Introduction to Real Analysis

Derek Li

Contents

1 Real Numbers

We define

$$\mathbb{N} = \{1, 2, \cdots\}.$$

If we take the closure of $\mathbb{N}$ under subtraction, we obtain

$$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}.$$

If we take the closure of $\mathbb{Z}$ under division by non-zero numbers, we obtain

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1 \right\},$$

where $(m, n) = 1$ means if $d \in \mathbb{N}$ divides both m, n, then $d = 1$.

Example 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for a contradiction that there are $m \in \mathbb{Z}, n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m, n) = 1$. Hence, $m^2 = 2n^2$, then m^2 is an even complete square. So $4|m^2$. But then $4|2n^2$ and thus $2|n^2$. So n has to be even. Hence both m, n are even, i.e., $2|m, 2|n$. This contradicts the fact that $(m, n) = 1$. □

1.1 Preliminaries

Definition 1.1. A ***function*** from A to B ($f : A \to B$) is the set of pairs $(x, y) \in A \times B$ s.t. (1) if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$; (2) $\forall x \in A, \exists y \in B$ s.t. $f(x) = y$.

Note that A is said to be the domain of f, but the range of f does not have to be B, and it is a subset of B.

Definition 1.2. Assume $f : A \to B$ is a function, f is said to be ***injective*** if

$$\forall x_1, x_2 \in A, f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2.$$

Property 1.1. If $f : A \to B, g : B \to C$ are injective, then $g \circ f : A \to C$ is injective.

Definition 1.3. f is said to be ***surjective*** if

$$\forall y \in B, \exists x \in A \text{ s.t. } f(x) = y.$$

Property 1.2. If there is a surjective map $g : A \to B$, then there is a injective map $f : B \to A$.

Definition 1.4. f is said to be ***bijective*** if f is injective and surjective.

Definition 1.5. For all x,

$$|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}.$$

Theorem 1.1 (Triangle Inequality)**.** $|x + y| \leqslant |x| + |y|$.

Proof. We have $(x + y)^2 = x^2 + y^2 + 2xy \leqslant |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2$. Thus,

$$|x + y| = \sqrt{(x + y)^2} \leqslant \sqrt{(|x| + |y|)^2} = |x| + |y|.$$

□

Definition 1.6. Assume $X \subseteq \mathbb{R}$, the ***maximum*** (***minimum***) of X is an element $a \in X$ s.t. $\forall x \in X, x \leqslant a$ ($x \geqslant a$).

Definition 1.7. The ***least upper bound*** of X, denoted by $\sup(X)$, is $a \in \mathbb{R}$ s.t. (1) $\forall x \in X, x \leqslant a$ (a is an upper bound for X); (2) if b is an upper bound for X, then $a \leqslant b$.

Example 1.2. $\max((0, 1))$ does not exist. $\sup((0, 1)) = 1$. $\sup(\mathbb{R})$ and $\sup(\mathbb{N})$ do not exist.

1.2 The Axiom of Completeness

Definition 1.8. $X \subseteq \mathbb{Q}$ is said to be an ***initial segment*** if (1) $X \neq \varnothing$; (2) $\forall x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$; (3) $X \neq \mathbb{Q}$.

Definition 1.9. $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$.

Property 1.3. $\mathbb{R}$ is an ordered field.

Property 1.4. If $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A, then $s = \sup(A)$ iff

$$\forall \varepsilon > 0, \exists a \in A \text{ s.t. } a + \varepsilon > s.$$

Proof. ($\Leftarrow$) Assume for a contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$. Let $\varepsilon = \frac{s-t}{2} > 0$, then

$$\forall a \in A, a + \varepsilon \leqslant t + \varepsilon = \frac{s+t}{2} < s,$$

which is a contradiction.

($\Rightarrow$) Assume for a contradiction that $\varepsilon_0 > 0$ and $\forall a \in A, a + \varepsilon_0 \leqslant s$. Thus $\forall a \in A, a \leqslant s - \varepsilon_0$, and $s - \varepsilon_0 < s$ is an upper bound for A, which is a contradiction. □

Theorem 1.2 (Axiom of Completeness)**.** If $X \subseteq \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let A_x be the initial segment of $\mathbb{Q}$ corresponding to x. Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \bigcup\limits_{x \in X} A_x$. Note that $\bigcup\limits_{x \in X} A_x$ is an initial segment of $\mathbb{Q}$ and thus $\sup(\bigcup\limits_{x \in X} A_x)$ is $\sup(X)$. □

1.3 Consequences of Completeness

Definition 1.10. Assume $\{A_n : n \in \mathbb{N}\}$ is a sequence of sets, $\{A_n : n \in \mathbb{N}\}$ is said to be ***nested*** if $A_n \supseteq A_{n+1}$.

Theorem 1.3 (Nested Interval Property)**.** Assume $\{I_n : n \in \mathbb{N}\}$ is a nested sequence of closed intervals of $\mathbb{R}$, then $\bigcap\limits_{n \in \mathbb{N}} I_n \neq \varnothing$.

Proof. Let $[a_n, b_n] = I_n$. Since $\{I_n : n \in \mathbb{N}\}$ is nested,

$$a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n, \forall n \in \mathbb{N}.$$

Let $A = \{a_n : n \in \mathbb{N}\}$.

Note that b_1 is an upper bound for A so A has supremum in $\mathbb{R}$. We have $\forall n \in \mathbb{N}, \sup(A) \leqslant b_n$ and $\sup(A) \geqslant a_n$. Thus, $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$, i.e., $\bigcap\limits_{n \in \mathbb{N}} I_n \neq \varnothing$. □

Theorem 1.4 (Archimedean Property)**.** (1) $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $y \leqslant n$;
(2) $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$.

Proof. (1) Assume for a contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$. Let $\alpha = \sup(\mathbb{N})$, then by lemma, $\exists n \in \mathbb{N}$ s.t. $n + 1 > \alpha$, which is a contradiction.

(2) From (1), we have $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{y} < n \Rightarrow \frac{1}{n} < y$. □

Theorem 1.5. $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., if $a < b, a, b \in \mathbb{R}$, then $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

Proof. Suppose $a < b, a, b \in \mathbb{R}$. By Archimedean Property, we can find $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, i.e., $1 < nb - na$. Hence we can find $m \in \mathbb{Z}$ s.t. $na < m < nb$. Therefore,

$$a < \frac{m}{n} < b,$$

and let $r = \frac{m}{n}$. $\square$

1.4 Cardinality

Definition 1.11. If there is a bijection $f : A \to B$, we say A, B are in ***one-to-one correspondence***, denoted $A \sim B$.

Property 1.5. If $A \sim B, B \sim C$, then $A \sim C$.

Definition 1.12. $\mathrm{Card}(A) \leqslant \mathrm{Card}(B)$ if there is a injective map $f : A \to B$.

Example 1.3. $\mathbb{N} \sim \mathbb{Z}, \mathbb{N} \sim \mathbb{N}^2, \mathbb{N} \sim \mathbb{Q}, \mathbb{N} \nsim \mathbb{R}$ $(\mathrm{Card}(\mathbb{N}) < \mathrm{Card}(\mathbb{R})), (-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$.

Theorem 1.6 (Schroeder-Bernstein Theorem)**.** If there are injective functions $f : A \to B$ and $h : B \to A$, then there is a bijection $g : A \to B$.

Definition 1.13. A is said to be a ***countable set*** if there is a bijective $f : A \to \mathbb{N}$.

Example 1.4. $\mathbb{N}^2$ is countable since there is an injective $f : \mathbb{N}^2 \to \mathbb{N}$ given by $f(m, n) = 2^m 3^n$.

Property 1.6. Countable union of countable sets is countable, i.e., if $\{A_n : n \in \mathbb{N}\}$ is a collection of countable sets, then $\bigcup\limits_{n \in \mathbb{N}} A_n$ is countable.

1.5 Cantor's Theorem

Theorem 1.7 (Cantor's Theorem)**.** If A is a set, then there is no map $g : A \to P(A)$ which is surjective.

2 Metric Spaces and Topology of Metric Spaces

2.1 Metric Spaces

Definition 2.1. A ***metric space*** is a pair (X, d), where $d : X^2 \to [0, \infty)$ s.t. $\forall x, y, z \in X$:
(1) $d(x, y) = 0$ iff $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leqslant d(z, y) + d(y, z)$.

