Mathematical Induction

Part Two

The **principle of mathematical induction** states that if for some property P(*n*), we have that

P(0) is true

and

For any natural number *n*, P(*n*) → P(*n* + 1)

Then

For any natural number *n*, P(*n*) is true.

One Major Catch

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0	1	2	3	4	5	6	7	8

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In an inductive proof, to prove P(5), we can only assume P(4). We cannot rely on any of our earlier results!

Strong Induction

The **principle of strong induction** states that if for some property P(n), we have that

P(0) is true

and

For any natural number n,
if P(n') is true for all n' ≤ n, then P(n + 1) is true

then

For any natural number n, P(n) is true.

The **principle of strong induction** states that if for some property P(n), we have that

Assume that P(n) holds for n and all smaller n.

P(0) is true

and

For any natural number n,
if P(n') is true for all n' ≤ n, then P(n + 1) is true

then

For any natural number n, P(n) is true.

Using Strong Induction

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Using Strong Induction

0 1 2 3 4 5 6 7 8

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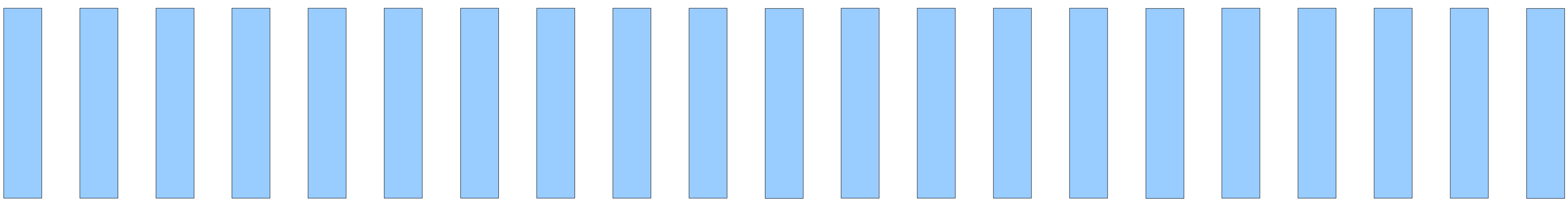
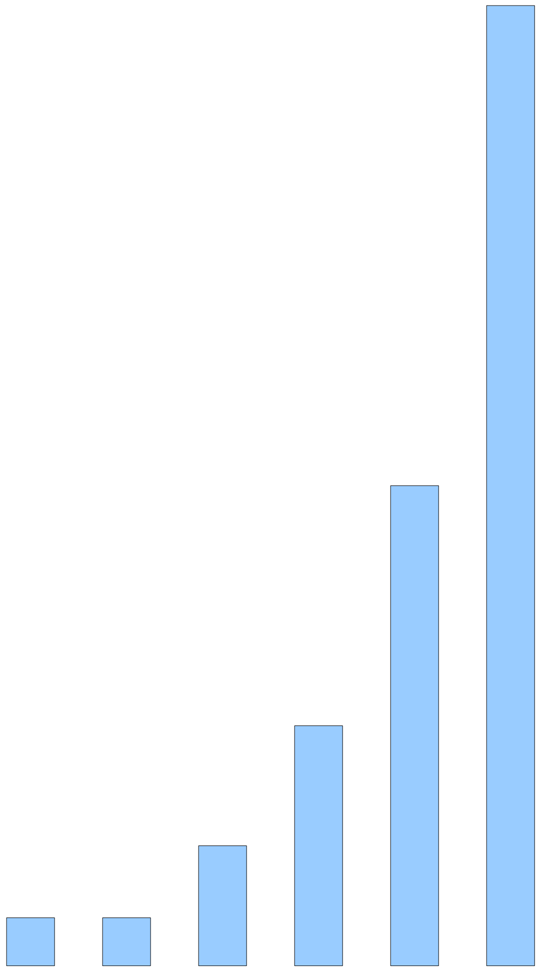
0	1	2	3	4	5	6	7	8

Induction and Dominoes



Strong Induction and Dominoes



Weak and Strong Induction

- **Weak induction** (regular induction) is good for showing that some property holds by incrementally adding in one new piece.
- **Strong induction** is good for showing that some property holds by breaking a large structure down into multiple small pieces.

Proof by Strong Induction

- State that you are attempting to prove something by strong induction.
- State what your choice of P(n) is.
- Prove the base case:
  - State what P(0) is, then prove it.
- Prove the inductive step:
  - State that you assume for all $0 \leq n' \leq n$, that P(n') is true.
  - State what P(n + 1) is. *(this is what you're trying to prove)*
  - Go prove P(n + 1).

Application: Binary Numbers

Binary Numbers

- The **binary number system** is base 2.
- Every number is represented as 1s and 0s encoding various powers of two.
- Examples:
  - $100_2 = 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 = 4$
  - $11011_2 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 27$
- Enormously useful in computing; almost all computers do computation on binary numbers.
- Question: How do we know that every natural number can be written in binary?

Justifying Binary Numbers

- To justify the binary representation, we will prove the following result:

Every natural number *n* can be expressed as the sum of distinct powers of two.

- This says that there's *at least* one way to write a number in binary; we'd need a separate proof to show that there's *exactly* one way to do it.
- So how do we prove this?

One Proof Idea

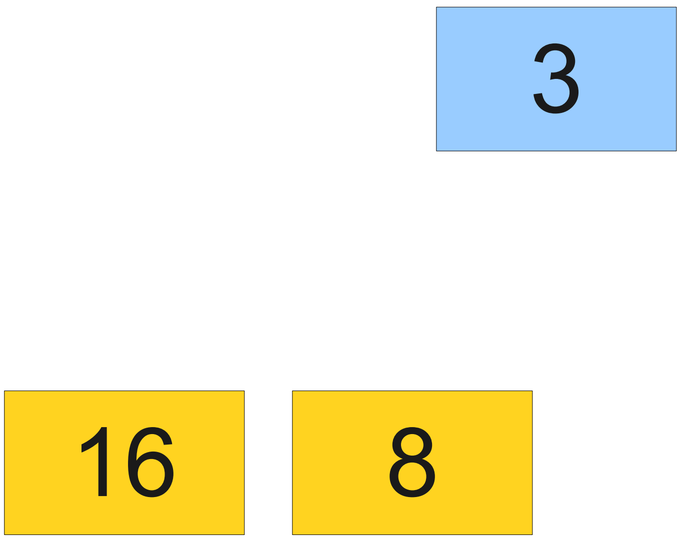
27

One Proof Idea

11

16

One Proof Idea



One Proof Idea

1

16 8 2

One Proof Idea

0

16 8 2 1

General Idea

- Repeatedly subtract out the largest power of two less than the number.
- Can't subtract 2^n twice for any *n*; otherwise, you could have subtracted 2^{n+1}.
- Eventually, we reach 0; the number is then the sum of the powers of two that we subtracted.
- How do we formalize this as a proof?

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Notice the stronger version of the induction hypothesis. We're now showing that P(n) is true for n and all smaller natural numbers. We're going to use this fact later on.

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Here's the key step of the proof. If we can show that

$$n + 1 - 2^k \leq n$$

then we can use the inductive hypothesis to claim that $n + 1 - 2^k$ is a sum of distinct powers of two.

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This step would fail in a normal inductive proof because we're talking about some number no greater than n, not necessarily n itself. Strong induction is extremely useful in cases like this.

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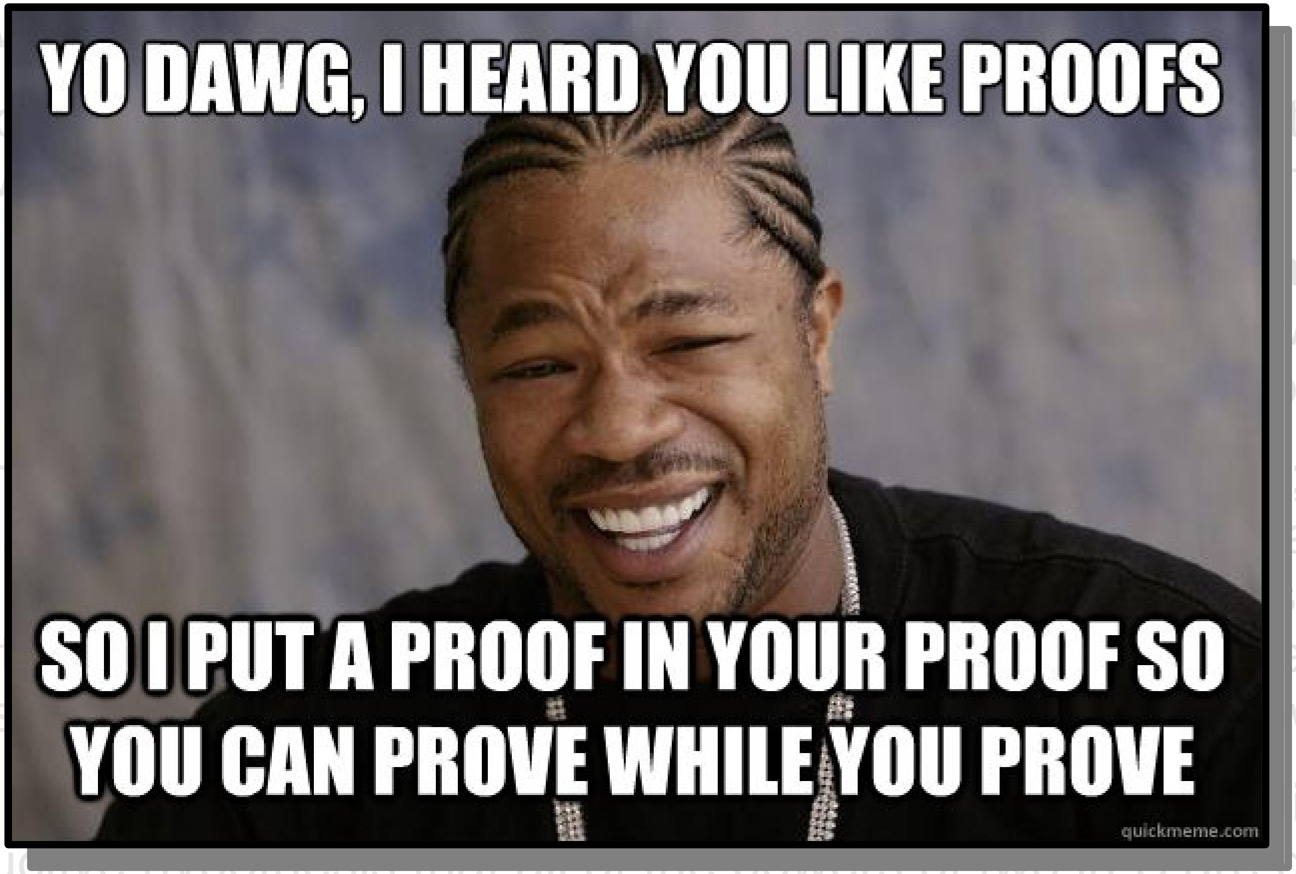

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We now need to show that these powers of two are all distinct. We know by the inductive hypothesis that all of the powers of two in S are distinct, so the only way that a power of two would be repeated would be if $2^k \in S$. We show that this is impossible by contradiction; assume that $2^k \in S$. Since $2^k \in S$ and the sum of the powers of two in S is $n + 1 – 2^k$, this means that $2^k \leq n + 1 – 2^k$. This means that $2(2^k) \leq n + 1$, so $2^{k+1} \leq n + 1$, contradicting the fact that 2^k is the largest power of two less than or equal to $n + 1$.

Theorem: Every natural number can be written as the sum of distinct powers of two.

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Reformulating Strong Induction

The **principle of strong induction** states that if for some property P(n), we have that

P(0) is true

and

For any natural number n,
if P(n') is true for all n' ≤ n, then P(n + 1) is true

then

For any natural number n, P(n) is true.

The **principle of strong induction** states that if for some property P(n), we have that

Assume that P(n) holds for n and all smaller n.

P(0) is true

and

For any natural number n,
if P(n') is true for all n' ≤ n, then P(n + 1) is true

then

For any natural number n, P(n) is true.

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and

For any natural number n,
if P(n') is true for all n' < n, then P(n) is true

then

For any natural number n, P(n) is true.

Application: **Continued Fractions**

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\boxed{1 + \cfrac{1}{2}}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\cfrac{3}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\cfrac{3}{2}}}$$

Continued Fractions

$$\frac{1}{4 + \frac{2}{3}}$$

Continued Fractions

$$\frac{1}{\boxed{4 + \frac{2}{3}}}$$

Continued Fractions

$$\frac{1}{\dfrac{14}{3}}$$

Continued Fractions

$$\cfrac{1}{\cfrac{14}{3}}$$

Continued Fractions

$$\frac{3}{14}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\boxed{4 + \cfrac{1}{2}}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{2}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\boxed{1 + \cfrac{2}{9}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$\frac{137}{42}$$

Continued Fractions

- A **continued fraction** is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\dots}}}}$$

- More formally, a continued fraction is either
  - An integer *n*, or
  - *n* + 1 / *F*, where *n* is an integer and *F* is a continued fraction.
- Continued fractions have numerous applications in number theory and computer science.
- (They're also really fun to write!)

Fun with Continued Fractions

- Every rational number (including negative numbers) has a continued fraction.
- Harder result: every *irrational* number has an (infinite) continued fraction.
- If we truncate an infinite continued fraction for an irrational number, we can get progressively better approximations of that number.

Pi as a Continued Fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}}}}}}$$

Approximating Pi

Approximating Pi

$\pi = 3$

3 = **3**.0000...

Approximating Pi

$\pi = 3$

3 = **3**.0000...

And he made the Sea of cast bronze, ten cubits from one brim to the other; it was completely round. [... A] line of thirty cubits measured its circumference.

1 Kings 7:23, New King James Translation

Approximating Pi

$$\pi = 3 + \frac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Approximating Pi

$$\pi = 3 + \cfrac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Archimedes knew of this approximation, circa 250 BCE

Approximating Pi

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

Approximating Pi

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.1415929**2...

Approximating Pi

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

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355/113 = **3.1415929**2...

Chinese mathematician Zu Chongzhi discovered this approximation in the early fifth century; this was the best approximation of pi for over a thousand years

Approximating Pi

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

3 = **3**.0000...

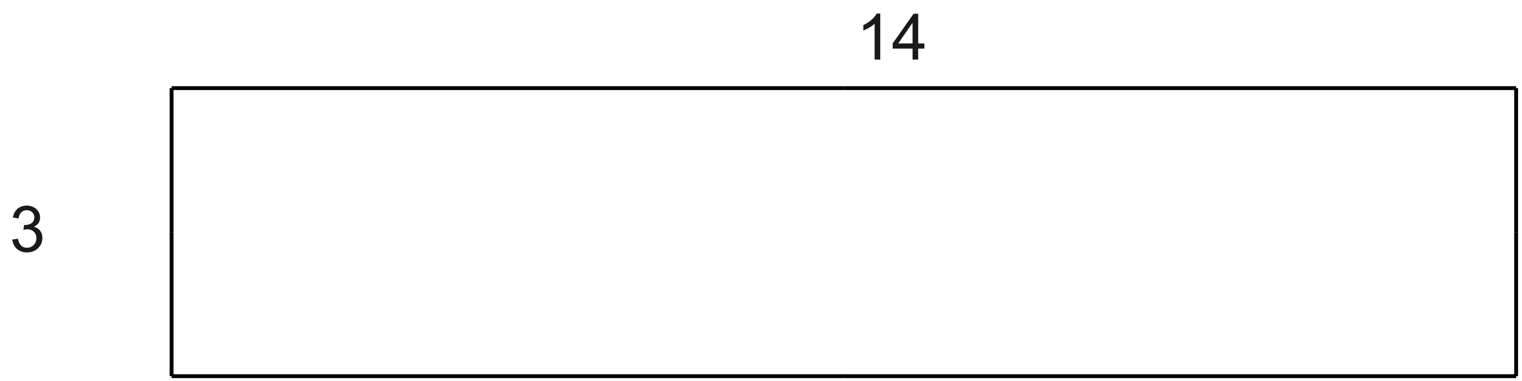
22/7 = **3.14**2857...

336/106 = **3.1415**094...

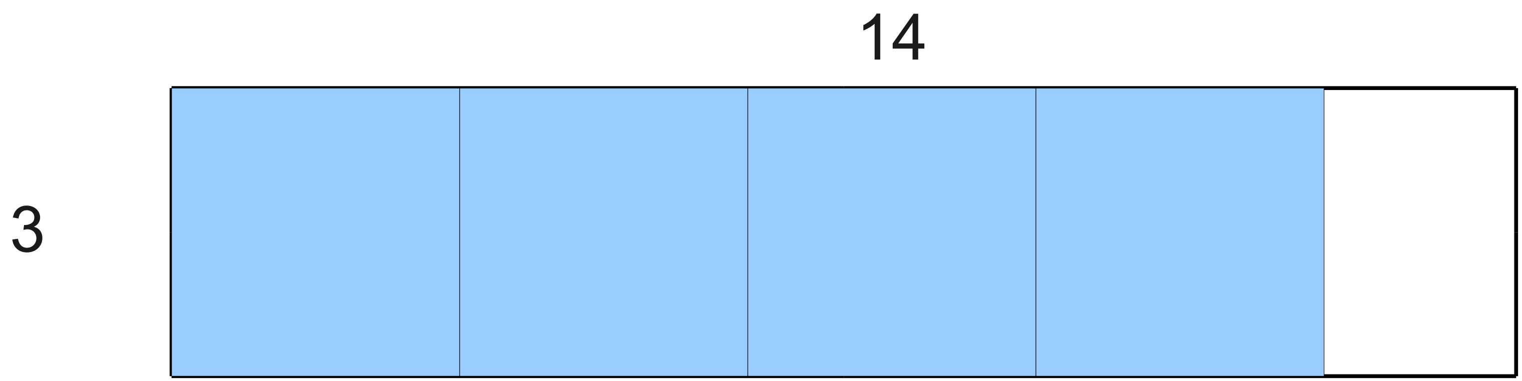
355/113 = **3.1415929**2...

103993/33102 = **3.141592653**0...

