University of Waterloo
ECON301 Winter 2018
Review

Consumption bundle:

A consumption bundle is a vector $x = (x_1, x_2) \in \mathbb{R}^2_+$, i.e. $x_1, x_2 \geq 0$. Given prices p and income m, consumer's budget set is

$$B = \{(x_1, x_2) \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq m\}$$

Preference relations:

- Strictly prefer: $\succ$
- Weakly prefer: $\succeq$
- Indifference: $\sim$

The preferene relation $\succeq$ on $\mathbb{R}^2_+$ is **complete** if for all $x, y \in \mathbb{R}^2_+$, either $x \succeq y$ or $y \succeq x$ (or both), **transitive** if for all $x, y \in \mathbb{R}^2_+$ s.t. $x \succeq y \succeq z$, we have $x \succeq z$.

The preference relation $\succeq$ on $\mathbb{R}^2_+$ is:

1. Montone if for all $x, y \in \mathbb{R}^2_+$ s.t. $x_1 > y_1$ and $x_2 > y_2$, we have that $x \succ y$.
2. Convex if for all $x, y \in \mathbb{R}^2_+$ s.t. $x \sim y$ and for all $0 \leq \alpha \leq 1$, we have that $\alpha x + (1 - \alpha) y \succeq x$.

Utility:

Consumer's utility maximization problem (UMP):

$$\max_{x_1, x_2 \geq 0} u(x_1, x_2) \text{ s.t. } p_1 x_1 + p_2 x_2 \leq m$$

Solutions to are demand functions $x_1(p, m)$ and $x_2(p, m)$.

Method of Lagrange:

To be used if u is differentiable.

1. Define Lagrangean.

$$\mathcal{L}(x_1, x_2, \lambda) = u(x_1, x_2) + \lambda(m - p_1 x_1 - p_2 x_2)$$

2. If the solution $x^\star$ to (PE) is s.t. $x_1^*, x_2^* \neq 0$, then it must solve the system of first-order conditions:

$$\begin{aligned}\frac{\partial}{\partial x_i}\mathcal{L}(x_1^*, x_2^*, \lambda) =& \frac{\partial}{\partial x_i}u(x_1^*, x_2^*) - \lambda p_i = 0 \text{ (L}i\text{) for } i = 1, 2\\ \frac{\partial}{\partial \lambda}\mathcal{L}(x_1^*, x_2^*, \lambda) =& m - p_1 x_1^* - p_2 x_2^* = 0 \text{ (L}\lambda\text{)}\end{aligned}$$

   The restriction that $x_1^*, opxt \neq 0$ implies that solution $x^\star$ is iterior.

$$\frac{\frac{\partial}{\partial x_1}u(x_1^*, x_2^*)}{\frac{\partial}{\partial x_2}u(x_1^*, x_2^*)} = \frac{p_1}{p_2} \text{ (MRS)}$$

3. Suppose that solution to (PE) is s.t. $x_2^* = 0$. Then we must have that $x_1^* = \frac{m}{p_1}$.
   A necessary condition for $(\frac{m}{p_1}, 0)$ to be optimal is that increasing consumption of good 2 while staying on budget line cannot increase consumer's utility, i.e. slope of indifference curve $\geq$ slope of budget line.

$$\frac{\frac{\partial}{\partial x_1}u(\frac{m}{p_1}, 0)}{\frac{\partial}{\partial x_2}u(\frac{m}{p_1}, 0)} \geq \frac{p_1}{p_2}$$

4. Similarly, necessary condition for $(0, \frac{m}{p_2})$ to be optimal is:

$$\frac{\frac{\partial}{\partial x_1}u(0, \frac{m}{p_2})}{\frac{\partial}{\partial x_2}u(0, \frac{m}{p_2})} \geq \frac{p_2}{p_1}$$

5. FOC (L1)-(Lλ) are only necessary for $x^\star$ to be optimal, i.e. any solution $x^\star$ to (PE) must be a solution to (L1)-(Lλ) but some solutions to (L1)-(Lλ) are not solutions to (PE).
   Result: If the consumer's preferences are *monotone* and *convex*, then any solution to (L1)-(Lλ) must be a solution to (PE).