Example 2.1. For $X = \mathbb{R}, d(x, y) = |x - y|$.

Example 2.2. For $X = \mathbb{R}^n, d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ (Euclidean distance).

Example 2.3. For $X = \mathbb{R}^n, d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|$.

Example 2.4. For $X = \mathbb{R}^n, d(\mathbf{x}, \mathbf{y}) = \max\limits_{1 \leqslant i \leqslant n} |x_i - y_i|$.

Example 2.5. For $X = l_\infty = \{\text{The collection of all } (x_n) \subseteq \mathbb{R} \text{ that are bounded}\} \subseteq \mathbb{R}^{\mathbb{N}}, d(\mathbf{x}, \mathbf{y}) = \sup\limits_{n \in \mathbb{N}} |x_n - y_n|$.

Example 2.6. For $X = C[0, 1] =$ All continuous functions $f : [0, 1] \to \mathbb{R}, d(f, g) = \sup\limits_{x \in [0,1]} |f(x) - h(x)|$

Example 2.7 (Discrete Metric)**.**

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}.$$

Definition 2.2. A metric space (X, d) is ***complete*** iff every Cauchy sequence is convergent.

Example 2.8. Here are some examples of complete metric space:

- $\mathbb{R}$ with $d(x, y) = |x - y|$.
- (X, d) with discrete metric $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$.
- $C[0, 1]$ with $d(f, g) = \sup\limits_{x \in [0,1]} |f(x) - g(x)| = \|f - g\|_\infty$.
- $(\mathbb{N}^{\mathbb{N}}, d)$ with $d((x_n), (y_n)) = \frac{1}{\min\{n : x_n \neq y_n\}}$.

2.2 Topology of Metric Spaces

Definition 2.3. ***Open ball*** with radius r and center x is

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

Definition 2.4. A set $U \subseteq X$ is ***open*** iff

$$\forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq U.$$

Example 2.9. $B_\varepsilon(x)$ is open.

Proof. Fix $x \in X$ and $\varepsilon > 0$. We want to show: $\forall y \in B_\varepsilon(x), \exists \delta > 0$ s.t. $B_\delta(y) \subseteq B_\varepsilon(x)$.

Take $y \in B_\varepsilon(x)$, then $d(x, y) < \varepsilon$. Take $\delta = \varepsilon - d(x, y) > 0$. Take any $z \in B_\delta(y)$, we have

$$d(x, z) \leqslant d(x, y) + d(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon.$$

Thus $z \in B_\varepsilon(x)$ and $B_\delta(y) \subseteq B_\varepsilon(x)$. □

Definition 2.5. A ***topological space*** is a pair (X, τ) where X is a set and τ is the subset of the power set of X which we call open s.t.

(1) $\varnothing, X \in \tau$;

(2) $U_1, \cdots, U_n \in \tau \Rightarrow \bigcap\limits_{i=1}^{n} U_i \in \tau$;

(3) $\{U_i : i \in I\} \subseteq \tau \Rightarrow \bigcup\limits_{i \in I} U_i \in \tau$.

Example 2.10. $\{X, \{\varnothing, X\}\}, \{X, P(X)\}$ are topological spaces and we call $\{X, P(X)\}$ as a discrete topological space.

Example 2.11. Given (X, d) is a metric space, define $\tau_d : U \in \tau_d$ iff $\forall x \in U, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$. Then τ_d is a topology.

Proof. First, $\varnothing, X \in \tau_d$ since $\forall x \in \varnothing, B_1(x) \subseteq \varnothing$ and $\forall x \in X, B_1(x) \subseteq X$.

Then suppose $U_1, \cdots, U_n \in \tau_d$, we want to show:

$$U = \bigcap_{i=1}^{n} U_i \in \tau_d \Leftrightarrow \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq U.$$

Since $U_1, \cdots, U_n \in \tau_d$ and $x \in U$, then $x \in U_i, \forall i = 1, \cdots, n, \exists \varepsilon_i > 0$ s.t. $B_{\varepsilon_i}(x) \subseteq U_i$. Take $\varepsilon = \min\limits_{1 \leqslant i \leqslant n} \varepsilon_i$ and thus $B_\varepsilon(x) \subseteq U_i, \forall i = 1, \cdots, n$. Hence, $B_\varepsilon(x) \subseteq U_i \subseteq U$.

Finally, let $\{U_i : i \in I\} \subseteq \tau_d$, we want to show:

$$U = \bigcup_{i \in I} \in \tau_d \Leftrightarrow \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq U.$$

Pick $i_0 \in I, x \in U_{i_0} \subseteq U$. Since $U_{i_0} \in \tau_d$ then $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U_{i_0} \subseteq U$.

Wherefore, τ_d is a topology. □

Definition 2.6. A subset F of a topological space (X, τ) is ***closed*** if $X \backslash F$ is open.

Property 2.1. Given a topological space (X, τ), we have:

(1) $\varnothing, X$ are closed;

(2) $F_1, \cdots, F_n$ are closed then $\bigcup\limits_{i=1}^{n} F_i$ is closed;

(3) $\{F_i, i \in I\}$ is a collection of closed set, then $\bigcap\limits_{i \in I} F_i$ is closed.

Definition 2.7. Given a topological space $(X, \tau), \tau \subseteq P(X)$ and $F \subseteq X$. Define the ***topological closure*** of F as the minimal closed superset of F, i.e.,

$$\overline{F} = \bigcap\{H : H \text{ is closed}, H \supseteq F\}.$$

Define the ***interior*** of F as the maximal open subset of F, i.e.,

$$F^\circ = \bigcup\{U : U \text{ is open}, U \subseteq F\}.$$

Example 2.12. Given (X, d) is a metric space, τ_d is the topology that $U \in \tau_d$ iff $\forall x \in U, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$. Suppose $F \subseteq X$, then

$$\overline{F} = \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap F \neq \varnothing\} = \left\{\lim_{n\to\infty} x_n : (x_n) \subseteq F, \lim_{n\to\infty} x_n \text{ exists}\right\}$$

and

$$F^\circ = \{x \in X : \exists \varepsilon > 0, B_\varepsilon(x) \subseteq F\} = \bigcup \{B_\varepsilon(x) : \varepsilon > 0, x \in F, B_\varepsilon(x) \subseteq F\}.$$

Property 2.2. If $K_1 \supseteq K_2 \subseteq \cdots$ are compact and nonempty subsets of X, then $K = \bigcap\limits_{n=1}^{\infty} K_n$ is compact and nonempty.

Definition 2.8. Let (X, d) be a metric space. $P \subseteq X$ is ***perfect*** if it is closed nonempty and for every open $U \subseteq X, U \cap P \neq \varnothing$, $U \cap P$ has at least two elements. Or $\forall x \in P, \forall \varepsilon > 0, B_\varepsilon(x) \cap P$ has at least one more element besides x.

Example 2.13. $S = [0, 1] \cup \left\{\frac{3}{2}\right\} \cup [2, 3]$ is not perfect.

Property 2.3. Perfect subsets P of complete metric space are not countable.

Example 2.14 (Cantor Set)**.** $C \subseteq [0, 1], C = \bigcap\limits_{n=0}^{\infty} C_n$, where $\varnothing \neq C_n \subseteq [0, 1]$ and C_n is closed and compact. $C_0 = [0, 1], C_1 = [0, 1] \backslash \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{1}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \cdots$.

Definition 2.9. Let (X, d) be a metric space, $A \neq \varnothing, B \subseteq X$. A and B are ***separated*** if $\overline{A} \cap B = \overline{B} \cap A = \varnothing$.

Definition 2.10. A set $C \subseteq X$ is ***connected*** if for every decomposition $C = A \cup B, A, B \neq \varnothing, A$ and B are not separated, i.e., $\overline{A} \cap B \neq \varnothing$ or $\overline{B} \cap A \neq \varnothing$.

Property 2.4. $C \subseteq \mathbb{R}$ is connected iff for all $a < b$ in C and every $c, a < c < b$ belongs to C, i.e., $\forall a, b \in C, [a, b] \subseteq C$.

Proof. Let $C = A \cup B, a_0 \in A, b_0 \in B, a_0 < b_0$. We define $I_0 = [a_0, b_0], c_0 = \frac{a_0+b_0}{2}$. Define $I_1 = [a_0, c_0], \cdots$. We have $x \in \overline{A} \cap B$ or $\overline{B} \cap A$. $\square$

Definition 2.11. A set $D \subseteq X$ is ***dense*** if $\overline{D} = X$, i.e., every point of X is a limit of a sequence of elements of D, or

$$\forall x \in X, \forall \varepsilon > 0, B_\varepsilon(x) \cap D \neq \varnothing \Leftrightarrow \forall U \subseteq X, U \neq \varnothing, U \cap D \neq \varnothing,$$

where U is open.