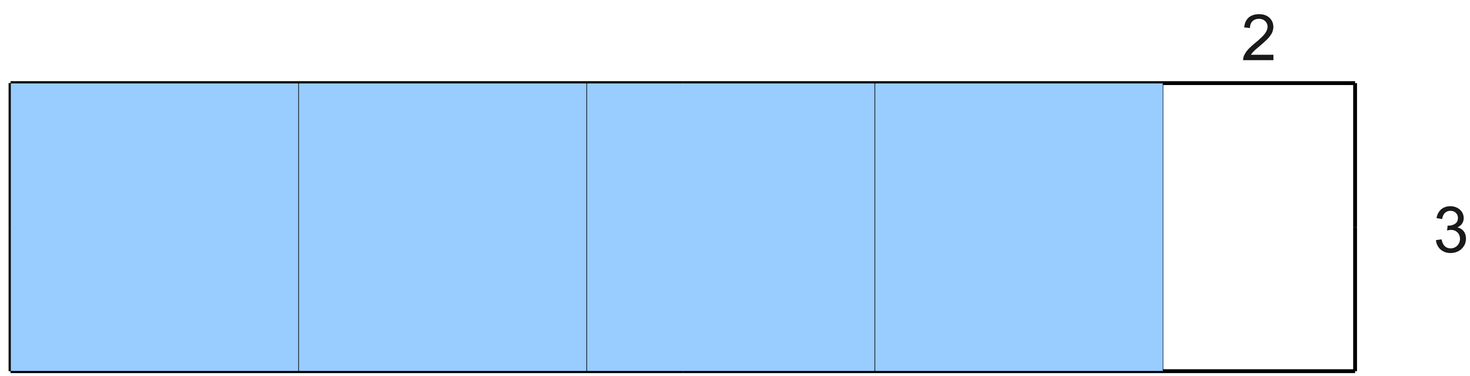
More Continued Fractions



More Continued Fractions



More Continued Fractions



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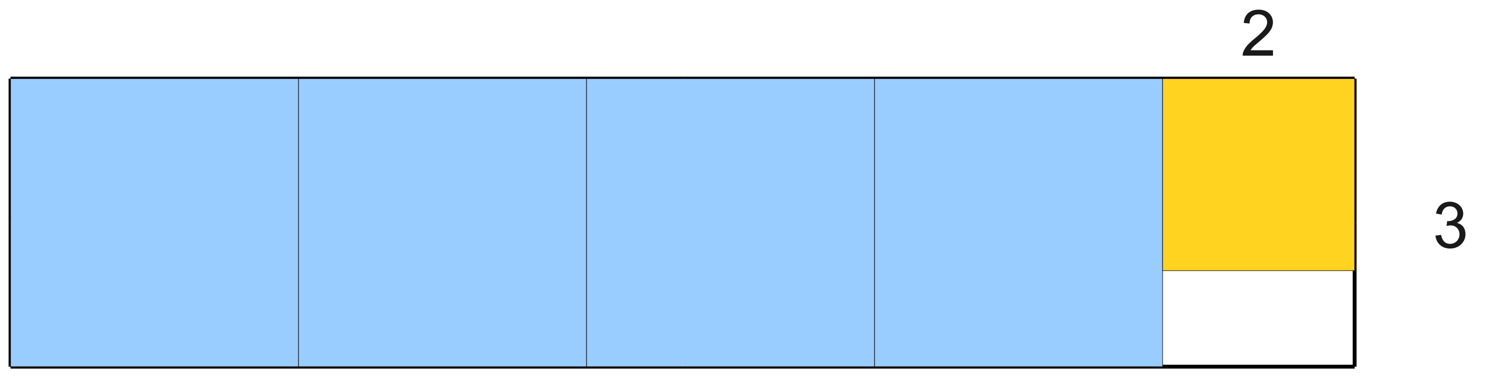

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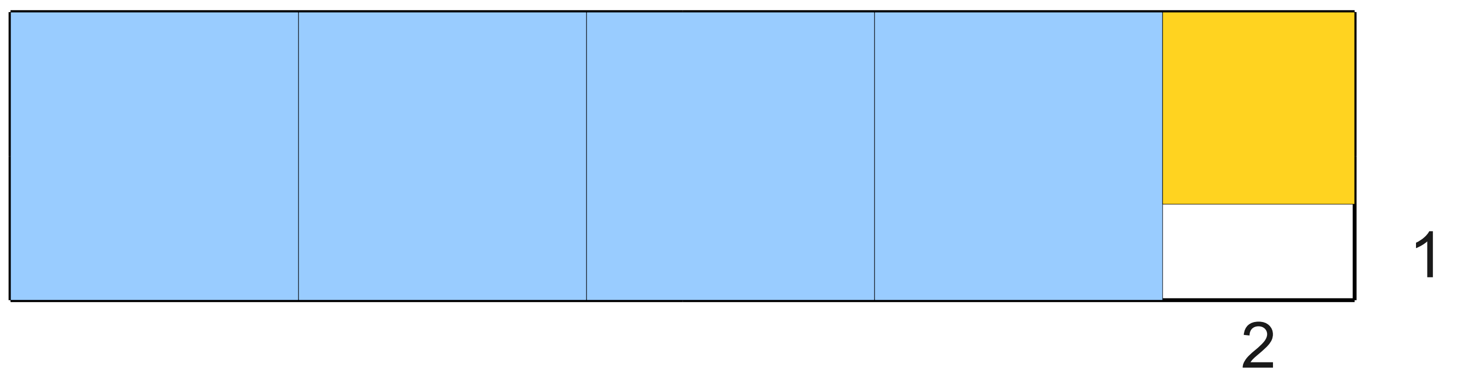

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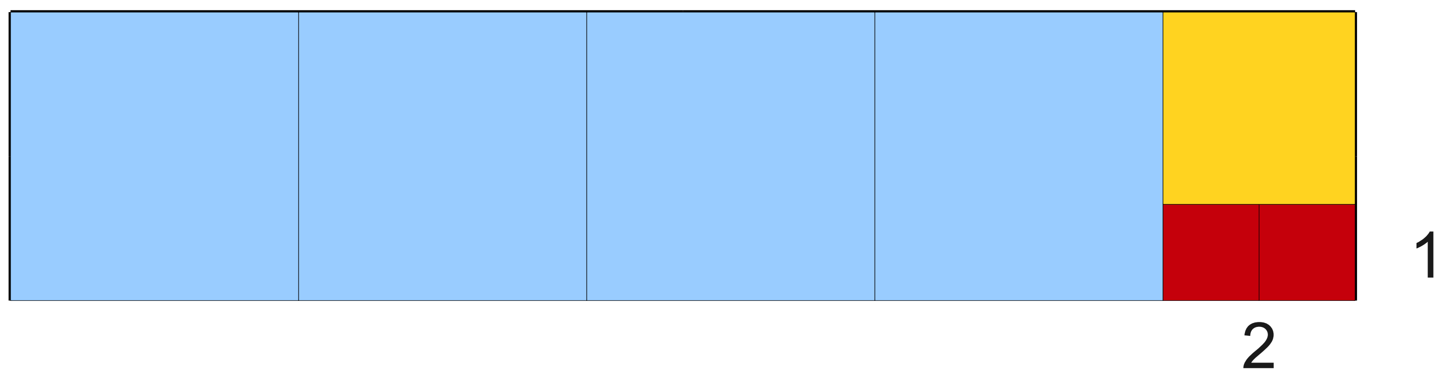

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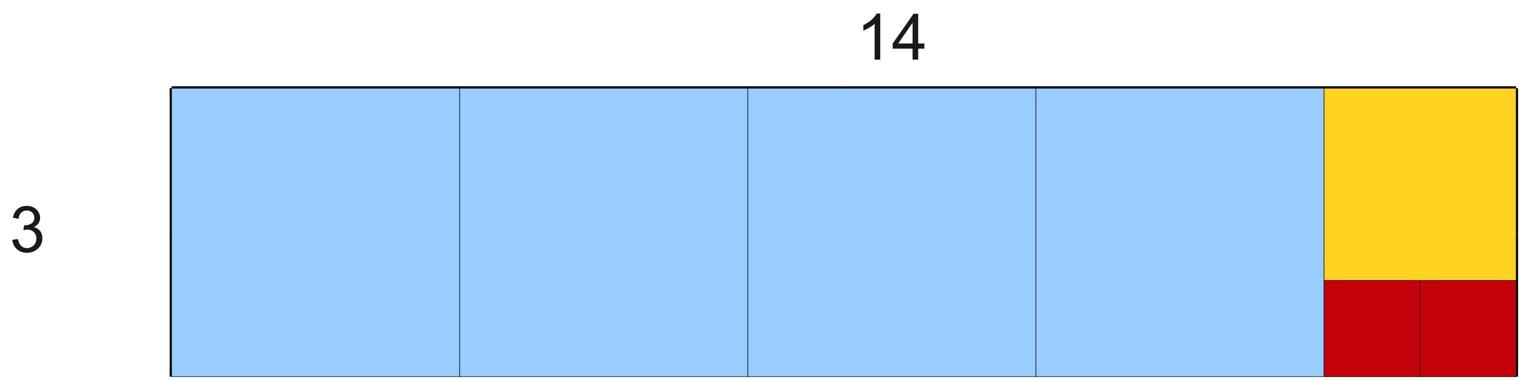
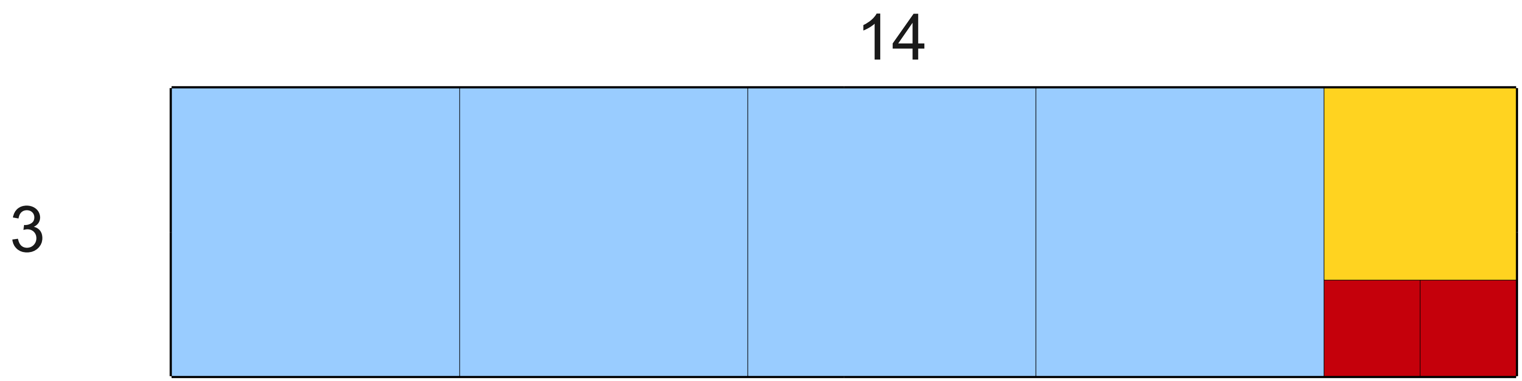
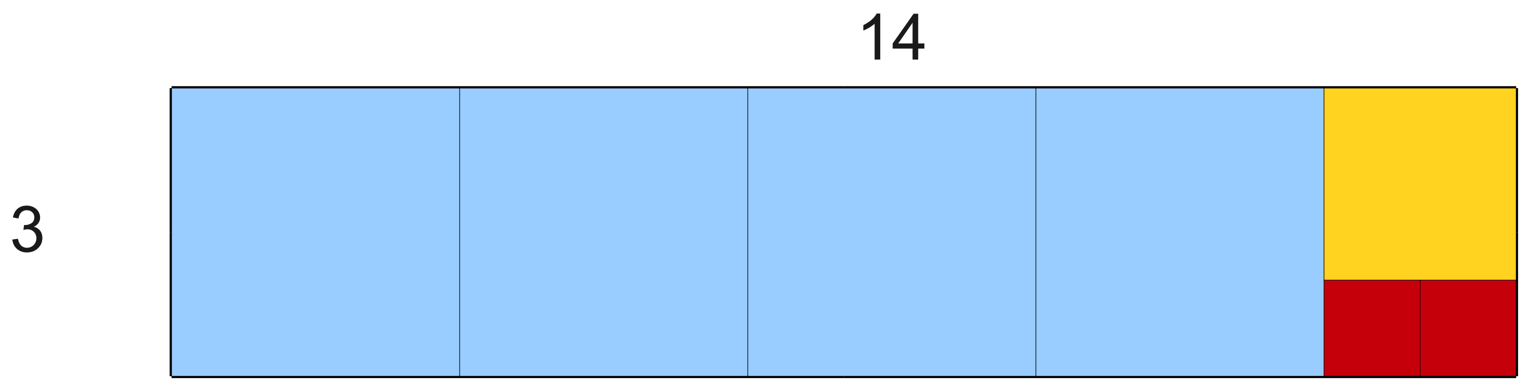


More Continued Fractions



$$\frac{3}{14} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

More Continued Fractions



$$\frac{3}{14} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}}}}}$$

An Interesting Continued Fraction

$x = 1$

1 / 1

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 \cdot}$$

1 / 1

2 / 1

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1}}$$

1 / 1

2 / 1

3 / 2

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

21 / 13

An Interesting Continued Fraction

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1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

21 / 13

34 / 21

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

21 / 13

34 / 21

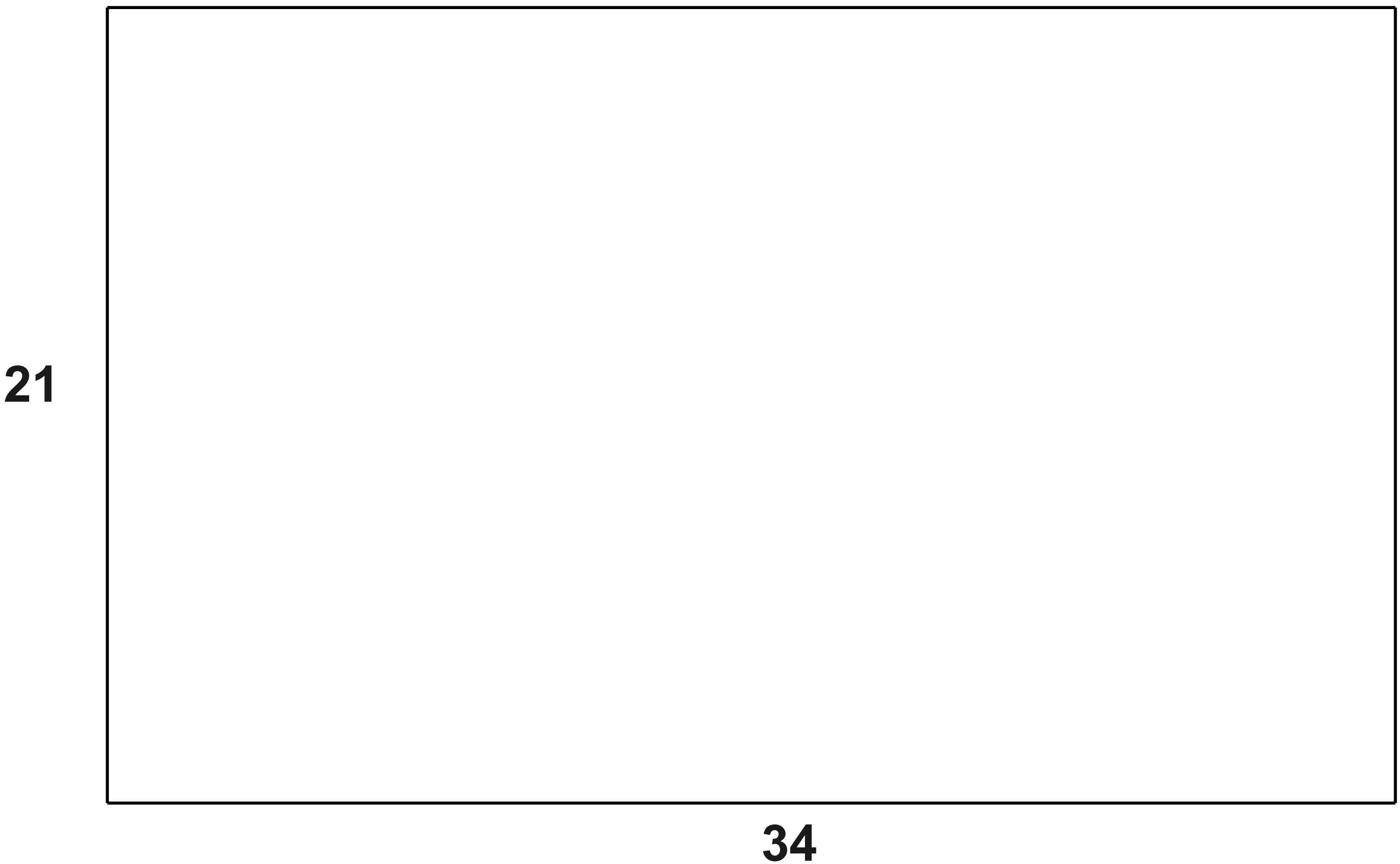
Each fraction is the ratio of consecutive Fibonacci numbers!

The Golden Ratio

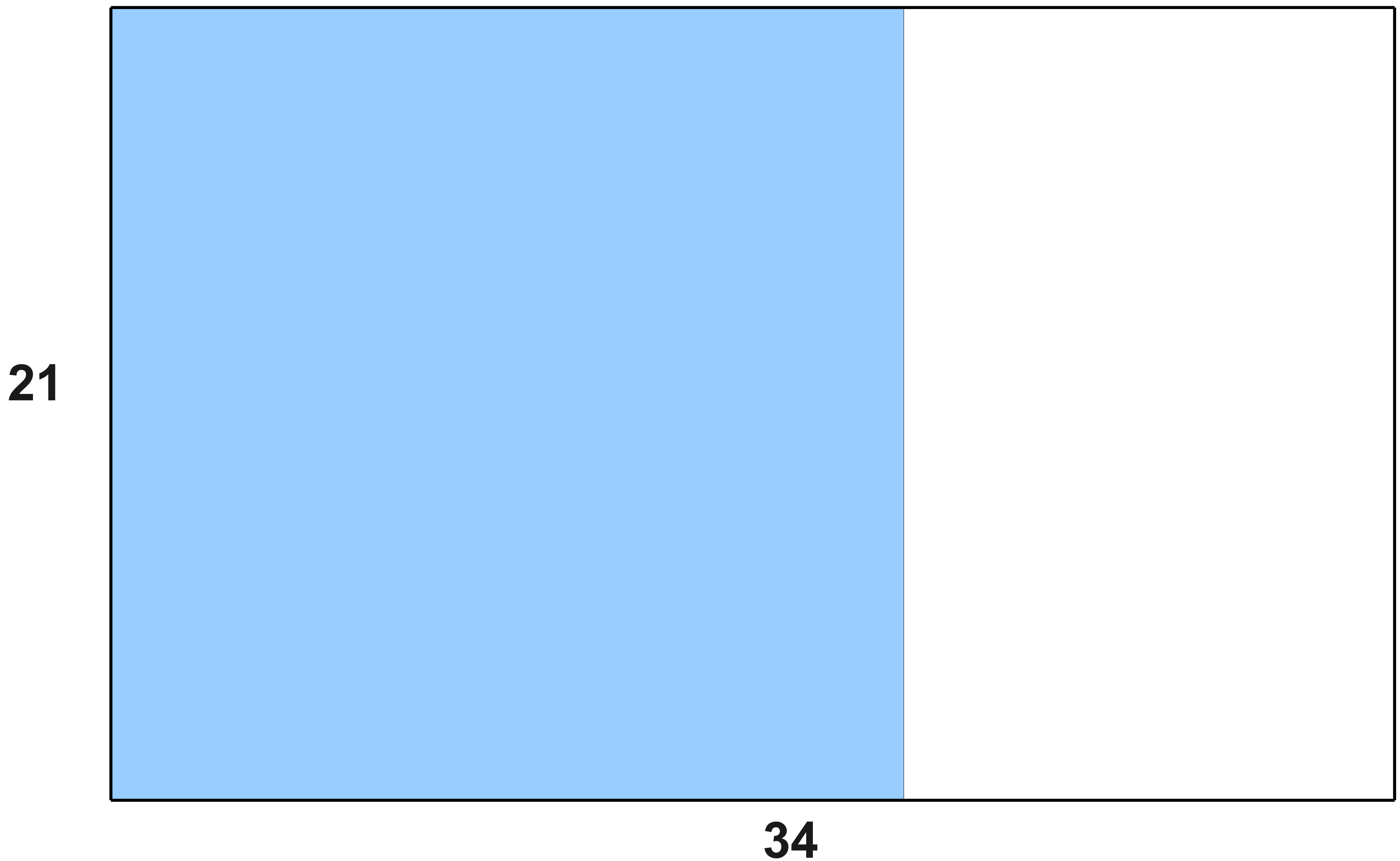
$$\phi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

