = ax, a \in \mathbb{R}$, it has a single fixed point 0. If $|a| < 1$, the fixed point 0 of ax is attracting. If $|a| > 1$, the fixed point 0 of ax is repelling.

Now, we can see the approximately linear functions. When we say a function F is differentiable at p, we mean its graph near p is close to a straight line, i.e., $F(p + \Delta x) \approx F(p) + F'(p) + \Delta x$, when $\Delta x \approx 0$.
Say p is a fixed point of F, F is differentiable on some I containing p, and we have: if $|F'(p)| < 1$, the fixed point p is attracting; if $|F'(p)| > 1$, the fixed point p is repelling.

- When F is differentiable near a fixed point $p, |F'(p)| \neq 1$, we say p is ***hyperbolic***.
- A hyperbolic fixed point is always either attracting or repelling.
- Non-hyperbolic fixed points could be attracting or repelling.

2.3 Orbits near a periodic orbit

Definition 2.3. A periodic orbit is attracting if every point on the orbit is an attracting fixed point of F^n, where n is the minimum period.

Definition 2.4. A periodic orbit is repelling if every point on the orbit is a repelling fixed point of F^n, where n is the minimum period.

Definition 2.5. A periodic orbit is hyperbolic if every point on the orbit is a hyperbolic fixed point of F^n, where n is the minimum period. If $|(F^n)'(p)| < 1$, the orbit of p is attracting. If $|(F^n)'(p)| > 1$, the orbit of p is repelling.

Example 2.1. $G(x) = x^2 - 1$, the orbit has minimum period 2. We have $(G \circ G)'(0) = G'(G(0)) \cdot G'(0) = 0 < 1$, then 0 is an attracting hyperbolic fixed point of G^2. Hence, the orbit of 0 is an attracting periodic orbit.
Similarly, $(G \circ G)'(-1) = 0$.

2.4 Application: approximating square roots

Let $H_a(x) = \frac{1}{2}(x + \frac{a}{x}), a \in (0, \infty), x \in (0, \infty)$. $\sqrt{a}$ is an attracting fixed point, with the whole state space as its basin of attraction.
We have $H_a(x) = x \Rightarrow a = x^2$. Since the state space is $(0, \infty)$, the only fixed point is $\sqrt{a}$.

3 Generalization of State Space

3.1 Measuring distance in a general state space

3.1.1 Distance functions

- Standard distance function on $\mathbb{R}$: The d between two points $x, y \in \mathbb{R}$ is $d(x,y) = |y - x|$ or $d(x, x+a) = |a|$.
- Standard distance function on $\mathbb{T}$: The d between two points on the unit circle is the length of the shortest path from one to the other, or $d(\theta, \theta + \alpha) \equiv |\alpha|, \alpha \in [-\pi, \pi]$.
- Standard distance function on $2^{\mathbb{N}}$: Given two different sequences $x, y \in 2^{\mathbb{N}}$, let m be the number of digits before the first place they differ, $d(x,y) = 2^{-m}$. When $x = y, d(x,y) = 0$.

  Example 3.1. $x = 010100101001\cdots, y = 010101010101\cdots$, then the distance is $d(x,y) = 2^{-5}$.

  Example 3.2. $x = 10000\cdots, y = 01111\cdots, d(x,y) = 2^0 = 1$.

3.1.2 General features of distance functions

A function that satisfies these properties below is called a ***metric***.

- The distance functions take a pair of points x, y in a state space Y and gives back a number $d(x,y) \in [0, \infty)$.
- $d(x,y) = d(y,x), \forall x, y \in Y$.
- $d(x,y) = 0 \Leftrightarrow x = y$.
- (Triangle inequality) $d(x,y) \leqslant d(x,p) + d(p,y), \forall x, y, p \in Y$.

3.2 Generalize open intervals and limits

Definition 3.1. Y is a state space with a metric d and an ***open ball*** of radius r around x as the set $B_x(r) = \{y \in Y | d(x,y) < r\}$.

Example 3.3. In $2^{\mathbb{N}}$, with the standard metric, the open ball $B_x(2^{-n})$ is the set of sequences that match x for at least the first $n+1$ digit. For instance, $x = 00100110000\cdots$, $B_x(2^{-4})$ consists of the sequences that look like $00100\cdots$.

Definition 3.2. p is the ***limit*** of $x_1, x_2, ...$ if $\forall r > 0, \exists x_n, x_{n+1}, ...$ that stays inside $B_x(r)$.

∘ Note: A sequence can have at most one limit, $\lim_{n\to\infty} x_n = p$.