Example 2.15. $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., $\mathbb{Q}$ has a point in any open nonempty interval.

Definition 2.12. $N \subseteq X$ is ***nowhere dense*** if $\forall U \neq \varnothing, \exists V \subseteq U$ s.t. $V \neq \varnothing$ and $V \cap N = \varnothing$, where U is open.

Definition 2.13. A metric space (X, d) is ***compact*** if every sequence has a converge subsequence, i.e.,

$$\forall (x_n) \subseteq X, \exists (x_{n_k}) \subseteq (x_n), \exists x \in X \text{ s.t. } \lim_{k\to\infty} x_{n_k} = x \in X.$$

Example 2.16. $(\mathbb{R}, |x - y|)$ is not compact (e.g., $x_n = n$) and $([0, 1], |x - y|)$ is compact.

Property 2.5. If (X, d) is compact, then it is bounded, i.e., $\exists M$ s.t. $\forall x, y \in X, d(x, y) \leqslant M$.

Property 2.6. If $Y \subseteq X, (X, d)$ is a metric space, and (Y, d) is compact, then Y is closed in X.

Theorem 2.1 (Baire Category Theorem)**.** If (X, d) is complete, then intersection of dense open subsets $\bigcap\limits_{n=1}^{\infty} D_n$ of X is dense in X.

Proof. **Claim**. Suppose $D_1, \cdots, D_n$ is a finite list of dense open subsets of $(X, d), D = \bigcap\limits_{i=1}^{n} D_n$ is also dense and open.

First note that D is open. Take $U \neq \varnothing$ be open. We need to show $U \cap D \neq \varnothing$. We have

$$
\begin{aligned}
U_1 &= U \cap D_1 \neq \varnothing \\
U_2 &= U_1 \cap D_2 \neq \varnothing \\
&\vdots \\
U_n &= U_{n-1} \cap D_n \neq \varnothing,
\end{aligned}
$$

and thus $U_n = U \cap D \neq \varnothing$.

We may assume that $D_1 \supseteq D_2 \supseteq \cdots$. Take $x_1 \in D_1$, then $\exists 0 < \varepsilon_1 < 1$ s.t. $\overline{B_{\varepsilon_1}(x_1)} \subseteq D_1$. Take $x_2 \in B_{\varepsilon_1}(x_1) \cap D_2 \neq \varnothing$, then $\exists 0 < \varepsilon_2 < \frac{1}{2}$ s.t. $\overline{B_{\varepsilon_2}(x_2)} \subseteq D_2, \cdots$. Suppose $n < m, x_m \in B_{\varepsilon_n}(x_n)$, i.e., $d(x_n, x_m) < \frac{1}{n}$. Thus $\{x_n\}$ is Cauchy. Thus, $x = \lim\limits_{n \to \infty} x_n = \lim\limits_{\substack{m \to \infty \\ m \geqslant n}} x_m \subseteq \overline{B_{\varepsilon_n}(x_m)}$. Hence, $x \in \bigcap\limits_{n=1}^{\infty} D_n$. ∎

Note that two categories of size for subsets are created in a metric space. A set of first category is one that can be written as a countable union of nowhere-dense sets. If our metric space is complete, then it is necessarily of second category, meaning it cannot be written as a countable union of nowhere-dense sets.

3 Sequences and Series

3.1 Sequences

Definition 3.1. Let (X, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x \in X$ iff

$$\forall\varepsilon > 0, \exists N \text{ s.t. } \forall n \geqslant N, d(x_n, x) < \varepsilon,$$

denoted $\lim\limits_{n\to\infty} x_n = x$.

Property 3.1. If $\lim\limits_{n\to\infty} x_n = x$, $\lim\limits_{n\to\infty} y_n = y$, then $x = y$.

Proof. We want to show: $d(x, y) = 0 \Leftrightarrow \forall\varepsilon > 0, d(x, y) < \varepsilon$.

Since $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$, then $\exists N_1$ s.t. $\forall n \geqslant N_1, d(x_n, x) < \frac{\varepsilon}{2}$ and $\exists N_2$ s.t. $\forall n \geqslant N_2, d(x_n, y) < \frac{\varepsilon}{2}$. Take $n \geqslant \max(N_1, N_2)$, then we have

$$d(x, y) \leqslant d(x_n, x) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Property 3.2. Let (X, d) be a metric space. Suppose $\lim\limits_{n\to\infty} x_n = x$, $(x_n) \subseteq F$ and F is closed, then $x \in F$.

Proof. Suppose $x \notin F$, i.e., $x \in X\backslash F$. Since F is closed then $X\backslash F$ is open, so $\exists\varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq X\backslash F$. Pick N s.t. $\forall n \geqslant N, d(x_n, x) < \varepsilon$, then $x_n \in B_\varepsilon(x) \Rightarrow (x_n) \subseteq X\backslash F$, which is a contradiction. □

Property 3.3. Let (X, d) be a metric space. Suppose $F \subseteq X$, and if F is not closed, then $\exists (x_n) \subseteq F$ and $x \notin F$ s.t. $\lim\limits_{n\to\infty} x_n = x$.

Proof. If F is not closed, then $U = X\backslash F$ is not open. So $\exists x \in U$ s.t. $\forall\varepsilon > 0, B_\varepsilon(x) \nsubseteq U$. Take $x_n \in B_{\frac{1}{n}}(x)\backslash U = B_{\frac{1}{n}}(x) \cap F, \forall n \in \mathbb{N}$. Then $(x_n) \subseteq F$.

Let $\varepsilon > 0, N = \left[\frac{1}{\varepsilon}\right] + 1$, and $n \geqslant N$. Since $x_n \in B_{\frac{1}{n}}(x)$, then $d(x_n, x) < \frac{1}{n} \leqslant \varepsilon$, i.e., $\lim\limits_{n\to\infty} x_n = x$. □

Definition 3.2. A sequence (x_n) in a metric space (X, d) is ***Cauchy*** if

$$\forall\varepsilon > 0, \exists N \text{ s.t. } \forall m, n \geqslant N, d(x_n, x_m) < \varepsilon.$$

Property 3.4. Convergent sequences are Cauchy.

Proof. Suppose $\lim\limits_{n\to\infty} x_n = x$. Let $\varepsilon > 0, \exists N$ s.t. $\forall n \geqslant N, d(x, x_n) < \frac{\varepsilon}{2}$. Take $m, n \geqslant N$,

$$d(x_n, x_m) \leqslant d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Note that when $X = \mathbb{R}$ with the usual metric, the converse is true. But in general, the converse is not. For example, $X = \mathbb{R}\backslash\{0\}$ with $d(x, y) = |x - y|$. Let $x_n = \frac{1}{n}$.

Property 3.5. Suppose (x_{n_k}) is a subsequence of (x_n) and $\lim\limits_{n\to\infty} x_n = x$, then $\lim\limits_{k\to\infty} x_{n_k} = x$.

Proof. Since $\lim_{n\to\infty} x_n = x$, then

$$\forall\varepsilon > 0, \exists N \text{ s.t. } \forall n \geqslant N, d(x, x_n) < \varepsilon.$$

Take $K = N$, let $k \geqslant N$, we have $n_k \geqslant k \geqslant N$, and thus $d(x, x_{n_k}) < \varepsilon$. $\square$

Theorem 3.1 (Bolzano-Weierstrass Theorem)**.** A subset Y of $\mathbb{R}$ is compact iff it is closed and bounded.

Note that the theorem is true for $\mathbb{R}^n$ but is false for infinite dimension.

Theorem 3.2 (Heine-Borel Theorem)**.** A subset Y of a metric space (X, d) is compact if every open cover $Y \subseteq \bigcup_{i\in I} U_i$ has a finite subcover $Y \subseteq \bigcup_{l=1}^{n} U_{i_l}$.

Definition 3.3. $(x_n) \subseteq \mathbb{R}$ is ***monotone*** if either $x_n \leqslant x_m, n \leqslant m$ or $x_n \geqslant x_m, n \leqslant m$.