$$\phi \approx 1.61803399$$

The Golden Ratio



The Golden Ratio



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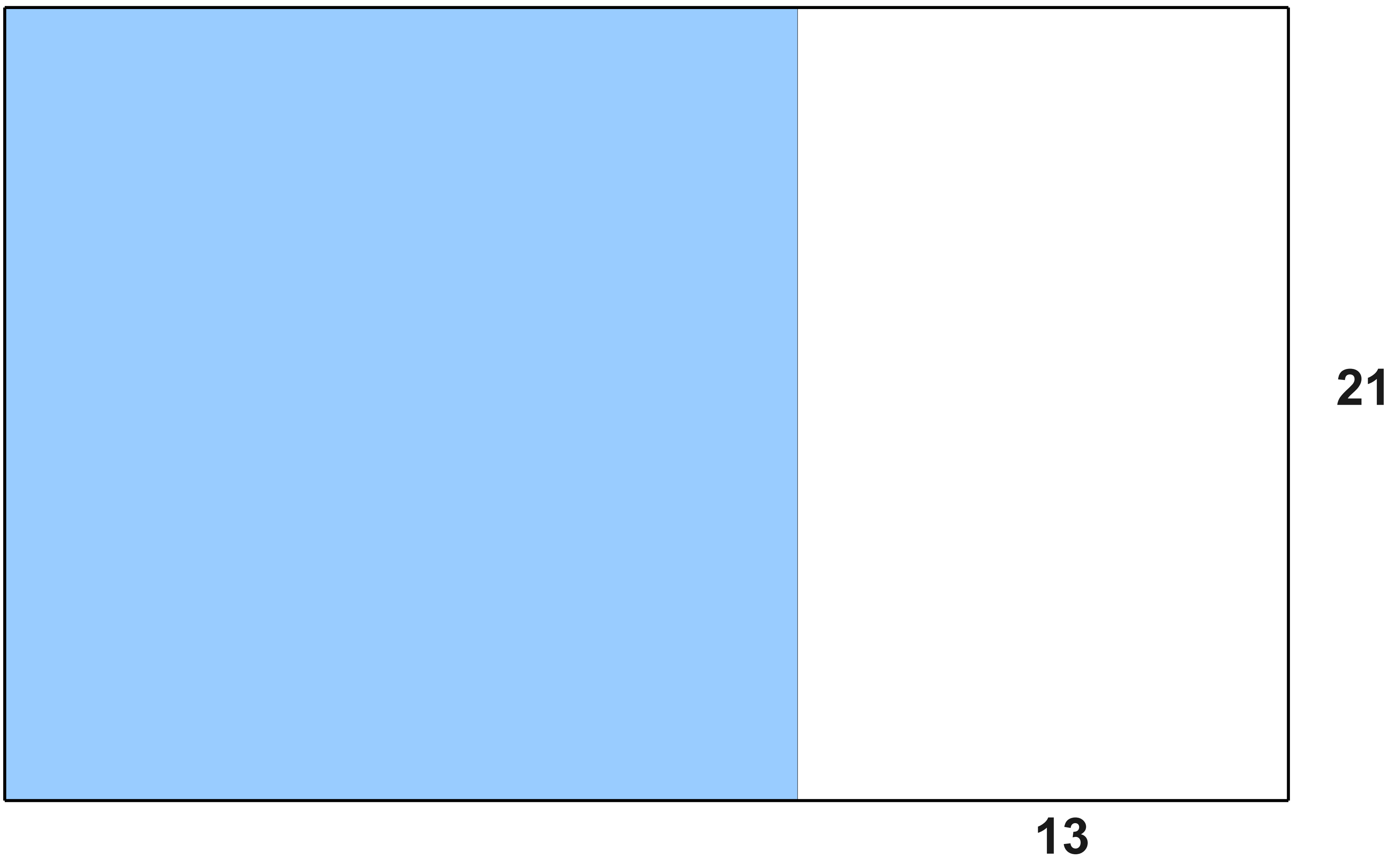

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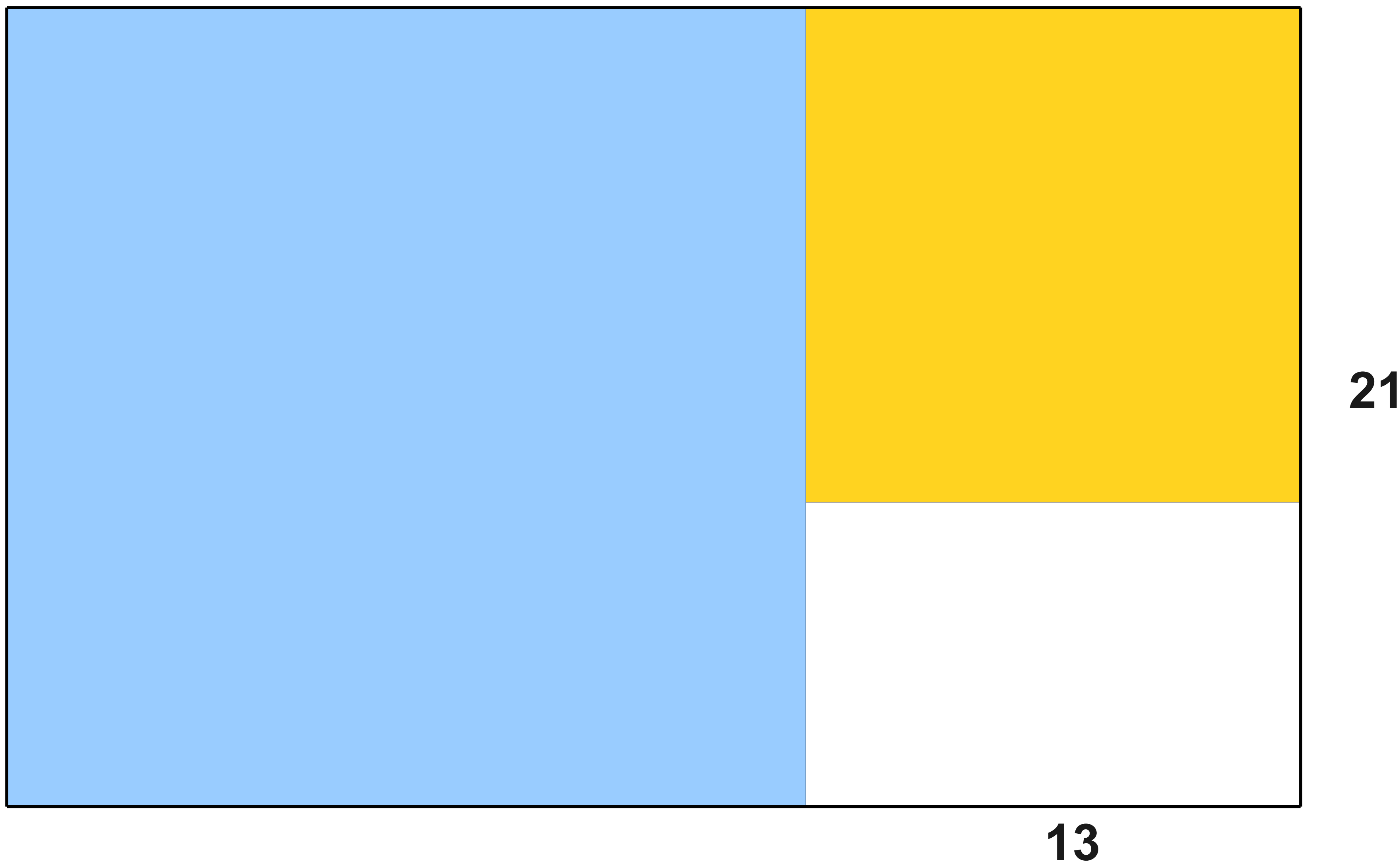

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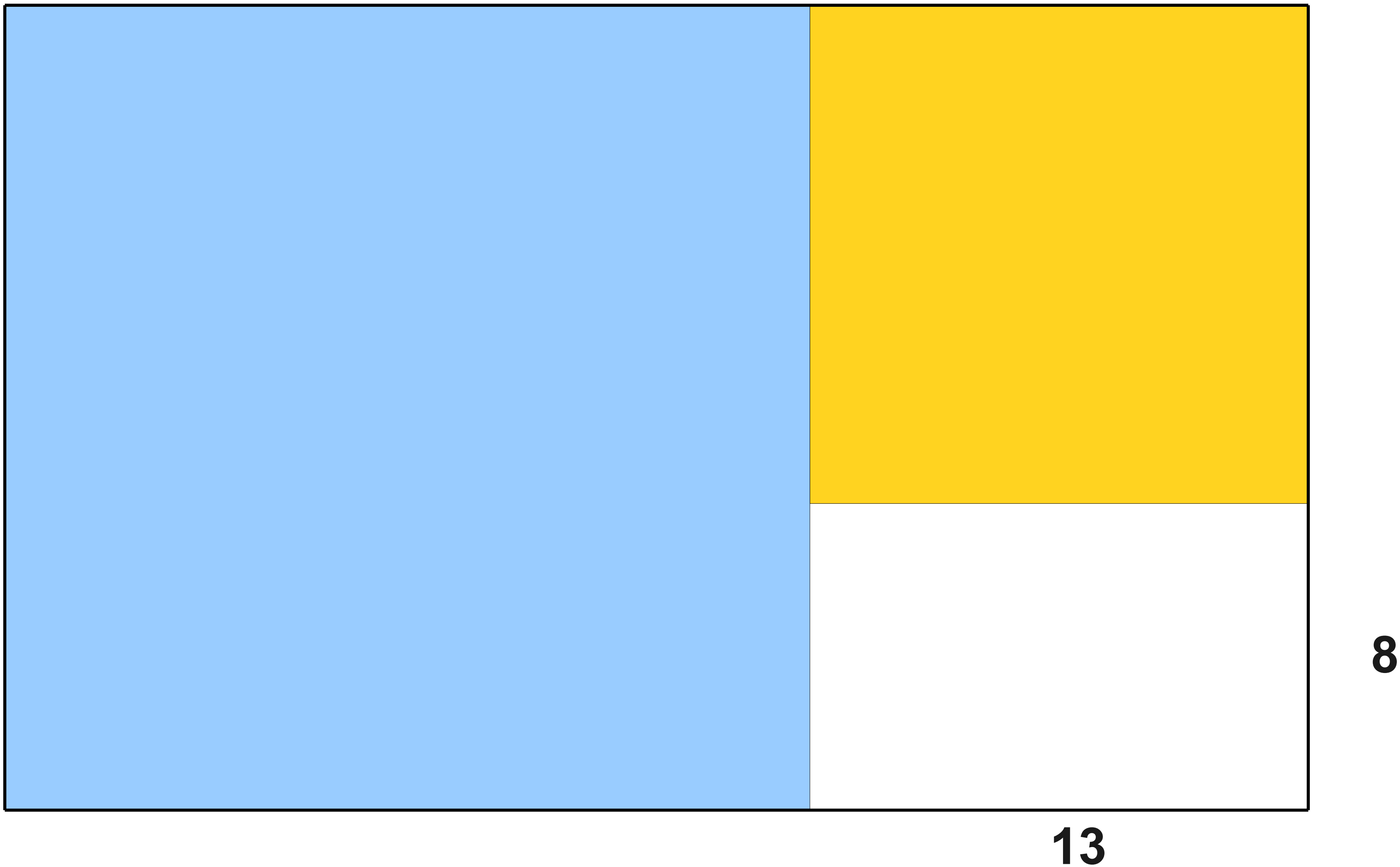

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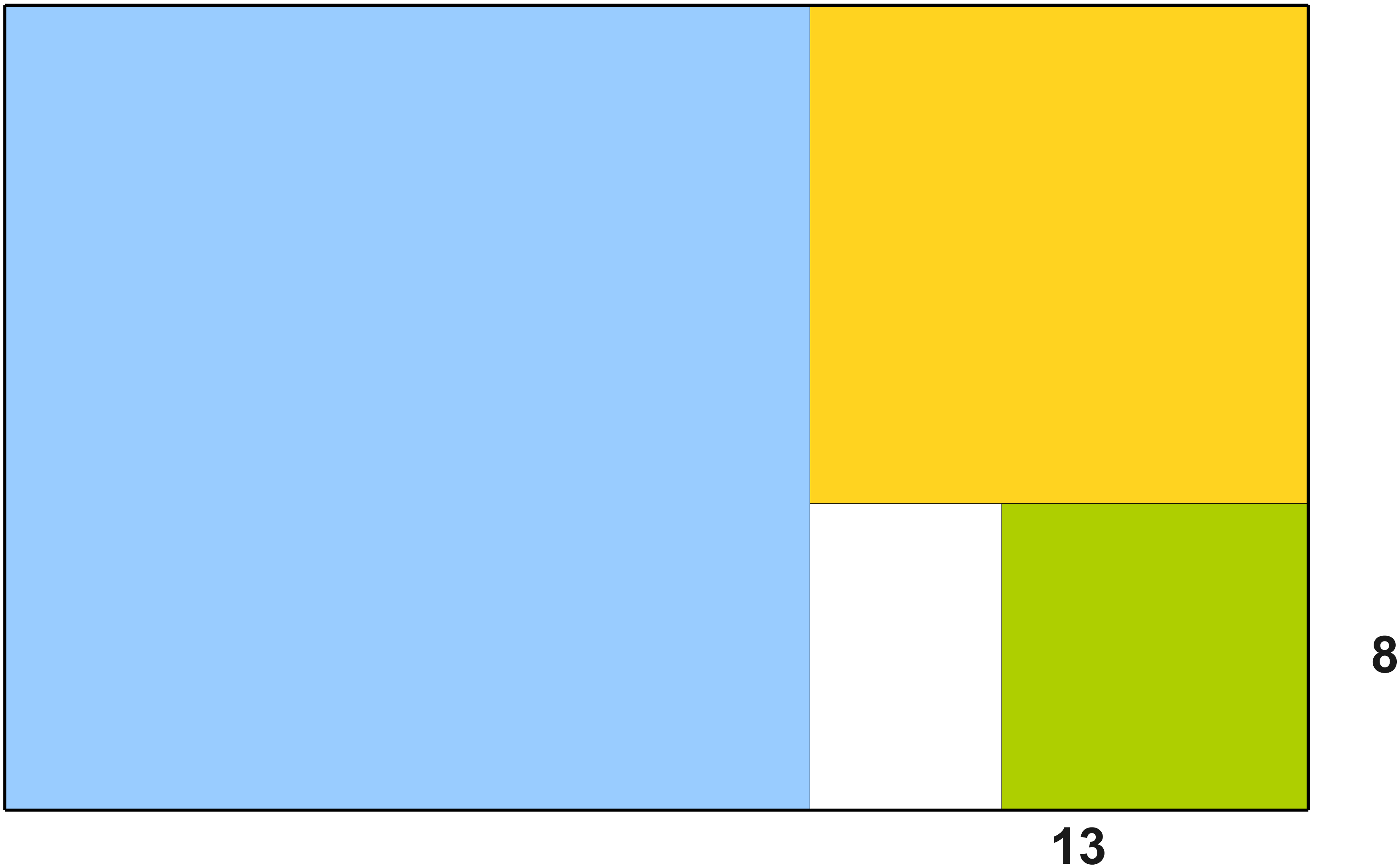

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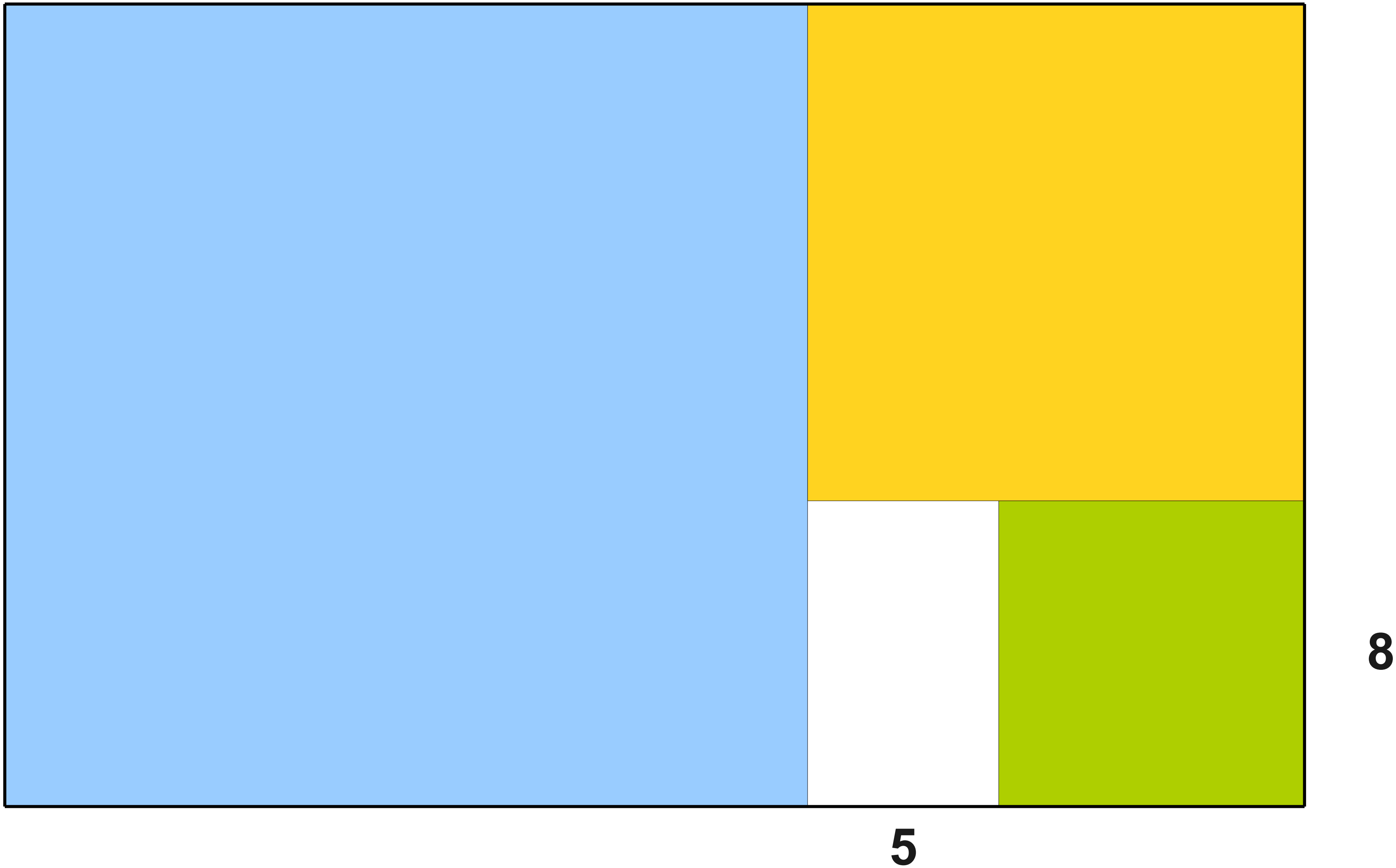

