

Introduction to Real Analysis

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1 Real Numbers

We define

$$\mathbb{N} = \{1, 2, \dots\}.$$

If we take the closure of \mathbb{N} under subtraction, we obtain

$$\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}.$$

If we take the closure of \mathbb{Z} under division by non-zero numbers, we obtain

$$\mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}, (m, n) = 1 \right\},$$

where $(m, n) = 1$ means if $d \in \mathbb{N}$ divides both m, n , then $d = 1$.

Example 1.1. There is no $r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Proof. Assume for a contradiction that there are $m \in \mathbb{Z}, n \in \mathbb{N}$ s.t. $\frac{m}{n} = \sqrt{2}$ and $(m, n) = 1$. Hence, $m^2 = 2n^2$, then m^2 is an even complete square. So $4|m^2$. But then $4|2n^2$ and thus $2|n^2$. So n has to be even. Hence both m, n are even, i.e., $2|m, 2|n$. This contradicts the fact that $(m, n) = 1$. \square

1.1 Preliminaries

Definition 1.1. A **function** from A to B ($f : A \rightarrow B$) is the set of pairs $(x, y) \in A \times B$ s.t. (1) if $(x, y_1) \in f$ and $(x, y_2) \in f$, then $y_1 = y_2$; (2) $\forall x \in A, \exists y \in B$ s.t. $f(x) = y$.

Note that A is said to be the domain of f , but the range of f does not have to be B , and it is a subset of B .

Definition 1.2. Assume $f : A \rightarrow B$ is a function, f is said to be **injective** if

$$\forall x_1, x_2 \in A, f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2.$$

Property 1.1. If $f : A \rightarrow B, g : B \rightarrow C$ are injective, then $g \circ f : A \rightarrow C$ is injective.

Definition 1.3. f is said to be **surjective** if

$$\forall y \in B, \exists x \in A \text{ s.t. } f(x) = y.$$

Property 1.2. If there is a surjective map $g : A \rightarrow B$, then there is an injective map $f : B \rightarrow A$.

Definition 1.4. f is said to be **bijective** if f is injective and surjective.

Definition 1.5. For all x ,

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

Theorem 1.1 (Triangle Inequality). $|x + y| \leq |x| + |y|$.

Proof. We have $(x + y)^2 = x^2 + y^2 + 2xy \leq |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2$. Thus,

$$|x + y| = \sqrt{(x + y)^2} \leq \sqrt{(|x| + |y|)^2} = |x| + |y|.$$

\square

Definition 1.6. Assume $X \subseteq \mathbb{R}$, the **maximum** (**minimum**) of X is an element $a \in X$ s.t. $\forall x \in X, x \leq a$ ($x \geq a$).

Definition 1.7. The **least upper bound** of X , denoted by $\sup(X)$, is $a \in \mathbb{R}$ s.t. (1) $\forall x \in X, x \leq a$ (a is an upper bound for X); (2) if b is an upper bound for X , then $a \leq b$.

Example 1.2. $\max((0, 1))$ does not exist. $\sup((0, 1)) = 1$. $\sup(\mathbb{R})$ and $\sup(\mathbb{N})$ do not exist.

1.2 The Axiom of Completeness

Definition 1.8. $X \subseteq \mathbb{Q}$ is said to be an *initial segment* if (1) $X \neq \emptyset$; (2) $\forall x, y \in \mathbb{Q}$, if $x < y$ and $y \in X$, then $x \in X$; (3) $X \neq \mathbb{Q}$.

Definition 1.9. $\mathbb{R} = \{\sup(X) : X \text{ is an initial segment of } \mathbb{Q}\}$.

Property 1.3. \mathbb{R} is an ordered field.

Property 1.4. If $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$ is an upper bound for A , then $s = \sup(A)$ iff

$$\forall \varepsilon > 0, \exists a \in A \text{ s.t. } a + \varepsilon > s.$$

Proof. (\Leftarrow) Assume for a contradiction that $t \in \mathbb{R}$ is an upper bound for A and $t < s$. Let $\varepsilon = \frac{s-t}{2} > 0$, then

$$\forall a \in A, a + \varepsilon \leq t + \varepsilon = \frac{s+t}{2} < s,$$

which is a contradiction.

(\Rightarrow) Assume for a contradiction that $\varepsilon_0 > 0$ and $\forall a \in A, a + \varepsilon_0 \leq s$. Thus $\forall a \in A, a \leq s - \varepsilon_0$, and $s - \varepsilon_0 < s$ is an upper bound for A , which is a contradiction. \square

Theorem 1.2 (Axiom of Completeness). If $X \subseteq \mathbb{R}$ is bounded above, then X has a least upper bound.

Proof. For $x \in X$, let A_x be the initial segment of \mathbb{Q} corresponding to x . Since X is bounded above, pick $b \in \mathbb{R}$ s.t. $\forall x \in X, x < b$. Then $b \notin \bigcup_{x \in X} A_x$. Note that $\bigcup_{x \in X} A_x$ is an initial segment of \mathbb{Q} and thus $\sup(\bigcup_{x \in X} A_x)$ is $\sup(X)$. \square

1.3 Consequences of Completeness

Definition 1.10. Assume $\{A_n : n \in \mathbb{N}\}$ is a sequence of sets, $\{A_n : n \in \mathbb{N}\}$ is said to be *nested* if $A_n \supseteq A_{n+1}$.

Theorem 1.3 (Nested Interval Property). Assume $\{I_n : n \in \mathbb{N}\}$ is a nested sequence of closed intervals of \mathbb{R} , then $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$.

Proof. Let $[a_n, b_n] = I_n$. Since $\{I_n : n \in \mathbb{N}\}$ is nested,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n, \forall n \in \mathbb{N}.$$

Let $A = \{a_n : n \in \mathbb{N}\}$.

Note that b_1 is an upper bound for A so A has supremum in \mathbb{R} . We have $\forall n \in \mathbb{N}, \sup(A) \leq b_n$ and $\sup(A) \geq a_n$. Thus, $\forall n \in \mathbb{N}, \sup(A) \in [a_n, b_n]$, i.e., $\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset$. \square

Theorem 1.4 (Archimedean Property). (1) $\forall y \in \mathbb{R}, \exists n \in \mathbb{N}$ s.t. $y \leq n$;
(2) $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$.

Proof. (1) Assume for a contradiction that \mathbb{N} is bounded in \mathbb{R} . Let $\alpha = \sup(\mathbb{N})$, then by lemma, $\exists n \in \mathbb{N}$ s.t. $n + 1 > \alpha$, which is a contradiction.

(2) From (1), we have $\forall y > 0, \exists n \in \mathbb{N}$ s.t. $\frac{1}{y} < n \Rightarrow \frac{1}{n} < y$. \square

Theorem 1.5. \mathbb{Q} is dense in \mathbb{R} , i.e., if $a < b, a, b \in \mathbb{R}$, then $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.

Proof. Suppose $a < b, a, b \in \mathbb{R}$. By Archimedean Property, we can find $n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, i.e., $1 < nb - na$. Hence we can find $m \in \mathbb{Z}$ s.t. $na < m < nb$. Therefore,

$$a < \frac{m}{n} < b,$$

and let $r = \frac{m}{n}$. □

1.4 Cardinality

Definition 1.11. If there is a bijection $f : A \rightarrow B$, we say A, B are in **one-to-one correspondence**, denoted $A \sim B$.

Property 1.5. If $A \sim B, B \sim C$, then $A \sim C$.