Theorem 3.3 (Monotone Subsequence Theorem)**.** Every sequence $(x_n) \subseteq \mathbb{R}$ has a monotone subsequence.

Proof. We define a peak term: given a (x_n), a particular term x_m is a peak term if $x_m \geqslant x_n, \forall n \geqslant m$. Now let (x_n) be any sequence, define set of peaks: $P = \{x_n : x_n \geqslant x_m, \forall m \geqslant n\}$. If P is infinite then $\exists(x_{n_k})$ s.t. $x_{n_k} \in P$ is a decreasing subsequence of (x_n). If P is finite then let $x_m = \min P$. Let $n_1 = m + 1$ then $x_{n_1} \notin P$ so that exists element is greater than x_{n_1}. Define $n_{k+1} = \min\{m \in \mathbb{N} : x_m > x_{n_k}\}$. Hence we get $x_{n_{k+1}} > x_{n_k}$ so (x_{n_k}) is an increasing subsequence of (x_n). $\square$

Theorem 3.4. Every bounded sequence contains a convergent subsequence.

Proof. If (x_n) is bounded sequence, then $\exists(x_{n_k})$ s.t. (x_{n_k}) is monotone and bounded sequence. Then by theorem, (x_{n_k}) must converge. $\square$

Property 3.6. If $a_n \leqslant b_n, \forall n, a = \lim_{n\to\infty} a_n, b = \lim_{n\to\infty} b_n$, then $a \leqslant b$.

Proof. Suppose $a > b$. Let $\varepsilon = \frac{a-b}{2}$. We know $\exists N_1$ s.t. $a_n \in B_\varepsilon(a)$ for $n \geqslant N_1$ and $\exists N_2$ s.t. $b_n \in B_\varepsilon(b)$ for $n \geqslant N_2$. Take $n > \max(N_1, N_2)$, then we have

$$b_n < \frac{a+b}{2} < a_n,$$

which is a contradiction. $\square$

Property 3.7 (Algebraic Limit Theorem)**.** Suppose $a = \lim_{n\to\infty} a_n, b = \lim_{n\to\infty} b_n$, then:

(1) $a + b = \lim_{n\to\infty}(a_n + b_n)$;
(2) $ab = \lim_{n\to\infty} a_n b_n$;
(3) $\frac{a}{b} = \lim_{n\to\infty} \frac{a_n}{b_n}$, and $b \neq 0$.

Property 3.8. Monotone bounded sequence (x_n) converges to its supremum or infimum.

Proof. We only prove one situation: Fix $\varepsilon > 0$. Let $s = \sup\{x_n : n \in \mathbb{N}\}$. We have $s - \varepsilon < s$ and thus $s - \varepsilon$ is not an upper bound of (x_n). Therefore, there is N s.t. $x_N > s - \varepsilon$. Take $n \geqslant N$, we have $x_n \geqslant x_N > s - \varepsilon$. Therefore, we have $|x_n - s| < \varepsilon$. $\square$

Definition 3.4. We define

$$\limsup_{n\to\infty} x_n = \inf\{y_m : m \in \mathbb{N}\},$$

where $y_m = \sup\{x_n : n \geqslant m\}$.

$$\liminf_{n\to\infty} x_n = \sup\{z_m : m \in \mathbb{N}\},$$

where $z_m = \inf\{x_n : n \geqslant m\}$.

3.2 Series

Definition 3.5. We define
$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n, S_n = \sum_{k=1}^{n} a_k.$$
We call $\sum\limits_{k=1}^{\infty} a_k$ is a summable series if the limit exists, i.e.,
$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geqslant N, |S_n - A| < \varepsilon.$$

Property 3.9 (Cauchy Criterion for Series)**.** $\sum\limits_{k=1}^{\infty} a_k$ is summable iff
$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geqslant m \geqslant N, |S_n - S_m| = \left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon.$$

Corollary 3.1. If $\sum\limits_{k=1}^{\infty} a_k$ is summable, then $|a_k| \to 0$.

Proof. We have $|a_k| = |s_k - s_{k-1}| < \varepsilon$ for $k > N$. □

Example 3.1. $\sum\limits_{k=1}^{\infty} \frac{1}{k^2}$ is summable.

Proof. Since $S_m \leqslant S_n, \forall m \leqslant n$, then it suffices to find $0 < M < \infty$ s.t. $S_m < M, \forall m$. We have
$$\begin{aligned} S_m &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2} < 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \cdots + \frac{1}{m(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) = 1 + 1 - \frac{1}{m} < 2. \end{aligned}$$
□

Example 3.2. $\sum\limits_{k=1}^{\infty} \frac{1}{k} = \infty$.

Proof. We have
$$\sum_{n=1}^{2^k} \frac{1}{n} \geqslant 1 + \frac{k}{2} \to \infty \text{ as } k \to \infty.$$
□

Theorem 3.5 (Algebraic Limit Theorem for Series)**.** Suppose $\sum\limits_{k=1}^{\infty} a_k = A$, $\sum\limits_{k=1}^{\infty} b_k = B, c \in \mathbb{R}$, then

(1) $\sum\limits_{k=1}^{\infty} ca_k = cA$;

(2) $\sum\limits_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof. (1) We want to show $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geqslant N, \left|\sum\limits_{k=1}^{\infty} ca_k - cA\right| < \varepsilon$. We know $\forall \varepsilon_0 > 0, \exists N_{\varepsilon_0}$ s.t. $\forall n \geqslant N_{\varepsilon_0}, \left|\sum\limits_{k=1}^{\infty} a_k - A\right| < \varepsilon_0$. Take $\varepsilon_0 = \frac{\varepsilon}{|c|}$, then we have
$$\left|\sum_{k=1}^{\infty} ca_k - cA\right| = |c| \left|\sum_{k=1}^{\infty} a_k - A\right| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$
□

Property 3.10 (Order Comparison Test)**.** Suppose $b_k \geqslant a_k \geqslant 0, \forall k$. If $\sum\limits_{k=1}^{\infty} b_k$ converges so does $\sum\limits_{k=1}^{\infty} a_k$. If $\sum\limits_{k=1}^{\infty} a_k$ diverges so does $\sum\limits_{k=1}^{\infty} b_k$.

Definition 3.6. We call a ***geometric series*** if it is

$$\sum_{k=1}^{\infty} ar^k.$$

Note that the geometric series converges to $\frac{a}{1-r}$ whenever $r^m \to 0$ iff $|r| < 1/$

Definition 3.7. $\sum\limits_{k=1}^{\infty} a_k$ is ***absolutely convergent*** if $\sum\limits_{k=1}^{\infty} |a_k|$ is convergent. $\sum\limits_{k=1}^{\infty} a_k$ is ***conditionally convergent*** if $\sum\limits_{k=1}^{\infty} a_k < \infty$ but $\sum\limits_{k=1}^{\infty} |a_k| = \infty$.

Example 3.3 (Alternating Series)**.** $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} < \infty$ but $\sum\limits_{k=1}^{\infty} \left|\frac{(-1)^{k+1}}{k}\right| = \sum\limits_{k=1}^{\infty} \frac{1}{k} = \infty$.

Property 3.11 (Absolute Convergence Test)**.** If $\sum\limits_{k=1}^{\infty} |a_k|$ converges so does $\sum\limits_{k=1}^{\infty} a_k$.

Proof. We use Cauchy test for $\sum\limits_{k=1}^{\infty} a_k$, i.e., we want to show $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geqslant m \geqslant N, |a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon$. We know $|a_{m+1} + \cdots + a_n| \leqslant |a_{m+1}| + \cdots + |a_n| = t_n - t_m$, where $t_n = \sum\limits_{k=1}^{n} |a_k|$. Given $\varepsilon > 0$ and N s.t. $|t_n - t_m| < \varepsilon$, then this N works for $|S_n - S_m| < \varepsilon$, where $S_n = \sum\limits_{k=1}^{n} a_k$. $\square$

Property 3.12 (Alternating Series Test)**.** Suppose $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$, $\lim\limits_{k\to\infty} a_k = 0$, then $\sum\limits_{k=1}^{\infty} (-1)^{k+1} a_k$ is convergent.

Proof. We want to show $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} (-1)^{k+1} a_k$ is Cauchy.