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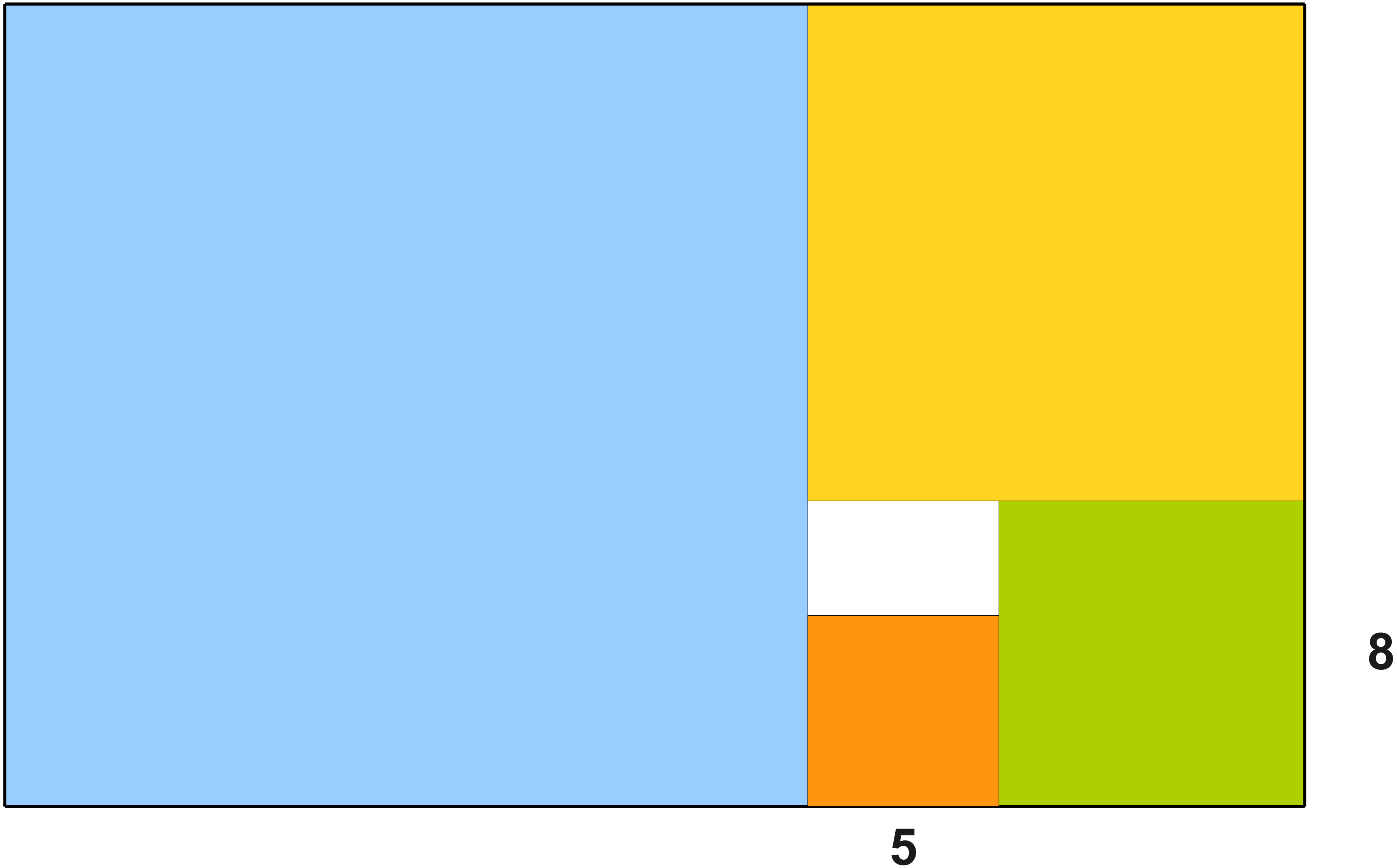

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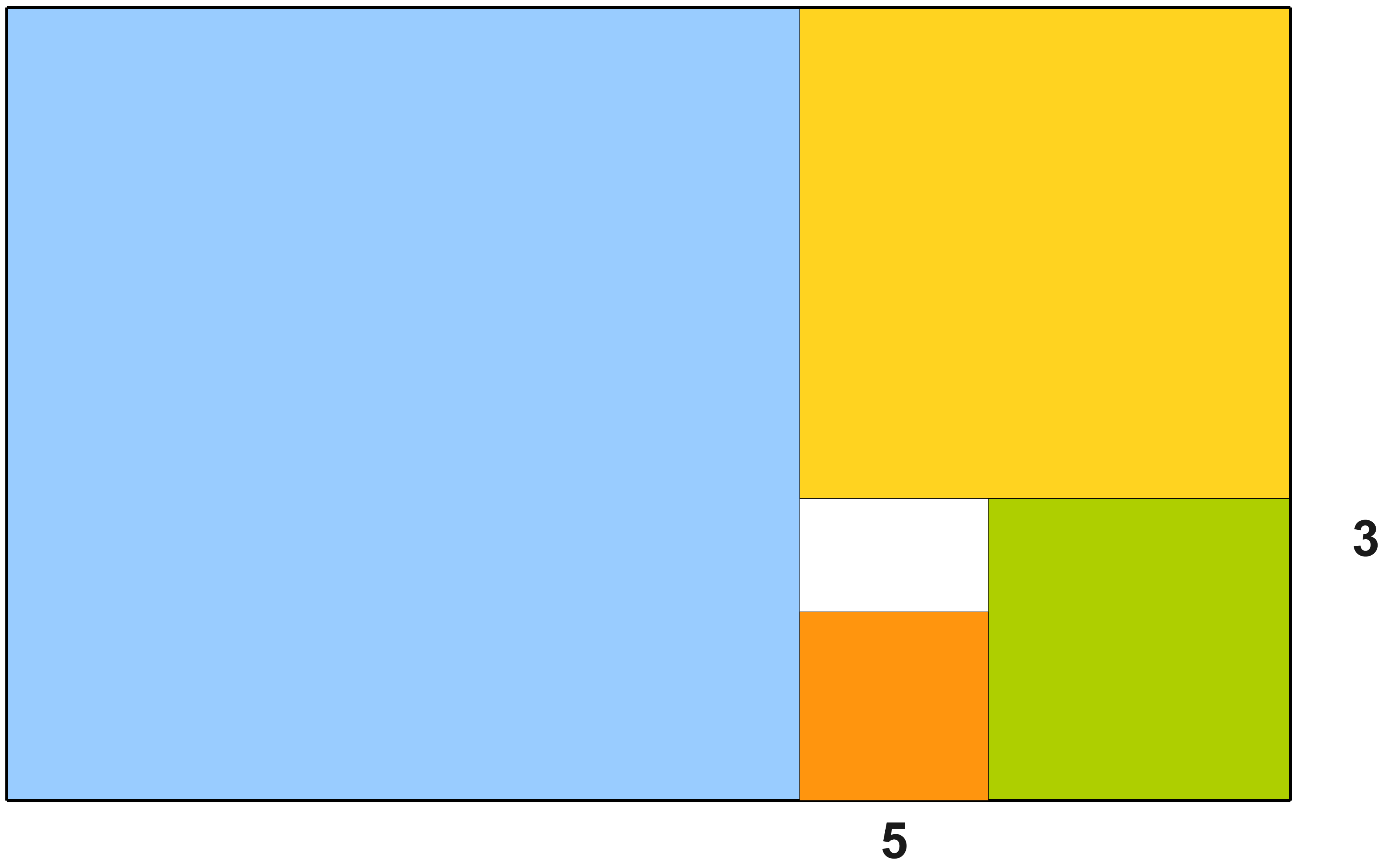

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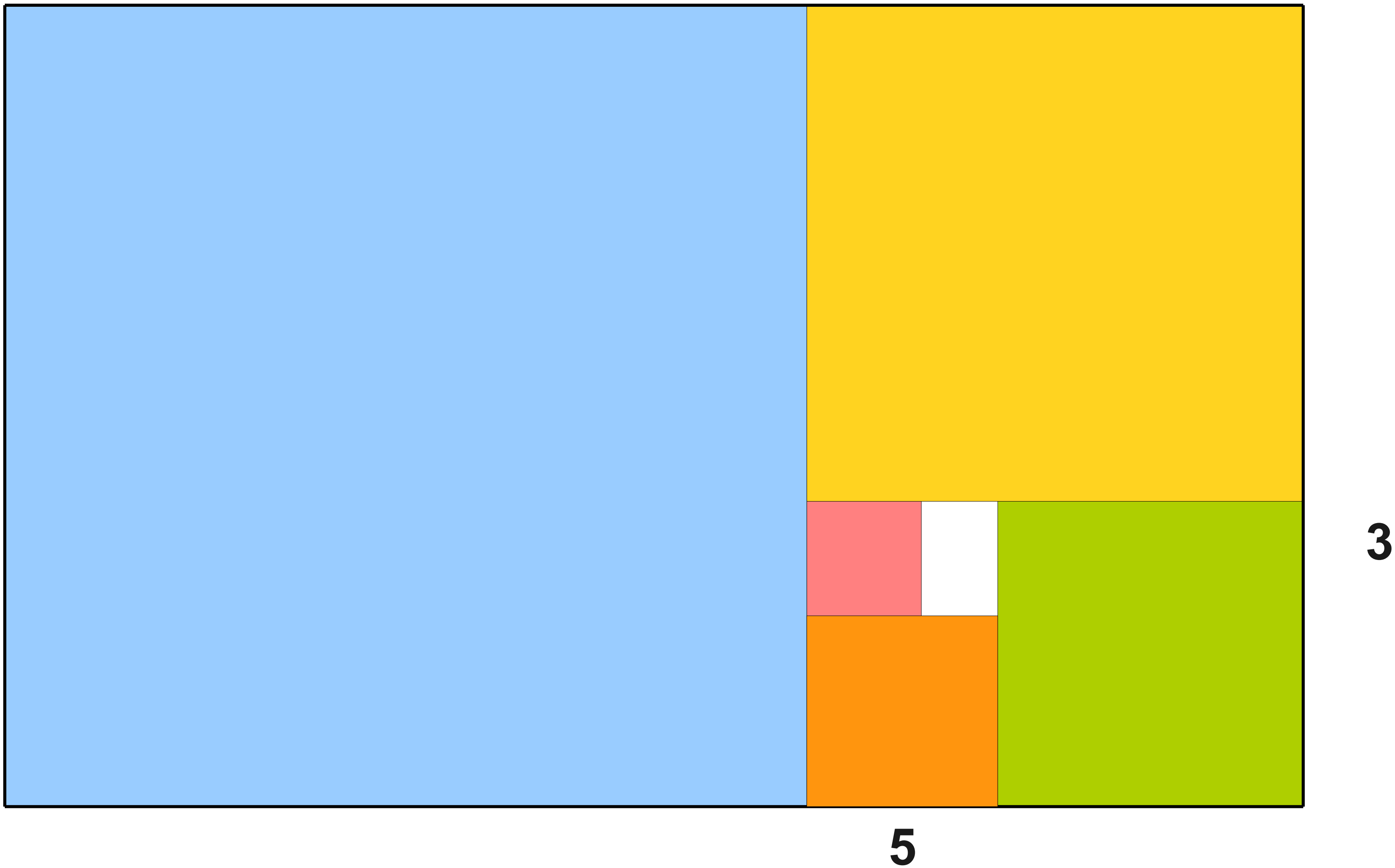

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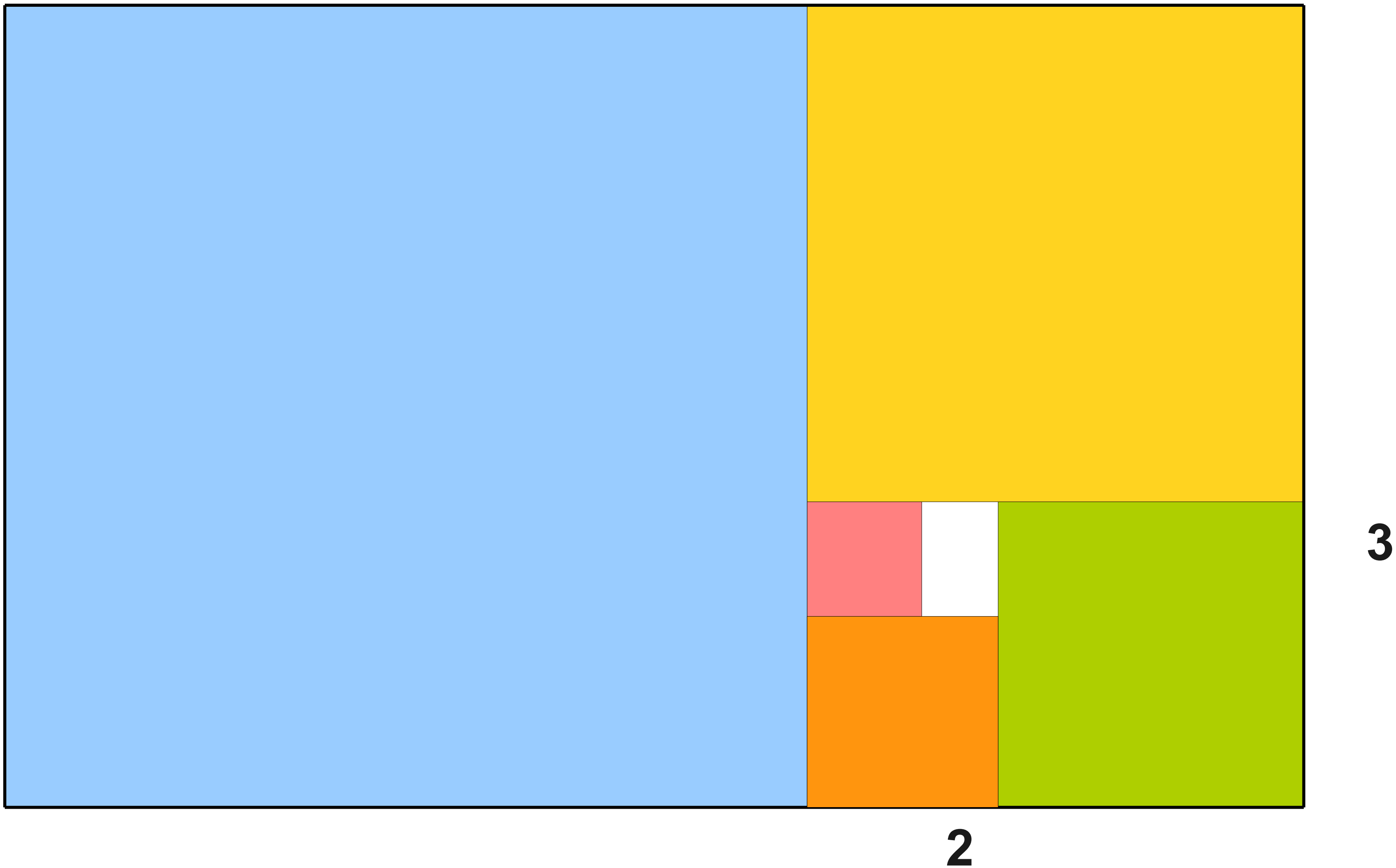

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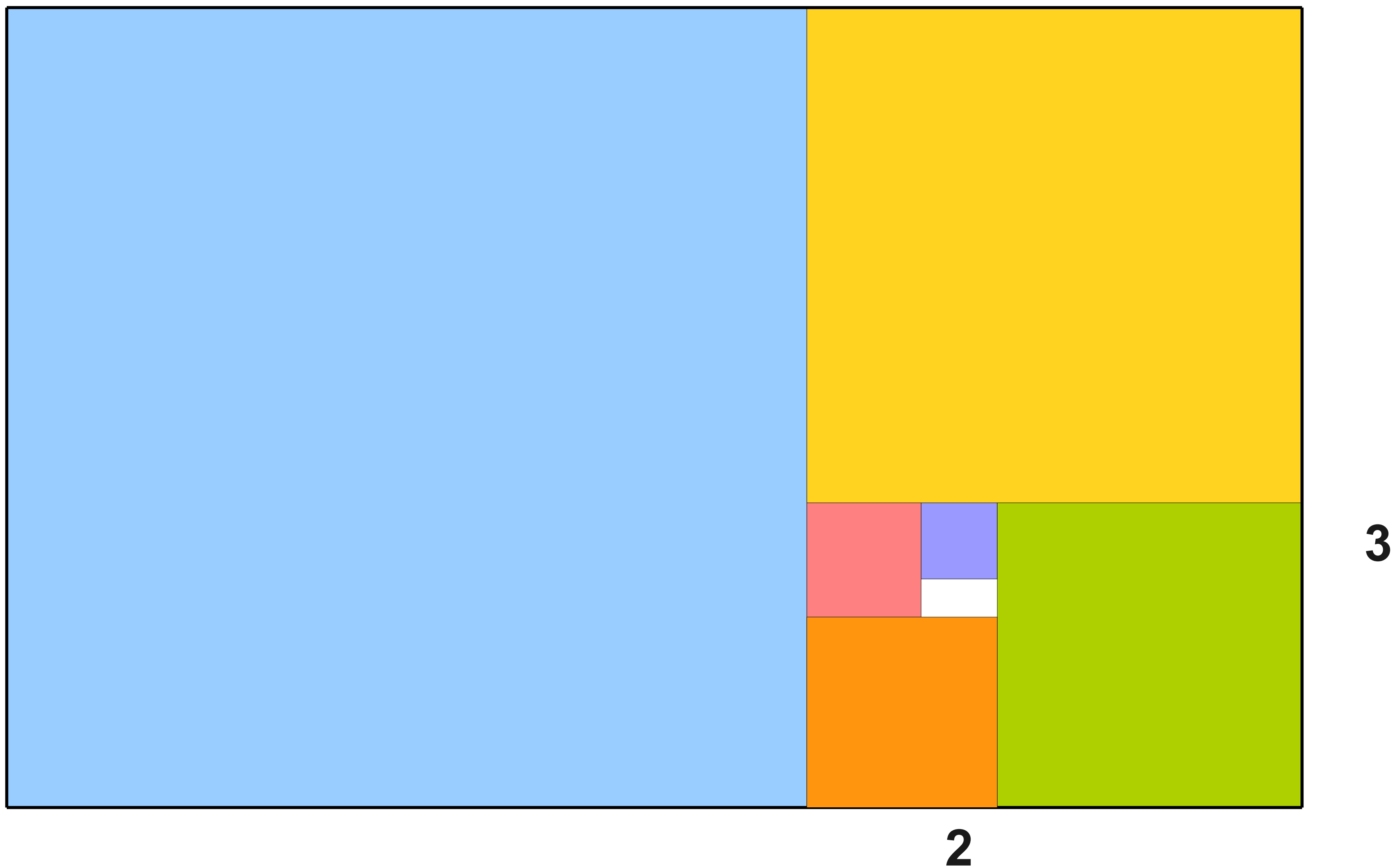

