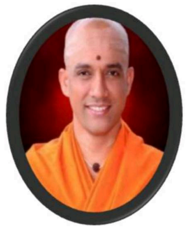

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DEPARTMENT OF MATHEMATICS
MATHEMATICS-3 FOR COMPUTER SCIENCE STREAM (BCS301)
MODULE - 2
JOINT DISTRIBUTION, STOCHASTIC PROCESS & MARKOV CHAIN

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Joint Probability:

Let $X = \{x_1, x_2, x_3, \ldots\ldots. x_m\}$ and $Y = \{y_1, y_2, y_3, \ldots\ldots. y_n\}$ are two discrete random variables , then the joint probability function of X and Y is defined as

$P(X = x_i, Y = y_j) = P(x_i, y_j) = f(x_i, y_j) = p_{ij} = f_{ij}$

where the function $f(x, y)$ satisfy the conditions

i) $f(x, y) \geq 0$ ii) $\sum_i \sum_j f(x_i, y_j) = 1$

The joint probability table as shown below,

X \ Y	y_1	y_2	y_3	.....	y_n	$f(x_i)$
x_1	p_{11}	p_{12}	p_{13}	.....	p_{1n}	$f(x_1)$
x_2	p_{21}	p_{22}	p_{23}	.....	p_{2n}	$f(x_2)$
x_3	p_{31}	p_{32}	p_{33}	.....	p_{3n}	$f(x_3)$
⋮	⋮	⋮	⋮	..... ⋮	⋮	⋮
x_m	p_{m1}	p_{m2}	p_{m3}	.....	p_{mn}	$f(x_m)$
$g(y_i)$	$g(y_1)$	$g(y_2)$	$g(y_3)$	.....	$g(y_n)$	1

Marginal Probability Distributions:

In the joint probability table $f(x_1), f(x_2), f(x_3), \dots\dots, f(x_m)$ and $g(y_1), g(y_2), g(y_3), \dots\dots, g(y_n)$ are called the marginal probability distributions respectively and represents the sum of all entries in all the rows and columns.

Independent Random Variables:

The discrete random variables X and Y are said to be independent if,

$P(X = x_i, Y = y_j) = P(X = x_i).P(Y = y_j)$, for every i, j and it is equivalent to

$P(X = x_i, Y = y_j) = P(X = x_i).P(Y = y_j) = f(x_i).g(y_j)$ **or** COV(X,Y) = 0

Expectation, Variance & Covariance:

Let X be the random variable taking the random values $x_1, x_2, x_3, \dots\dots x_m$, having the probability function $f(x)$. Then,

a) The expectation of X is denoted by E(X) and is defined as, $\mu_X = E(X) = \sum_{i=1}^{m} x_i f(x_i)$

b) The expectation of Y is denoted by E(Y) and is defined as, $\mu_Y = E(Y) = \sum_{j=1}^{n} y_j f(y_j)$

c) The variance of X is denoted by ${\sigma_X}^2$ and is defined as ${\sigma_X}^2 = E(X^2) - [E(X)]^2$

$$\Rightarrow {\sigma_X}^2 = \sum_{i=1}^{m} {x_i}^2 f(x_i) - {\mu_X}^2$$

d) The variance of Y is denoted by ${\sigma_Y}^2$ and is defined as ${\sigma_Y}^2 = E(Y^2) - [E(Y)]^2$

$$\Rightarrow {\sigma_Y}^2 = \sum_{i=1}^{n} {y_j}^2 f(y_j) - {\mu_Y}^2$$

e) The covariance of X and Y is denoted by COV(X,Y) and defined as $COV(X,Y) = E(XY) - E(X).E(Y)$

$$COV(X,Y) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j f(x_i, y_j) - \mu_X.\mu_Y$$

f) The correlation between X and Y is $\rho(X,Y) = \frac{COV(X,Y)}{\sigma_X \sigma_Y}$

PROBLEMS

1) The joint distribution of two random variables X and Y are as follows:

X \ Y	-4	2	7
1	$1/8$	$1/4$	$1/8$
5	$1/4$	$1/8$	$1/8$

Compute the following,

i) E(X) and E(Y)

ii) E(XY)

iii) σ_X & σ_Y

iv) $\rho(X,Y)$

Sol^n: Given,

$x_1 = 1,\ x_2 = 5,\ y_1 = -4,\ y_2 = 2,\ y_3 = 7$

And the probabilities are

$p_{11} = \frac{1}{8},\ p_{12} = \frac{1}{4},\ p_{13} = \frac{1}{8},\ p_{21} = \frac{1}{4},\ p_{22} = \frac{1}{8},\ p_{23} = \frac{1}{8}$

Given the joint probability distribution is follows as

Y \ X	-4	2	7	$f(x_i)$
1	$1/8$	$1/4$	$1/8$	$1/2$
5	$1/4$	$1/8$	$1/8$	$1/2$
$g(y_i)$	$3/8$	$3/8$	$1/4$	1

The marginal distribution of X and Y are

x_i	1	5
$f(x_i)$	$1/2$	$1/2$

y_i	-4	2	7
$g(y_i)$	$3/8$	$3/8$	$1/4$

i) $\mu_X = E(X) = \sum_{i=1}^{2} x_i f(x_i) = \left(1 \times \frac{1}{2}\right) + \left(5 \times \frac{1}{2}\right) = 3$

$\mu_Y = E(Y) = \sum_{j=1}^{3} y_j g(y_j) = \left(-4 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(7 \times \frac{1}{4}\right) = 1$

ii) $E(XY) = \sum_{i=1}^{2}\sum_{j=1}^{3} x_i y_j f(x_i, y_j)$

$= \left(1 \times (-4) \times \frac{1}{8}\right) + \left(1 \times 2 \times \frac{1}{4}\right) + \left(1 \times 7 \times \frac{1}{8}\right) + \left(5 \times (-4) \times \frac{1}{4}\right) + \left(5 \times 2 \times \frac{1}{8}\right) + \left(5 \times 7 \times \frac{1}{8}\right)$

$= \frac{3}{2}$

iii) $\sigma_X{}^2 = E(X^2) - \mu_X{}^2 = \sum_{i=1}^{2} x_i{}^2 f(x_i) - \mu_X{}^2 = \left(1^2 \times \frac{1}{2}\right) + \left(5^2 \times \frac{1}{2}\right) - 9 = 13 - 9 = 4 \Rightarrow \sigma_X = 2$

$\sigma_Y{}^2 = E(Y^2) - \mu_Y{}^2 = \sum_{j=1}^{3} y_j{}^2 g(y_j) - \mu_Y{}^2 = \left((-4)^2 \times \frac{3}{8}\right) + \left(2^2 \times \frac{3}{8}\right) + \left(7^2 \times \frac{1}{4}\right) - 1^2 = \frac{75}{4} \Rightarrow \sigma_Y = 4.33$

iv) $COV(X,Y) = E(XY) - \mu_X\mu_Y = \frac{3}{2} - (3)(1) = -\frac{3}{2}$

v) $\rho(X,Y) = \frac{COV(X,Y)}{\sigma_X\sigma_Y} = \frac{-3/2}{2\times 4.33} = -0.1732$

Hence the given random variables are not independent

2) The joint distribution of two random variables X and Y are as follows:

Y \ X	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Find the marginal distribution of X and Y. Also find the covariance of X and Y.