Suppose $n > m$, then $|s_n - s_m| = |a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m+1} a_n|$. Since (a_n) is a non-negative decreasing sequence, then

$$a_{m+1} - a_{m+2} + \cdots + (-1)^{n-m-1} a_n = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \cdots \leqslant a_{m+1}.$$

Thus, $0 \leqslant |s_n - s_m| \leqslant a_{m+1}$. Since $a_{m+1} \to 0$ then $\forall \varepsilon > 0, \exists N$ s.t. $\forall m \geqslant N, |a_{m+1}| = a_{m+1} < \varepsilon$. Take such N and thus $\forall n > m \geqslant N, |s_n - s_m| < \varepsilon$. $\square$

Property 3.13 (Ratio Test)**.** Given $\sum\limits_{k=1}^{\infty} a_k, a_k \neq 0, \forall k$. If $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r < 1$, then $\sum\limits_{k=1}^{\infty} |a_k|$ is convergent.

Proof. Define $S = \left\{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n}\right| \geqslant r'\right\}$, then S contains finitely many elements of $\mathbb{N}$. If S were to be infinite set, if we take $\varepsilon = r' - r$, then $\left|\frac{a_{n+1}}{a_n}\right| - r \geqslant r' - r$ for infinitely many terms which contradicts that r point of convergence. Therefore, $S' = \left\{n \in \mathbb{N} : \left|\frac{a_{n+1}}{a_n}\right| < r'\right\}$ contains all but finitely many elements of $\mathbb{N}$. Let $N = 1 + \max S$, then $\forall n \geqslant N, \left|\frac{a_{n+1}}{a_n}\right| < r' \Rightarrow |a_{n+1}| < r'|a_n|$. Since $0 < r' < 1$, $\sum\limits_{n=1}^{\infty}(r')^n$ converges which implies $|a_N|\sum\limits_{n=1}^{\infty}(r')^n$ converges. We have $\sum\limits_{n=1}^{\infty}|a_n| = \sum\limits_{n=1}^{N}|a_n| + \sum\limits_{n=N+1}^{\infty}|a_n| < C + |a_N|\sum\limits_{n=N+1}^{\infty}(r')^{n-N}$ converges by comparison test. Hence $\sum\limits_{n=1}^{\infty}|a_n|$ converges. $\square$

Definition 3.8. Let $\sum\limits_{k=1}^{\infty} a_k$ be a series. A series $\sum\limits_{k=1}^{\infty} b_k$ is called a ***rearrangement*** of $\sum\limits_{k=1}^{\infty} a_k$ is for all n, there is unique k s.t. $b_k = a_n$.

Example 3.4. $a_k = \frac{1}{k}, b_{2k} = \frac{1}{2k+1}, b_{2k+1} = \frac{1}{2k}$.

4 Functional Limits and Continuity

4.1 Functional Limits

Definition 4.1. Let $A \subseteq \mathbb{R}, a \in \overline{A \backslash \{a\}}$, i.e., a is an accumulation point of A. Let $f : A \to \mathbb{R}$, define $\lim\limits_{x \to a} f(x) = L$ iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Example 4.1. $f(x) = cx$ on $A = \mathbb{R}, a \in \overline{A \backslash \{a\}}, \lim\limits_{x \to a} f(x) = ca$.

Proof. Let $\varepsilon > 0, \delta = \frac{\varepsilon}{|c|}$, then we have $|cx - ca| = |c||x - a| < \varepsilon$. □

Example 4.2. $f(x) = x^2$ on $A = \mathbb{R}$, $\lim\limits_{x \to \sqrt{2}} = f(x) = 2$.

Proof. Let $\varepsilon > 0$. Let $\delta = \min\left(\frac{\varepsilon}{2+\sqrt{2}}, 2 - \sqrt{2}\right)$. Let $0 < |x - \sqrt{2}| < \delta$, we have $|x^2 - 2| = |x - \sqrt{2}||x + \sqrt{2}| < (2 + \sqrt{2})|x - \sqrt{2}| = \varepsilon$. □

Property 4.1 (Sequential Criterion for Functional Limits)**.** Suppose $a \in \overline{A \backslash \{a\}}, f : A \to \mathbb{R}$. The following are equivalent:

(1) $\lim\limits_{x \to a} f(x) = L$;

(2) $\forall (x_n) \subseteq A \backslash \{a\}, x_n \to a \Rightarrow f(x_n) \to L$.

Proof. We prove (1) $\Rightarrow$ (2) : Take arbitrary $(x_n) \subseteq A \backslash \{a\}, x_n \to a$. Let $\varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Also, $\exists N$ s.t. $n \geqslant N \Rightarrow |x_n - a| < \delta$. Let $n \geqslant N$, then $|x_n - a| < \delta$ and thus $|f(x_n) - L| < \varepsilon$. □

Theorem 4.1 (Algebraic Limit Theorem for Functional Limits)**.** Suppose $f, g : A \to \mathbb{R}, a \in \overline{A \backslash \{a\}}$. Suppose $\lim\limits_{x \to a} f(x) = L, \lim\limits_{x \to a} g(x) = M, c \in \mathbb{R}$. We have

(1) $\lim\limits_{x \to a} cf(x) = cL$;

(2) $\lim\limits_{x \to a}(f(x) + g(x)) = L + M$;

(3) $\lim\limits_{x \to a}(f(x)g(x)) = LM$;

(4) $\lim\limits_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ when $M \neq 0$.

Property 4.2 (Divergence Criterion)**.** Suppose $f : A \to \mathbb{R}, a \in \overline{A \backslash \{a\}}$. $\lim\limits_{x \to a} f(x)$ does not exist if there are two sequences $(x_n).(y_n) \subseteq A \backslash \{a\}$ s.t. $x_n \to a, y_n \to a$, $\lim\limits_{n \to \infty} f(x_n) = L, \lim\limits_{n \to \infty} f(y_n) = M$ exist but $L \neq M$.

Example 4.3. Let $A = \mathbb{R}^+, a = 0, f(x) = \sin\left(\frac{1}{x}\right)$. Let $a_n = \frac{1}{2n\pi}, b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. We have $a_n, b_n \to a$. Besides, $\lim\limits_{n \to \infty} f(a_n) = 0, \lim\limits_{n \to \infty} f(b_n) = 1$. Hence $\lim\limits_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist.

Definition 4.2. Suppose $f : A \to \mathbb{R}, a \in A \backslash \{a\}$. We define $\lim\limits_{x \to a} f(x) = \infty$ iff

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

Definition 4.3. we define $\lim\limits_{x \to \infty} f(x) = L$ iff

$$\forall \varepsilon > 0, \exists M > 0 \text{ s.t. } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

4.2 Continuous Functions

Definition 4.4. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. $f : X \to Y$ is ***continuous*** at $a \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } x \in B_\delta^X(a) \Rightarrow f(x) \in B_\varepsilon^Y(f(a)).$$

Note that for $X = Y = \mathbb{R}, d(x, y) = |x - y|$, we can write $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - f(a)| > \varepsilon$, i.e., $\lim_{x \to a} f(x) = f(a)$.

Definition 4.5. $f : X \to Y$ is ***continuous*** if it is continuous at every point $a \in X$.

Property 4.3. The following are equivalent:
(1) f is continuous at a;
(2) $\lim_{x \to a} f(x) = f(a)$;
(3) $\forall (x_n) \subseteq A, x_n \to a \Rightarrow f(x_n) \to f(a)$.

Corollary 4.1. f is discontinuous at a if there is sequence $(x_n) \to a$ s.t. $\lim_{n \to \infty} f(x_n) \neq f(a)$.

Note that we may have $\lim_{x \to a} f(x)$ exists but f is discontinuous at a.

Theorem 4.2 (Algebraic Continuous Theorem)**.** Suppose $f, g : A \to \mathbb{R}$ are continuous at $a \in A, c \in \mathbb{R}$. We have
(1) $cf(x)$ is continuous at a;
(2) $f(x) \pm g(x)$ is continuous at a;
(3) $f(x)g(x)$ is continuous at a;
(4) $\frac{f(x)}{g(x)}$ is continuous at a if $g(a) \neq 0$.

Property 4.4. Suppose $f : A \to B \subseteq \mathbb{R}, g : B \to \mathbb{R}$. $(g \circ f)(x) = g(f(x))$ is continuous at $a \in A$ whenever f is continuous at a and g is continuous at $f(a)$.

4.3 Continuous Functions on Compact Sets

Property 4.5. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces, $f : X \to Y$ is continuous. If $K \subseteq X$ is compact, so is its image $f[K] = \{f(x) : x \in K\}$.