Definition 1.12. $\text{Card}(A) \leq \text{Card}(B)$ if there is an injective map $f : A \rightarrow B$.

Example 1.3. $\mathbb{N} \sim \mathbb{Z}, \mathbb{N} \sim \mathbb{N}^2, \mathbb{N} \sim \mathbb{Q}, \mathbb{N} \not\sim \mathbb{R}$ ($\text{Card}(\mathbb{N}) < \text{Card}(\mathbb{R})$), $(-\frac{\pi}{2}, \frac{\pi}{2}) \sim \mathbb{R}$.

Theorem 1.6 (Schroeder-Bernstein Theorem). If there are injective functions $f : A \rightarrow B$ and $h : B \rightarrow A$, then there is a bijection $g : A \rightarrow B$.

Definition 1.13. A is said to be a **countable set** if there is a bijective $f : A \rightarrow \mathbb{N}$.

Example 1.4. \mathbb{N}^2 is countable since there is an injective $f : \mathbb{N}^2 \rightarrow \mathbb{N}$ given by $f(m, n) = 2^m 3^n$.

Property 1.6. Countable union of countable sets is countable, i.e., if $\{A_n : n \in \mathbb{N}\}$ is a collection of countable sets, then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

1.5 Cantor's Theorem

Theorem 1.7 (Cantor's Theorem). If A is a set, then there is no map $g : A \rightarrow P(A)$ which is surjective.

2 Metric Spaces and Topology of Metric Spaces

2.1 Metric Spaces

Definition 2.1. A **metric space** is a pair (X, d) , where $d : X^2 \rightarrow [0, \infty)$ s.t. $\forall x, y, z \in X$:

- (1) $d(x, y) = 0$ iff $x = y$;
- (2) $d(x, y) = d(y, x)$;
- (3) $d(x, z) \leq d(z, y) + d(y, z)$.

Example 2.1. For $X = \mathbb{R}$, $d(x, y) = |x - y|$.

Example 2.2. For $X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$ (Euclidean distance).

Example 2.3. For $X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| + \cdots + |x_n - y_n|$.

Example 2.4. For $X = \mathbb{R}^n$, $d(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq n} |x_i - y_i|$.

Example 2.5. For $X = l_\infty = \{\text{The collection of all } (x_n) \subseteq \mathbb{R} \text{ that are bounded}\} \subseteq \mathbb{R}^\mathbb{N}$, $d(\mathbf{x}, \mathbf{y}) = \sup_{n \in \mathbb{N}} |x_n - y_n|$.

Example 2.6. For $X = C[0, 1] = \text{All continuous functions } f : [0, 1] \rightarrow \mathbb{R}$, $d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$.

Example 2.7 (Discrete Metric).

$$d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}.$$

Definition 2.2. A metric space (X, d) is **complete** iff every Cauchy sequence is convergent.

Example 2.8. Here are some examples of complete metric space:

- \mathbb{R} with $d(x, y) = |x - y|$.
- (X, d) with discrete metric $d(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$.
- $C[0, 1]$ with $d(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| = \|f - g\|_\infty$.
- $(\mathbb{N}^\mathbb{N}, d)$ with $d((x_n), (y_n)) = \frac{1}{\min\{n : x_n \neq y_n\}}$.

2.2 Topology of Metric Spaces

Definition 2.3. **Open ball** with radius r and center x is

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

Definition 2.4. A set $U \subseteq X$ is **open** iff

$$\forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq U.$$

Example 2.9. $B_\varepsilon(x)$ is open.

Proof. Fix $x \in X$ and $\varepsilon > 0$. We want to show: $\forall y \in B_\varepsilon(x), \exists \delta > 0$ s.t. $B_\delta(y) \subseteq B_\varepsilon(x)$.

Take $y \in B_\varepsilon(x)$, then $d(x, y) < \varepsilon$. Take $\delta = \varepsilon - d(x, y) > 0$. Take any $z \in B_\delta(y)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + \varepsilon - d(x, y) = \varepsilon.$$

Thus $z \in B_\varepsilon(x)$ and $B_\delta(y) \subseteq B_\varepsilon(x)$. □

Definition 2.5. A **topological space** is a pair (X, τ) where X is a set and τ is the subset of the power set of X which we call open s.t.

- (1) $\emptyset, X \in \tau$;
- (2) $U_1, \dots, U_n \in \tau \Rightarrow \bigcap_{i=1}^n U_i \in \tau$;
- (3) $\{U_i : i \in I\} \subseteq \tau \Rightarrow \bigcup_{i \in I} U_i \in \tau$.

Example 2.10. $\{X, \{\emptyset, X\}\}, \{X, P(X)\}$ are topological spaces and we call $\{X, P(X)\}$ as a discrete topological space.

Example 2.11. Given (X, d) is a metric space, define $\tau_d : U \in \tau_d$ iff $\forall x \in U, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$. Then τ_d is a topology.

Proof. First, $\emptyset, X \in \tau_d$ since $\forall x \in \emptyset, B_1(x) \subseteq \emptyset$ and $\forall x \in X, B_1(x) \subseteq X$.

Then suppose $U_1, \dots, U_n \in \tau_d$, we want to show:

$$U = \bigcap_{i=1}^n U_i \in \tau_d \Leftrightarrow \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq U.$$

Since $U_1, \dots, U_n \in \tau_d$ and $x \in U$, then $x \in U_i, \forall i = 1, \dots, n, \exists \varepsilon_i > 0$ s.t. $B_{\varepsilon_i}(x) \subseteq U_i$. Take $\varepsilon = \min_{1 \leq i \leq n} \varepsilon_i$ and thus $B_\varepsilon(x) \subseteq U_i, \forall i = 1, \dots, n$. Hence, $B_\varepsilon(x) \subseteq U_i \subseteq U$.

Finally, let $\{U_i : i \in I\} \subseteq \tau_d$, we want to show:

$$U = \bigcup_{i \in I} U_i \in \tau_d \Leftrightarrow \forall x \in U, \exists \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \subseteq U.$$

Pick $i_0 \in I, x \in U_{i_0} \subseteq U$. Since $U_{i_0} \in \tau_d$ then $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U_{i_0} \subseteq U$.

Wherefore, τ_d is a topology. □

Definition 2.6. A subset F of a topological space (X, τ) is **closed** if $X \setminus F$ is open.

Property 2.1. Given a topological space (X, τ) , we have:

- (1) \emptyset, X are closed;
- (2) F_1, \dots, F_n are closed then $\bigcup_{i=1}^n F_i$ is closed;
- (3) $\{F_i, i \in I\}$ is a collection of closed set, then $\bigcap_{i \in I} F_i$ is closed.

Definition 2.7. Given a topological space $(X, \tau), \tau \subseteq P(X)$ and $F \subseteq X$. Define the **topological closure** of F as the minimal closed superset of F , i.e.,

$$\overline{F} = \bigcap \{H : H \text{ is closed, } H \supseteq F\}.$$

Define the **interior** of F as the maximal open subset of F , i.e.,

$$F^\circ = \bigcup \{U : U \text{ is open, } U \subseteq F\}.$$

Example 2.12. Given (X, d) is a metric space, τ_d is the topology that $U \in \tau_d$ iff $\forall x \in U, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq U$. Suppose $F \subseteq X$, then

$$\overline{F} = \{x \in X : \forall \varepsilon > 0, B_\varepsilon(x) \cap F \neq \emptyset\} = \left\{ \lim_{n \rightarrow \infty} x_n : (x_n) \subseteq F, \lim_{n \rightarrow \infty} x_n \text{ exists} \right\}$$

and

$$F^\circ = \{x \in X : \exists \varepsilon > 0, B_\varepsilon(x) \subseteq F\} = \bigcup \{B_\varepsilon(x) : \varepsilon > 0, x \in F, B_\varepsilon(x) \subseteq F\}.$$

Property 2.2. If $K_1 \supseteq K_2 \subseteq \dots$ are compact and nonempty subsets of X , then $K = \bigcap_{n=1}^{\infty} K_n$ is compact and nonempty.