Example 3.4. $x_n = \underbrace{000\cdots0}_{n \text{ zeros}}111\cdots$ in $2^{\mathbb{N}}$, $\lim_{n\to\infty} x_n = \overline{0}$.

Proof. We know $d(x_n, \overline{0}) = 2^{-n}$, we need $2^{-n} < \varepsilon$, i.e., $n < \log_2(\frac{1}{\varepsilon})$. Thus, we take $N = \log_2(\frac{1}{\varepsilon})$.
Therefore, $\forall \varepsilon > 0, \exists N > 0$ s.t. $n > N \Rightarrow x_n \in B_{\overline{0}}(\varepsilon)$. □

3.3 Generalize attraction and repulsion to state spaces

Consider a dynamical system with state space Y and dynamical map F and we have a metric d on Y. Suppose $p \in Y$ is a fixed point of F.

- A basin of attraction for p is an open ball U with the following properties:

  (1) $p \in U$.

  (2) Every orbit starting in U stays in U forever.

  (3) Every orbit starting in U limits to p.

  If there is a b.o.a. for p, we say p is attracting.

- A region of repulsion for p is an open ball U with the following properties:

  (1) $p \in U$.

  (2) Every orbit starting in U eventually leaves U unless it starts at p.

  If there is a r.o.r. for p, we say p is repelling.

Example 3.5. The doubling map $D : \mathbb{T} \to \mathbb{T}$ defined by $D(\theta) \equiv 2\theta$ has a single fixed point 0, which is repelling.

Example 3.6. The shift map $S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ has two fixed points $\overline{0}$ and $\overline{1}$, both of which are repelling.

Solution. We can find $B_{\overline{0}}(1)$ is a r.o.r. for $\overline{0}$.
$B_{\overline{0}}$ consists of all sequences that look like $0 \cdots$. Pick any point $x \in B_{\overline{0}}(1)$ other than $\overline{0}$ and we know x has at least a 1, say

$$x = \underbrace{0 \cdots}_{n \text{ digits}} 1 \cdots ,$$

and then we have $S^n(x) = 1 \cdots \notin B_{\overline{0}}(1)$.

Example 3.7. The dynamical map A is defined on $2^{\mathbb{N}}$ as each 1 that is followed by a 0 turns into a 0. Classifying fixed points as attracting, repelling or neither.

Solution. Fixed points are $p_n = \underbrace{0 \cdots 0}_{n \text{ zeros}} \overline{1}, q = \overline{0}.p_n$ is repelling, $\forall n \geqslant 0$.
Consider any $x \in B_{p_n}(2^{-n})$ other than p_n itself, then after first 1 occurs, there must be a 0 somewhere, say

$$x = \underbrace{0 \cdots 0}_{n} 1 \cdots 0 \cdots .$$

Hence, $A(x) = 0 \cdots 01 \cdots 00 \cdots, A^2(x) = 0 \cdots 01 \cdots 000 \cdots$, ..., and thus $A^n(x) = \underbrace{0 \cdots 0}_{n+1} \cdots 0 \cdots \notin B_{p_n}(2^{-n})$.

4 Semiconjugacy

4.1 The doubling map and the shift map

We can represent number as binary sequences. For example,

$$\frac{1}{6} = 0.1666... = \frac{1}{10} + \frac{6}{100} + \frac{6}{1000} + \cdots ,$$

or

$$\frac{1}{6} = 0.001010... = \frac{0}{2} + \frac{0}{4} + \frac{1}{8} + \frac{0}{16} + \frac{1}{32} + \cdots .$$

Define $\phi : 2^{\mathbb{N}} \to \mathbb{T}$ given by $\phi(w) \equiv 2\pi w$, where w is the sequence of binary digits.

Actually, ϕ is an example of a semiconjugacy from S to D. When doubling a number, each binary digit moves one place to the left. For example, $D(2\pi \cdot 0.00\overline{10}) \equiv 2 \cdot 2\pi \cdot 0.00\overline{10} \equiv 2\pi \cdot 0.0\overline{10}$.

If a 1 moves into the 1's place, we can change it back into a 0, because that changes the angle by 2π. For example, $D(2\pi \cdot 0.11\overline{10}) \equiv 2\pi \cdot 1.1\overline{10} \equiv 2\pi + 2\pi \cdot 0.1\overline{10} \equiv 2\pi \cdot 0.1\overline{10}$.

We can express the relation map between the shift map and the doubling map in a formula $D(\phi(w)) = \phi(S(w))$, i.e., to double the angle with binary map representation w, first shift w and see what angle the result represents.

4.1.1 Finding fixed points

Theorem 4.1. If $w \in 2^{\mathbb{N}}$ is a fixed point of S, then $\phi(w) \in \mathbb{T}$ is a fixed point of D.