Sol^n: Given,

$x_1 = 1,\ x_2 = 2,\ y_1 = -2,\ y_2 = -1,\ y_3 = 4, y_4 = 5$

And the probabilities are

$p_{11} = 0.1,\ p_{12} = 0.2,\ p_{13} = 0,\ p_{14} = 0.3,\ p_{21} = 0.2,\ p_{22} = 0.1,\ p_{23} = 0.1, p_{13} = 0$

Given the joint probability distribution is follows as

Y \ X	-4	2	7		$f(x_i)$
1	0.1	0.2	0	0.3	0.6
2	0.2	0.1	0.1	0	0.4
$g(y_i)$	0.3	0.3	0.2	0.3	1

The marginal distribution of X and Y are

x_i	1	2
$f(x_i)$	0.6	0.4

y_i	-2	-1	4	5
$g(y_i)$	0.3	0.3	0.1	0.3

i) $\mu_X = E(X) = \sum_{i=1}^{2} x_i f(x_i) = (1 \times 0.6) + (2 \times 0.4) = 1.4$

$\mu_Y = E(Y) = \sum_{j=1}^{3} y_j g(y_j) = (-2 \times 0.3) + (-1 \times 0.3) + (4 \times 0.1) + (5 \times 0.3) = 1$

ii) $E(XY) = \sum_{i=1}^{2} \sum_{j=1}^{3} x_i y_j f(x_i, y_j)$

$= (1)(-2)(0.1) + (1)(-1)(0.2) + (1)(4)(0) + (1)(5)(0.3) + (2)(-2)(0.2) + (2)(-1)(0.1) + (2)(4)(0.1) + (2)(5)(0)$

$= -0.2 - 0.2 + 0 + 1.5 - 0.8 - 0.2 + 0.8 + 0 = 2.3 - 1.4 = 0.9$

iii) ${\sigma_X}^2 = E(X^2) - {\mu_X}^2 = \sum_{i=1}^{2} {x_i}^2 f(x_i) - {\mu_X}^2$

$= (1^2 \times 0.6) + (2^2 \times 0.4) - (1.4)^2 = 2.2 - 1.96 = 0.24 \Rightarrow \sigma_X = 0.4898$

${\sigma_Y}^2 = E(Y^2) - {\mu_Y}^2 = \sum_{j=1}^{3} {y_j}^2 g(y_j) - {\mu_Y}^2$

$= ((-2)^2 \times 0.3) + ((-1)^2 \times 0.3) + (4^2 \times 0.1) + (5^2 \times 0.3) - 1^2 = 9.6 \Rightarrow \sigma_Y = 3.0983$

iv) $COV(X,Y) = E(XY) - \mu_X \mu_Y = 0.9 - (1.4)(1) = -0.5$

vi) $\rho(X,Y) = \frac{COV(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.5}{0.4898 \times 3.0983} = -0.3294$

Hence the given random variables are not independent

3) Determine,
i) Marginal distribution.
ii) Covariance between the discrete random variables X and Y, using the joint probability distribution.

X \ Y	3	4	5
2	$^1/_6$	$^1/_6$	$^1/_6$
5	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$
7	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$

Sol^n: Given,

$x_1 = 2, x_2 = 5, x_3 = 7, y_1 = 3, y_2 = 4, y_3 = 5$

And the probabilities are

$p_{11} = \frac{1}{6}, p_{12} = \frac{1}{6}, p_{13} = \frac{1}{6}, p_{21} = \frac{1}{12}, p_{22} = \frac{1}{12}, p_{23} = \frac{1}{12} p_{31} = \frac{1}{12}, p_{32} = \frac{1}{12}, p_{33} = \frac{1}{12}$

The joint distribution table is as follows:

X \ Y	3	4	5	$f(x_i)$
2	$^1/_6$	$^1/_6$	$^1/_6$	$^1/_2$
5	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_4$
7	$^1/_{12}$	$^1/_{12}$	$^1/_{12}$	$^1/_4$
$g(y_i)$	$^1/_3$	$^1/_3$	$^1/_3$	1

The marginal distributions of X and Y are

x_i	2	5	7
$f(x_i)$	$1/2$	$1/4$	$1/4$

y_i	3	4	5
$g(y_i)$	$1/3$	$1/3$	$1/3$

$$\mu_X = E(X) = \sum_{i=1}^{3} x_i f(x_i)$$
$$\Rightarrow \mu_X = \left(2\times\frac{1}{2}\right)+\left(5\times\frac{1}{4}\right)+\left(7\times\frac{1}{4}\right)$$
$$\Rightarrow \mu_X = 4$$
$$\mu_Y = E(Y) = \sum_{j=1}^{3} y_j g(y_j)$$
$$\Rightarrow \mu_Y = \left(3\times\frac{1}{3}\right)+\left(4\times\frac{1}{3}\right)+\left(5\times\frac{1}{3}\right)$$
$$\Rightarrow \mu_Y = 4$$
$$E(XY) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j f(x_i, y_j)$$
$$\Rightarrow E(XY) = \left(2\times 3\times\frac{1}{6}\right)+\left(2\times 4\times\frac{1}{6}\right)+\left(2\times 5\times\frac{1}{6}\right)+\left(5\times 3\times\frac{1}{12}\right)+\left(5\times 4\times\frac{1}{12}\right)+\left(5\times 5\times\frac{1}{12}\right)$$
$$+\left(7\times 3\times\frac{1}{12}\right)+\left(7\times 4\times\frac{1}{12}\right)+\left(7\times 5\times\frac{1}{12}\right) = 16$$

$$\therefore Cov(X,Y) = E(XY) - \mu_X\mu_Y$$
$$\Rightarrow Cov(X,Y) = 16 - 4\times 4$$
$$\Rightarrow Cov(X,Y) = 0$$

Hence the given random variables are independent

4) The joint probability distribution of discrete random variables X and Y is given below:

X \ Y	1	3	6
1	$1/9$	$1/6$	$1/18$
3	$1/6$	$1/4$	$1/12$
6	$1/18$	$1/12$	$1/36$

Determine,
i) Marginal distribution of X and Y.
ii) Are X and Y statistically independent?