Definition 2.8. Let (X, d) be a metric space. $P \subseteq X$ is **perfect** if it is closed nonempty and for every open $U \subseteq X, U \cap P \neq \emptyset, U \cap P$ has at least two elements. Or $\forall x \in P, \forall \varepsilon > 0, B_\varepsilon(x) \cap P$ has at least one more element besides x .

Example 2.13. $S = [0, 1] \cup \{\frac{3}{2}\} \cup [2, 3]$ is not perfect.

Property 2.3. Perfect subsets P of complete metric space are not countable.

Example 2.14 (Cantor Set). $C \subseteq [0, 1], C = \bigcap_{n=0}^{\infty} C_n$, where $\emptyset \neq C_n \subseteq [0, 1]$ and C_n is closed and compact. $C_0 = [0, 1], C_1 = [0, 1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1], C_2 = [0, \frac{1}{9}] \cup [\frac{1}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \dots$

Definition 2.9. Let (X, d) be a metric space, $A \neq \emptyset, B \subseteq X$. A and B are **separated** if $\overline{A} \cap B = \overline{B} \cap A = \emptyset$.

Definition 2.10. A set $C \subseteq X$ is **connected** if for every decomposition $C = A \cup B, A, B \neq \emptyset, A$ and B are not separated, i.e., $\overline{A} \cap B \neq \emptyset$ or $\overline{B} \cap A \neq \emptyset$.

Property 2.4. $C \subseteq \mathbb{R}$ is connected iff for all $a < b$ in C and every $c, a < c < b$ belongs to C , i.e., $\forall a, b \in C, [a, b] \subseteq C$.

Proof. Let $C = A \cup B, a_0 \in A, b_0 \in B, a_0 < b_0$. We define $I_0 = [a_0, b_0], c_0 = \frac{a_0 + b_0}{2}$. Define $I_1 = [a_0, c_0], \dots$. We have $x \in \overline{A} \cap B$ or $\overline{B} \cap A$. \square

Definition 2.11. A set $D \subseteq X$ is **dense** if $\overline{D} = X$, i.e., every point of X is a limit of a sequence of elements of D , or

$$\forall x \in X, \forall \varepsilon > 0, B_\varepsilon(x) \cap D \neq \emptyset \Leftrightarrow \forall U \subseteq X, U \neq \emptyset, U \cap D \neq \emptyset,$$

where U is open.

Example 2.15. \mathbb{Q} is dense in \mathbb{R} , i.e., \mathbb{Q} has a point in any open nonempty interval.

Definition 2.12. $N \subseteq X$ is **nowhere dense** if $\forall U \neq \emptyset, \exists V \subseteq U$ s.t. $V \neq \emptyset$ and $V \cap N = \emptyset$, where U is open.

Definition 2.13. A metric space (X, d) is **compact** if every sequence has a converge subsequence, i.e.,

$$\forall (x_n) \subseteq X, \exists (x_{n_k}) \subseteq (x_n), \exists x \in X \text{ s.t. } \lim_{k \rightarrow \infty} x_{n_k} = x \in X.$$

Example 2.16. $(\mathbb{R}, |x - y|)$ is not compact (e.g., $x_n = n$) and $([0, 1], |x - y|)$ is compact.

Property 2.5. If (X, d) is compact, then it is bounded, i.e., $\exists M$ s.t. $\forall x, y \in X, d(x, y) \leq M$.

Property 2.6. If $Y \subseteq X$, (X, d) is a metric space, and (Y, d) is compact, then Y is closed in X .

Theorem 2.1 (Baire Category Theorem). If (X, d) is complete, then intersection of dense open subsets $\bigcap_{n=1}^{\infty} D_n$ of X is dense in X .

Proof. Claim. Suppose D_1, \dots, D_n is a finite list of dense open subsets of (X, d) , $D = \bigcap_{i=1}^n D_i$ is also dense and open.

First note that D is open. Take $U \neq \emptyset$ be open. We need to show $U \cap D \neq \emptyset$. We have

$$\begin{aligned} U_1 &= U \cap D_1 \neq \emptyset \\ U_2 &= U_1 \cap D_2 \neq \emptyset \\ &\vdots \\ U_n &= U_{n-1} \cap D_n \neq \emptyset, \end{aligned}$$

and thus $U_n = U \cap D \neq \emptyset$.

We may assume that $D_1 \supseteq D_2 \supseteq \dots$. Take $x_1 \in D_1$, then $\exists 0 < \varepsilon_1 < 1$ s.t. $B_{\varepsilon_1}(x_1) \subseteq D_1$. Take $x_2 \in B_{\varepsilon_1}(x_1) \cap D_2 \neq \emptyset$, then $\exists 0 < \varepsilon_2 < \frac{1}{2}$ s.t. $\overline{B_{\varepsilon_2}(x_2)} \subseteq D_2, \dots$. Suppose $n < m, x_m \in B_{\varepsilon_n}(x_n)$, i.e., $d(x_n, x_m) < \frac{1}{n}$. Thus $\{x_n\}$ is Cauchy. Thus, $x = \lim_{n \rightarrow \infty} x_n = \lim_{\substack{m \rightarrow \infty \\ m \geq n}} x_m \subseteq \overline{B_{\varepsilon_n}(x_n)}$. Hence,

$$x \in \bigcap_{n=1}^{\infty} D_n. \quad \square$$

Note that two categories of size for subsets are created in a metric space. A set of first category is one that can be written as a countable union of nowhere-dense sets. If our metric space is complete, then it is necessarily of second category, meaning it cannot be written as a countable union of nowhere-dense sets.

3 Sequences and Series

3.1 Sequences

Definition 3.1. Let (X, d) be a metric space. A sequence $(x_n)_{n \in \mathbb{N}}$ converges to $x \in X$ iff

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geq N, d(x_n, x) < \varepsilon,$$

denoted $\lim_{n \rightarrow \infty} x_n = x$.

Property 3.1. If $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} y_n = y$, then $x = y$.

Proof. We want to show: $d(x, y) = 0 \Leftrightarrow \forall \varepsilon > 0, d(x, y) < \varepsilon$.

Since $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\exists N_1$ s.t. $\forall n \geq N_1, d(x_n, x) < \frac{\varepsilon}{2}$ and $\exists N_2$ s.t. $\forall n \geq N_2, d(x_n, y) < \frac{\varepsilon}{2}$. Take $n \geq \max(N_1, N_2)$, then we have

$$d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Property 3.2. Let (X, d) be a metric space. Suppose $\lim_{n \rightarrow \infty} x_n = x$, $(x_n) \subseteq F$ and F is closed, then $x \in F$.

