Monte Carlo Methods

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Pseudorandom Numbers

We first generate an i.i.d. sequence $U_i \sim \text{Uniform}[0,1]$.

Method (Linear Congruential Generator/LCG)**.**

- Choose large positive integers m, a, and b.
- Start with a seed value x_0, e.g., the current time in milliseconds.
- Recursively, $x_n = (ax_{n-1} + b) \mod m$, i.e., x_n is the remainder when $ax_{n-1} + b$ is divided by m. Hence $0 \leqslant x_n \leqslant m - 1$.
- Let $U_n = \dfrac{x_n}{m}$, $\{U_n\}$ will seem to be approximately i.i.d. Uniform$[0,1]$.

Note. We need m large so many possible values; a large enough that no obvious pattern between U_{n-1} and U_n; b to avoid short cycles of numbers. We want large period, i.e., number of iterations before repeat. One common choice: $m = 2^{32}, a = 69069, b = 23606797$.

Theorem. The LCG has full period (m) iff both $\gcd(b,m) = 1$, and every "prime or 4" divisor of m also divides $a - 1$.

Once we have $U_i \sim \text{Uniform}[0,1]$, we can generate other distributions with transformations, using change of variable theorem.

Example. To make $X \sim \text{Uniform}[L,R]$, set $X = (R-L)U_1 + L$.

Example. To make $X \sim \text{Bernoulli}(p)$, set

$$X = \begin{cases} 1, & U_1 \leqslant p \\ 0, & U_1 > p \end{cases}$$

Example. To make $Y \sim \text{Binomial}(n,p)$, either set $Y = X_1 + \cdots + X_n$ where

$$X_i = \begin{cases} 1, & U_i \leqslant p \\ 0, & U_i > p \end{cases}$$

or set

$$Y = \max\left\{ j : \sum_{k=0}^{j-1} \binom{n}{k} p^k (1-p)^{n-k} \leqslant U_1 \right\}$$

Generally, to make $P(Y = x_i) = p_i$ for some $x_1 < x_2 < \cdots$, where $p_i \geqslant 0$ and $\sum_i p_i = 1$, set

$$Y = \max\left\{ x_j ; \sum_{k=1}^{j-1} p_k \leqslant U_1 \right\}$$

Example. To make $Z \sim \text{Exponential}(1)$, set $Z = -\ln(U_1)$. Generally, to make $W \sim \text{Exponential}(\lambda)$, set $W = \dfrac{Z}{\lambda} = \dfrac{-\ln(U_1)}{\lambda}$ so that W has density $\lambda e^{-\lambda x}$ for $x > 0$.

Example. If

$$X = \sqrt{2\ln\left(\frac{1}{U_1}\right)}\cos(2\pi U_2)$$

$$Y = \sqrt{2\ln\left(\frac{1}{U_1}\right)}\sin(2\pi U_2)$$

then $X, Y \sim \mathcal{N}(0,1)$ and $X \perp Y$.

Method (Inverse CDF Method)**.**

- We want CDF $P(X \leqslant x) = F(x)$.
- For $0 < t < 1$, set $F^{-1}(t) = \min\{x; F(x) \geqslant t\}$ and $X = F^{-1}(U_1)$.
- $X \leqslant x$ iff $U_1 \leqslant F(x)$ and thus $P(X \leqslant x) = P(U_1 \leqslant F(x)) = F(x)$.