Sol^n: Given,

$x_1 = 1, x_2 = 3, x_3 = 6, y_1 = 1, y_2 = 3, y_3 = 6$

And the probabilities are

$p_{11} = \frac{1}{9}, p_{12} = \frac{1}{6}, p_{13} = \frac{1}{18}, p_{21} = \frac{1}{6}, p_{22} = \frac{1}{4}, p_{23} = \frac{1}{12}, p_{31} = \frac{1}{18}, p_{32} = \frac{1}{12}, p_{33} = \frac{1}{36}$

The joint distribution table is as follows

X \ Y	1	3	6	$f(x_i)$
1	$1/9$	$1/6$	$1/18$	$1/3$
3	$1/6$	$1/4$	$1/12$	$1/2$
6	$1/18$	$1/12$	$1/36$	$3/18$
$g(y_i)$	$1/3$	$1/2$	$3/18$	1

i) The marginal distributions of X and Y are,

x_i	1	3	6
$f(x_i)$	$1/3$	$1/2$	$3/18$

y_i	1	3	6
$g(y_i)$	$1/3$	$1/2$	$3/18$

ii) $\mu_X = E(X) = \sum_{i=1}^{3} x_i f(x_i) = \frac{1}{3} + \frac{3}{2} + 1 = 2.8333$

$\mu_Y = E(Y) = \sum_{j=1}^{3} y_j g(y_j) = \frac{1}{3} + \frac{3}{2} + 1 = 2.8333$

$E(XY) = \sum_{i=1}^{3} \sum_{j=1}^{3} x_i y_j p_{ij} = \frac{1}{9} + \frac{3}{6} + \frac{6}{18} + \frac{3}{6} + \frac{9}{4} + \frac{18}{12} + \frac{6}{18} + \frac{18}{12} + \frac{36}{36} = 8.0278$

$COV(X,Y) = E(XY) - \mu_X \mu_Y = 8.0278 - (2.8333)(2.8333) = 8.0278 - 8.0276 = 0.0002$

$\therefore$ The given random variables X and Y are not statistically independent.

5) Determine,
i) Marginal distribution.
ii) Covariance between the discrete random variables X and Y along with corelation using the joint probability distribution.

X \ Y	1	3	9
2	$1/8$	$1/24$	$1/12$
4	$1/4$	$1/4$	0
6	$1/8$	$1/24$	$1/12$

Sol^n: Given

$x_1 = 2, x_2 = 4, x_3 = 6, y_1 = 1, y_2 = 3, y_3 = 9$

And the probabilities are

$p_{11} = \frac{1}{8}, p_{12} = \frac{1}{24}, p_{13} = \frac{1}{12}, p_{21} = \frac{1}{4}, p_{22} = \frac{1}{4}, p_{23} = 0, p_{31} = \frac{1}{8}, p_{32} = \frac{1}{24}, p_{33} = \frac{1}{12}$

The joint distribution table is as follows

X \ Y	1	3	9	$f(x_i)$
2	$1/8$	$1/24$	$1/12$	$1/4$
4	$1/4$	$1/4$	0	$1/2$
6	$1/8$	$1/24$	$1/12$	$1/4$
$g(y_i)$	$1/2$	$1/3$	$1/6$	1

The marginal distributions of X and Y are

x_i	2	4	6
$f(x_i)$	$1/4$	$1/2$	$1/4$

y_i	3	4	5
$g(y_i)$	$1/2$	$1/3$	$1/6$

$\mu_X = E(X) = \sum_{i=1}^{3} x_i f(x_i) = \left(2 \times \frac{1}{4}\right) + \left(4 \times \frac{1}{2}\right) + \left(6 \times \frac{1}{4}\right) = 0.5 + 2 + 1.5 = 4$

$\mu_Y = E(Y) = \sum_{j=1}^{3} y_j g(y_j) = \left(1 \times \frac{1}{2}\right) + \left(3 \times \frac{1}{3}\right) + \left(9 \times \frac{1}{6}\right) = 0.5 + 1 + 1.5 = 3$

$E(XY) = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j f(x_i, y_j)$

$= (1)(-2)(0.1) + (1)(-1)(0.2) + (1)(4)(0) + (1)(5)(0.3) + (2)(-2)(0.2) + (2)(-1)(0.1) + (2)(4)(0.1) + (2)(5)(0)$

$= -0.2 - 0.2 + 0 + 1.5 - 0.8 - 0.2 + 0.8 + 0 = 2.3 - 1.4 = 0.9$

$\sigma_X{}^2 = E(X^2) - \mu_X{}^2 = \sum_{i=1}^{3} x_i{}^2 f(x_i) - \mu_X{}^2$

$= \left(2^2 \times \frac{1}{4}\right) + \left(4^2 \times \frac{1}{2}\right) + \left(6^2 \times \frac{1}{4}\right) - 4^2 = 18 - 16 = 2 \Rightarrow \sigma_X = 1.4142$

$\sigma_Y{}^2 = E(Y^2) - \mu_Y{}^2 = \sum_{j=1}^{3} y_j{}^2 g(y_j) - \mu_Y{}^2$

$= \left(1^2 \times \frac{1}{2}\right) + \left(3^2 \times \frac{1}{3}\right) + \left(9^2 \times \frac{1}{6}\right) - 3^2 = 17 - 9 = 8 \Rightarrow \sigma_Y = 2.8284$

$COV(X,Y) = E(XY) - \mu_X\mu_Y = 12 - (4)(3) = 12 - 12 = 0$

$\rho(X,Y) = \frac{COV(X,Y)}{\sigma_X\sigma_Y} = \frac{0}{1.4142\times2.8284} = 0$

∴The given random variables X and Y are not statistically independent.

6) Determine,

i) Marginal distribution.

ii) Covariance between the discrete random variables X and Y along with corelation using the joint probability distribution.

X \ Y	-3	2	4
1	0.1	0.2	0.2
2	0.3	0.1	0.1

Sol^n: Given

$x_1 = 1, x_2 = 2, y_1 = -3, y_2 = 2, y_3 = 4$

And the probabilities are

$p_{11} = 0.1, p_{12} = 0.2, p_{13} = 0.2, p_{21} = 0.3, p_{22} = 0.1, p_{23} = 0.1$

The joint distribution table is as follows

X \ Y	-3	2	4	$f(x_i)$
1	0.1	0.2	0.2	0.5
2	0.3	0.1	0.1	0.5
$g(y_i)$	0.4	0.3	0.3	1

The marginal distributions of X and Y are

x_i	1	2
$f(x_i)$	0.5	0.5

y_i	-3	2	4
$g(y_i)$	0.4	0.3	0.3

$\mu_X = E(X) = \sum_{i=1}^{2} x_i f(x_i) = (1 \times 0.5) + (2 \times 0.5) = 0.5 + 1 = 1.5$

$\mu_Y = E(Y) = \sum_{j=1}^{3} y_j g(y_j) = (-3 \times 0.4) + (2 \times 0.3) + (4 \times 0.3) = -1.2 + 0.6 + 1.2 = 0.6$

$E(XY) = \sum_{i=1}^{2}\sum_{j=1}^{3} x_i y_j f(x_i, y_j)$

$= (1)(-3)(0.1) + (1)(2)(0.2) + (1)(4)(0.2) + (2)(-3)(0.3) + (2)(2)(0.1) + (2)(4)(0.1)$