Proof. Suppose $x \notin F$, i.e., $x \in X \setminus F$. Since F is closed then $X \setminus F$ is open, so $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq X \setminus F$. Pick N s.t. $\forall n \geq N, d(x_n, x) < \varepsilon$, then $x_n \in B_\varepsilon(x) \Rightarrow (x_n) \subseteq X \setminus F$, which is a contradiction. □

Property 3.3. Let (X, d) be a metric space. Suppose $F \subseteq X$, and if F is not closed, then $\exists (x_n) \subseteq F$ and $x \notin F$ s.t. $\lim_{n \rightarrow \infty} x_n = x$.

Proof. If F is not closed, then $U = X \setminus F$ is not open. So $\exists x \in U$ s.t. $\forall \varepsilon > 0, B_\varepsilon(x) \not\subseteq U$. Take $x_n \in B_{\frac{1}{n}}(x) \setminus U = B_{\frac{1}{n}}(x) \cap F, \forall n \in \mathbb{N}$. Then $(x_n) \subseteq F$.

Let $\varepsilon > 0, N = \left[\frac{1}{\varepsilon}\right] + 1$, and $n \geq N$. Since $x_n \in B_{\frac{1}{n}}(x)$, then $d(x_n, x) < \frac{1}{n} \leq \varepsilon$, i.e., $\lim_{n \rightarrow \infty} x_n = x$. □

Definition 3.2. A sequence (x_n) in a metric space (X, d) is **Cauchy** if

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall m, n \geq N, d(x_n, x_m) < \varepsilon.$$

Property 3.4. Convergent sequences are Cauchy.

Proof. Suppose $\lim_{n \rightarrow \infty} x_n = x$. Let $\varepsilon > 0, \exists N$ s.t. $\forall n \geq N, d(x, x_n) < \frac{\varepsilon}{2}$. Take $m, n \geq N$,

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Note that when $X = \mathbb{R}$ with the usual metric, the converse is true. But in general, the converse is not. For example, $X = \mathbb{R} \setminus \{0\}$ with $d(x, y) = |x - y|$. Let $x_n = \frac{1}{n}$.

Property 3.5. Suppose (x_{n_k}) is a subsequence of (x_n) and $\lim_{n \rightarrow \infty} x_n = x$, then $\lim_{k \rightarrow \infty} x_{n_k} = x$.

Proof. Since $\lim_{n \rightarrow \infty} x_n = x$, then

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geq N, d(x, x_n) < \varepsilon.$$

Take $K = N$, let $k \geq N$, we have $n_k \geq k \geq N$, and thus $d(x, x_{n_k}) < \varepsilon$. \square

Theorem 3.1 (Bolzano-Weierstrass Theorem). A subset Y of \mathbb{R} is compact iff it is closed and bounded.

Note that the theorem is true for \mathbb{R}^n but is false for infinite dimension.

Theorem 3.2 (Heine-Borel Theorem). A subset Y of a metric space (X, d) is compact if every open cover $Y \subseteq \bigcup_{i \in I} U_i$ has a finite subcover $Y \subseteq \bigcup_{l=1}^n U_{i_l}$.

Definition 3.3. $(x_n) \subseteq \mathbb{R}$ is **monotone** if either $x_n \leq x_m, n \leq m$ or $x_n \geq x_m, n \leq m$.

Theorem 3.3 (Monotone Subsequence Theorem). Every sequence $(x_n) \subseteq \mathbb{R}$ has a monotone subsequence.

Proof. We define a peak term: given a (x_n) , a particular term x_m is a peak term if $x_m \geq x_n, \forall n \geq m$. Now let (x_n) be any sequence, define set of peaks: $P = \{x_n : x_n \geq x_m, \forall m \geq n\}$. If P is infinite then $\exists (x_{n_k})$ s.t. $x_{n_k} \in P$ is a decreasing subsequence of (x_n) . If P is finite then let $x_m = \min P$. Let $n_1 = m + 1$ then $x_{n_1} \notin P$ so that exists element is greater than x_{n_1} . Define $n_{k+1} = \min\{m \in \mathbb{N} : x_m > x_{n_k}\}$. Hence we get $x_{n_{k+1}} > x_{n_k}$ so (x_{n_k}) is an increasing subsequence of (x_n) . \square

Theorem 3.4. Every bounded sequence contains a convergent subsequence.

Proof. If (x_n) is bounded sequence, then $\exists (x_{n_k})$ s.t. (x_{n_k}) is monotone and bounded sequence. Then by theorem, (x_{n_k}) must converge. \square

Property 3.6. If $a_n \leq b_n, \forall n, a = \lim_{n \rightarrow \infty} a_n, b = \lim_{n \rightarrow \infty} b_n$, then $a \leq b$.

Proof. Suppose $a > b$. Let $\varepsilon = \frac{a-b}{2}$. We know $\exists N_1$ s.t. $a_n \in B_\varepsilon(a)$ for $n \geq N_1$ and $\exists N_2$ s.t. $b_n \in B_\varepsilon(b)$ for $n \geq N_2$. Take $n > \max(N_1, N_2)$, then we have

$$b_n < \frac{a+b}{2} < a_n,$$

which is a contradiction. \square

Property 3.7 (Algebraic Limit Theorem). Suppose $a = \lim_{n \rightarrow \infty} a_n, b = \lim_{n \rightarrow \infty} b_n$, then:

- (1) $a + b = \lim_{n \rightarrow \infty} (a_n + b_n)$;
- (2) $ab = \lim_{n \rightarrow \infty} a_n b_n$;
- (3) $\frac{a}{b} = \lim_{n \rightarrow \infty} \frac{a_n}{b_n}$, and $b \neq 0$.

Property 3.8. Monotone bounded sequence (x_n) converges to its supremum or infimum.

Proof. We only prove one situation: Fix $\varepsilon > 0$. Let $s = \sup\{x_n : n \in \mathbb{N}\}$. We have $s - \varepsilon < s$ and thus $s - \varepsilon$ is not an upper bound of (x_n) . Therefore, there is N s.t. $x_N > s - \varepsilon$. Take $n \geq N$, we have $x_n \geq x_N > s - \varepsilon$. Therefore, we have $|x_n - s| < \varepsilon$. \square

Definition 3.4. We define

$$\limsup_{n \rightarrow \infty} x_n = \inf\{y_m : m \in \mathbb{N}\},$$

where $y_m = \sup\{x_n : n \geq m\}$.

$$\liminf_{n \rightarrow \infty} x_n = \sup\{z_m : m \in \mathbb{N}\},$$

where $z_m = \inf\{x_n : n \geq m\}$.

3.2 Series

Definition 3.5. We define

$$\sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n, S_n = \sum_{k=1}^n a_k.$$

We call $\sum_{k=1}^{\infty} a_k$ is a summable series if the limit exists, i.e.,

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geq N, |S_n - A| < \varepsilon.$$

Property 3.9 (Cauchy Criterion for Series). $\sum_{k=1}^{\infty} a_k$ is summable iff

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geq m \geq N, |S_n - S_m| = \left| \sum_{k=m+1}^n a_k \right| < \varepsilon.$$

Corollary 3.1. If $\sum_{k=1}^{\infty} a_k$ is summable, then $|a_k| \rightarrow 0$.

Proof. We have $|a_k| = |s_k - s_{k-1}| < \varepsilon$ for $k > N$. □

Example 3.1. $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is summable.