Endowments:

An endowment is a consumption bundle $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2_+$, where ω_i is the quantity of good $i = 1, 2$ that belong to consumer. Given prices p and endowment ω, budget set is:

$$B = \{(x_1, x_2) \in \mathbb{R}^2_+ : p_1 x_1 + p_2 x_2 \leq p_1 \omega_1 + p_2 \omega_2\}$$

Competitive Equilibrium:

A competitive equilibrium (x^{A*}, x^{B*}, p^*) consists of an allocation of goods $x^{J*} = (x_1^{J*}, x_2^{J*})$ for each consumer $J = A, B$ along with prices $p^* = p_1^*, p_2^*$ which satisfy:

1. Given prices $p^* = (p_1^*, p_2^*)$, the allocations x^{J*} for consumer $J = A, B$ is a solution to UMP.

$$\max_{x^{J*} \in \mathbb{R}_+^2} u^J(x_1^J, x_2^J) \text{ s.t. } p_1 x_1^J + p_2 x_2^J \leq p_1 \omega_1^J + p_2 \omega_2^J$$

2. For each good $i = 1, 2$, the aggregate allocations exhaust aggregate endowments:

$$x_i^{A*} + x_i^{B*} = \omega_i^A + \omega_i^B \text{ (MCi)}$$

Result: If consumers' preferences are monotone, and if (MC1) holds, then (MC2) also holds.

Welfare:

Allocations x^A and x^B Pareto dominate allocations y^A and y^B if $u^J(x_1^J, x_2^J) \geq u^J(y_1^J, y_2^J)$ for all $J = A, B$ with (at least) one inequality strict.
Pareto-efficient allocations x^A and x^B are not Pareto dominated by any feasible allocations y^A and y^B.
Result: Allocations consistent with bargaining between consumers are Pareto efficient allocations x^A and x^B s.t. $u^J(x_1^J, x_2^J) \geq u^J(\omega_1^J, \omega_2^J)$ for all $J = A, B$.

Finding Pareto-efficient allocations:

1. Start with allocations y^A and y^B.

2. Find optimal allocations x^A and x^B for consumer A s.t. consumer B is indifferent between x^B and y^B.

$$\max_{\{0 \leq x_i^A \leq \omega_i^A + \omega_i^B\}_{i=1,2}} u^A(x_1^A, x_2^A) \text{ s.t. } u^B(\omega_1^A + \omega_1^B - x_1^A, \omega_2^A + \omega_2^B - x_2^A) = u^B(y_1^B, y_2^B)$$

Result: If both consumers' preferences are monotone and convex, then solutions to FOC are Pareto-efficient allocations.

$$\frac{\frac{\partial}{\partial x_1^A} u^A(x_1^{A*}, x_2^{A*})}{\frac{\partial}{\partial x_2^A} u^A(x_1^{A*}, x_2^{A*})} = \frac{\frac{\partial}{\partial x_1^B} u^B(x_1^{B*}, x_2^{B*})}{\frac{\partial}{\partial x_2^B} u^B(x_1^{B*}, x_2^{B*})}$$

Allocations y^A and y^B are arbitrary, can find more Pareto-efficient allocations by considering different initial allocations. The set of all Pareto-efficient allocations is the Pareto set, or contract curve.

First Welfare Theorem: Suppose that consumers' preferences are monotone and that price p^* and allocations x^{A*} and x^{B*} form a competitive equilibrium. Then x^{A*} and x^{B*} are Pareto-efficient.

