Introduction to Real Analysis

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1 Real Numbers

We define

$$\mathbb{N} = \{1, 2, \cdots\}.$$

If we take the closure of $\mathbb{N}$ under subtraction, we obtain

$$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}.$$

If we take the closure of $\mathbb{Z}$ under division by non-zero numbers, we obtain

$$\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1\},$$

where $(m, n) = 1$ means if $d \in \mathbb{N}$ divides both m, n, then $d = 1$.

Theorem 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for a contradiction that there are $m \in \mathbb{Z}, n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m, n) = 1$. Hence, $m^2 = 2n^2$, then m^2 is an even complete square. So $4|m^2$. But then $4|2n^2$ and thus $2|n^2$. So n has to be even. Hence both m, n are even, i.e., $2|m, 2|n$. This contradicts the fact that $(m, n) = 1$. $\square$

1.1 Preliminaries

Definition 1.1. A ***function*** from A to B ($f : A \to B$) is the set of pairs $(x, y) \in A \times B$ s.t. (1) if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$; (2) $\forall x \in A, \exists y \in B$ s.t. $f(x) = y$.

Note that A is said to be the domain of f, but the range of f does not have to be B, and it is a subset of B.

Definition 1.2. Assume $f : A \to B$ is a function, f is said to be ***injective*** if

$$\forall x_1, x_2 \in A, f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2.$$

Definition 1.3. f is said to be ***surjective*** if

$$\forall y \in B, \exists x \in A \text{ s.t. } f(x) = y.$$

Definition 1.4. f is said to be ***bijective*** if f is injective and surjective.

Definition 1.5. $\forall x$,

$$|x| = \begin{cases} x, & x \geqslant 0 \\ -x, & x < 0 \end{cases}.$$

Theorem 1.2 (Triangle Inequality)**.** $|x + y| \leqslant |x| + |y|$.

Proof. We have $(x + y)^2 = x^2 + y^2 + 2xy \leqslant |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2$. Thus,

$$|x + y| = \sqrt{(x + y)^2} \leqslant \sqrt{(|x| + |y|)^2} = |x| + |y|.$$

$\square$

Definition 1.6. Assume $X \subseteq \mathbb{R}$, the ***maximum*** (***minimum***) of X is an element $a \in X$ s.t. $\forall x \in X, x \leqslant a$ ($x \geqslant a$).

Definition 1.7. The ***least upper bound*** of X, denoted by $\sup(X)$, is $a \in \mathbb{R}$ s.t. (1) $\forall x \in X, x \leqslant a$ (a is an upper bound for X); (2) if b is an upper bound for X, then $a \leqslant b$.

Example 1.1. $\max((0, 1))$ does not exist. $\sup((0, 1)) = 1.$ $\sup(\mathbb{R})$ and $\sup(\mathbb{N})$ do not exist.

1.2 The Axiom of Completeness

Definition 1.8. $X \subseteq \mathbb{Q}$ is said to be an ***initial segment*** if (1) $X \neq \varnothing$; (2) $\forall x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$; (3) $X \neq \mathbb{Q}$.

Definition 1.9. $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$.

Property 1.1. $\mathbb{R}$ is an ordered field.

Lemma 1. If $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A, then $s = \sup(A)$ iff

$$\forall \varepsilon > 0, \exists a \in A \text{ s.t. } a + \varepsilon > s.$$

Proof. ($\Leftarrow$) Assume for a contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$. Let $\varepsilon = \frac{s-t}{2} > 0$, then

$$\forall a \in A, a + \varepsilon \leqslant t + \varepsilon = \frac{s+t}{2} < s,$$

which is a contradiction.

($\Rightarrow$) Assume for a contradiction that $\varepsilon_0 > 0$ and $\forall a \in A, a + \varepsilon_0 \leqslant s$. Thus $\forall a \in A, a \leqslant s - \varepsilon_0$, and $s - \varepsilon_0 < s$ is an upper bound for A, which is a contradiction. $\square$

Theorem 1.3 (Axiom of Completeness)**.** If $X \subseteq \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let A_x be the initial segment of $\mathbb{Q}$ corresponding to x. Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \bigcup_{x \in X} A_x$. Note that $\bigcup_{x \in X} A_x$ is an initial segment of $\mathbb{Q}$ and thus $\sup(\bigcup_{x \in X} A_x)$ is $\sup(X)$. $\square$

1.3 Consequences of Completeness

Definition 1.10. Assume $\{A_n : n \in \mathbb{N}\}$ is a sequence of sets, $\{A_n : n \in \mathbb{N}\}$ is said to be ***nested*** if $A_n \supseteq A_{n+1}$.

Theorem 1.4 (Nested Interval Property)**.** Assume $\{I_n : n \in \mathbb{N}\}$ is a nested sequence of closed intervals of $\mathbb{R}$, then $\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing$.

Proof. Let $[a_n, b_n] = I_n$. Since $\{I_n : n \in \mathbb{N}\}$ is nested,

$$a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n, \forall n \in \mathbb{N}.$$

Let $A = \{a_n : n \in \mathbb{N}\}$.

Note that b_1 is an upper bound for A so A has supremum in $\mathbb{R}$. We have $\forall n \in \mathbb{N}, \sup(A) \leqslant b_n$ and $\sup(A) \geqslant a_n$. Thus, $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$, i.e., $\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing$. $\square$

Theorem 1.5 (Archimedean Property)**.** (1) $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $y \leqslant n$;
(2) $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$.

Proof. (1) Assume for a contradiction that $\mathbb{N}$ is bounded in $\mathbb{R}$. Let $\alpha = \sup(\mathbb{N})$, then by lemma, $\exists n \in \mathbb{N}$ s.t. $n + 1 > \alpha$, which is a contradiction.

(2) From (1), we have $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{y} < n \Rightarrow \frac{1}{n} < y$. $\square$

Theorem 1.6. $\mathbb{Q}$ is dense in $\mathbb{R}$, i.e., if $a < b, a, b \in \mathbb{R}$, then $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

Proof. Suppose $a < b, a, b \in \mathbb{R}$. By Archimedean Property, we can find $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, i.e., $1 < nb - na$. Hence we can find $m \in \mathbb{Z}$ s.t. $na < m < nb$. Therefore,

$$a < \frac{m}{n} < b,$$

and let $r = \frac{m}{n}$. $\square$

1.4 Cantor's Theorem

Definition 1.11. If there is a bijection $f : A \to B$, we say A, B are in ***one-to-one correspondence***, denoted $A \sim B$.

Definition 1.12. $\mathrm{Card}(A) \leqslant \mathrm{Card}(B)$ if there is a injective map $f : A \to B$.

Example 1.2. $\mathbb{N} \sim \mathbb{Z}, \mathbb{N} \sim \mathbb{N}^2, \mathbb{N} \sim \mathbb{Q}, \mathbb{N} \nsim \mathbb{R}$ $(\mathrm{Card}(\mathbb{N}) < \mathrm{Card}(\mathbb{R})), (-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$.

Theorem 1.7 (Schroeder-Bernstein Theorem)**.** Assume there are injective maps from A to B and from B to A, then there is a bijection from A to B.