Proof. Since $S_m \leq S_n, \forall m \leq n$, then it suffices to find $0 < M < \infty$ s.t. $S_m < M, \forall m$. We have

$$\begin{aligned} S_m &= 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2} < 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{m(m-1)} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) = 1 + 1 - \frac{1}{m} < 2. \end{aligned}$$

□

Example 3.2. $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

Proof. We have

$$\sum_{n=1}^{2^k} \frac{1}{n} \geq 1 + \frac{k}{2} \rightarrow \infty \text{ as } k \rightarrow \infty.$$

□

Theorem 3.5 (Algebraic Limit Theorem for Series). Suppose $\sum_{k=1}^{\infty} a_k = A, \sum_{k=1}^{\infty} b_k = B, c \in \mathbb{R}$, then

- (1) $\sum_{k=1}^{\infty} ca_k = cA$;
- (2) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof. (1) We want to show $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N, \left| \sum_{k=1}^n ca_k - cA \right| < \varepsilon$. We know $\forall \varepsilon_0 > 0, \exists N_{\varepsilon_0}$ s.t.

$\forall n \geq N_{\varepsilon_0}, \left| \sum_{k=1}^n a_k - A \right| < \varepsilon_0$. Take $\varepsilon_0 = \frac{\varepsilon}{|c|}$, then we have

$$\left| \sum_{k=1}^n ca_k - cA \right| = |c| \left| \sum_{k=1}^n a_k - A \right| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

□

Property 3.10 (Order Comparison Test). Suppose $b_k \geq a_k \geq 0, \forall k$. If $\sum_{k=1}^{\infty} b_k$ converges so does $\sum_{k=1}^{\infty} a_k$. If $\sum_{k=1}^{\infty} a_k$ diverges so does $\sum_{k=1}^{\infty} b_k$.

Definition 3.6. We call a *geometric series* if it is

$$\sum_{k=1}^{\infty} ar^k.$$

Note that the geometric series converges to $\frac{a}{1-r}$ whenever $r^m \rightarrow 0$ iff $|r| < 1$ /

Definition 3.7. $\sum_{k=1}^{\infty} a_k$ is **absolutely convergent** if $\sum_{k=1}^{\infty} |a_k|$ is convergent. $\sum_{k=1}^{\infty} a_k$ is **conditionally convergent** if $\sum_{k=1}^{\infty} a_k < \infty$ but $\sum_{k=1}^{\infty} |a_k| = \infty$.

Example 3.3 (Alternating Series). $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} < \infty$ but $\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

Property 3.11 (Absolute Convergence Test). If $\sum_{k=1}^{\infty} |a_k|$ converges so does $\sum_{k=1}^{\infty} a_k$.

Proof. We use Cauchy test for $\sum_{k=1}^{\infty} a_k$, i.e., we want to show $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq m \geq N, |a_{m+1} + a_{m+2} + \dots + a_n| < \varepsilon$. We know $|a_{m+1} + \dots + a_n| \leq |a_{m+1}| + \dots + |a_n| = t_n - t_m$, where $t_n = \sum_{k=1}^n |a_k|$. Given $\varepsilon > 0$ and N s.t. $|t_n - t_m| < \varepsilon$, then this N works for $|S_n - S_m| < \varepsilon$, where $S_n = \sum_{k=1}^n a_k$. \square

Property 3.12 (Alternating Series Test). Suppose $a_1 \geq a_2 \geq \dots \geq 0, \lim_{k \rightarrow \infty} a_k = 0$, then $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ is convergent.

Proof. We want to show $\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^n (-1)^{k+1} a_k$ is Cauchy.

Suppose $n > m$, then $|s_n - s_m| = |a_{m+1} - a_{m+2} + \dots + (-1)^{n-m+1} a_n|$. Since (a_n) is a non-negative decreasing sequence, then

$$a_{m+1} - a_{m+2} + \dots + (-1)^{n-m-1} a_n = a_{m+1} - (a_{m+2} - a_{m+3}) - (a_{m+4} - a_{m+5}) - \dots \leq a_{m+1}.$$

Thus, $0 \leq |s_n - s_m| \leq a_{m+1}$. Since $a_{m+1} \rightarrow 0$ then $\forall \varepsilon > 0, \exists N$ s.t. $\forall m \geq N, |a_{m+1}| = a_{m+1} < \varepsilon$. Take such N and thus $\forall n > m \geq N, |s_n - s_m| < \varepsilon$. \square

Property 3.13 (Ratio Test). Given $\sum_{k=1}^{\infty} a_k, a_k \neq 0, \forall k$. If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1$, then $\sum_{k=1}^{\infty} |a_k|$ is convergent.

Proof. Define $S = \left\{ n \in \mathbb{N} : \left| \frac{a_{n+1}}{a_n} \right| \geq r' \right\}$, then S contains finitely many elements of \mathbb{N} . If S were to be infinite set, if we take $\varepsilon = r' - r$, then $\left| \frac{a_{n+1}}{a_n} \right| - r \geq r' - r$ for infinitely many terms which contradicts that r point of convergence. Therefore, $S' = \left\{ n \in \mathbb{N} : \left| \frac{a_{n+1}}{a_n} \right| < r' \right\}$ contains all but

finitely many elements of \mathbb{N} . Let $N = 1 + \max S$, then $\forall n \geq N, \left| \frac{a_{n+1}}{a_n} \right| < r' \Rightarrow |a_{n+1}| < r'|a_n|$. Since $0 < r' < 1$, $\sum_{n=1}^{\infty} (r')^n$ converges which implies $|a_N| \sum_{n=1}^{\infty} (r')^n$ converges. We have $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^N |a_n| + \sum_{n=N+1}^{\infty} |a_n| < C + |a_N| \sum_{n=N+1}^{\infty} (r')^{n-N}$ converges by comparison test. Hence $\sum_{n=1}^{\infty} |a_n|$ converges. \square

Definition 3.8. Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a *rearrangement* of $\sum_{k=1}^{\infty} a_k$ if for all n , there is unique k s.t. $b_k = a_n$.

Example 3.4. $a_k = \frac{1}{k}, b_{2k} = \frac{1}{2k+1}, b_{2k+1} = \frac{1}{2k}$.

4 Functional Limits and Continuity

4.1 Functional Limits

Definition 4.1. Let $A \subseteq \mathbb{R}, a \in \overline{A \setminus \{a\}}$, i.e., a is an accumulation point of A . Let $f : A \rightarrow \mathbb{R}$, define $\lim_{x \rightarrow a} f(x) = L$ iff

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Example 4.1. $f(x) = cx$ on $A = \mathbb{R}, a \in \overline{A \setminus \{a\}}, \lim_{x \rightarrow a} f(x) = ca$.

Proof. Let $\varepsilon > 0, \delta = \frac{\varepsilon}{|c|}$, then we have $|cx - ca| = |c||x - a| < \varepsilon$. □

Example 4.2. $f(x) = x^2$ on $A = \mathbb{R}, \lim_{x \rightarrow \sqrt{2}} f(x) = 2$.

Proof. Let $\varepsilon > 0$. Let $\delta = \min\left(\frac{\varepsilon}{2+\sqrt{2}}, 2 - \sqrt{2}\right)$. Let $0 < |x - \sqrt{2}| < \delta$, we have $|x^2 - 2| = |x - \sqrt{2}||x + \sqrt{2}| < (2 + \sqrt{2})|x - \sqrt{2}| = \varepsilon$. □

Property 4.1 (Sequential Criterion for Functional Limits). Suppose $a \in \overline{A \setminus \{a\}}, f : A \rightarrow \mathbb{R}$. The following are equivalent:

- (1) $\lim_{x \rightarrow a} f(x) = L$;
- (2) $\forall (x_n) \subseteq A \setminus \{a\}, x_n \rightarrow a \Rightarrow f(x_n) \rightarrow L$.