$= -0.3 + 0.4 + 0.8 - 1.8 + 0.4 + 0 = 2.3 - 1.4 = 0.9$

$\sigma_X{}^2 = E(X^2) - \mu_X{}^2 = \sum_{i=1}^{2} x_i{}^2 f(x_i) - \mu_X{}^2$

$= (1^2 \times 0.5) + (2^2 \times 0.5) - 1.5^2 = 2.5 - 2.25 = 0.25 \Rightarrow \sigma_X = 0.5$

$\sigma_Y{}^2 = E(Y^2) - \mu_Y{}^2 = \sum_{j=1}^{3} y_j{}^2 g(y_j) - \mu_Y{}^2$

$= (-3^2 \times 0.4) + (2^2 \times 0.3) + (4^2 \times 0.3) - 0.6^2 = 9.6 - 0.36 = 9.24 \Rightarrow \sigma_Y = 3.0397$

$COV(X,Y) = E(XY) - \mu_X\mu_Y = 0.9 - (1.5)(0.6) = 0.9 - 0.9 = 0$

$\rho(X,Y) = \frac{COV(X,Y)}{\sigma_X\sigma_Y} = \frac{0}{0.5\times 3.0397} = 0$

$\therefore$ The given random variables X and Y are not statistically independent.

7) X and Y are independent random variables. X takes the values 2,5 and 7 with probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$ respectively. Y takes the values 3,4 and 5 with the probabilities $\frac{1}{3}, \frac{1}{3} \& \frac{1}{3}$.

a) Find the JPD of X and Y

b) Show that COV (X, Y) = 0

Sol^n:

Given X & Y are independent random variables follows the marginal probabilities as below.

x	2	5	7
$f(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

y	3	4	5
$g(y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

The joint distribution table is as follows

X \ Y	3	4	5	$f(x_i)$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$
5	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
7	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$
$g(y_i)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

$\therefore \boldsymbol{\mu_X = E(X) = \sum x_i f(x_i) = \left(2\times\frac{1}{2}\right) + \left(5\times\frac{1}{4}\right) + \left(7\times\frac{1}{4}\right) = 4}$

$\therefore \boldsymbol{\mu_Y = E(X) = \sum y_j g(y_j) = \left(3\times\frac{1}{3}\right) + \left(4\times\frac{1}{3}\right) + \left(5\times\frac{1}{3}\right) = 4}$

$\therefore E(XY) = \sum_{i=1}^{2}\sum_{j=1}^{3} x_i y_j f(x_i, y_j) = 1 + \frac{8}{6} + \frac{10}{6} + \frac{15}{12} + \frac{20}{12} + \frac{25}{12} + \frac{21}{12} + \frac{28}{12} + \frac{35}{12} = \frac{192}{12} = 16$

$\therefore COV(X,Y) = E(XY) - \mu_X\mu_Y = 16 - (4)(4) = 16 - 16 = 0$

Stochastic Process

Stochastic process consists of sequence of experiments in which each experiment has a finite number of outcomes with the given probabilities.

Probability Vector

A vector $V = [v_1, v_2, v_3, \ldots\ldots, v_n]$ is called the probability vector if each one of its components are non-negative and their sum is equal to unity or 1.

Ex: $= [0.1, 0.6, 0.3]$, $V = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$, etc…

Stochastic Matrix

A square matrix P is called a stochastic matrix if all the entries of P are non-negative and the sum of all the entries of any row is 1

(or)

A square matrix P is called a stochastic matrix where each row is in the form of the probability vector.

Ex: $= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Regular Stochastic Matrix

A matrix P is said to be a Regular Stochastic Matrix, if all the entries of some power (P^n) are positive. The Regular Stochastic Matrix P has a unique probability vector Q such that QP=Q and all the sum of the probabilities of a fixed vector matrix should be equal to 1.

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn} \end{bmatrix}$$

Transition Matrix

A transition matrix is also known as a stochastic or probability matrix, is a square matrix (n x n) representing the transition probabilities of a stochastic system.

Ex: $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

PROBLEMS

1) Verify that the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}$ is a regular stochastic matrix.

Sol^n: Given matrix A , each element is nonnegative and the sum of the elements in each row is equal to 1.

∴A is stochastic matrix.

$$\text{Let } A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 5/16 & 9/16 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 0 & 1 & 0 \\ 1/8 & 5/16 & 9/16 \\ 1/2 & 1/4 & 1/4 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 5/32 & 41/64 & 13/64 \\ 1/8 & 5/16 & 9/16 \end{bmatrix}$$

∴Hence, all the entries in A^3 are nonnegative or positive and the sum of each row =1.

∴ The given matrix A is regular stochastic matrix.

2) Prove that the Markov Chain with Transition matrix $A = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ **is irreducible.**

Sol^n: Given matrix A is a stochastic matrix (Being a transition matrix).

Also, all the elements of given matrix does have non-negative and the sum of each row=1.

$$\therefore A^2 = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/16 \\ 1/4 & 1/3 & 5/12 \end{bmatrix}$$

∴Hence, all the entries in A^2 are nonnegative or positive and the sum of each row =1.

Hence the given transition matrix A is regular, consequently it follows that the given Markov Chain is irreducible.

3) Find the fixed probability vector for the regular stochastic matrix $A = \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$.

Sol^n: Given $A = \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix}$

Since, the given matrix A is of second order.

Let $Q = [x \quad y]$ be the fixed probability vector, for every $x \geq 0, y \geq 0 \& x + y = 1$

$$\therefore \boldsymbol{QA} = [\boldsymbol{x} \quad \boldsymbol{y}] \cdot \begin{bmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\boldsymbol{x} + \frac{1}{4}\boldsymbol{y} & \frac{2}{3}\boldsymbol{x} + \frac{3}{4}\boldsymbol{y} \end{bmatrix}$$

Since QA=Q

$$\Rightarrow \begin{bmatrix} \frac{1}{3}x + \frac{1}{4}y & \frac{2}{3}x + \frac{3}{4}y \end{bmatrix} = [x \quad y]$$

$$\Rightarrow \frac{1}{3}x + \frac{1}{4}y = x, \frac{2}{3}x + \frac{3}{4}y = y$$

$$\Rightarrow \frac{2}{3}x = \frac{1}{4}y, \frac{2}{3}x = \frac{1}{4}y \ldots (1), We have x + y = 1 \Rightarrow y = 1 - x$$

$$\therefore (1) \Rightarrow \frac{2}{3}x = \frac{1}{4}(1 - x)$$

$$\Rightarrow \frac{2}{3}x + \frac{1}{4}x = \frac{1}{4}$$

$$\Rightarrow \frac{2}{3}x + \frac{1}{4}x = \frac{1}{4}$$

$$\Rightarrow \frac{8x + 3x}{12} = \frac{1}{4}$$

$$\Rightarrow \frac{11}{3}x = 1 \Rightarrow x = \frac{3}{11}$$

$$\therefore y = 1 - x \Rightarrow y = 1 - \frac{3}{11} \Rightarrow y = \frac{8}{11}$$

Thus, the required fixed probability vector is $Q = [x \quad y] = \begin{bmatrix} \frac{3}{11} & \frac{8}{11} \end{bmatrix}$

4) Find the fixed probability vector of the regular stochastic matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$.