Proof. Suppose $S(w) = w$, then $D(\phi(w)) = \phi(S(w)) = \phi(w)$. □

4.1.2 Finding periodic points

Theorem 4.2. If $S^n(w) = w$, then $D^n(\phi(w)) \equiv \phi(w)$.

Theorem 4.3. If $S^n(w)$ is a fixed point of S, then $D^n(\phi(w))$ is a fixed point of D.

Example 4.1. $\overline{01}$ is 2-periodic of S, then $\phi(\overline{01})$ is a 2-periodic of D.

$$\begin{aligned}\phi(\overline{01}) &\equiv 2\pi\left(\frac{0}{2}+\frac{1}{2^2}+\frac{0}{2^3}+\frac{1}{2^4}+\cdots\right)\equiv 2\pi\cdot\frac{1}{4}\left(1+\frac{1}{4}+\frac{1}{4^2}+\cdots\right)\\ &\equiv 2\pi\cdot\frac{1}{4}\cdot\frac{1}{1-1/4}\equiv\frac{2}{3}\pi.\end{aligned}$$

Example 4.2. $\overline{000111}$ is 6-periodic for S. First, calculate $0.000111 = \frac{7}{64}$, then

$$\begin{aligned}\phi(\overline{000111}) &\equiv 2\pi\cdot\left(\frac{7}{64}+\frac{7}{64^2}+\frac{7}{64^3}+\cdots\right)\equiv 2\pi\cdot\frac{7}{64}\left(1+\frac{1}{64}+\frac{1}{64^2}+\cdots\right)\\ &\equiv 2\pi\cdot\frac{7}{64}\cdot\frac{64}{63}\equiv\frac{2}{9}\pi.\end{aligned}$$

4.1.3 Finding eventually fixed points

The eventually fixed points of S are the sequences that end with $\overline{0}$ and $\overline{1}$. Theses sequences describe the angles $2\pi t$ where t is a faction with a power of 2 in the denominator: $2\pi\cdot\frac{\alpha}{2^\beta}$.

4.2 Formal definition of semiconjugacy

Definition 4.1. Let X be a space, consider a map $d : X\times X\to[0,\infty)$ s.t.

$$\begin{aligned}&d(x,y)=d(y,x),\forall x,y\in X.\\ &d(x,y)=0\Leftrightarrow x=y.\\ &d(x,y)\leqslant d(x,w)+d(w,y),\forall x,y,w\in X.\end{aligned}$$

This map d is called a ***metric*** on X and the pair (X,d) is a ***metric space***.

Definition 4.2. Let $(X,d_X),(Y,d_Y)$ be metric spaces. Consider a map $f : X\to Y, f$ is ***continuous*** at $x_0\in X$ iff $\forall\varepsilon>0,\exists\delta>0$ s.t. $\forall x\in B(x_0,\delta)\Rightarrow f(x)\in B(f(x_0),\varepsilon)$. If f is continuous at all $x_0\in X$, then f is called continuous in X.

Definition 4.3. Let X and Y be metric spaces. Consider two maps $f : X\to X, g : Y\to Y$. A ***semiconjugacy*** is a map $\psi : X\to Y$ s.t.
(1) ψ is surjective, i.e., every point in Y has a preimage.
(2) There si an integer $m>0$ s.t. ψ is at most m-to-one.
(3) ψ is continuous.
(4) $\psi(f(x))=g(\psi(x))$, or $\psi\circ f=g\circ\psi$.

Example 4.3. $D : \mathbb{T} \to \mathbb{T}$ is given by $\theta \mapsto 2\theta \mod 2\pi, S : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is given by $x_1x_2x_3 \cdots \mapsto x_2x_3 \cdots$. Let $\theta = 2\pi t, t \in [0,1], t = \sum\limits_{n=1}^{\infty} \frac{x_n}{2^n}, x_n = 0$ or 1. Define $\phi : 2^{\mathbb{N}} \to \mathbb{T}$, given by $(x_n) \mapsto 2\pi \sum\limits_{n=1}^{\infty} \frac{x_n}{2^n} \mod 2\pi$. ϕ is a semiconjugacy from S to D.

Proof. (1) ϕ is surjective, i.e., every $\alpha \in \mathbb{T}$ can be written as $\alpha \equiv \phi(w)$ for some w. This is true because every α can be written as $\alpha \equiv 2\pi \cdot x, x \in [0,1]$ and every x admits a binary expansion $x = 0.w_1w_2....$

(2) For any $\alpha \in \mathbb{T}$, there are only finitely many $w \in 2^{\mathbb{N}}$ s.t. $\phi(w) \equiv \alpha$. This is true because most $x \in [0,1]$ only have one binary representation. The only case when having more than one representation is when x is rational with denominator a power of 2 (for example, $\frac{1}{2} = 0.1\overline{0} = 0.0\overline{1}$) and thus ϕ is at most 2-to-1.

