Probability

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1 Review

1.1 Set

Definition 1.1 (Power Set)**.** For a given set Ω, the power set is the set of all of its subsets

$$\mathcal{P}(\Omega) = \{A | A \subset \Omega\}.$$

The power set is closed w.r.t. all the usual set-theoretic operations.

Definition 1.2 (Symmetric Difference)**.** Define the symmetric difference of any two sets

$$A\Delta B := (A - B) + (B - A).$$

Actually, $A\Delta B = A \cup B - AB$.

Theorem 1.1 (De Morgan's Laws)**.** For any collection of sets $A^t, t \in T$ all in $\mathcal{P}(\Omega)$,

$$\left(\bigcup_{t\in T} A_t\right)^C = \bigcap_{t\in T} A_t^C \text{ and } \left(\bigcap_{t\in T} A_t\right)^C = \bigcup_{t\in T} A_t^C.$$

With the notation of set, one way to consider whole number could be:
$0 = \varnothing, 1 = \{\varnothing\}, 2 = \{0, 1\}, \cdots$, and thus

$$\begin{aligned} n + 1 &= n \cup \{n\} \\ &= \{0, 1, \cdots, n - 1\} \cup \{n\} \\ &= \{0, 1, \cdots, n\}. \end{aligned}$$

We can also define number systems with set:

$$\begin{aligned} &\mathbb{N} = \{1, 2, \cdots\}, \\ &\mathbb{W} = \mathbb{N} \cup \{0\}, \\ &\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}, \\ &\mathbb{Q} = \left\{ \frac{n}{m} \middle| n \in \mathbb{Z}, m \in \mathbb{N} \right\}, \\ &\mathbb{R} = \left\{ x = \lim_{n\to\infty} r_n \middle| r_n \in \mathbb{Q}, n \in \mathbb{N} \right\}, \\ &\mathbb{C} = \{z = x + iy | x, y \in \mathbb{R}\}. \end{aligned}$$

In multi-variable calculus, we define

$$\mathbb{R}^n = \{\mathbf{x} | x_i \in \mathbb{R}. i = 1, \cdots, n\},$$

where $\mathbf{x} = (x_i, i = 1, \cdots, n)$ and

$$\mathbb{R}^\infty = \{\mathbf{x} = (x_i, i = 1, 2, \cdots) | x_i \in \mathbb{R}, i \in \mathbb{N}\}.$$

1.2 Functions

Before we define a function, we look at the product $A \times B$ of any two sets A and B, which is defined as the set of all ordered pairs that may be formed of the elements of the first set A, with the second set B :

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

Definition 1.3 (Ordered Pairs)**.** An ordered pair is $(a, b) = \{\{a\}, \{a, b\}\}$.

Definition 1.4 (Function)**.** A function f with domain A and range B, denoted by $f : A \to B$, is any $f \subset A \times B$ s.t. $\forall a \in A, \exists! b \in B$ with $(a, b) \in f$.

From the definition, b is uniquely determined by a and we may write $b = f(a)$.

The collection of all functions from a particular domain A to a certain B is denoted by

$$B^A = \{f \subset A \times B | f : A \to B\}.$$

1.3 Inverse Image

Definition 1.5 (Inverse Image)**.** For any function say $X : \Omega \to \mathcal{X}$, the inverse image of any $A \subset \mathcal{X}$ is defined as

$$X^{-1}(A) := \{\omega \in \Omega | X(\omega) \in A\}.$$

1.4 Indicator Functions and Indicator Map

Definition 1.6 (Indicator Function)**.** For any $A \subset \Omega$, we define $I_A \in \{0, 1\}^\Omega$ by

$$I_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \in A^C \end{cases}.$$

Indicator function defines a bijective correspondence between subsets of Ω and their indicator functions, that is referred to as the indicator map

$$\begin{aligned} I : \mathcal{P}(\Omega) &\xrightarrow{\cong} 2^{\Omega} \\ A &\mapsto I_A. \end{aligned}$$

1.5 Series

Recall that when $|a| < 1$,

$$\sum_{i=0}^{\infty} a^i := \lim_{n\to\infty} \sum_{i=0}^{n} a^i = \lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \frac{1}{1-a}.$$

2 Random Variables

Definition 2.1 (Finite Discrete Uniform Distribution)**.** $U \sim \text{unif}(\Omega)$ with ${}^{\#}\Omega < {}^{\#}\mathbb{N}$ iff

$$P(U = \omega) = \frac{1}{\Omega} \Leftrightarrow P(U \in A) = \frac{{}^{\#}A}{{}^{\#}\Omega}.$$

Example 2.1. $U \sim \text{unif}\{1, \cdots, n\}$ iff $P(U = i) = \frac{1}{n}, i = 1, \cdots, n$.

Note. $-U \sim \text{unif}\{-n, \cdots, -1\}$ and $n + 1 - U \sim \text{unif}\{1, \cdots, n\}$. Hence we say $n + 1 - U \overset{\text{d}}{=} U$ and thus

$$n + 1 - \mathbb{E}[U] = \mathbb{E}[U] \Rightarrow \mathbb{E}[U] = \frac{n+1}{2} = \frac{1 + \cdots + n}{n}.$$

Definition 2.2 (Uniform Distribution)**.** $U \sim \text{unif}[0, 1]$ iff

$$P(U \leqslant u) = u, \forall 0 \leqslant u \leqslant 1.$$

2.1 Distribution Functions in General

Theorem 2.1 (Sequential Continuity)**.** $A_n \to A \Rightarrow P(A_n) \to P(A)$.

2.2 Fundamental Theorem of Applied Probability

For any $p \in \mathbb{N}$ with $p \geqslant 2$ we define the p-adic series

$$U = \sum_{i=1}^{\infty} p^{-i} Z_i.$$

Lemma 2.1. Let $p^\infty = \{\mathbf{x} | x_i \in p, i \in \mathbb{N}\}$, where $p = \{0, 1, \cdots, p-1\}$ and $\dot{p}^\infty = \{\mathbf{x} | x_i \in p, i \in \mathbb{N},$ but not allowed to end in $p - 1$ repeated$\}$. Then $u = \sum\limits_{i=1}^{\infty} p^{-i} z_i$ defines a bijective function $\Phi : \dot{p}^\infty \to [0, 1)$.

Note. The range cannot include 1, because it is not allowed to end in $p - 1$ repeated and

$$\sum_{i=1}^{\infty} p^{-i}(p-1) = \frac{p-1}{p} \sum_{i=0}^{\infty} p^{-i} = \frac{p-1}{p} = 1.$$

Theorem 2.2 (Fundamental Theorem of Applied Probability)**.** For $U = \sum_{i=1} p^{-i} Z_i, p \geqslant 2$, we have

$$U \sim \mathrm{unif}[0,1] \Leftrightarrow Z_i \overset{\text{i.i.d.}}{\sim} \mathrm{unif}\{0, 1, \cdots, p-1\}.$$