Sol^n: Since the given matrix P is of order 3x3, the required fixed probability vector Q must be also order of 3x3.

Let $Q = [x \quad y \quad z]$, $\quad For\ every\ x \geq 0, y \geq 0, z \geq 0 \& x + y + z = 1$

Also, QP=Q

$$\therefore [x \quad y \quad z]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{z}{2} & x+\frac{z}{2} & y \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{z}{2} & x+\frac{z}{2} & y \end{bmatrix} = [x \quad y \quad z]$$

$$\Rightarrow \frac{z}{2} = x, x + \frac{z}{2} = y, y = z$$

$$\Rightarrow \frac{1}{2}(1-x-y) = x \ldots\ldots (1), x + \frac{1}{2}(1-x-y) = y \ldots (2), y = 1-x-y \ldots (3)$$

$$\Rightarrow 3x + y = 1, x - 3y = -1, x + 2y = 1$$

$$\Rightarrow x = \frac{1}{5}, y = \frac{2}{5} \Rightarrow z = 1 - \frac{1}{5} - \frac{2}{5} = \frac{2}{5}$$

Hence the required fixed probability vector is $Q = [x \quad y \quad z] = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$

5) Find the fixed probability vector of the regular stochastic matrix $\mathbf{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$.

Sol^n:

Given, $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$

Since the given matrix P is of order 3x3, the required fixed probability vector Q must be also order of 3x3.

Let $Q = [x \quad y \quad z] For every x \geq 0, y \geq 0, z \geq 0 \& x + y + z = 1$

Also, QP=Q

$$\therefore QP = [x \quad y \quad z]\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow QP = \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y & \frac{1}{4}x + z & \frac{1}{4}x + \frac{1}{2}y \end{bmatrix}$$

WKT

$QP = Q$

$$\Rightarrow \begin{bmatrix} \frac{1}{2}x + \frac{1}{2}y & \frac{1}{4}x + z & \frac{1}{4}x + \frac{1}{2}y \end{bmatrix} = [x \quad y \quad z]$$

$$\Rightarrow x = \frac{x}{2} + \frac{y}{2}, y = \frac{x}{4} + z, z = \frac{1}{4}x + \frac{1}{2}y$$

$$\Rightarrow \frac{x}{2} + \frac{y}{2} = 0, \frac{x}{4} + (1-x-y) - y = 0, \frac{1}{4}x + \frac{1}{2}y = 1 - x - y$$

$$\Rightarrow x + y = 0, 3x + 8y = 4 \ldots\ldots (1), 5x + 6y = 4 \ldots\ldots (2)$$

By solving eq (1) & (2)

$$\Rightarrow x = \frac{4}{11}, y = \frac{4}{11}, z = 1 - \frac{4}{11} - \frac{4}{11} = 1 - \frac{8}{11} = \frac{3}{11}$$

$$Q = [x \quad y \quad z] = \begin{bmatrix} \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \end{bmatrix}$$

6) Find the fixed probability vector of the regular stochastic matrix $\mathbf{P} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$.

Sol^n:

Given, $P = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

Since the given matrix P is of order 3x3, the required fixed probability vector Q must be also order of 3x3.

Let $Q = [x \quad y \quad z] For every x \geq 0, y \geq 0, z \geq 0 \& x + y + z = 1$

Also, QP=Q

$$\therefore QP = [x \quad y \quad z]\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\Rightarrow QP = \begin{bmatrix} \frac{1}{6}y & x + \frac{1}{2}y + \frac{2}{3}z & \frac{1}{3}y + \frac{1}{3}z \end{bmatrix}$$

$WKT,\ QP = Q$

$$\Rightarrow \begin{bmatrix} \frac{1}{6}y & x + \frac{1}{2}y + \frac{2}{3}z & \frac{1}{3}y + \frac{1}{3}z \end{bmatrix} = [x \quad y \quad z]$$

$$\Rightarrow x = \frac{1}{6}y, y = x + \frac{1}{2}y + \frac{2}{3}z, z = \frac{1}{3}y + \frac{1}{3}z$$

$$\Rightarrow 6x - y = 0, 2x - 7y = -4 \ldots (1) 2x + 3y = 2 \ldots\ldots (2)$$

By solving eq (1) & (2)

$$\Rightarrow x = \frac{1}{10}, y = \frac{6}{10}, z = 1 - \frac{1}{10} - \frac{6}{10} = 1 - \frac{7}{10} = \frac{3}{10}$$

$$\therefore Q = [x \quad y \quad z] = \begin{bmatrix} \frac{1}{10} & \frac{6}{10} & \frac{3}{10} \end{bmatrix}$$

7) Find the fixed probability vector of the regular stochastic matrix P $= \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$.

Sol^n:

Given, $P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

Since the given matrix P is of order 3x3, the required fixed probability vector Q must be also order of 3x3.

Let $Q = [x \quad y \quad z] For every x \geq 0, y \geq 0, z \geq 0 \& x + y + z = 1$

Also, QP=Q

$$\therefore QP = [x \quad y \quad z]\begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\Rightarrow QP = \begin{bmatrix} \frac{1}{2}y + \frac{1}{2}z & \frac{2}{3}x + \frac{1}{2}z & \frac{1}{3}x + \frac{1}{2}y \end{bmatrix}$$

WKT

$QP = Q$

$$\Rightarrow \begin{bmatrix} \frac{1}{2}y + \frac{1}{2}z & \frac{2}{3}x + \frac{1}{2}z & \frac{1}{3}x + \frac{1}{2}y \end{bmatrix} = [x \quad y \quad z]$$

$$\Rightarrow x = \frac{1}{2}y + \frac{1}{2}z, y = \frac{2}{3}x + \frac{1}{2}z, z = \frac{1}{3}x + \frac{1}{2}y$$

$$\Rightarrow 3x - 1 = 0, x - 9y = -3, 8x + 9y = 6$$

$$\Rightarrow x = \frac{9}{27}, y = \frac{10}{27}, z = \frac{8}{27}$$

$$\therefore Q[x \quad y \quad z] = \begin{bmatrix} \frac{9}{27} & \frac{10}{27} & \frac{8}{27} \end{bmatrix}$$

8) If $P_1 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ **and** $P_2 = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$. **Show that P_1, P_2 and $P_1 P_2$ are stochastic matrices.**

Sol^n: In P_1 we have $a + (1-a) = 1$ and $b + (1-b) = 1$

In P_2 we have $b + (1-b) = 1$ and $a + (1-a) = 1$

Thus, P_1 and P_2 are stochastic matrices.