- Competitive equilibrium must exhaust gains from trade.
- Competitive equilibrium allocations reproduce the outcomes of some bargaining protocal.
- In bargaining, computing outcomes requires a lot of information about consumers' preferences and aggregate endowments.
- Markets only require consumers to know their own preferences and endowments.
- Prices aggregate economy-wide information.

No-envy: Allocations x^{A*} and x^{B*} have no-envy if:

$$\begin{aligned} u^A(x_1^A, x_2^A) \geq & u^A(x_1^B, x_2^B) \text{ and} \\ u^B(x_1^B, x_2^B) \geq & u^B(x_1^A, x_2^A) \end{aligned}$$

Fairness: Allocations x^A and x^B are fair if they are Pareto-efficient and satisfy no-envy.

Second Welfare Theorem: If consumers' preferences are monotone and convex, then for any Pareto-efficient allocations x^A and x^B, there exist endowments ω^A and ω^B along with a competitive equilibrium that generates x^A and x^B.

- Competitive equilibria take no stand on final distribution of goods.
- SWT leaves room for governement intervention that is consistent with Pareto-efficiency (e.g. taxation).

Externalities:

- An advantage of competitive markets is their decentralization.
- A problem can arise if consumption choices of some consumers impact well-being of other consumers: these are called consumption externalities.
- A positive externality is present if some consumers benefit from the consumption of a good by other consumers.
- A negative externality is the reverse.

Example:
$\omega^A = (2,1), \omega^B = (1,1), u^A(x_1^A, x_2^A) = x_1^{A\frac{1}{2}} x_2^{A\frac{1}{2}}, u^B(x_1^B, x_2^A) = x_1^{B\frac{1}{2}}(2 - x_2^A)^{\frac{1}{2}}$.
Consumption of good 2 by consumer A imposes a negative externality on consumer B.

Markets only for good 1 and good 2:
Demand functions:

$$(x_1^A(p), x_2^A(p)) = (\frac{2p_1 + p_2}{2p_1}, \frac{2p_1 + p_2}{2p_2})$$
$$(x_1^B(p), x_2^B(p)) = (\frac{p_1 + p_2}{p_1}, 0)$$

Prices $p^* = (1, \frac{2}{3})$ and allocations $x^{A*} = (\frac{4}{3}, 2), x^{B*} = (\frac{5}{3}, 0)$ form a comptetitive equilibrium.

By calculating slope of indifference curve, these allocations are **not** Pareto-efficient.
In equilibrium, consumer A consumes "too much" of good 2 relative to its impact on consumer B. Consumer B is willing to exchange good 1 against reduction in consumption of good 2, but no market for this trade exists.

Establish a market for property right:
Competitive equilibrium:
A competitive equilibrium consists of prices $p^* = (p_1^*, p_2^*, p_R^*)$ and allocations $x^{A*} = (x_1^{A*}, x_2^{A*}, x_R^{A*})$ and $x^{B*} = (x_1^{B*}, x_2^{B*}, x_R^{B*})$ which satisfy:

1. Given price p^*, allocation x^{A*} solves:

$$\max_{x_1^A, x_2^A, x_R^A \geq 0} x_1^{A\frac{1}{2}} x_2^{A\frac{1}{2}} \text{ s.t. } p_1^* x_1^A + p_2^* x_2^A + p_R^* x_R^A \leq 2p_1^* + p_2^* + p_R^* \omega_R^A,\ x_2^A \leq x_R^A$$

   Allocation x^{B*} satisfies

$$\max_{x_1^B, x_2^B, x_R^B \geq 0} x_1^{B\frac{1}{2}} x_R^{B\frac{1}{2}} \text{ s.t. } p_1^* x_1^B + p_2^* x_2^B + p_R^* x_R^B \leq p_1^* + p_2^* + p_R^* \omega_R^B,\ x_2^B \leq x_R^B$$