Proof. We prove (1) \Rightarrow (2) : Take arbitrary $(x_n) \subseteq A \setminus \{a\}, x_n \rightarrow a$. Let $\varepsilon > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$. Also, $\exists N$ s.t. $n \geq N \Rightarrow |x_n - a| < \delta$. Let $n \geq N$, then $|x_n - a| < \delta$ and thus $|f(x_n) - L| < \varepsilon$. □

Theorem 4.1 (Algebraic Limit Theorem for Functional Limits). Suppose $f, g : A \rightarrow \mathbb{R}, a \in \overline{A \setminus \{a\}}$. Suppose $\lim_{x \rightarrow a} f(x) = L, \lim_{x \rightarrow a} g(x) = M, c \in \mathbb{R}$. We have

- (1) $\lim_{x \rightarrow a} cf(x) = cL$;
- (2) $\lim_{x \rightarrow a} (f(x) + g(x)) = L + M$;
- (3) $\lim_{x \rightarrow a} (f(x)g(x)) = LM$;
- (4) $\lim_{x \rightarrow a} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$ when $M \neq 0$.

Property 4.2 (Divergence Criterion). Suppose $f : A \rightarrow \mathbb{R}, a \in \overline{A \setminus \{a\}}$. $\lim_{x \rightarrow a} f(x)$ does not exist if there are two sequences $(x_n), (y_n) \subseteq A \setminus \{a\}$ s.t. $x_n \rightarrow a, y_n \rightarrow a, \lim_{n \rightarrow \infty} f(x_n) = L, \lim_{n \rightarrow \infty} f(y_n) = M$ exist but $L \neq M$.

Example 4.3. Let $A = \mathbb{R}^+, a = 0, f(x) = \sin\left(\frac{1}{x}\right)$. Let $a_n = \frac{1}{2n\pi}, b_n = \frac{1}{2n\pi + \frac{\pi}{2}}$. We have $a_n, b_n \rightarrow a$. Besides, $\lim_{n \rightarrow \infty} f(a_n) = 0, \lim_{n \rightarrow \infty} f(b_n) = 1$. Hence $\lim_{x \rightarrow 0^+} \sin\left(\frac{1}{x}\right)$ does not exist.

Definition 4.2. Suppose $f : A \rightarrow \mathbb{R}, a \in A \setminus \{a\}$. We define $\lim_{x \rightarrow a} f(x) = \infty$ iff

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow f(x) > M.$$

Definition 4.3. we define $\lim_{x \rightarrow \infty} f(x) = L$ iff

$$\forall \varepsilon > 0, \exists M > 0 \text{ s.t. } x > M \Rightarrow |f(x) - L| < \varepsilon.$$

4.2 Continuous Functions

Definition 4.4. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. $f : X \rightarrow Y$ is **continuous** at $a \in X$ if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } x \in B_\delta^X(a) \Rightarrow f(x) \in B_\varepsilon^Y(f(a)).$$

Note that for $X = Y = \mathbb{R}, d(x, y) = |x - y|$, we can write $\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$, i.e., $\lim_{x \rightarrow a} f(x) = f(a)$.

Definition 4.5. $f : X \rightarrow Y$ is **continuous** if it is continuous at every point $a \in X$.

Property 4.3. The following are equivalent:

- (1) f is continuous at a ;
- (2) $\lim_{x \rightarrow a} f(x) = f(a)$;
- (3) $\forall (x_n) \subseteq A, x_n \rightarrow a \Rightarrow f(x_n) \rightarrow f(a)$.

Corollary 4.1. f is discontinuous at a if there is sequence $(x_n) \rightarrow a$ s.t. $\lim_{n \rightarrow \infty} f(x_n) \neq f(a)$.

Note that we may have $\lim_{x \rightarrow a} f(x)$ exists but f is discontinuous at a .

Theorem 4.2 (Algebraic Continuous Theorem). Suppose $f, g : A \rightarrow \mathbb{R}$ are continuous at $a \in A, c \in \mathbb{R}$. We have

- (1) $cf(x)$ is continuous at a ;
- (2) $f(x) \pm g(x)$ is continuous at a ;
- (3) $f(x)g(x)$ is continuous at a ;
- (4) $\frac{f(x)}{g(x)}$ is continuous at a if $g(a) \neq 0$.

Property 4.4. Suppose $f : A \rightarrow B \subseteq \mathbb{R}, g : B \rightarrow \mathbb{R}$. $(g \circ f)(x) = g(f(x))$ is continuous at $a \in A$ whenever f is continuous at a and g is continuous at $f(a)$.

4.3 Continuous Functions on Compact Sets

Property 4.5. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces, $f : X \rightarrow Y$ is continuous. If $K \subseteq X$ is compact, so is its image $f[K] = \{f(x) : x \in K\}$.

Property 4.6. Suppose $(X, d_X), (Y, d_Y)$ are metric spaces. $f^{-1}(F)$ is closed in X whenever $F \subseteq Y$ is closed in Y .

Theorem 4.3 (Extreme Value Theorem). If $f : K \rightarrow \mathbb{R}$ is continuous, K is compact, then $\exists x_1, x_2 \in K$ s.t. $\forall x \in K, f(x_1) \leq f(x) \leq f(x_2)$.

Proof. Let $H = f[K] = \{f(x) : x \in K\} \subseteq \mathbb{R}$, which is compact. Since compact subsets of \mathbb{R} are bounded, then let $y_2 = \sup(H)$. We have $y \leq y_2, \forall y \in H$ and $\forall \varepsilon > 0, \exists y \in H$ s.t. $y_2 - \varepsilon < y \leq y_2$. Take $\varepsilon = \frac{1}{n}, z_n \in H$, then $y_2 - \frac{1}{n} < z_n \leq y_2$. Now we find $a_n \in K$ s.t. $f(a_n) = z_n, n = 1, 2, \dots$. By theorem, we have $a_{n_k} \rightarrow x_2$, then $f(x_2) = \lim_{k \rightarrow \infty} f(a_{n_k}) = y_2$. \square

Definition 4.6. Assume $f : A \rightarrow \mathbb{R}$ is a function. We say f is **uniformly continuous** on A if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $x, y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Example 4.4. $f(x) = x^2$ is not uniformly continuous.

Proof. Let $\varepsilon = 1$ and $\forall \delta > 0$, let $I_\delta = \left[\frac{2}{\delta} + 1, \frac{2}{\delta} + 1 + \delta\right]$, then we have $|f(x) - f(y)| \geq 1$. \square

Property 4.7. Assume $f : A \rightarrow \mathbb{R}$ is a function, then f fails to be uniformly continuous iff $\exists \varepsilon_0 > 0$ and $(x_n), (y_n) \subseteq A$ s.t. $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n)| \geq \varepsilon_0$.

Proof. (\Leftarrow) It is obvious.

(\Rightarrow) Assume f is not uniformly continuous. Fix $\varepsilon_0 > 0$ s.t. the definition of uniformly continuous fails for ε_0 , i.e., $\forall \delta > 0, \exists x_\delta, y_\delta$ s.t. $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$. For each n , pick x_n, y_n as above, then it is obvious that $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n)| \geq \varepsilon_0$. \square

Property 4.8. Assume $f : K \rightarrow \mathbb{R}$ is continuous and K is compact, then f is uniformly continuous on K , i.e., continuous functions on compact sets are uniformly continuous.