Now, $\mathbf{P_1P_2} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$

$$= \begin{bmatrix} (1-a)(1-b)+a^2 & b(1-a)+a(1-a) \\ b(1-b)+a(1-b) & (1-a)(1-b)+b^2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} (say)$$

We shall know that $a_1 + b_1 = 1$ and $a_2 + b_2 = 1$

Now,

$$\begin{aligned} a_1 + b_1 &= (1-a)(1-b) + a^2 + b(1-a) + a(1-a) \\ &= (1-a)\{1-b+b\} + a\{a+1-a\} \\ &= 1-a+a \\ &= 1 \end{aligned}$$

$\therefore a_1 + b_1 = 1$

Also,

$$\begin{aligned} a_2 + b_2 &= b(1-b) + a(1-b) + (1-b)(1-a) + b^2 \\ &= b\{1-b+b\} + (1-b)\{a+1-a\} \\ &= b+1-b \\ &= 1 \end{aligned}$$

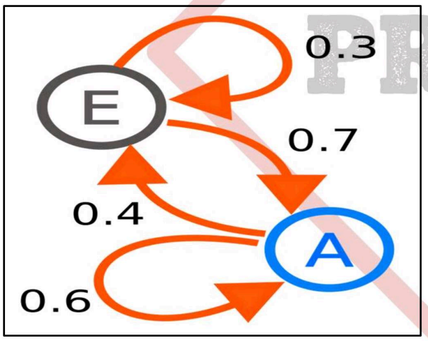
$\therefore a_2 + b_2 = 1$

Thus, $\mathbf{P_1P_2}$ is a stochastic matrix.

Markov Chain

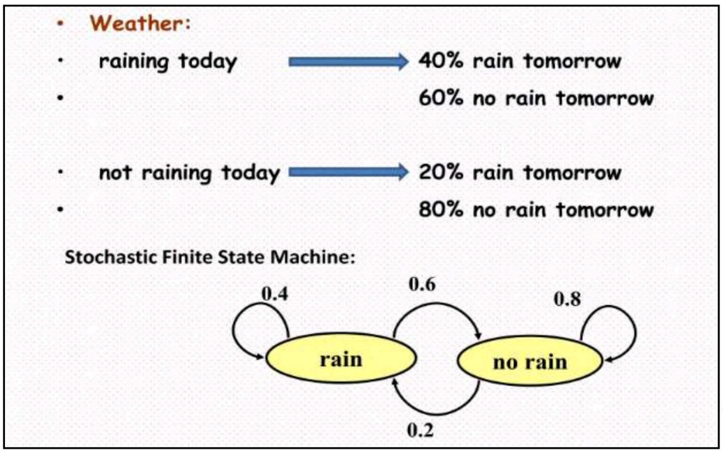
A Markov Chian or Markov process is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event.

Ex (1).



$$A = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}$$

Ex (2).



$$A = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{bmatrix}$$

Higher transition probabilities:

Let P be a $n \times n$ transition probability matrix of the Markov chain with the probabilities $p_{ij}, 1 \le i, j \le n$, is called changes from a_i to the state a_j, that is $a_i \to a_j$.The probabilities that the system changes from a_i to the state a_j in exists n steps is denoted by $p^{(n)}{}_{ij}$ and the matrix formed by the probabilities $p^{(n)}{}_{ij}$is called the n - step transition matrix , denoted by $P^{(n)}$ and initial probabilities are defined as,

$$p^{(0)} = [p_1{}^{(0)}, p_2{}^{(0)}, p_3{}^{(0)}, p_4{}^{(0)} \ldots\ldots p_n{}^{(0)}]$$
$$p^{(1)} = [p_1{}^{(1)}, p_2{}^{(1)}, p_3{}^{(1)}, p_4{}^{(1)} \ldots\ldots p_n{}^{(1)}]$$
$$p^{(2)} = [p_1{}^{(2)}, p_2{}^{(2)}, p_3{}^{(2)}, p_4{}^{(2)} \ldots\ldots p_n{}^{(2)}]$$
$$-------------------$$
$$p^{(n)} = [p_1{}^{(n)}, p_2{}^{(n)}, p_3{}^{(n)}, p_4{}^{(n)} \ldots\ldots p_n{}^{(n)}]$$

And $p^{(1)}, p^{(2)}, p^{(3)}, p^{(4)} \ldots.$ will be evaluated as

$$p^{(1)} = p^{(0)}P,\ p^{(2)} = p^{(1)}P = p^{(0)}P^2 ,\ p^{(3)} = p^{(2)}P = p^{(0)}P^3 \ldots.. p^{(n)} = p^{(n-1)}P = p^{(0)}P^n$$

1) Consider the $t.p.m.$ of the $P = \begin{array}{c} t \\ b \end{array}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$, hence find P^2 , P^3, also find $p^{(3)}$take the initial probability distribution the person rolled a die and decided that he will go by bus if the number appeared on the face is divisible by 3.

Sol^n: Given,

$$P = \begin{array}{c} t \\ b \end{array}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} p_{tt} & p_{tb} \\ p_{tb} & p_{bb} \end{bmatrix}$$

$$\therefore P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} p^{(2)}{}_{tt} & p^{(2)}{}_{tb} \\ p^{(2)}{}_{tb} & p^{(2)}{}_{bb} \end{bmatrix}$$

$$\therefore P^3 = P^2.P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow P^3 = \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix} = \begin{bmatrix} p^{(3)}{}_{tt} & p^{(3)}{}_{tb} \\ p^{(3)}{}_{tb} & p^{(3)}{}_{bb} \end{bmatrix}$$

$\therefore p^{(2)}{}_{tb} = \frac{1}{2}$ Means that the probability that the system changes from the state $t \to b$ in exactly 2 steps is $\frac{1}{2}$.

$\therefore p^{(3)}{}_{bt} = \frac{3}{8}$ Means that the probability that the system changes from the state $b \to t$ in exactly 3 steps is $\frac{3}{8}$.

Given, the probability distribution is evaluated from the person rolled a die and decided that he will go by bus if the number appeared on the face is divisible by 3.

$$\therefore p(b) = \frac{2}{6} = \frac{1}{3} ,\ \Rightarrow p(t) = 1 - p(b) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$p^{(0)} = [p(t) \quad p(b)] = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

$$\therefore p^{(2)} = p^{(0)}P^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} . \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} = \begin{bmatrix} \frac{5}{12} & \frac{7}{12} \end{bmatrix}$$

$$\therefore p^{(3)} = p^{(0)}P^3 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} . \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix} = \begin{bmatrix} \frac{7}{24} & \frac{17}{24} \end{bmatrix} = [p_t{}^{(3)} \quad p_b{}^{(3)}]$$

$\therefore$ The probability of travelling by train after 3 days $=\frac{7}{24}$

$\therefore$ The probability of travelling by bus after 3 days $=\frac{17}{24}$

2) The transition matrix P of a Markov chain is given by $\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with initial probability distribution $p^{(0)} = [1/4 \quad 3/4]$. Define and find the following i) $p_{21}{}^{(2)}$ ii) $p_{12}{}^{(2)}$ iii) $p^{(2)}$ iv) $p_1{}^{(2)}$ v) the vector $p^{(0)}P^n$ approaches. Vi) The matrix approaches.