Property 4.6. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. $f^{-1}(F)$ is closed in X whenever $F \subseteq Y$ is closed in Y.

Theorem 4.3 (Extreme Value Theorem)**.** If $f : K \to \mathbb{R}$ is continuous, K is compact, then $\exists x_1, x_2 \in K$ s.t. $\forall x \in K, f(x_1) \leqslant f(x) \leqslant f(x_2)$.

Proof. Let $H = f[K] = \{f(x) : x \in K\} \subseteq \mathbb{R}$, which is compact. Since compact subsets of $\mathbb{R}$ are bounded, then let $y_2 = \sup(H)$. We have $y \leqslant y_2, \forall y \in H$ and $\forall \varepsilon > 0, \exists y \in H$ s.t. $y_2 - \varepsilon < y \leqslant y_2$. Take $\varepsilon = \frac{1}{n}, z_n \in H$, then $y_2 - \frac{1}{n} < z_n \leqslant y_2$. Now we find $a_n \in K$ s.t. $f(a_n) = z_n, n = 1, 2, \cdots$. By theorem, we have $a_{n_k} \to x_2$, then $f(x_2) = \lim_{k \to \infty} f(a_{n_k}) = y_2$. □

Definition 4.6. Assume $f : A \to \mathbb{R}$ is a function. We say f is ***uniformly continuous*** on A if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Example 4.4. $f(x) = x^2$ is not uniformly continuous.

Proof. Let $\varepsilon = 1$ and $\forall \delta > 0$, let $I_\delta = \left[\frac{2}{\delta} + 1, \frac{2}{\delta} + 1 + \delta\right]$, then we have $|f(x) - f(y)| \geqslant 1$. □

Property 4.7. Assume $f : A \to \mathbb{R}$ is a function, then f fails to be uniformly continuous iff $\exists \varepsilon_0 > 0$ and $(x_n), (y_n) \subseteq A$ s.t. $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n)| \geqslant \varepsilon_0$.

Proof. ($\Leftarrow$) It is obvious.

($\Rightarrow$) Assume f is not uniformly continuous. Fix $\varepsilon_0 > 0$ s.t. the definition of uniformly continuous fails for ε_0, i.e., $\forall \delta > 0, \exists x_\delta, y_\delta$ s.t. $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geqslant \varepsilon_0$. For each n, pick x_n, y_n as above, then it is obvious that $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n) \geqslant \varepsilon_0$. $\square$

Property 4.8. Assume $f : K \to \mathbb{R}$ is continuous and K is compact, then f is uniformly continuous on K, i.e., continuous functions on compact sets are uniformly continuous.

Proof. Assume for a contradiction that $f : K \to \mathbb{R}$ is continuous, K is compact, and f is not uniformly continuous. Then $\exists \varepsilon_0 > 0, (x_n), (y_n) \subseteq K$ s.t. $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n)| \geqslant \varepsilon_0$. Since K is compact, x_n has a subsequence x_{n_k} s.t. $\lim\limits_{k\to\infty} x_{n_k} = x \in K$. Moreover, (y_{n_k}) has a subsequence s.t. $y_{n_{k_m}} \to y \in f(K)$. Let $x'_m = x_{n_{k_m}}, y'_m = y_{n_{k_m}}$, then $x'_m \to x, y'_m \to y$. On the one hand, $\lim\limits_{m\to\infty} |x'_m - y'_m| = 0$ and thus $x = y$. On the other hand, $|f(x'_m) - f(y'_m)| \geqslant \varepsilon_0 \Rightarrow \lim\limits_{m\to\infty} |f(x'_m) - f(y'_m)| \geqslant \varepsilon_0 \Rightarrow |f(x) - f(x)| \geqslant \varepsilon_0$ which is a contradiction. $\square$

Definition 4.7. A function $f : A \to \mathbb{R}$ is said to be ***Lipschitz*** if $\exists M \in \mathbb{N}$ s.t.

$$\forall x \neq y \in A, \left|\frac{f(x) - f(y)}{x - y}\right| < M.$$

Property 4.9. Lipschitz functions are uniformly continuous.

Proof. Assume f is Lipschitz on A, then for every $\varepsilon > 0$, take $\delta < \frac{\varepsilon}{M}$. $\square$

Remark: The converse does not hold, for example $f(x) = \sqrt{x^2 - 1}$.

Property 4.10. Assume $f : E \to \mathbb{R}$ is continuous and E is connected, then $f(E)$ is connected, i.e., continuous image of connected sets is connected.

Proof. Assume $f(E)$ is not connected. Fix $A, B \subseteq f(E)$ s.t. $\overline{A} \cap B = \varnothing = \overline{B} \cap A$ and $f(E) = A \cup B$. Let $C = f^{-1}(A), D = f^{-1}(B)$. Note that $E = C \cup D, C \cap D = \varnothing$ because f is a function. We now show that $\overline{C} \cap D = \varnothing$: Assume not, then $\exists (x_n) \subseteq C$ s.t. (x_n) is convergent and $\lim\limits_{n\to\infty} f(x_n) = f(x) \in B$, i.e., $\lim\limits_{n\to\infty} f(x_n) \in \overline{A} \cap B$, which is a contradiction. Similarly, $\overline{D} \cap C = \varnothing$ and thus E can be separated by C and D, which is a contradiction. $\square$

4.4 Sets of Discontinuities

Let $f : \mathbb{R} \to \mathbb{R}, D_f = \{x \in \mathbb{R} : f \text{ is not continuous at } x\}$.

Example 4.5 ($D_f = \varnothing$)**.** f is continuous.

Example 4.6 ($D_f = \mathbb{R}$)**.** $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$.

Example 4.7. Given a countable set $A = \{a_1, \cdots\}$, define $f(a_n) = \frac{1}{n}$ and $f(x) = 0, \forall x \notin A$. We have $D_f = A$.

Example 4.8. There is no $f : \mathbb{R} \to \mathbb{R}$ s.t. $D_f = \mathbb{R} \backslash \mathbb{Q}$.

Definition 4.8. A subset F of $\mathbb{R}$ is a F_σ***-set*** if $F = \bigcup_{n=1}^{\infty} F_n$ s.t. F_n is closed for all n.

Definition 4.9. Let $\alpha > 0, f : \mathbb{R} \to \mathbb{R}, a \in \mathbb{R}$. f is α***-continuous*** at a if

$$\exists \delta > 0 \text{ s.t. } x, y \in (a - \delta, a + \delta) \Rightarrow |f(x) - f(y)| < \alpha.$$

Note that f is continuous at a iff f is α-continuous at a for all $\alpha > 0$.

Property 4.11. For every $f : \mathbb{R} \to \mathbb{R}$, the set D_f is F_δ-subset of $\mathbb{R}$.

Definition 4.10. Let $f : \mathbb{R} \to \mathbb{R}$. f is ***removable discontinuous*** if $\lim_{x\to a} f(x)$ exists but does not equal $f(a)$. f has a ***jump*** at a if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$. If $\lim_{x\to a} f(x)$ does not exist for other reasons, we say f is ***essential discontinuous***.

Definition 4.11. $f : \mathbb{R} \to \mathbb{R}$ is ***monotone*** if either $x \leqslant y \Rightarrow f(x) \leqslant f(y)$ or $x \leqslant y \Rightarrow f(x) \geqslant f(y)$.

Property 4.12. Discontinuity of a monotone function f is a jump. Moreover, D_f is countable.

5 The Derivative

5.1 Derivatives and the Intermediate Value Property

Definition 5.1. Let $f : \mathbb{R} \to \mathbb{R}, c \in \mathbb{R}$. Define the ***derivative*** of f at c :

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

If $f'(c)$ exists, we say f is ***differentiable*** at c. If f' exists for all $a \in \mathbb{R}$, we say g is ***differentiable*** on $\mathbb{R}$.

Property 5.1. If f is differentiable at c, then f is continuous at c.

Proof. We have

$$\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) = f'(c) \cdot 0 = 0.$$

□

Theorem 5.1 (Algebraic Differentiability Theorem)**.** Suppose f, g are differentiable, $a, c \in \mathbb{R}$. We have

(1) $(cf)'(a) = cf'(a)$;
(2) $(f + g)'(a) = f'(a) + g'(a)$;
(3) $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$;
(4) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}, g(a) \neq 0$.