Proof. Assume for a contradiction that $f : K \rightarrow \mathbb{R}$ is continuous, K is compact, and f is not uniformly continuous. Then $\exists \varepsilon_0 > 0, (x_n), (y_n) \subseteq K$ s.t. $\lim_{n \rightarrow \infty} |x_n - y_n| = 0$ and $\forall n, |f(x_n) - f(y_n)| \geq \varepsilon_0$. Since K is compact, x_n has a subsequence x_{n_k} s.t. $\lim_{k \rightarrow \infty} x_{n_k} = x \in K$. Moreover, (y_{n_k}) has a subsequence s.t. $y_{n_{k_m}} \rightarrow y \in f(K)$. Let $x'_m = x_{n_{k_m}}, y'_m = y_{n_{k_m}}$, then $x'_m \rightarrow x, y'_m \rightarrow y$. On the one hand, $\lim_{m \rightarrow \infty} |x'_m - y'_m| = 0$ and thus $x = y$. On the other hand, $|f(x'_m) - f(y'_m)| \geq \varepsilon_0 \Rightarrow \lim_{m \rightarrow \infty} |f(x'_m) - f(y'_m)| \geq \varepsilon_0 \Rightarrow |f(x) - f(x)| \geq \varepsilon_0$ which is a contradiction. \square

Definition 4.7. A function $f : A \rightarrow \mathbb{R}$ is said to be **Lipschitz** if $\exists M \in \mathbb{N}$ s.t.

$$\forall x \neq y \in A, \left| \frac{f(x) - f(y)}{x - y} \right| < M.$$

Property 4.9. Lipschitz functions are uniformly continuous.

Proof. Assume f is Lipschitz on A , then for every $\varepsilon > 0$, take $\delta < \frac{\varepsilon}{M}$. \square

Remark: The converse does not hold, for example $f(x) = \sqrt{x^2 - 1}$.

Property 4.10. Assume $f : E \rightarrow \mathbb{R}$ is continuous and E is connected, then $f(E)$ is connected, i.e., continuous image of connected sets is connected.

Proof. Assume $f(E)$ is not connected. Fix $A, B \subseteq f(E)$ s.t. $\overline{A} \cap B = \emptyset = \overline{B} \cap A$ and $f(E) = A \cup B$. Let $C = f^{-1}(A), D = f^{-1}(B)$. Note that $E = C \cup D, C \cap D = \emptyset$ because f is a function. We now show that $\overline{C} \cap D = \emptyset$: Assume not, then $\exists (x_n) \subseteq C$ s.t. (x_n) is convergent and $\lim_{n \rightarrow \infty} f(x_n) = f(x) \in B$, i.e., $\lim_{n \rightarrow \infty} f(x_n) \in \overline{A} \cap B$, which is a contradiction. Similarly, $\overline{D} \cap C = \emptyset$ and thus E can be separated by C and D , which is a contradiction. \square

4.4 Sets of Discontinuities

Let $f : \mathbb{R} \rightarrow \mathbb{R}, D_f = \{x \in \mathbb{R} : f \text{ is not continuous at } x\}$.

Example 4.5 ($D_f = \emptyset$). f is continuous.

Example 4.6 ($D_f = \mathbb{R}$). $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$.

Example 4.7. Given a countable set $A = \{a_1, \dots\}$, define $f(a_n) = \frac{1}{n}$ and $f(x) = 0, \forall x \notin A$. We have $D_f = A$.

Example 4.8. There is no $f : \mathbb{R} \rightarrow \mathbb{R}$ s.t. $D_f = \mathbb{R} \setminus \mathbb{Q}$.

Definition 4.8. A subset F of \mathbb{R} is a F_σ -**set** if $F = \bigcup_{n=1}^{\infty} F_n$ s.t. F_n is closed for all n .

Definition 4.9. Let $\alpha > 0, f : \mathbb{R} \rightarrow \mathbb{R}, a \in \mathbb{R}$. f is α -**continuous** at a if

$$\exists \delta > 0 \text{ s.t. } x, y \in (a - \delta, a + \delta) \Rightarrow |f(x) - f(y)| < \alpha.$$

Note that f is continuous at a iff f is α -continuous at a for all $\alpha > 0$.

Property 4.11. For every $f : \mathbb{R} \rightarrow \mathbb{R}$, the set D_f is F_δ -subset of \mathbb{R} .

Definition 4.10. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. f is **removable discontinuous** if $\lim_{x \rightarrow a} f(x)$ exists but does not equal $f(a)$. f has a **jump** at a if $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$. If $\lim_{x \rightarrow a} f(x)$ does not exist for other reasons, we say f is **essential discontinuous**.

Definition 4.11. $f : \mathbb{R} \rightarrow \mathbb{R}$ is **monotone** if either $x \leq y \Rightarrow f(x) \leq f(y)$ or $x \leq y \Rightarrow f(x) \geq f(y)$.

Property 4.12. Discontinuity of a monotone function f is a jump. Moreover, D_f is countable.

5 The Derivative

5.1 Derivatives and the Intermediate Value Property

Definition 5.1. Let $f : \mathbb{R} \rightarrow \mathbb{R}, c \in \mathbb{R}$. Define the **derivative** of f at c :

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}.$$

If $f'(c)$ exists, we say f is **differentiable** at c . If f' exists for all $a \in \mathbb{R}$, we say g is **differentiable** on \mathbb{R} .

Property 5.1. If f is differentiable at c , then f is continuous at c .

Proof. We have

$$\lim_{x \rightarrow c} (f(x) - f(c)) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \cdot (x - c) = f'(c) \cdot 0 = 0.$$

□

Theorem 5.1 (Algebraic Differentiability Theorem). Suppose f, g are differentiable, $a, c \in \mathbb{R}$. We have

- (1) $(cf)'(a) = cf'(a)$;
- (2) $(f + g)'(a) = f'(a) + g'(a)$;
- (3) $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$;
- (4) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}, g(a) \neq 0$.

Theorem 5.2 (Chain Rule). Let $f : A \rightarrow B, g : B \rightarrow \mathbb{R}, f(A) \subseteq B$ so that $g \circ f$ is defined. If f is differentiable at c and if g is differentiable at $f(c)$, then $g \circ f$ is differentiable at a with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

Theorem 5.3 (Interior Extremum Theorem). If f is differentiable on (a, b) , f attains maximum at some $c \in (a, b)$, then $f'(c) = 0$.

Proof. We have

$$f'(c) = \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \leq 0$$

and

$$f'(c) = \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0,$$

then $f'(c) = 0$.

□

Theorem 5.4 (Darboux's Theorem). If f is differentiable on $[a, b]$ and $f'(a) < \alpha < f'(b)$ or $f'(a) > \alpha > f'(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = \alpha$.

5.2 The Mean Value Theorem

Theorem 5.5 (Rolle's Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$, then $\exists c \in (a, b)$ s.t. $f'(c) = 0$.

Proof. Since f is continuous on a compact set, f attains a maximum and a minimum. If both the maximum and minimum occur at the endpoints, then f is necessarily a constant function and $f'(x) = 0$ on (a, b) . On the other hand, if either the maximum or minimum occurs at some point $c \in (a, b)$, then it follows from the interior extremum theorem that $f'(c) = 0$.

□

Theorem 5.6 (Mean Value Theorem). If $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , then $\exists c \in (a, b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Consider

$$d(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right].$$

We know d is continuous on $[a, b]$ and differentiable on (a, b) . Also, $d(a) = d(b) = 0$. By Rolle's Theorem, $\exists c \in (a, b)$ s.t. $d'(c) = 0$. \square

Corollary 5.1. If $f : (a, b) \rightarrow \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b) .