2. Allocations x^{A*} and x^{B*} clear all markets:

$$x_i^{A*} + x_i^{B*} = \omega_i^A + \omega_i^B \text{ for all } i = 1, 2, R$$

Simplifying UMP:
In any equilibrium, $p_R^* > 0$. If $p_R^* = 0$, consumer B's demand for rights is undefined. By the same argument, $p_1^* > 0$.
In any equilibrium, $p_2^* = 0$. If $p_2^* > 0$, then we must have $x_2^{B*} = 0$. By (MC2), $x_2^{A*} = 2$. Since $x_2^{A*} \leq x_R^{A*}$, we have $x_R^{A*} \geq 2$. By (MCR), $x_R^{A*} = 2$. Because $p_1^*, p_2^* > 0$, $x_R^{B*} = 0$ is never optimal.

In any equilibrium, $x_2^{A*} = x_R^{A*}$. $x_2^{A*} < x_R^{A*}$ cannot be optimal because u^A is increasing in x_2^A and $p_2^* = 0$.
Rewrite consumers' UMP:

$$\max_{x_1^J, x_R^J \geq 0} x_1^{J\frac{1}{2}} x_R^{J\frac{1}{2}} \text{ s.t. } p_1^* x_1^J + p_R^* x_R^J \leq p_1^* \omega^J + p_R^* \omega_R^J$$

Prices $p^* = (1, 0, \frac{3}{2})$ and allocations $x^{A*} = (1 + \frac{3}{4}\omega_R^A, \frac{2}{3} + \frac{1}{2}\omega_R^A, \frac{2}{3} + \frac{1}{2}\omega_R^A), x^{B*} = (\frac{1}{2} + \frac{3}{4}\omega_R^B, \frac{1}{3} + \frac{1}{2}\omega_R^B, \frac{1}{3} + \frac{1}{2}\omega_R^B)$ form a comptetitive equilibrium.

By calculating slope of indifference curve, these allocations are Pareto-efficient.
Externalities create missing markets problem, but if property rights are established over externality, then welfare theorems apply.

Government intervention through permit system:
Government expropriates all endowments of good 2 in the economy and establish a permit system: to consume one unit of good 2, consumers need to pay cost $c > 0$ for the permit. Assume that revenue is returned to consumers in equal shares. Then there is a competitive market for good 1 that determines its equilibrium price.
Given permit price c, a competitive equilibrium is price p_1^*, allocatioons x^{A*} and x^{B*} and per-capital tax return T^* that satisfy:

1. Given p_1^* and c, x^{A*} is a solution to

$$\max_{x_1^A, x_2^A \geq 0} x_1^{A\frac{1}{2}} x_2^{A\frac{1}{2}} \text{ s.t. } p_1^* x_1^A + c x_2^A \leq 2p_1^* + T^*$$

   x^{B*} is a solution to

$$\max_{x_1^B, x_2^B \geq 0} x_1^{B\frac{1}{2}} (2 - x_2^A)^{\frac{1}{2}} \text{ s.t. } p_1^* x_1^B + c x_2^B \leq p_1^* + T^*$$

2. $x_1^{A*} + x_1^{B*} = 3$ (MC1)

3. Government's budget is balanced.
   $2T^* = c(x_2^{A*} + x_2^{B*})$ (BB)

Demand functions:

$$(x_1^A(p_1, c, T), x_2^A(p_1, c, T)) = \left(\frac{2p_1 + T}{2p_1}, \frac{2p_1 + T}{2c}\right)$$
$$(x_1^B(p_1, c, T), x_2^B(p_1, c, T)) = \left(\frac{p_1 + T}{p_1}, 0\right)$$

Given c, price $p_1^* = 1$, tax return $T^* = \frac{2}{3}$ and allocations $x^{A*} = (\frac{4}{3}, \frac{4}{3c})$, $x^{B*} = (\frac{5}{3}, 0)$ form a competitive equilibrium.
These allocations are Pareto-efficient only for a certain value of c ($c = \frac{3}{2}$).