Sol^n:

Given transition matrix $P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$

$$P^2 = P.P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} . \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$$

$$\Rightarrow P^2 = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix} = \begin{bmatrix} p_{11}{}^{(2)} & p_{12}{}^{(2)} \\ p_{21}{}^{(2)} & p_{22}{}^{(2)} \end{bmatrix}$$

$$\therefore p_{21}{}^{(2)} = \frac{9}{16} \ , p_{12}{}^{(2)} = \frac{3}{8}$$

Given initial probability distribution is $p^{(0)} = [1/4 \quad 3/4]$

$$\therefore p^{(2)} = p^{(0)}P^2 = [1/4 \quad 3/4] . \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$$

$$\Rightarrow p^{(2)} = \begin{bmatrix} \frac{37}{64} & \frac{27}{64} \end{bmatrix} = [p_1{}^{(2)} \quad p_2{}^{(2)}]$$

$$\therefore p_1{}^{(2)} = \frac{37}{64}$$

$p^{(0)}P^n$ Approaches the unique probability vector $Q = [x \quad y]$ for which $QP = Q$

$$\Rightarrow [x \quad y] \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = [x \quad y]$$

$$\Rightarrow \begin{bmatrix} \frac{x}{2} + \frac{3y}{4} & \frac{x}{2} + \frac{y}{4} \end{bmatrix} = [x \quad y]$$

$$\Rightarrow \frac{x}{2} + \frac{3y}{4} = x, \frac{x}{2} + \frac{y}{4} = y$$

$$\Rightarrow \frac{x}{2} + \frac{3y}{4} = x, \frac{x}{2} + \frac{y}{4} = y$$

$$\Rightarrow -\frac{x}{2} + \frac{3(1-x)}{4} = 0$$

$$\Rightarrow -\frac{x}{2} - \frac{3x}{4} = -\frac{3}{4}$$

$$\Rightarrow \frac{5}{4}x = \frac{3}{4}$$

$$\Rightarrow x = \frac{3}{5} \Rightarrow y = \frac{2}{5}$$

$$\therefore Q[x \quad y] = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$$

Therefore, the vector $p^{(0)}P^n$ approaches the vector $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix}$

Therefore, the vector P^n approaches the matrix $\begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$

3) The t.p.m. of a Markov chain is given by $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$ and the initial probability distribution is $p^{(0)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$. Find $p_{13}{}^{(2)}$, $p_{23}{}^{(2)}$, $p^{(2)} p_1{}^{(2)}$.

Sol^n:

Given transition matrix of Markov chain $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$

And the initial probability distribution $p^{(0)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$

$$\therefore P^2 = P.P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} . \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$\Rightarrow P^2 == \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix} = \begin{bmatrix} p^{(2)}{}_{11} & p^{(2)}{}_{12} & p^{(2)}{}_{13} \\ p^{(2)}{}_{21} & p^{(2)}{}_{22} & p^{(2)}{}_{23} \\ p^{(2)}{}_{31} & p^{(2)}{}_{32} & p^{(2)}{}_{33} \end{bmatrix}$$

$$\therefore p^{(2)}{}_{13} = 3/8 , p^{(2)}{}_{23} = 1/2$$

$$\therefore p^{(2)} = p^{(0)} P^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} . \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$\therefore p^{(2)} = [p_1{}^{(2)} \quad p_2{}^{(2)} \quad p_3{}^{(2)}] = \begin{bmatrix} \frac{7}{16} & \frac{1}{8} & \frac{7}{16} \end{bmatrix}$$

$$\therefore p_1{}^{(2)} = \frac{7}{16}$$

4) Prove that the Markov chain whose t.p.m is $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible. Find the corresponding stationary probability vector.

Sol^n:

Given transition matrix of Markov chain

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$\Rightarrow P = \frac{1}{6} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\Rightarrow P^2 = P.P = \frac{1}{6} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix} . \frac{1}{6} \begin{bmatrix} 0 & 4 & 2 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{bmatrix}$$

$$\Rightarrow P^2 == \frac{1}{36} \begin{bmatrix} 18 & 6 & 12 \\ 9 & 21 & 6 \\ 9 & 12 & 15 \end{bmatrix}$$

Since all the entries of P^2 are non- negative, thus the given t.p.m P is regular and hence the Markov chain having t.p.m P is irreducible.

Let the unique probability vector $Q = [x \quad y \quad z]$ for which $QP = Q$, $\forall x + y + z = 1$

$$QP = [x \quad y \quad z]\begin{bmatrix} 0 & ^2/_3 & ^1/_3 \\ ^1/_2 & 0 & ^1/_2 \\ ^1/_2 & ^1/_2 & 0 \end{bmatrix} = [x \quad y \quad z]$$

$\Rightarrow \left[\frac{y}{2}+\frac{z}{2} \quad \frac{2x}{3}+\frac{z}{2} \quad \frac{x}{3}+\frac{y}{2}\right] = [x \quad y \quad z]$

$\Rightarrow \frac{y}{2}+\frac{z}{2} = x, \frac{2x}{3}+\frac{z}{2} = y, \frac{x}{3}+\frac{y}{2} = z$

$\Rightarrow 2x - y - z = 0, 4x - 6y + 3z = 0, 2x + 3y - 6z = 0$

$\Rightarrow 2x - y - (1 - x - y) = 0, 4x - 6y + 3(1 - x - y) = 0$

$\Rightarrow 3x = 1 \& x - 9y = -3$

$\Rightarrow x = \frac{1}{3}, \frac{1}{3} - 9y = -3 \Rightarrow 9y = \frac{10}{3} \Rightarrow y = \frac{10}{27}$

$\therefore z = 1 - x - y = 1 - \frac{1}{3} - \frac{10}{27} = \frac{8}{27}$

$\therefore Q[x \quad y \quad z] = \left[\frac{1}{3} \quad \frac{10}{27} \quad \frac{8}{27}\right]$ is the required stationary probability vector.

5) A student's study habits are as follows. If he studies one night, he is 30% sure to study the next night. On the other hand, if he does not study one night, he is 40% sure to study the next night. Find the transition matrix for the chain of his study.