Proof. Assume $x < y, x, y \in (a, b)$. We set $c \in (x, y)$, then by mean value theorem,

$$0 = f'(c) = \frac{f(y) - f(x)}{y - x} \Rightarrow f(y) - f(x) = 0.$$

\square

Corollary 5.2. If $f : (a, b) \rightarrow \mathbb{R}$ is differentiable and $f'(x) = g'(x)$ for all $x \in (a, b)$, then $f(x) = g(x) + c$.

Proof. Apply the previous corollary to the function $h(x) = f(x) - g(x)$. \square

Theorem 5.7 (Generalized Mean Value Theorem). If $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b) , then $\exists c \in (a, b)$ s.t.

$$[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c).$$

If g' is never zero on (a, b) then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof. Apply the mean value theorem to the function $h(x) = [f(b) - f(a)]g(x) - [g(b) - g(a)]f(x)$. \square

Theorem 5.8 (L'Hospital's Rule: 0/0 Case). Suppose f, g are continuous on I with $a \in I$ and are differentiable on $I \setminus \{a\}$. If $f(a) = g(a) = 0, g'(x) \neq 0, \forall x \neq a$, then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

Proof. Since $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$, then for all $\varepsilon > 0, \exists \delta > 0$ s.t.

$$x \in (a - \delta, a + \delta) \Rightarrow \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

By the generalized mean value theorem, for every $y \in (a, a + \delta), \exists x \in (a, y)$ s.t.

$$\frac{f'(x)}{g'(x)} = \frac{f(y) - f(a)}{g(y) - g(a)} = \frac{f(y)}{g(y)}$$

and thus

$$\left| \frac{f(y)}{g(y)} - L \right| = \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

\square

Theorem 5.9 (L'Hospital's Rule: ∞/∞ Case). Suppose f, g are differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim_{x \rightarrow a} g(x) = \infty$ or $-\infty$, then

$$\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = L.$$

6 Sequences and Series of Functions

6.1 Uniform Convergence of a Sequence of Functions

Definition 6.1. For each $n \in \mathbb{N}$, let f_n be a function defined on a set $A \subseteq \mathbb{R}$. The sequence (f_n) of functions **converges pointwise** on A to a function f if $f_n(x) \rightarrow f(x), \forall x \in A$. We can write $f_n \rightarrow f, \lim f_n = f$, or $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

Example 6.1. Consider

$$f_n(x) = \frac{x^2 + nx}{n}$$

on \mathbb{R} . We can compute

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x^2 + nx}{n} = \lim_{n \rightarrow \infty} \frac{x^2}{n} + x = x.$$

Thus, (f_n) converges pointwise to $f(x) = x$ on \mathbb{R} .

Example 6.2. Consider

$$f_n(x) = x^n$$

on $[0, 1]$. If $0 \leq x < 1$, $x^n \rightarrow 0$. If $x = 1$, $x^n \rightarrow 1$. It follows that $f_n \rightarrow f$ pointwise on $[0, 1]$ where

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}.$$

Note that pointwise convergent sequence of continuous functions may converge to a non-continuous function.

Definition 6.2. Let (f_n) be a sequence of functions defined on a set $A \subseteq \mathbb{R}$, then (f_n) **converges uniformly** on A to a limit function f defined on A if

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall n \geq N, \forall x \in A, |f(x) - f_n(x)| < \varepsilon.$$

Example 6.3. Consider

$$f_n(x) = \frac{x^2 + nx}{n}$$

where converges pointwise on \mathbb{R} to $f(x) = x$. But on \mathbb{R} , the convergence is not uniform. We have

$$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \frac{x^2}{n}.$$

In order to force $|f_n(x) - f(x)| < \varepsilon$, we need $N > \frac{x^2}{\varepsilon}$. Although it is possible to do for each $x \in \mathbb{R}$, there is no way to choose a single value of N that will work for all values of x at the same time.

On the other hand, we can show that $f_n \rightarrow f$ uniformly on the set $[-b, b]$.

Property 6.1 (Cauchy Criterion for Uniform Convergence). A sequence of functions (f_n) defined on a set $A \subseteq \mathbb{R}$ converges uniformly on A iff

$$\forall \varepsilon > 0, \exists N \text{ s.t. } \forall x \in A, \forall m, n \geq N, |f_n(x) - f_m(x)| < \varepsilon.$$

Theorem 6.1 (Continuous Limit Theorem). Let (f_n) be a sequence of functions defined on $A \subseteq \mathbb{R}$ that converges uniformly on A to a function f . If each f_n is continuous at $c \in A$, then f is continuous at c .

Proof. Let $\varepsilon > 0$ and fix $c \in A$. Choose N s.t.

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3}, \forall x \in A.$$

Since f_N is continuous, then $\exists \delta > 0$ s.t.

$$|x - c| < \delta \Rightarrow |f_N(x) - f_N(c)| < \frac{\varepsilon}{3}.$$

Thus,

$$\begin{aligned} |f(x) - f(c)| &= |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence, f is continuous at $c \in A$. □

Property 6.2 (Algebraic Limit Theorem for Uniform Convergence). Suppose $(f_n), (g_n)$ are uniformly convergent on A , then:

- (1) $(cf_n + g_n)$ is uniformly convergent;
- (2) If $\exists M > 0$ s.t. $|f_n| \leq M, |g_n| \leq M$, then $(f_n g_n)$ is uniformly convergent.

Proof. Using Cauchy criterion, we have

$$\begin{aligned} |f_m(x)g_m(x) - f_n(x)g_n(x)| &= |f_m(x)g_m(x) - f_m(x)g_n(x) + f_m(x)g_n(x) - f_n(x)g_n(x)| \\ &\leq |f_m(x)||g_m(x) - g_n(x)| + |g_n(x)||f_m(x) - f_n(x)| \\ &\leq M(|g_m(x) - g_n(x)| + |f_m(x) - f_n(x)|) \end{aligned}$$

□

6.2 Uniform Convergence and Differentiation

Theorem 6.2 (Differentiable Limit Theorem). Let $f_n \rightarrow f$ pointwise on $[a, b]$ and assume each f_n is differentiable. If (f'_n) converges uniformly on $[a, b]$ to a function g , then the function f is differentiable and $f' = g$.

Theorem 6.3. Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume (f'_n) converges uniformly on $[a, b]$. If $\exists x_0 \in [a, b]$ s.t. $f_n(x_0)$ is convergent, then (f_n) converges uniformly on $[a, b]$.

Theorem 6.4. Let (f_n) be a sequence of differentiable functions defined on $[a, b]$ and assume (f'_n) converges uniformly to a function g on $[a, b]$. If $\exists x_0 \in [a, b]$ s.t. $f_n(x_0)$ is convergent, then (f_n) converges uniformly. Moreover, the limit function $f = \lim f_n$ is differentiable and $f' = g$.

6.3 Series of Functions

Definition 6.3. For each $n \in \mathbb{N}$, let f_n and f be functions defined on a set $A \subseteq \mathbb{R}$. The infinite series

$$\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + \cdots$$

converges pointwise on A to $f(x)$ if the sequence $s_k(x)$ of partial sums defined by

$$s_k(x) = f_1(x) + \cdots + f_k(x)$$

converges pointwise to $f(x)$. The series **converges uniformly** on A to f if the sequence $s_k(x)$ converges uniformly on A to $f(x)$.