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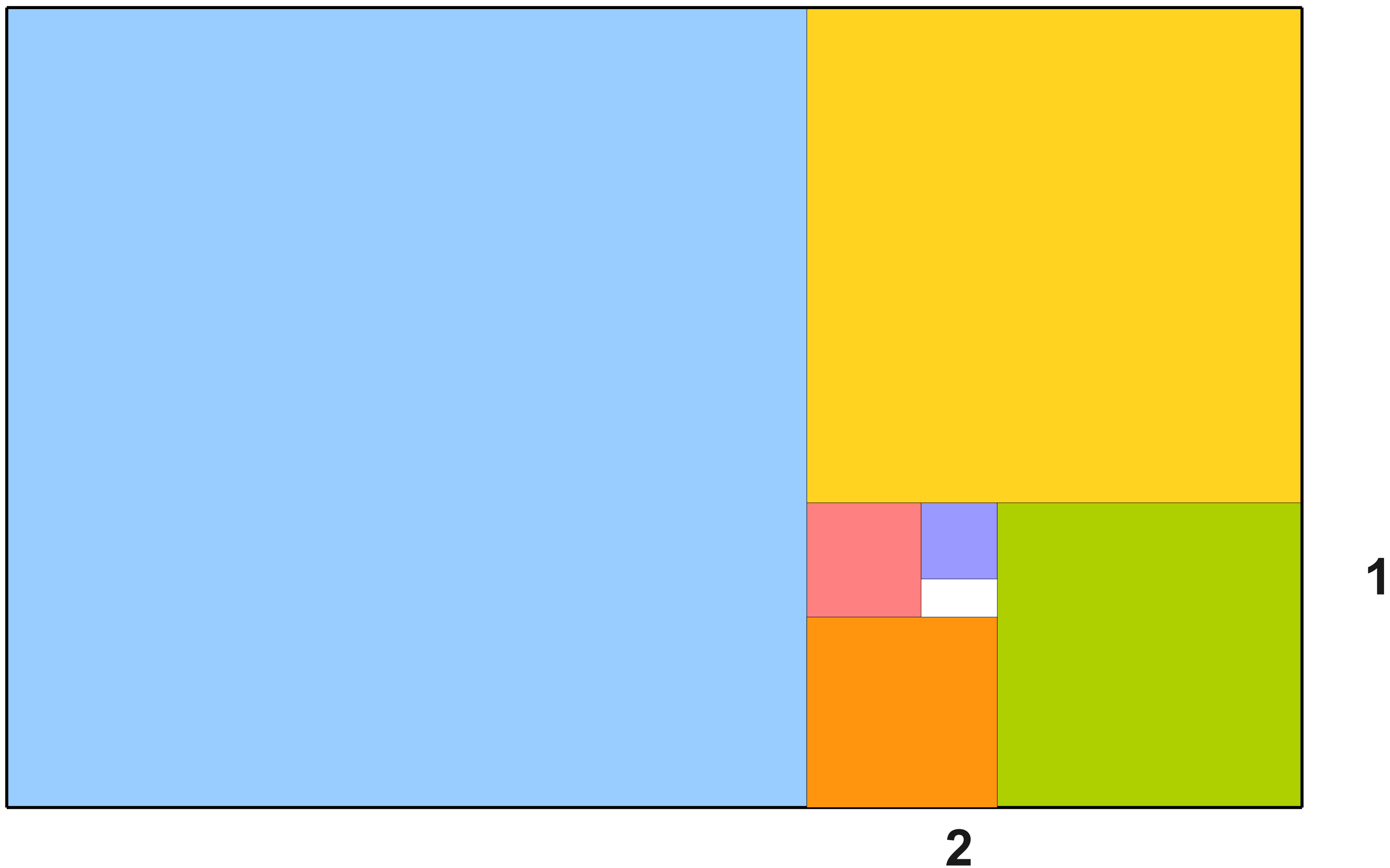

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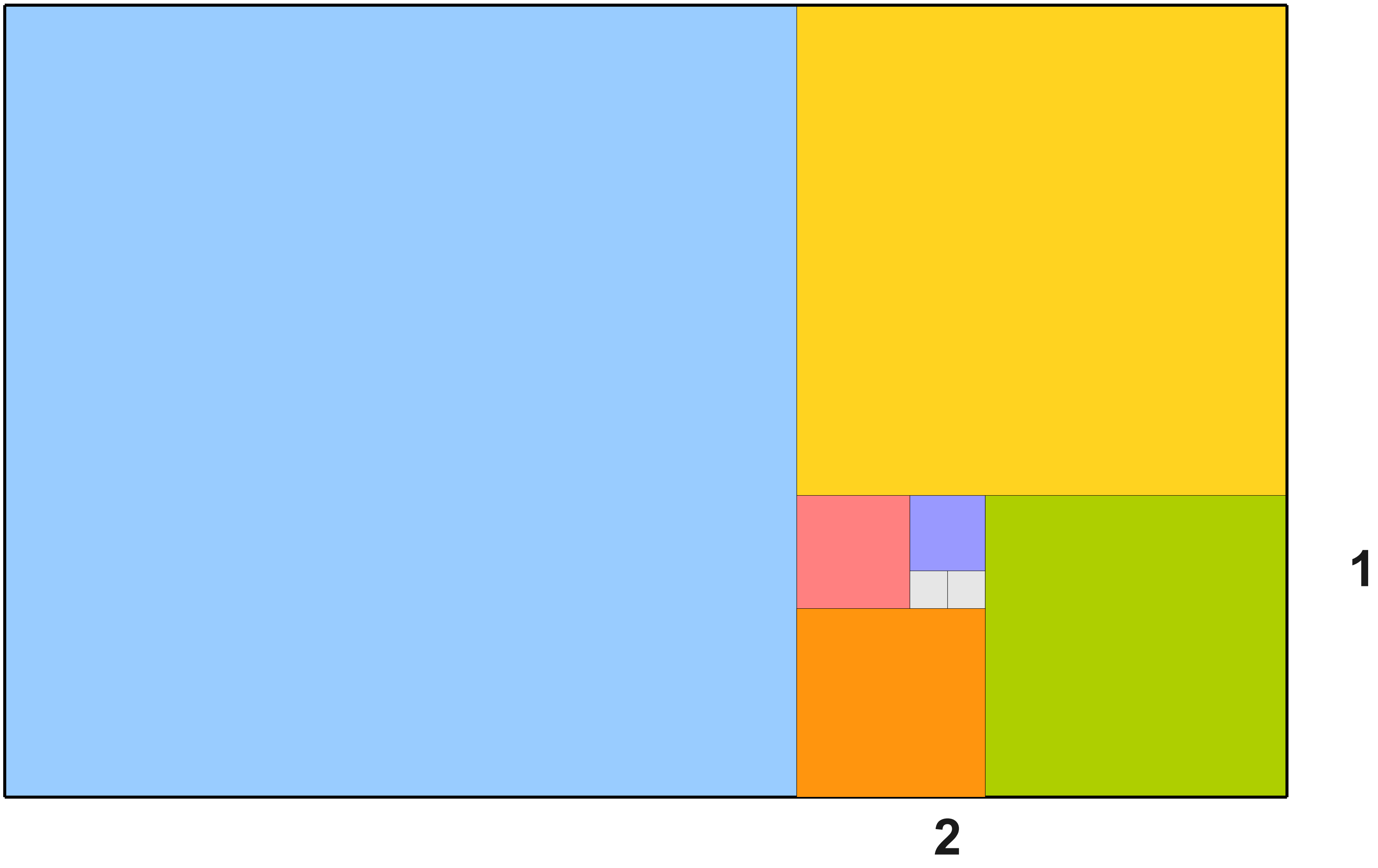

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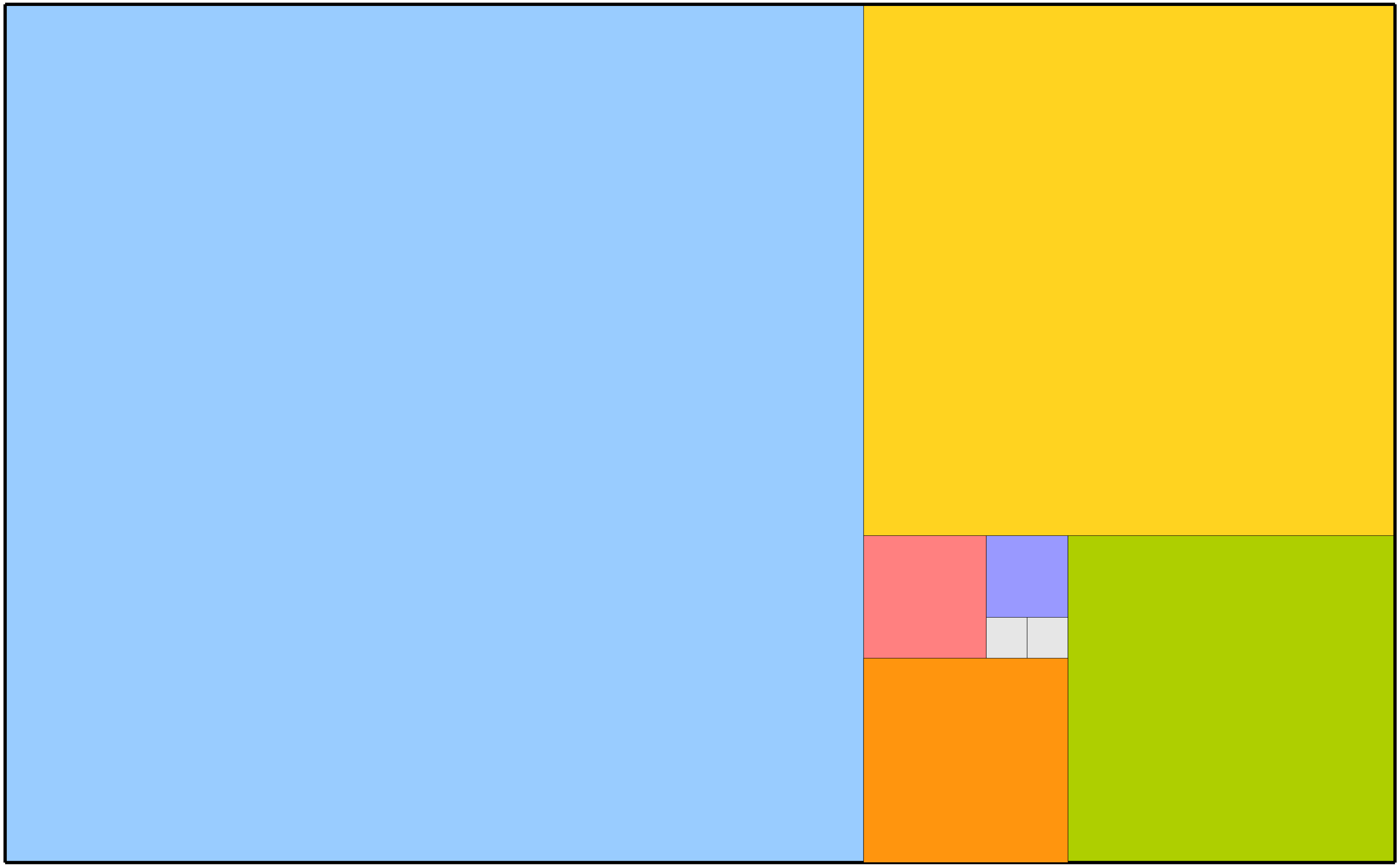
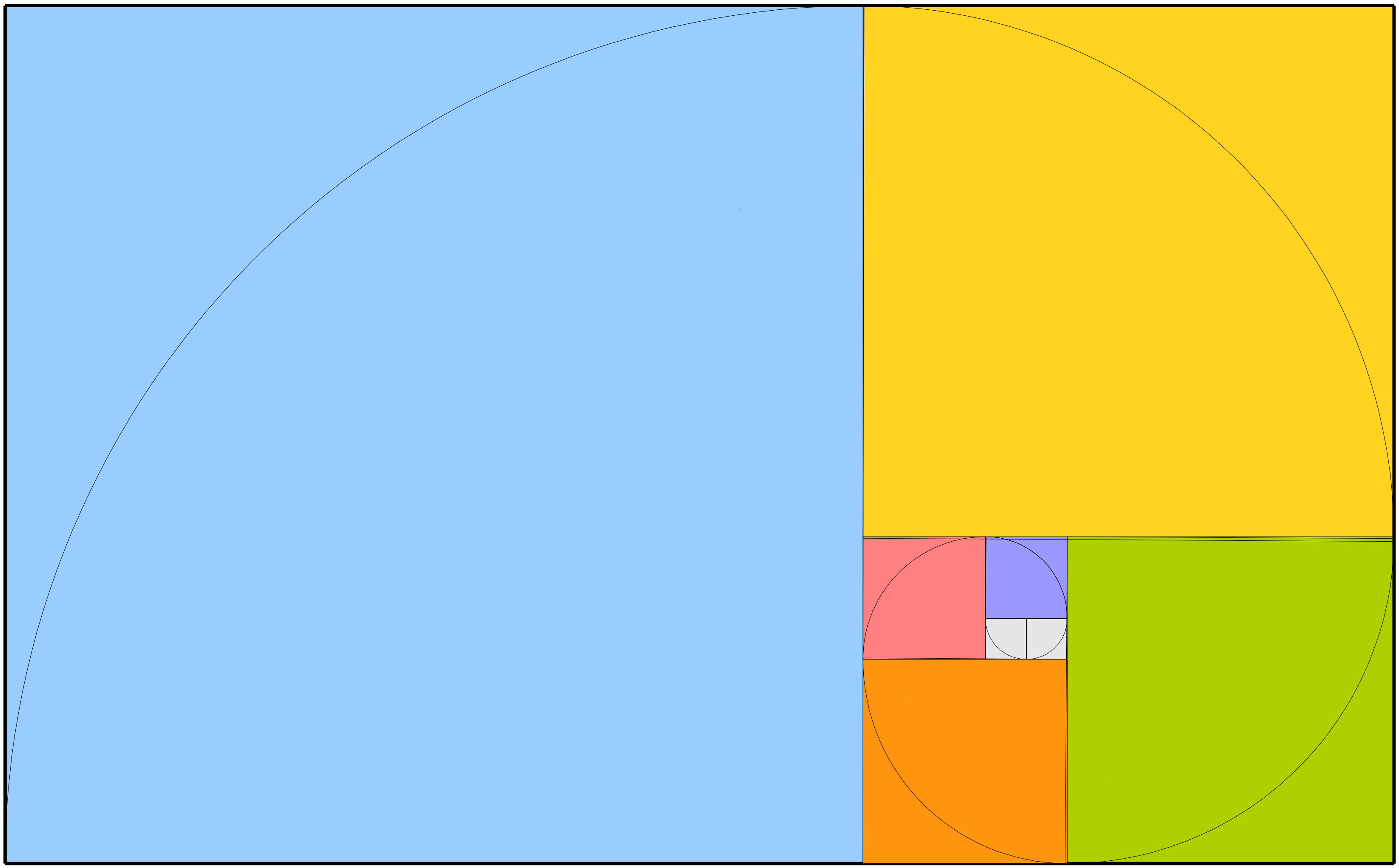


The Golden Spiral



How do we prove all rational numbers have continued fractions?

Constructing a Continued Fraction

$$\frac{25}{9}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \frac{7}{9}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \cfrac{1}{\cfrac{9}{7}}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \frac{2}{7}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\cfrac{7}{2}}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \frac{1}{\frac{9}{7}}$$

$$\frac{9}{7} = 1 + \frac{1}{\frac{7}{2}}$$

$$\frac{7}{2}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\cfrac{7}{2}}$$

$$\frac{7}{2} = 3 + \frac{1}{2}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \cfrac{1}{\cfrac{9}{7}}$$

$$\frac{9}{7} = 1 + \cfrac{1}{\boxed{\cfrac{7}{2}}}$$

$$\boxed{\frac{7}{2} = 3 + \frac{1}{2}}$$

Constructing a Continued Fraction

$$\frac{25}{9} = 2 + \cfrac{1}{\cfrac{9}{7}}$$

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$$\frac{7}{2}$$

$$\frac{2}{1}$$

Constructing a Continued Fraction

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$$\frac{9}{\mathbf{7}}$$

$$\frac{7}{\mathbf{2}}$$

$$\frac{2}{\mathbf{1}}$$

Constructing a Continued Fraction

$$\frac{25}{\color{red}{\mathbf{9}}} = 2 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}$$