Theorem 5.2 (Chain Rule)**.** Let $f : A \to B, g : B \to \mathbb{R}, f(A) \subseteq B$ so that $g \circ f$ is defined. If f is differentiable at c and if g is differentiable at $f(c)$, then $g \circ f$ is differentiable at a with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Theorem 5.3 (Interior Extremum Theorem)**.** If f is differentiable on (a, b), f attains maximum at some $c \in (a, b)$, then $f'(c) = 0$.

Proof. We have

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leqslant 0$$

and

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geqslant 0,$$

then $f'(c) = 0$. □

Theorem 5.4 (Darboux's Theorem)**.** If f is differentiable on $[a, b]$ and $f'(a) < \alpha < f'(b)$ or $f'(a) > \alpha > f'(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = \alpha$.

5.2 The Mean Value Theorem

Theorem 5.5 (Rolle's Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

Proof. Since f is continuous on a compact set, f attains a maximum and a minimum. If both the maximum and minimum occur at the endpoints, then f is necessarily a constant function and $f'(x) = 0$ on (a, b). On the other hand, if either the maximum or minimum occurs at some point $c \in (a, b)$, then it follows from the interior extremum theorem that $f'(c) = 0$. □

Theorem 5.6 (Mean Value Theorem)**.** If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider

$$d(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)\right].$$

We know d is continuous on $[a, b]$ and differentiable on (a, b). Also, $d(a) = d(b) = -0$. By Rolle's Theorem, $\exists c \in (a, b)$ s.t. $d'(c) = 0$. ☐

Corollary 5.1. If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Proof. Assume $x < y, x, y \in (a, b.)$ We set $c \in (x, y)$, then by mean value theorem,

$$0 = f'(c) = \frac{f(y) - f(x)}{y - x} \Rightarrow f(y) - f(x) = 0.$$

☐

Corollary 5.2. If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + c$.

Proof. Apply the previous corollary to the function $h(x) = f(x) - g(x)$. ☐

Theorem 5.7 (Generalized Mean Value Theorem)**.** If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ s.t.

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b) then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Apply the mean value theorem to the function $h(x) = [f(b)-f(a)]g(x)-[g(b)-g(a)]f(x)$. ☐

Theorem 5.8 (L'Hospital's Rule: 0/0 Case)**.** Suppose f, g are continuous on I with $a \in I$ and are differentiable on $I\backslash\{a\}$. If $f(a) = g(a) = 0, g'(x) \neq 0, \forall x \neq a$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

Proof. Since $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$, then for all $\varepsilon > 0, \exists \delta > 0$ s.t.

$$x \in (a - \delta, a + \delta) \Rightarrow \left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon.$$

By the generalized mean value theorem, for every $y \in (a, a + \delta), \exists x \in (a, y)$ s.t.

$$\frac{f'(x)}{g'(x)} = \frac{f(y) - f(a)}{g(y) - g(a)} = \frac{f(y)}{g(y)}$$

and thus

$$\left|\frac{f(y)}{g(y)} - L\right| = \left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon.$$

☐

Theorem 5.9 (L'Hospital's Rule: ∞/∞ Case)**.** Suppose f, g are differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim\limits_{x \to a} g(x) = \infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

6 Sequences and Series of Functions

6.1 Uniform Convergence of a Sequence of Functions

Definition 6.1. For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) of functions ***converges pointwise*** on A to a function f if $f_n(x) \to f(x), \forall x \in A$. We can write $f_n \to f, \lim f_n = f$, or $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

Example 6.1. Consider

$$f_n(x) = \frac{x^2 + nx}{n}$$

on $\mathbb{R}$. We can compute

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2 + nx}{n} = \lim_{n\to\infty} \frac{x^2}{n} + x = x.$$

Thus, (f_n) converges pointwise to $f(x) = x$ on $\mathbb{R}$.

Example 6.2. Consider

$$f_n(x) = x^n$$

on $[0,1]$. If $0 \leqslant x < 1, x^n \to 0$. If $x = 1, x^n \to 1$. It follows that $f_n \to f$ pointwise on $[0,1]$ where

$$f(x) = \begin{cases} 0, & 0 \leqslant x < 1 \\ 1, & x = 1 \end{cases}.$$

Note that pointwise convergent sequence of continuous functions may converge to a non-continuous function.

Definition 6.2. Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$, then (f_n) ***converges uniformly*** on A to a limit function f defined on A if

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geqslant N, \forall x \in A, |f(x) - f_n(x)| < \varepsilon.$$

Example 6.3. Consider

$$f_n(x) = \frac{x^2 + nx}{n}$$

where converges pointwise on $\mathbb{R}$ to $f(x) = x$. But on $\mathbb{R}$, the convergence is not uniform. We have

$$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \frac{x^2}{n}.$$

In order to force $|f_n(x) - f(x)| < \varepsilon$, we need $N > \frac{x^2}{\varepsilon}$. Although it is possible to do for each $x \in \mathbb{R}$, there is no way to choose a single value of N that will work for all values of x at the same time.

On the other hand, we can show that $f_n \to f$ uniformly on the set $[-b, b]$.

Property 6.1 (Cauchy Criterion for Uniform Convergence)**.** A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A iff

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall x \in A, \forall m, n \geqslant N, |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem 6.1 (Continuous Limit Theorem)**.** Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f. If each f_n is continuous at $c \in A$, then f is continuous at c.

Proof. Let $\varepsilon > 0$ and fix $c \in A$. Choose N s.t.

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3}, \forall x \in A.$$

Since f_N is continuous, then $\exists \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f_N(x) - f_N(c)| < \frac{\varepsilon}{3}.$$

Thus,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leqslant |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, f is continuous at $c \in A$. □

Property 6.2 (Algebraic Limit Theorem for Uniform Convergence)**.** Suppose $(f_n).(g_n)$ are uniformly convergent on A, then:

(1) $(cf_n + g_n)$ is uniformly convergent;
(2) If $\exists M > 0$ s.t. $|f_n| \leqslant M, |g_n| \leqslant M$, then $(f_n g_n)$ is uniformly convergent.

Proof. Using Cauchy criterion, we have

$$\begin{aligned} |f_m(x)g_m(x) - f_n(x)g_n(x)| &= |f_m(x)g_m(x) - f_m(x)g_n(x) + f_m(x)g_n(x) - f_n(x)g_n(x)| \\ &\leqslant |f_m(x)||g_m(x) - g_n(x)| + |g_n(x)||f_m(x) - f_n(x)| \\ &\leqslant M(|g_m(x) - g_n(x)| + |f_m(x) - f_n(x)|) \end{aligned}$$

□

6.2 Uniform Convergence and Differentiation

Theorem 6.2 (Differentiable Limit Theorem)**.** Let $f_n \to f$ pointwise on $[a, b]$ and assume each f_n is differentiable. If (f_n') converges uniformly on $[a, b]$ to a function g, then the function f is differentiable and $f' = g$.

Theorem 6.3. Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume (f_n') converges uniformly on $[a, b]$. If $\exists x_0 \in [a, b]$ s.t. $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$.

Theorem 6.4. Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume (f_n') converges uniformly to a function g on $[a, b]$. If $\exists x_0 \in [a, b]$ s.t. $f_n(x_0)$ is convergent, then (f_n) converges uniformly. Moreover, the limit function $f = \lim f_n$ is differentiable and $f' = g$.

6.3 Series of Functions

Definition 6.3. For each $n \in \mathbb{N}$, let f_n and f be functions defined on a set $A \subseteq \mathbb{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence $s_k(x)$ of partial sums defined by

$$s_k(x) = f_1(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$. The series ***converges uniformly*** on A to f if the sequence $s_k(x)$ converges uniformly on A to $f(x)$.

Theorem 6.5 (Term-by-Term Continuity Theorem)**.** Let f_n be continuous functions defined on $A \subseteq \mathbb{R}$ and $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on A to f. Then f is continuous on A.