(3) ϕ is continuous.
Define d_1 is on $2^{\mathbb{N}}, d_2$ is on $\mathbb{T}$. Pick any $w_0 = (x_n^0) \in 2^{\mathbb{N}}$. Let $\varepsilon > 0$.
For $w \in 2^{\mathbb{N}}$,

$$d_2(\phi(w), \phi(w_0)) \equiv \left| 2\pi \left(\sum_{n=1}^{\infty} \frac{x_n}{2^n} - \sum_{n=1}^{\infty} \frac{x_n^0}{2^n} \right) \right| \mod 2\pi \leqslant 2\pi \cdot \sum_{n=1}^{\infty} \frac{|x_n - x_n^0|}{2^n} \mod 2\pi.$$

Pick an open ball $B(w_0, 2^{-N})$, for all $w \in B(w_0, 2^{-N})$, we know

$$d_2(\phi(w), \phi(w_0)) \leqslant 2\pi \cdot \sum_{n=N+2}^{\infty} \frac{|x_n - x_n^0|}{2^n} \mod 2\pi \leqslant 2\pi \cdot \sum_{n=N+2}^{\infty} \frac{1}{2^n} = \frac{\pi}{2^N}.$$

We want $d_2(\phi(w), \phi(w_0)) \leqslant \frac{\pi}{2^N} \leqslant \varepsilon$, i.e., $N > \log_2(\frac{\pi}{\varepsilon})$. Hence, take $\delta = 2^{-N}$ for $N > \log_2(\frac{\pi}{\varepsilon})$, if $w \in B(w_0, \delta)$, then $d_2(\phi(w), \phi(w_0)) < \varepsilon$. Therefore, ϕ is continuous at w_0. Since w_0 is arbitrary, ϕ is continuous.

(4) $\phi(S(w)) = D(\phi(w))$. We have

$$w = w_1w_2w_3 \cdots, S(w) = w_2w_3 \cdots, \phi(S(w)) = 2\pi(0.w_2w_3...).$$

Besides,

$$\begin{aligned} \phi(w) = 2\pi(0.w_1w_2w_3...), D(\phi(w)) &= 2 \cdot 2\pi(0.w_1w_2w_3...) = 2\pi(w_1.w_2w_3...) \\ &= 2\pi w_1 + 2\pi(0.w_2w_3...) \\ &= 2\pi(0.w_2w_3...)\phi(S(w)). \end{aligned}$$

Thus, ϕ is a semiconjugacy from S to D. $\square$

4.3 Semiconjugacy toolbox

Let $E : W \to W, F : X \to X, \psi : W \to X$ be a semiconjugacy.

Theorem 4.4. If w is a fixed point of E, then $\psi(w)$ is a fixed point of F.

Proof. $F(\psi(w)) = \psi(E(w)) = \psi(w)$. $\square$

Theorem 4.5. $F^n(\psi(w)) = \psi(E^n(w)), \forall w \in W, n \in \mathbb{N}$.

Corollary 1. If $\psi : W \to X$ is a semiconjugacy from E to F, then ψ is also a semiconjugacy from E^n to $F^n, \forall n \in \mathbb{N}$.

Theorem 4.6. If w is an n-periodic point of E, then $\psi(w)$ is an n-periodic point of F.

Proof. If $E^n(w) = w$, then $F^n(\psi(w)) = \psi(E^n(w)) = \psi(w)$. $\square$

Theorem 4.7. If w is an eventually fixed point of E, then $\psi(w)$ is an eventually fixed point of F.

Proof. We have $E^{n+1}(w) = E^n(w)$. Then,

$$F^{n+1}(\psi(w)) = \psi(E^{n+1}(w)) = \psi(E^n(w)) = F^n(\psi(w)).$$

$\square$

Theorem 4.8. Suppose ψ is at most m-to-one. If $\psi(w)$ is a fixed point of F, the orbit of w mush eventually reach a periodic point of E, with a minimum period of at most m.

4.4 The quadratic map and the doubling map

Consider the quadratic map $F(x) = x^2 - 2, F : [-2, 2] \to [-2, 2]$. Define $\psi : \mathbb{T} \to [-2, 2]$ given by $\psi(\theta) \equiv 2\cos\theta$. After checking, we can draw a conclusion that ψ is a semiconjugacy from D to F.

5 Dynamics of Quadratic Maps