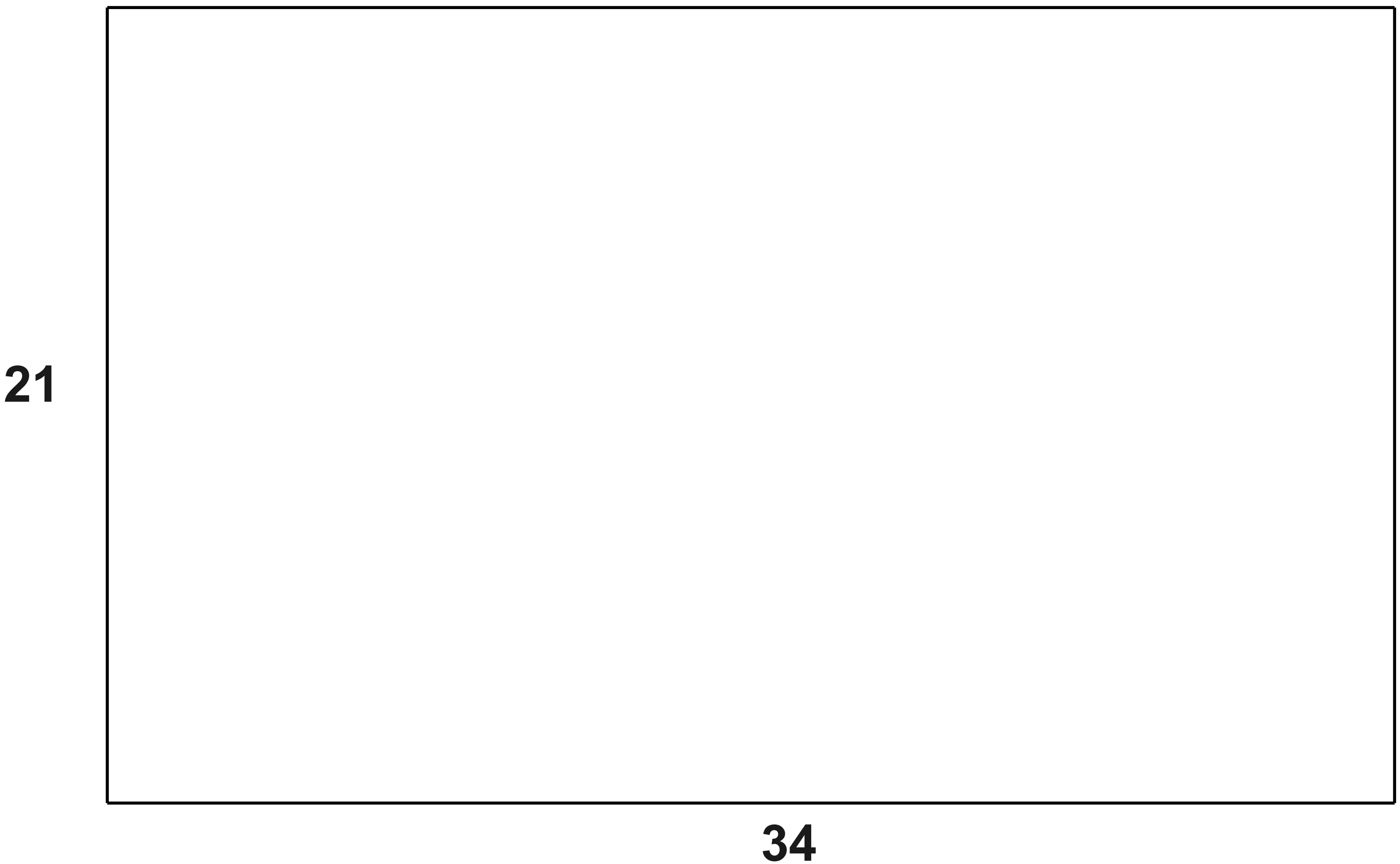
$$\frac{9}{\color{red}{\mathbf{7}}}$$

$$\frac{7}{\color{red}{\mathbf{2}}}$$

$$\frac{2}{\color{red}{\mathbf{1}}}$$

$$\color{red}{\mathbf{9 > 7 > 2 > 1}}$$

The Golden Ratio



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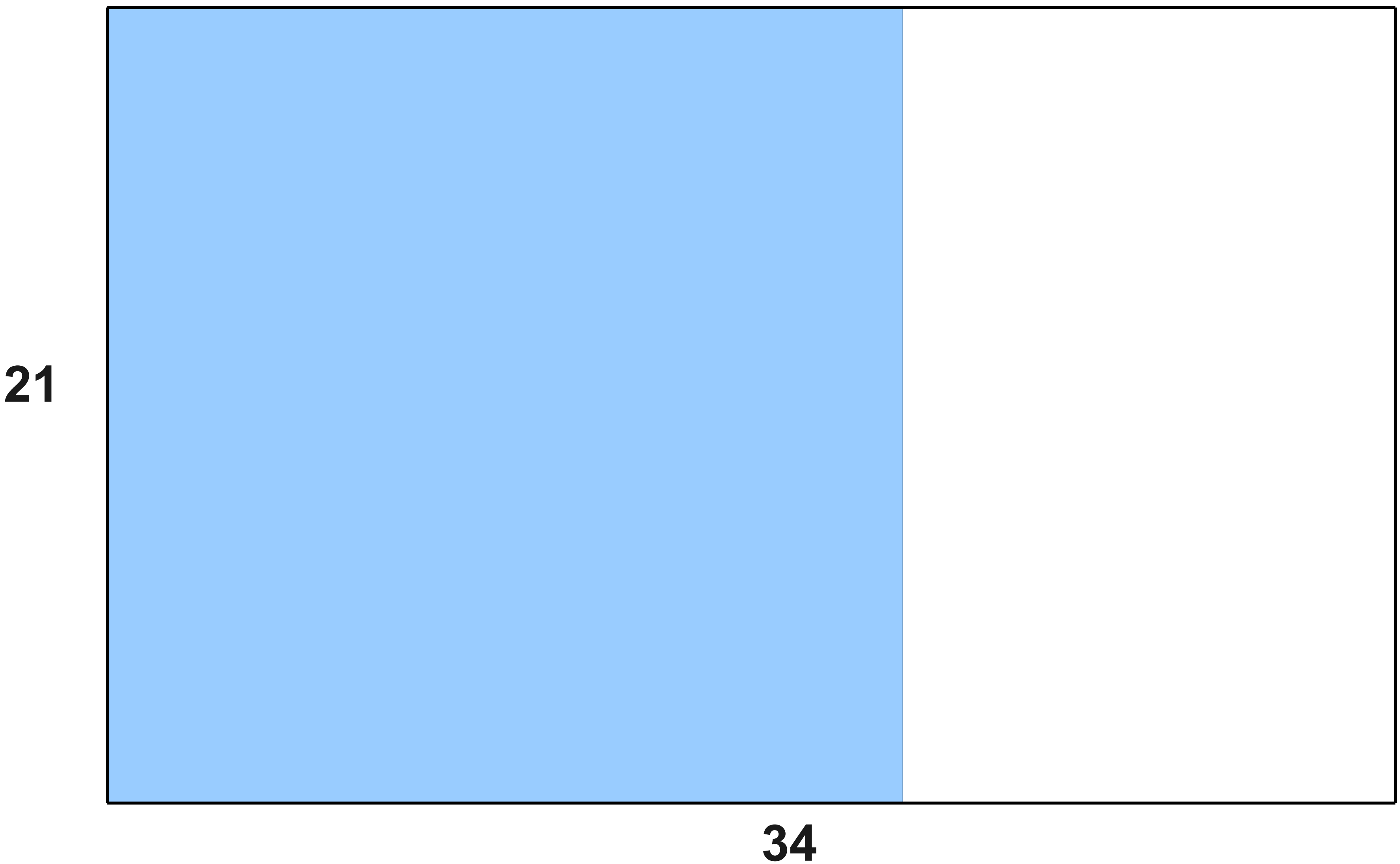

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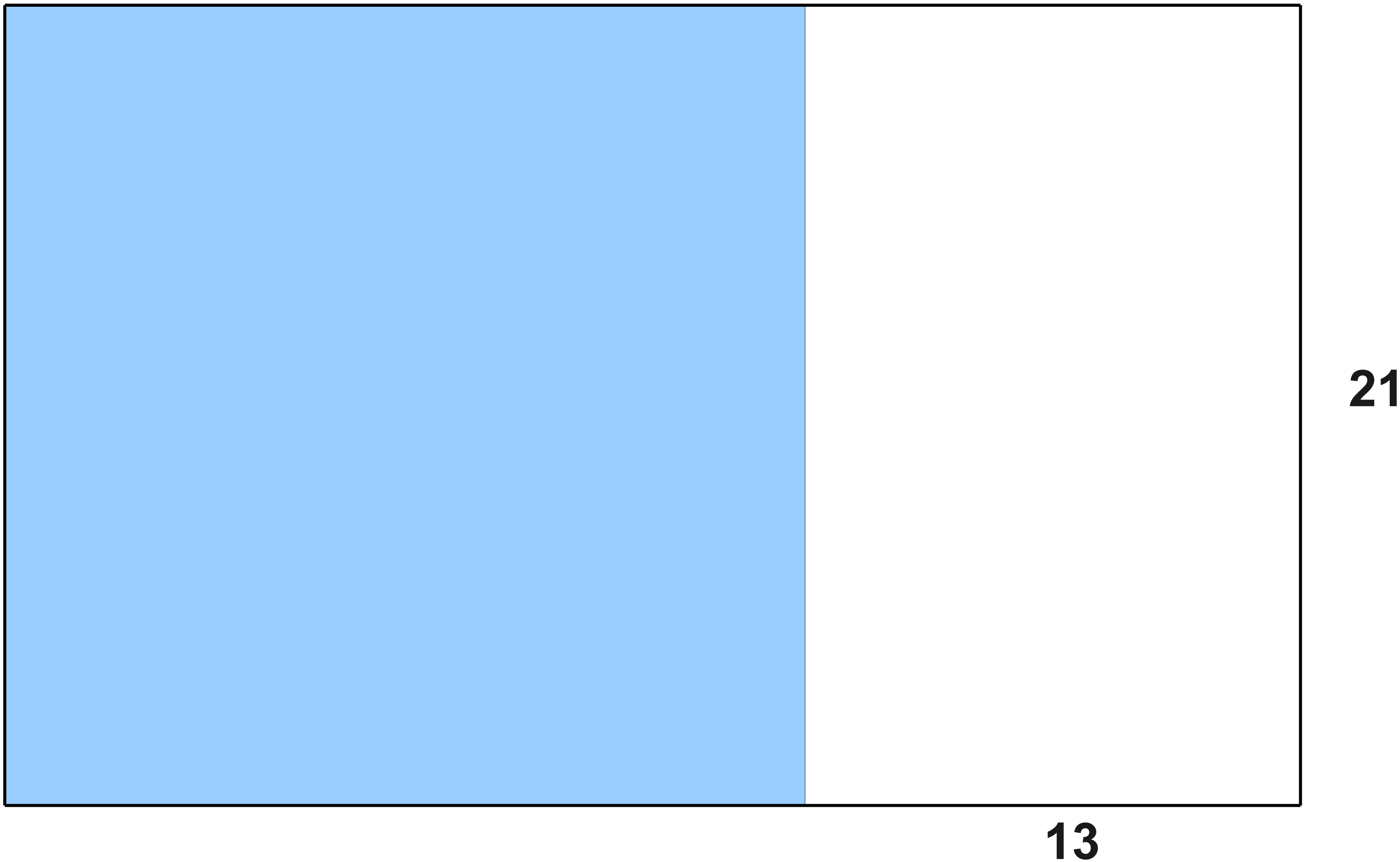

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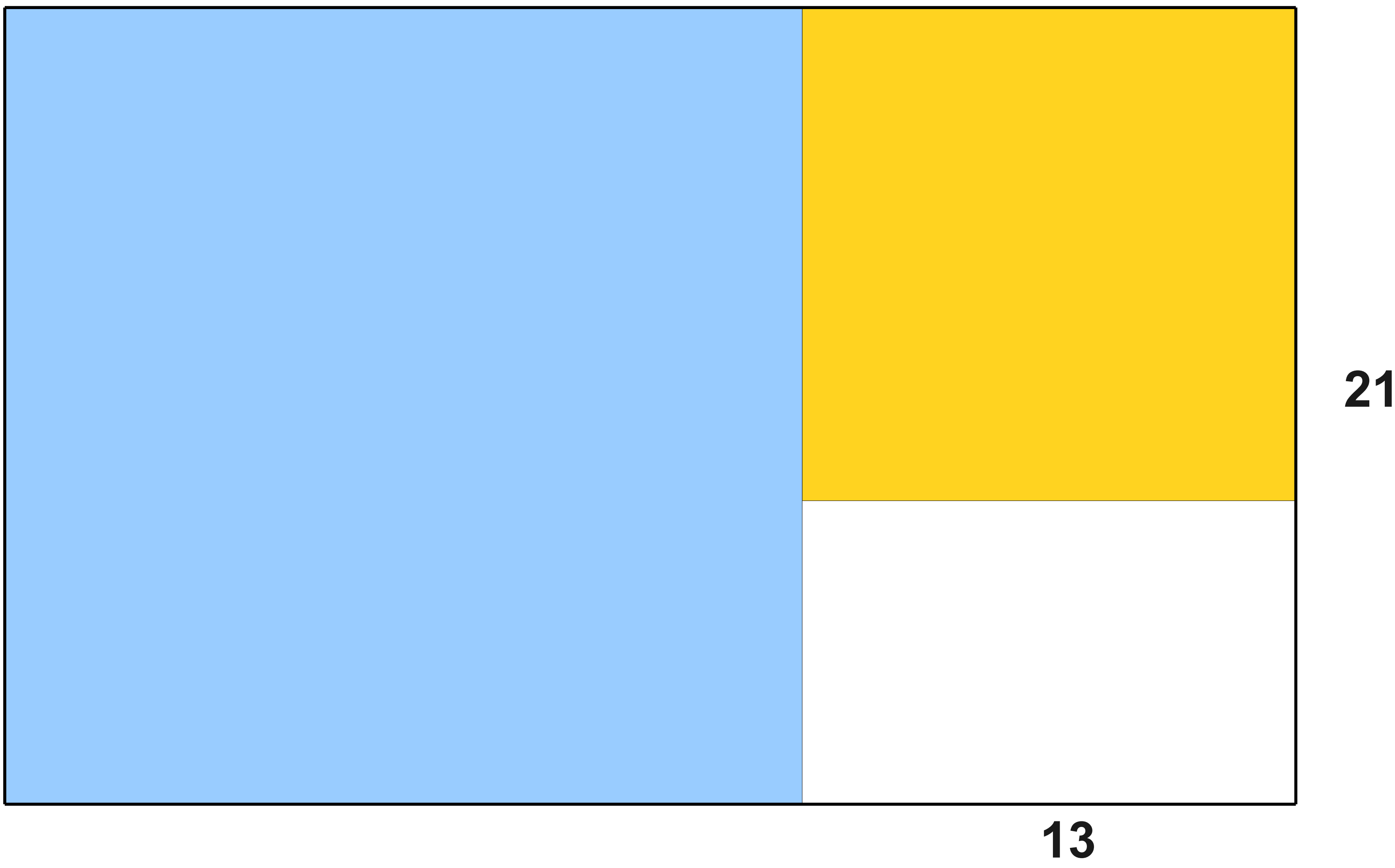

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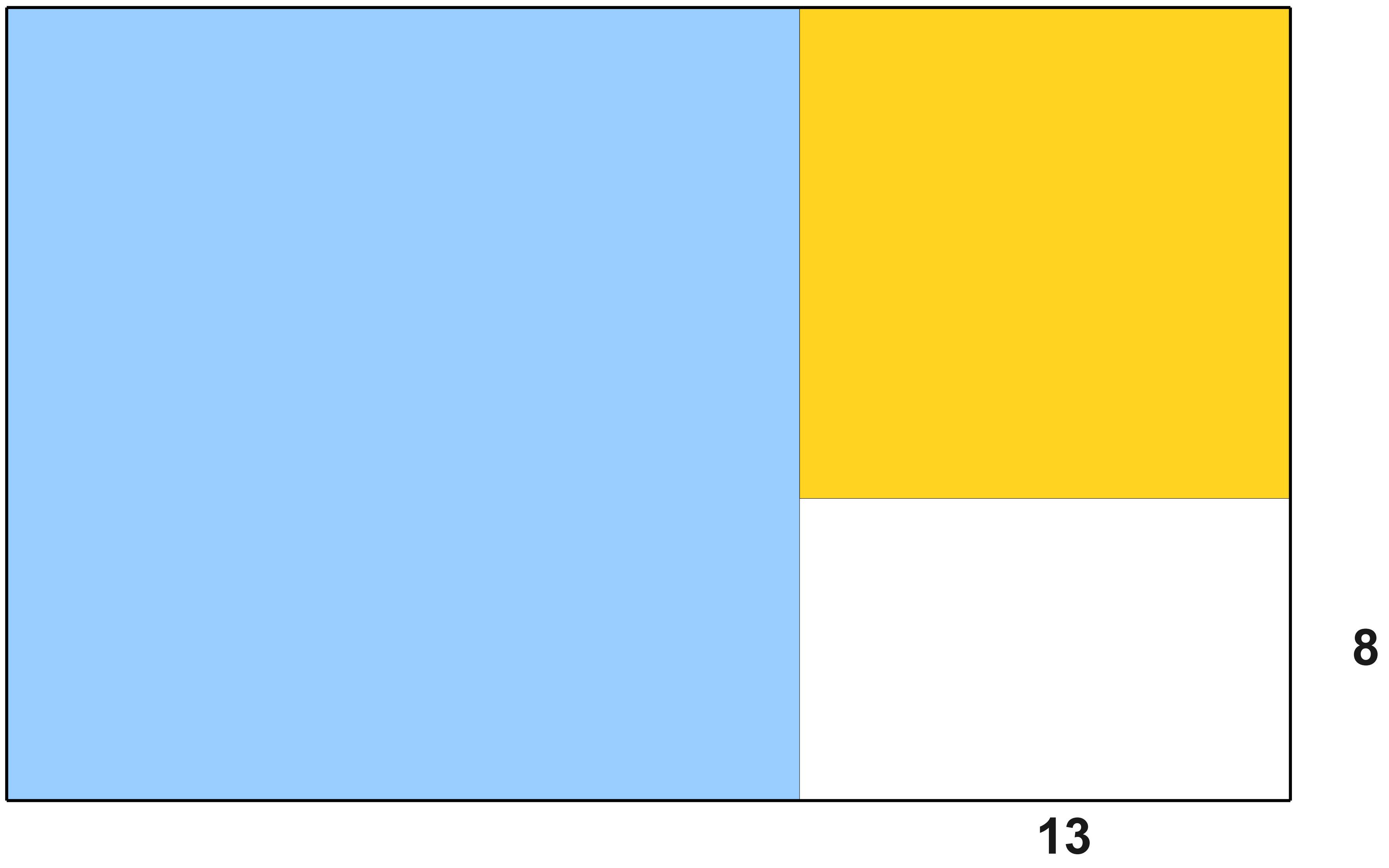

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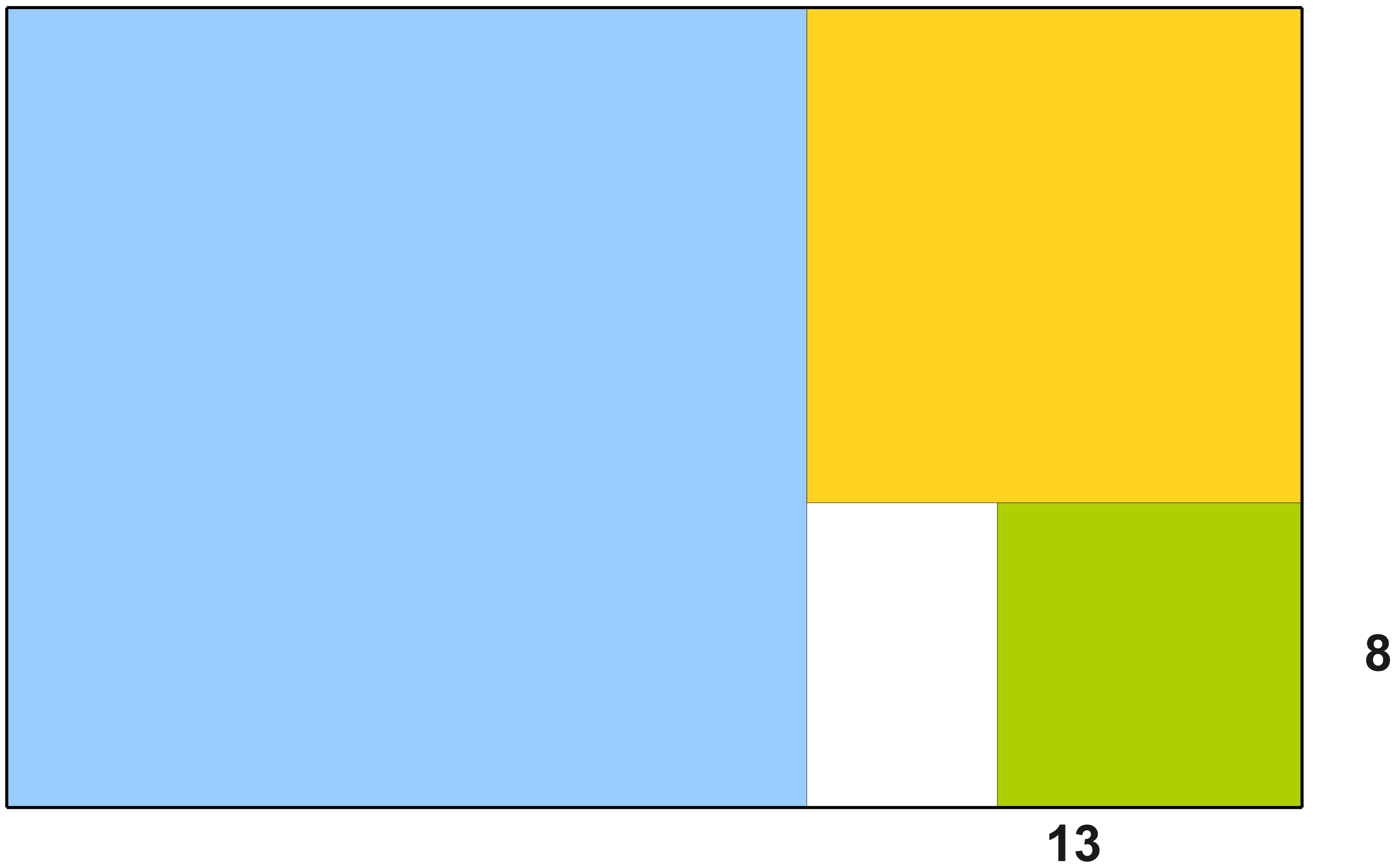