Theorem 6.6 (Term-by-Term Differentiability Theorem)**.** Let f_n be differentiable functions defined on an interval A and $\sum\limits_{n=1}^{\infty} f_n'(x)$ converges uniformly to $g(x)$ on A. If $\exists x_0 \in [a,b]$ s.t. $\sum\limits_{n=1}^{\infty} f_n(x_0)$ converges, then $\sum\limits_{n=1}^{\infty} f_n(x)$ converges uniformly to a differentiable function $f(x)$ s.t. $f'(x) = g(x)$ on A, i.e.,

$$f(x) = \sum_{n=1}^{\infty} f_n(x), f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

Property 6.3 (Cauchy Criterion for Uniform Convergence of Series)**.** $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$ iff $\forall \varepsilon > 0, \exists N$ s.t.

$$n > m \geqslant N, x \in A \Rightarrow |f_{m+1}(x) + \cdots + f_n(x)| < \varepsilon.$$

Corollary 6.1 (Weierstrass M-Test)**.** For each $n \in \mathbb{N}$, let f_n be a function defined on $A \subseteq \mathbb{R}$, and let $M_n > 0$ be a real number satisfying $|f_n(x)| \leqslant M_n$ for all $x \in A$. If $\sum\limits_{n=1}^{\infty} M_n$ converges, then $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on A.

6.4 Power Series

Property 6.4. If a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ converges at $x_0 \in \mathbb{R}$, then it converges absolutely for any x satisfying $|x| < |x_0|$.

Property 6.5. If a power series $\sum\limits_{n=0}^{\infty} a_n x^n$ converges absolutely at x_0, then it converges uniformly on the closed interval $[-c, c]$, where $c = |x_0|$.

Lemma 6.1 (Abel's Lemma)**.** Let b_n satisfy $b_1 \geqslant b_2 \geqslant \cdots \geqslant 0$, and $\sum\limits_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded, i.e., $\exists A > 0$ s.t. $|a_1 + \cdots + a_n| \leqslant A, \forall n \in \mathbb{N}$. Then for all $n \in \mathbb{N}$,

$$|a_1 b_1 + \cdots + a_n b_n| \leqslant A b_1.$$

Theorem 6.7 (Abel's Theorem)**.** Let $g(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ be a power series that converges at $x = R > 0$, then the series converges uniformly on $[0, R]$. A similar result holds if the series converges at $x = -R$.

Property 6.6. If a power series converges pointwise on $A \subseteq \mathbb{R}$, then it converges uniformly on any compact set $K \subseteq A$.

Property 6.7. If $\sum\limits_{n=0}^{\infty} a_n x^n$ converges for all $x \in (-R, R)$, then the differentiated series $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ converges at each $x \in (-R, R)$. Consequently, the convergence is uniform on compact sets contained in $(-R, R)$.

Property 6.8. Suppose $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ converges on an interval $A \subseteq \mathbb{R}$. The function f is continuous on A and differentiable on any open interval $(-R, R) \subseteq A$. The derivative is given by $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$. Moreover, f is infinitely differentiable on $(-R, R)$ and the successive derivatives can be obtained via term-by-term differentiation of the appropriate series.

7 The Riemann Integral

7.1 The Definition of the Riemann Integral

Definition 7.1. A ***partition*** P of $[a, b]$ is a finite set of points from $[a, b]$ that includes both a and $b : P = \{x_0, x_1, \cdots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$.

Definition 7.2. For each sub-interval $[x_{k-1}, x_k]$ of P, let $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$ and $M_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$. The ***lower sum*** of f w.r.t. P is given by

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

and the ***upper sum*** of f w.r.t. P is given by

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}).$$

Definition 7.3. A partition Q is a ***refinement*** of a partition P if Q contains all of the points of P, i.e., $P \subseteq Q$.

Lemma 7.1. If $P \subseteq Q$, then $L(f, P) \leqslant L(f, Q)$ and $U(f, P) \geqslant U(f, Q)$.

Lemma 7.2. If P_1 and P_2 are any two partitions of $[a, b]$, then $L(f, P_1) \leqslant U(f, P_2)$.

Definition 7.4. Let $\mathcal{P}$ be the collection of all possible partitions of the interval $[a, b]$. The ***upper integral*** of f is

$$U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$$

and the ***lower integral*** of f is

$$L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}.$$

Lemma 7.3. For any bounded function f on $[a, b]$, $U(f) \geqslant L(f)$.

Definition 7.5. A bounded function f defined on $[a, b]$ is ***Riemann integrable*** if $U(f) = L(f)$:

$$\int_a^b f = U(f) = L(f).$$

Property 7.1 (Integrability Criterion)**.** A bounded function f is integrable on $[a, b]$ iff

$$\forall \varepsilon > 0, \exists P_\varepsilon \text{ of } [a, b] \text{ s.t. } U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Property 7.2. If f is continuous on $[a, b]$, then it is integrable.

7.2 Integrating Functions with Discontinuities

Property 7.3. If $f : [a, b] \to \mathbb{R}$ is bounded and integrable on $[c, b]$ for all $c \in (a, b)$, then f is integrable on $[a, b]$. An analogous result holds at the other endpoint.

7.3 Properties of the Integral

Property 7.4. Assume $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. Then, f is integrable on $[a, b]$ iff f is integrable on $[a, c]$ and $[c, b]$. We have

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Property 7.5. Assume f and g are integrable functions on $[a, b]$, then

(1) $f + g$ is integrable on $[a, b]$ with

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g;$$

(2) for $k \in \mathbb{R}$, kf is integrable with

$$\int_a^b kf = f \int_a^b f;$$

(3) if $m \leqslant f(x) \leqslant M$ on $[a, b]$, then

$$m(b - a) \leqslant \int_a^b f \leqslant M(b - a);$$

(4) if $f(x) \leqslant g(x)$ on $[a, b]$, then

$$\int_a^b f \leqslant \int_a^b g;$$

(5) $|f|$ is integrable and

$$\left| \int_a^b f \right| \leqslant \int_a^b |f|.$$

Definition 7.6. If f is integrable on $[a, b]$, define

$$\int_b^a f = -\int_a^b f.$$

For $c \in [a, b]$, define

$$\int_c^c f = 0.$$

Theorem 7.1 (Integrable Limit Theorem)**.** Assume $f_n \to f$ uniformly on $[a, b]$ and each f_n is integrable. Then, f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

7.4 The Fundamental Theorem of Calculus

Theorem 7.2 (Fundamental Theorem of Calculus)**.** (i) If $f : [a, b] \to \mathbb{R}$ is integrable and $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ for all $x \in [a, b]$, then

$$\int_a^b f = F(b) - F(a).$$

(ii) Let $g : [a, b] \to \mathbb{R}$ be integrable and for $x \in [a, b]$ define

$$G(x) = \int_a^x g.$$

Then G is continuous on $[a, b]$. If g is continuous at some point $c \in [a, b]$, then G is differentiable at c and $G'(c) = g(c)$.

7.5 Lebesgue's Criterion for Riemann Integrability

Definition 7.7. A set $A \subseteq \mathbb{R}$ has ***measure zero*** if, for all $\varepsilon > 0$, there exists a countable collection of open intervals O_n s.t.

$$A \subseteq \bigcup_{n=1}^{\infty} O_n \text{ and } \sum_{n=1}^{\infty} |O_n| \leqslant \varepsilon.$$

Definition 7.8. Let f be defined on $[a, b]$ and let $\alpha > 0$. The function f is ***α-continuous*** at $x \in [a, b]$ if

$$\exists \delta > 0 \text{ s.t. } \forall y, z \in (x - \delta, x + \delta) \Rightarrow |f(y) - f(z)| < \alpha.$$

Definition 7.9. Let f be a bounded function on $[a, b]$. For each $\alpha > 0$, define D^α to be the set of points in $[a, b]$ where the function f fails to be α-continuous, i.e.,

$$D^\alpha = \{x \in [a, b] : f \text{ is not } \alpha\text{-continuous at } x\}.$$

Property 7.6. Let $K \subseteq \mathbb{R}$. The following statements are all equivalent:

(1) Every sequence contained in K has a convergent subsequence that converges to a limit in K.

(2) K is closed and bounded.

(3) Given a collection of open intervals $\{G_\lambda : \lambda \in \Lambda\}$ that covers K, i.e., $K \subseteq \bigcup_{\lambda \in \Lambda} G_\lambda$ there exists a finite sub-collection $\{G_{\lambda_1}, \cdots, G_{\lambda_N}\}$ of the original set that also covers K.

Theorem 7.3 (Lebesgue's Theorem)**.** Let f be a bounded function defined on the interval $[a, b]$, then f is Riemann-integrable iff the set of points where f is not continuous has measure zero.

Corollary 7.1. Suppose f, g are bounded on $[a, b]$, f is continuous and $D = \{x \in [a, b] : g(x) \neq f(x)\}$ has measure zero, then g is integrable and

$$\int_a^b f = \int_a^b g.$$