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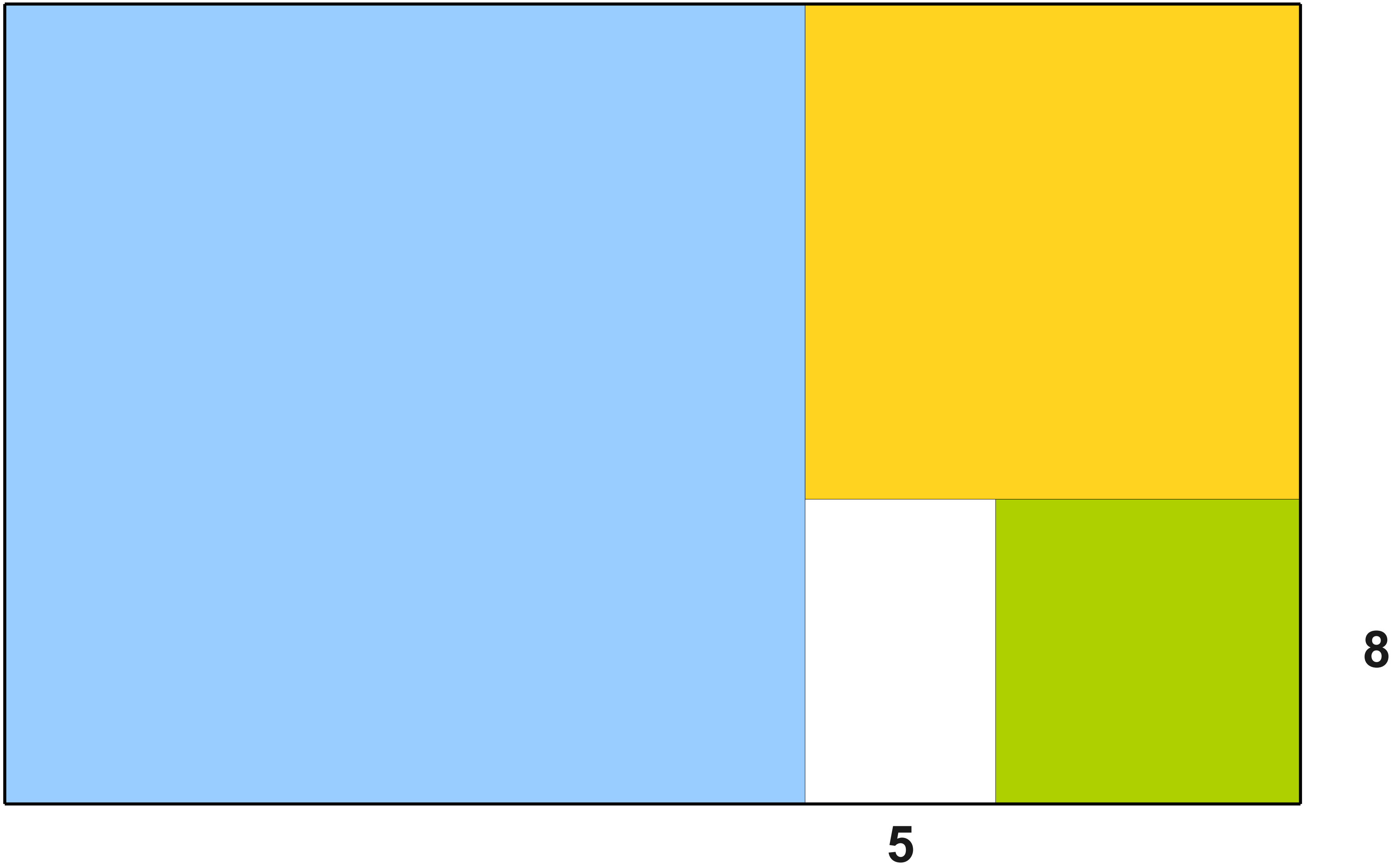

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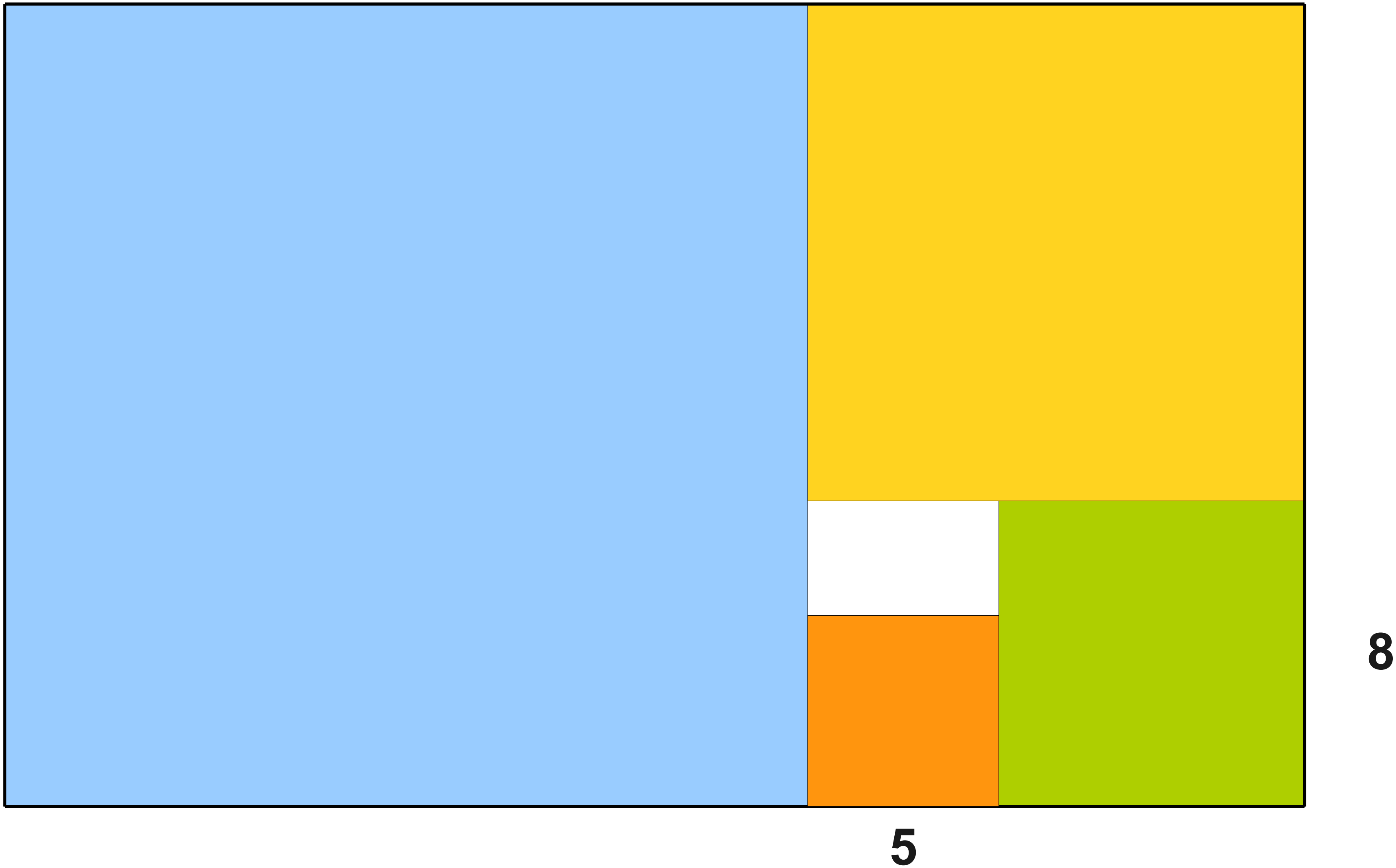

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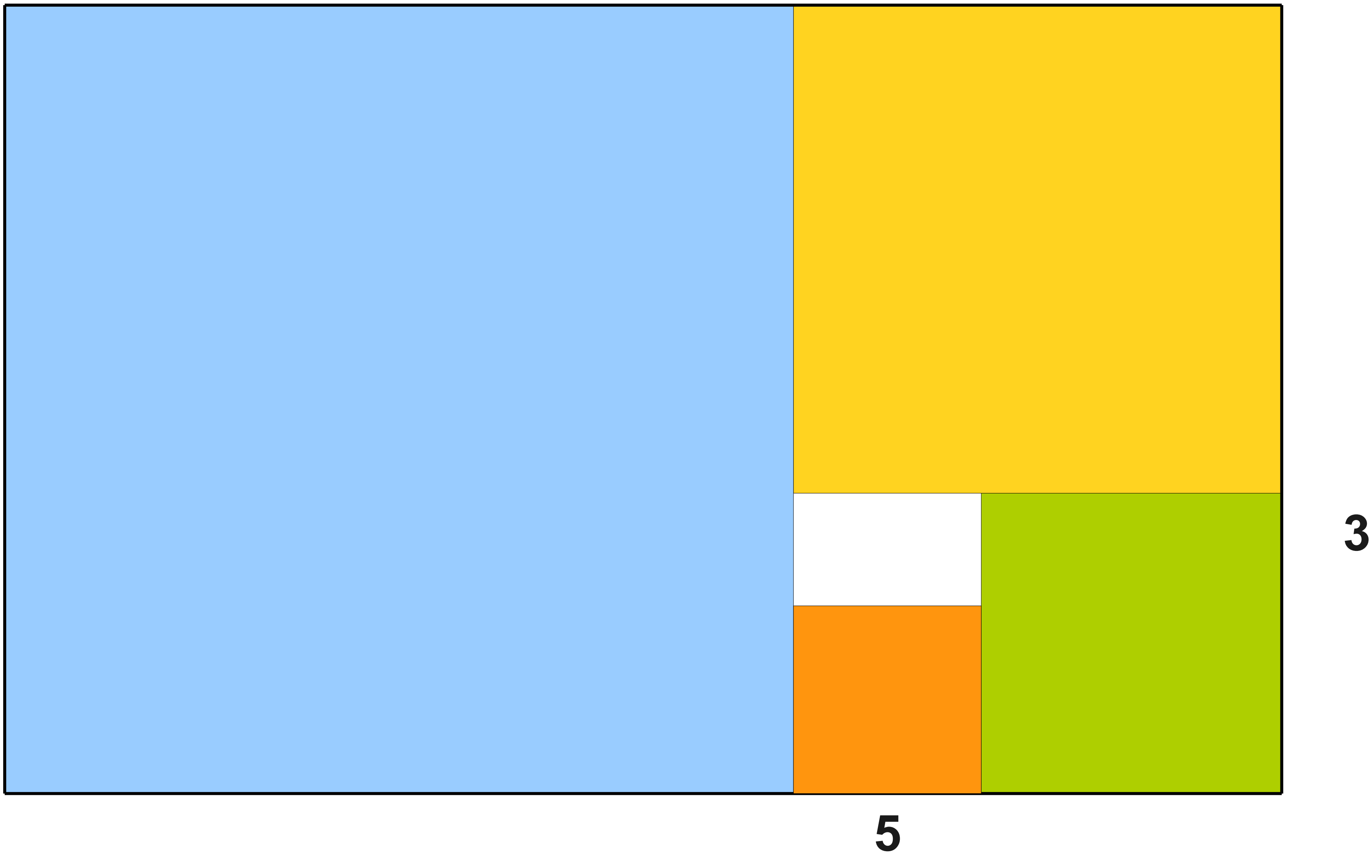

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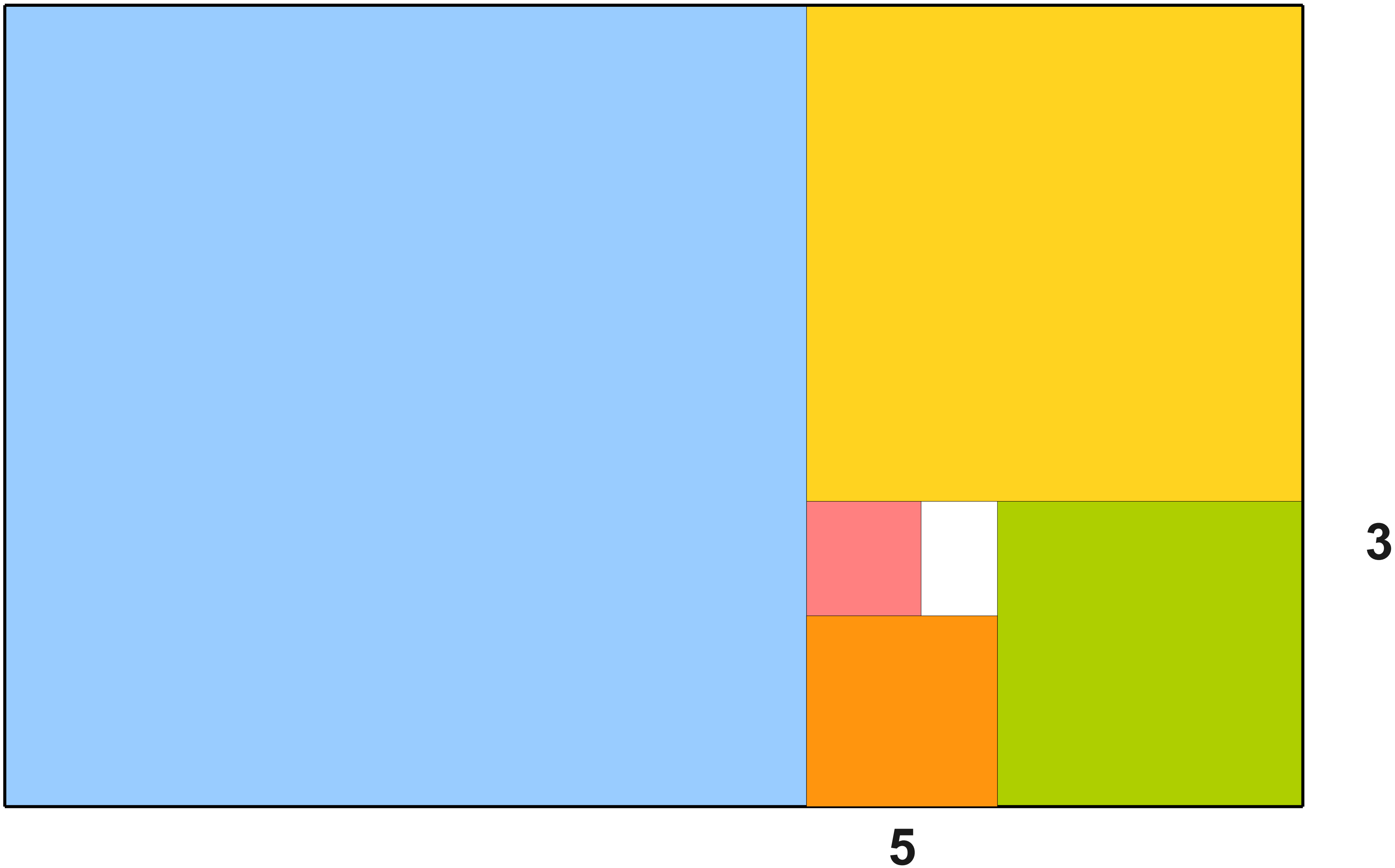

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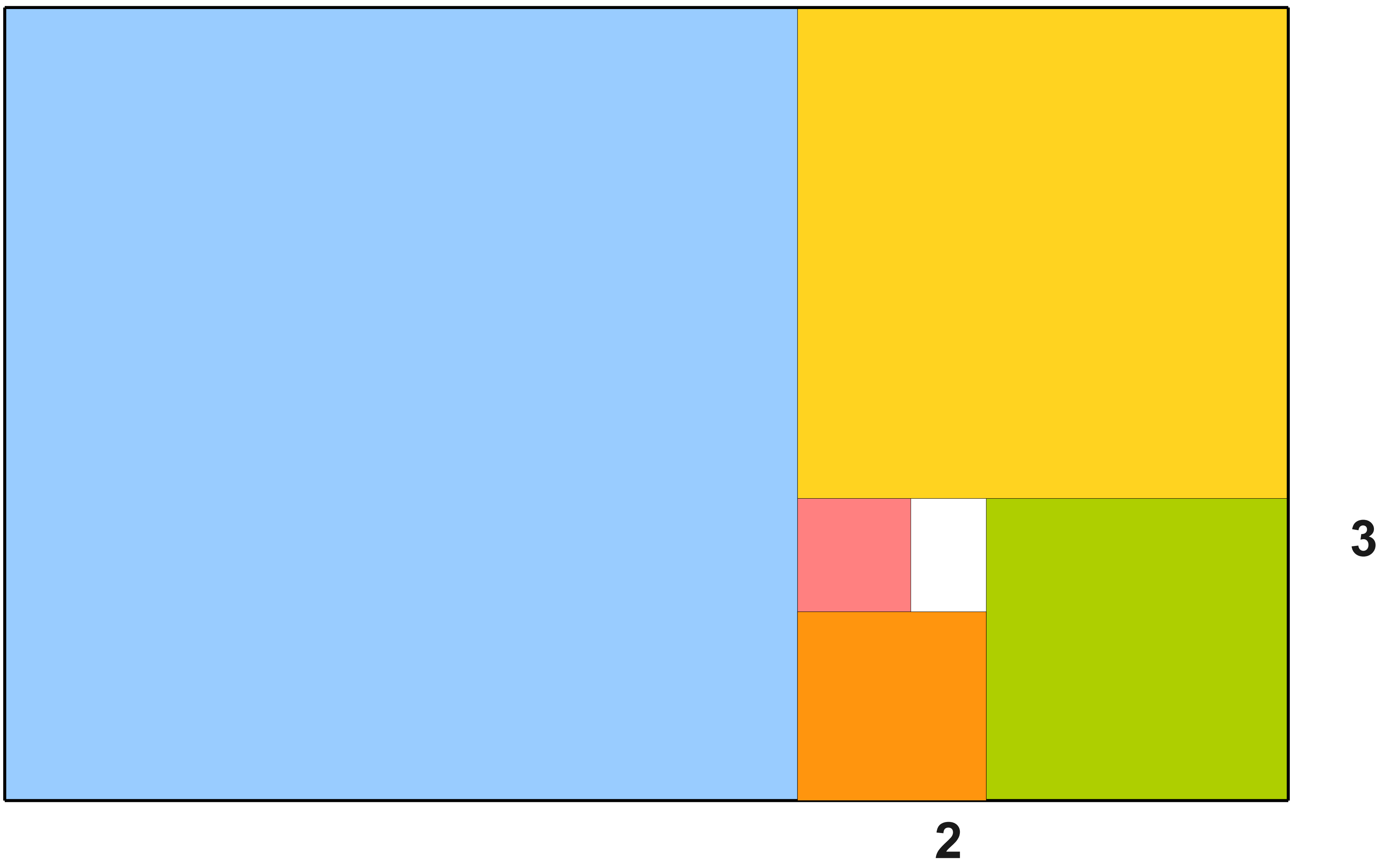

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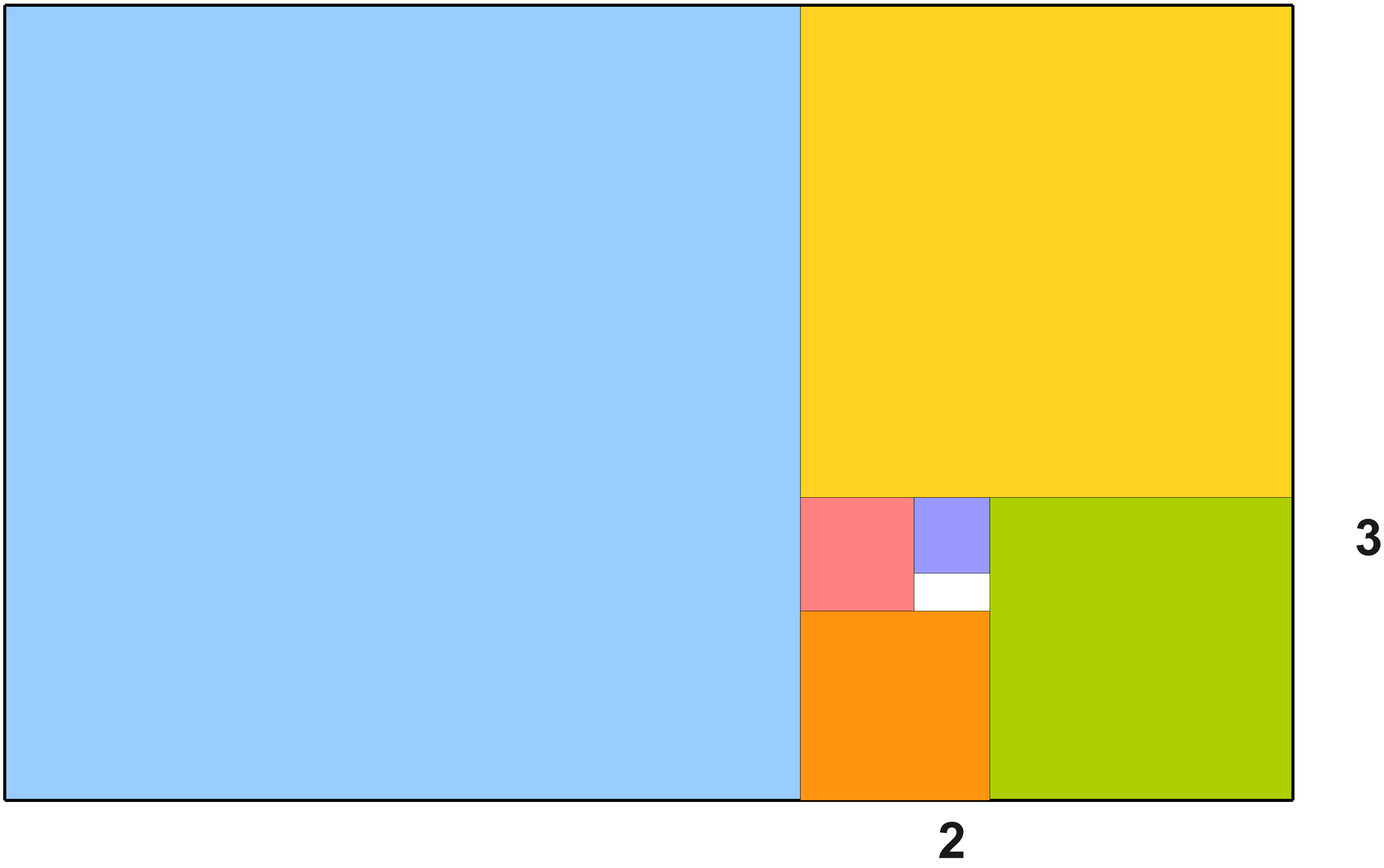

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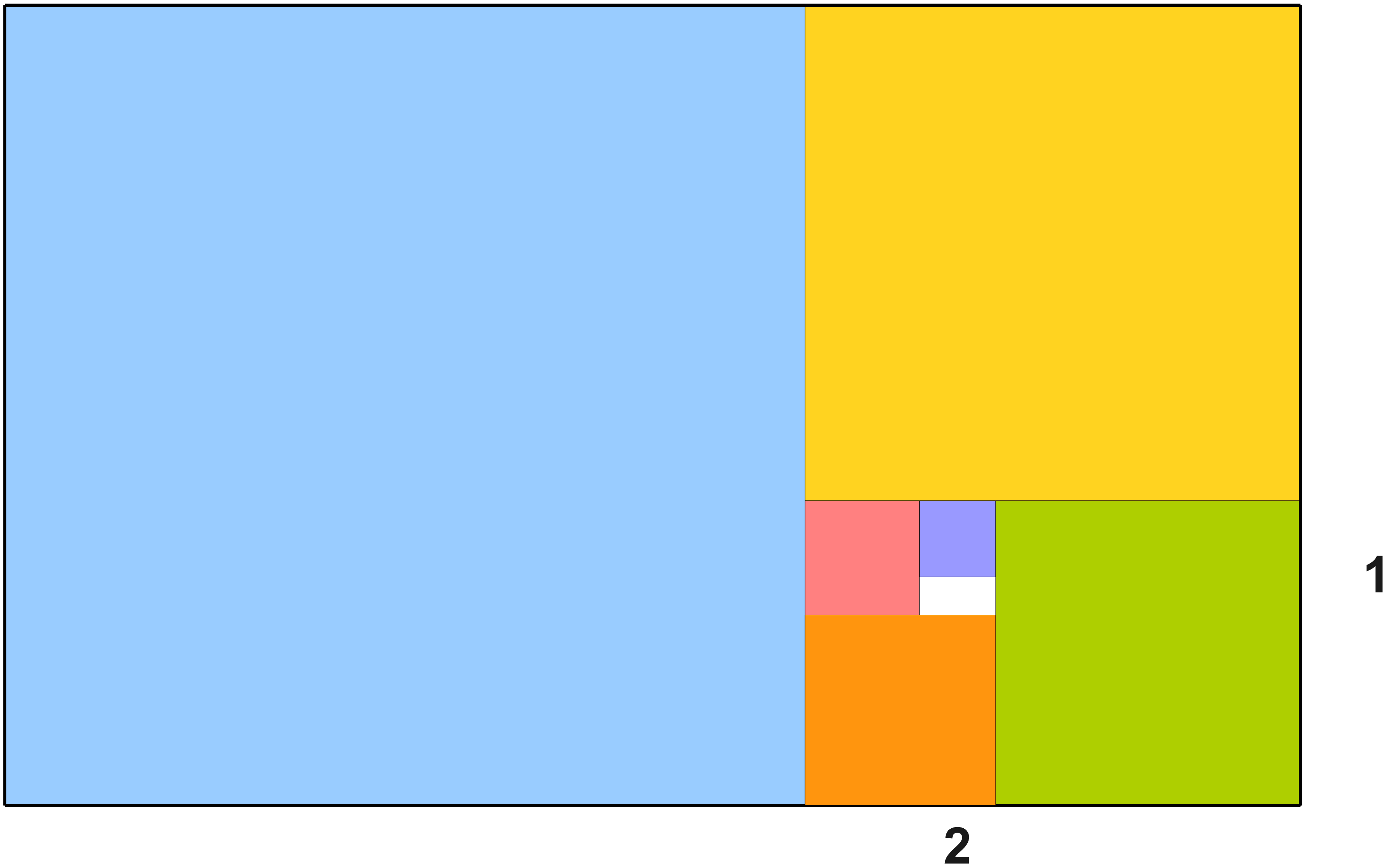

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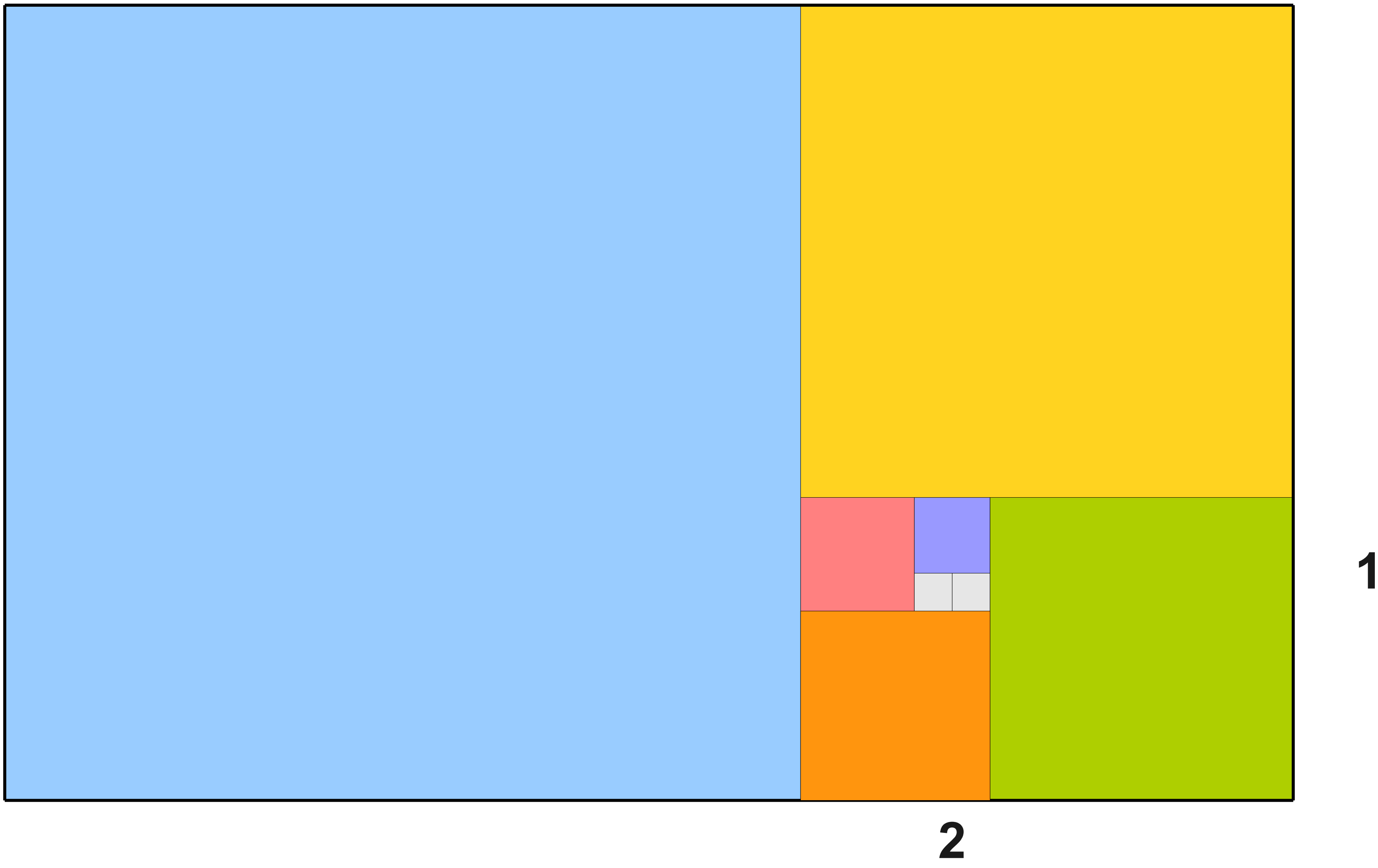

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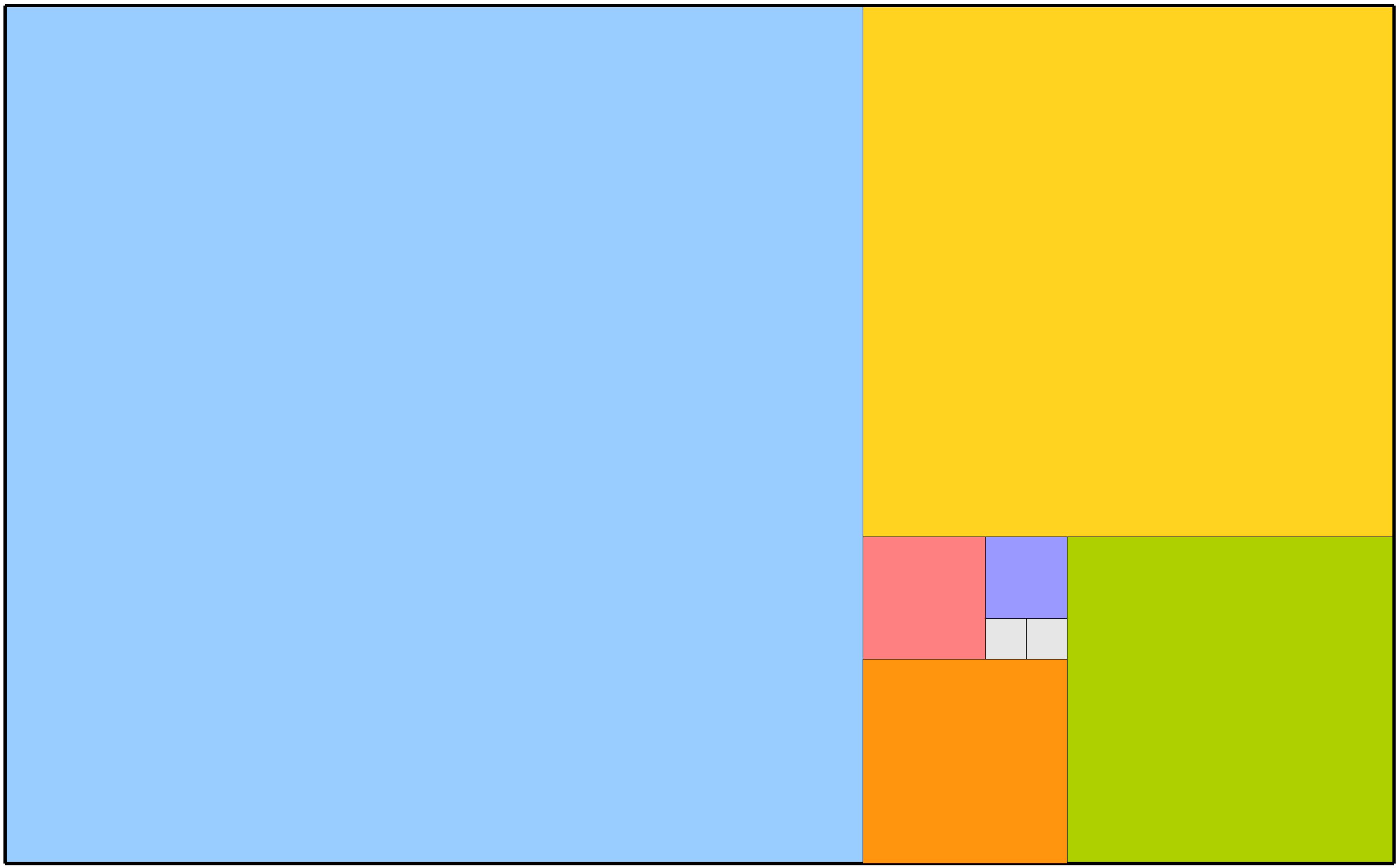

The Division Algorithm

- For any integers a and b, with $b \neq 0$, there exists **unique** integers **q** and **r** such that

$$a = \mathbf{q}b + \mathbf{r}$$

and

$$0 \leq \mathbf{r} < b$$

- **q** is the **quotient** and **r** is the **remainder**.
- If both a and b are nonnegative, then both **q** and **r** are nonnegative.
- Given $a = 11$ and $b = 4$: $11 = \mathbf{2} \cdot 4 + \mathbf{3}$
- Given $a = 137$ and $b = 42$: $137 = \mathbf{3} \cdot 42 + \mathbf{11}$

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The division algorithm is the mathematically rigorous way to justify getting a quotient and a remainder.

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For our base … with denominator … rational number with … fraction and m = m /

For our indu… nonnegative … We prove that P(n) is t… minator n has a contin…

Consider an… . Using the division … ses:

Notice how we're using the fact that $r < n$ to justify using the inductive hypothesis.

Since our induction starts at 1, we also have to show that $r \geq 1$. Otherwise we might be out of the range of where the inductive hypothesis holds.

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For more on continued fractions:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html

The Well-Ordering Principle

Extremal Cases

- Our proof about powers of two relied on a key step:

Let 2^k be the largest power of two less than or equal to $n + 1$.

- Many proofs work by picking some *extremal objects* (the largest *x* such that..., the smallest *y* such that..., etc.)

The Well-Ordering Principle

- The **well-ordering principle** is the following:

  Any nonempty set of natural numbers has a least element.

- Examples:
  - The least element of $\{1, 2, 3\}$ is 1.
  - The least element of $\mathbb{N}$ is 0.
  - There is no least element of $\mathbb{Z}$, but $\mathbb{Z}$ is not a set of natural numbers.
  - There is no least element of $\emptyset$, but $\emptyset$ is empty.

Proof by Well-Ordering

- Many proofs by induction or strong induction can be rewritten as proofs using the well-ordering principle.
- To prove that P(*n*) is true for all natural numbers *n*:
  - Consider the set $S = \{ n \mid n \in \mathbb{N}$ and $P(n)$ is false$\}$ of all natural numbers for which P(*n*) is false.
  - Assume, for the sake of contradiction, that *S* is nonempty.
  - Using the well-ordering principle, take the smallest element of *S*, call it n_0.
  - Derive a contradiction with n_0.
  - Conclude that *S* must be empty, so P(*n*) is always true.

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This set S is the set of all natural numbers n where the theorem isn't true. If this set is empty, we're done. So our goal now is to show that it has to be empty.

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An important detail here is that we're picking the smallest element of S, not just any arbitrary element of S. We'll use this fact later on.

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This is notationally dense, but we're just pulling off the last term of the sum. Since we know that $n_0 > 0$, the upper bound on this sum is still a natural number.

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This means that the theorem is false for $n_0 - 1$, which in turn means that $n_0 - 1$ has to be in the set S.

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One Simplifying Assumption

- For general rational numbers, p and q can be integers.
- We will assume that the square root of two is positive.
- Because of this, p and q can be assumed to be natural numbers.

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Notationally this is quite dense, but it just says that S is the set of denominators in an expression of the square root of two as a ratio.

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Next Time

- **Graphs and Relations**
  - Representing structured data.
  - Categorizing how objects are connected.