Sol^n: We have two possible states

a_1= Studying a_2=Not studying

Therefore, given that

p_{11} =Probability of studying on night, given that he has studied in the previous night=30%=0.3

p_{12} =Probability of not studying on night, given that he has studied the previous night=70%=0.7

p_{21} =Probability of studying on night, given that he has not studied the previous night=40%=0.4

p_{22} =Probability of not studying on night, given that he has not studied the previous night=60%=0.6

Accordingly, the transition matrix of the chain of study is $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$

Let the unique probability vector $Q = [x \quad y]$ for which $QP = Q$

$\therefore [x \quad y]\begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \quad y]$

$\Rightarrow [0.3x + 0.4y \quad 0.7x + 0.6y] = [x \quad y]$

$\Rightarrow 0.3x + 0.4y = x, 0.7x + 0.6y = y$

$\Rightarrow 0.7x - 0.4y = 0$

$\Rightarrow 0.7x - 0.4(1 - x) = 0$

$\Rightarrow 1.1x - 0.4 = 0$

$\Rightarrow x = \frac{0.4}{1.1} = \frac{4}{11} \Rightarrow y = 1 - \frac{4}{11} = \frac{7}{11}$

$\therefore Q[x \quad y] = \left[\frac{4}{11} \quad \frac{7}{11}\right] = [p_{a_1} p_{a_2}]$

Thus, we conclude that in the long run the student will study $\frac{4}{11}$ of the time or 36.36 % of the time.

6) A software engineer goes to his work-place every day by motor bike or by car. He never goes by a bike on two consecutive days; but if he goes by car on a day then he is equally likely to go by car or bike on the next day. Find the transition matrix for the chain of the mode of transport he uses. If car is used on the first day of a week, find the probability that, (i) Bike is used, (ii) Car is used on the fifth day.

Sol^n: Given the Markov chain of the mode of transport has the following two states:

a_1= Using bike a_2=Using car

And to find,

p_{11} =Probability of using bike on a day, given that bike has been used on the previous day=0 (Because bike is not used on two consecutive days)

p_{12} =Probability of using car on a day, given that the bike has been used on the previous day=1 (Because it is certain that car is used on a day if bike is used on the previous day)

p_{21} =Probability of using bike on a day, given that car is used on the previous day= ½ (Because using car or bike on a day are equally likely if car is used on the previous day)

p_{22} =Probability of using car on a day, given that car is used on the previous day=1/2

Hence the transition matrix for the chain of the mode of transport is $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

∴ The initial probability distribution vector of the mode of transport is given by $p^{(0)} = [p_1{}^{(0)} \quad p_2{}^{(0)}] = [0 \quad 1]$

$$\therefore P^2 = P.P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} . \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$\Rightarrow P^4 = P^2.P^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} . \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

$$\Rightarrow P^4 = \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

$$\therefore p^{(4)} = p^{(0)} P^4 = [0 \quad 1] . \begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix}$$

$$\Rightarrow p^{(4)} = [p_1{}^{(4)} \quad p_2{}^{(4)}] = \begin{bmatrix} \frac{5}{16} & \frac{11}{16} \end{bmatrix}$$

Therefore, on the fifth day the probability of using the bike is $p_1{}^{(4)} = \frac{5}{16}$, the probability of using the car is $p_2{}^{(4)} = \frac{11}{16}$.

7) A man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non-filter cigarettes the next week with the probability 0.2. On the other hand, if he smokes non filter cigarettes one week there is a probability of 0.7that he will smoke non filter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes?

Sol^n:

Let A= Smoking filter cigarettes   B= Smoking non filter cigarettes

Therefore, the associated transition probability matrix is as follows

$$P = \begin{bmatrix} p_{AA}{}^{(1)} & p_{AB}{}^{(1)} \\ p_{BA}{}^{(1)} & p_{BB}{}^{(1)} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

Let the unique probability vector $Q = [x \quad y]$ for which $QP = Q, \forall x + y = 1$

$$\therefore [x \quad y] \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = [x \quad y]$$

$$\Rightarrow [0.8x + 0.3y \quad 0.2x + 0.7y] = [x \quad y]$$

$$\Rightarrow 0.8x + 0.3y = x, 0.2x + 0.7y = y$$

$$\Rightarrow 0.2x - 0.3y = 0, 0.2x - 0.3y = 0$$

$$\Rightarrow 0.2x - 0.3(1 - x) = 0$$

$$\Rightarrow 0.2x + 0.3x - 0.3 = 0$$

$$\Rightarrow 0.5x = 0.3$$

$$\Rightarrow x = \frac{0.3}{0.5} = \frac{3}{5} \Rightarrow y = \frac{0.2}{0.5} = \frac{2}{5}$$

$$\therefore Q = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \end{bmatrix} = [p_A \quad p_B]$$

Thus, in the long run, he will smoke filter cigarettes $\frac{3}{5}$ or 60% of the time.

(SMOKING IS INJURIOUS TO HEALTH, IT CAUSES CANCER AND TOBACCO CAUSES PAINFUL DEATH)

8) Three boys A, B, C are throwing ball to each other. "A" always throws the ball to "B" and "B" always throws ball to "C". "C" is just as likely to throw the ball to "B" as to "A". If, "C" was the first person to throw the ball, find the probabilities that after three throws.

i) A has the ball

ii) B has the ball

iii) C has the ball

Sol^n: Given three boys A, B, C are throwing a ball associated with the transition probability matrix of the Markov chain as below,

$$P = \begin{bmatrix} p_{AA}^{(1)} & p_{AB}^{(1)} & p_{AC}^{(1)} \\ p_{BA}^{(1)} & p_{BB}^{(1)} & p_{BC}^{(1)} \\ p_{CA}^{(1)} & p_{CB}^{(1)} & p_{CC}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$\Rightarrow P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$\Rightarrow P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\therefore P^3 = P^2.P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} . \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$\Rightarrow P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Initially if C has the ball, associated with the initial probability vector is given by $p^{(0)} = [0 \quad 0 \quad 1]$

$$\therefore p^{(3)} = p^{(0)} P^3 = [0 \quad 0 \quad 1] . \begin{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \end{bmatrix} = [1/4 \quad 1/4 \quad 1/2]$$

$$\therefore p^{(3)} = [p_A^{(3)} \quad p_B^{(3)} \quad p_C^{(3)}] = [1/4 \quad 1/4 \quad 1/2]$$

Thus, after three throws, the probability that the ball is with A is $p_A^{(3)} = \frac{1}{4}$, with B is $p_B^{(3)} = \frac{1}{4}$ and with C is $p_C^{(3)} = \frac{1}{2}$

9) A gambler's luck follows a pattern: if he wins a game, the probability of winning next game is 0.6. However, he loses the game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game if so,

i) What is the probability of winning second game.

ii) What is the probability of winning the third game.

iii) In the long run, how often he will win.

Sol^n: Let W = Win the game , L = Lose the game

The transition probability matrix is given,

$$P = \begin{bmatrix} p_{ww} & p_{wl} \\ p_{lw} & p_{ll} \end{bmatrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}$$

And we know that the probability of winning and losing have the equal priority.

$\therefore$ The initial probability vector $p^{(0)} = [p_w^{(0)} \quad p_l^{(0)}] = [0.5 \quad 0.5]$

$$\therefore p^{(1)} = p^{(0)} P = [0.5 \quad 0.5] . \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = [0.45 \quad 0.55]$$

$\therefore p^{(2)} = p^{(1)}P = [0.45 \quad 0.55].\begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix} = [0.435 \quad 0.565]$

$\therefore p^{(3)} = p^{(2)}P = [0.435 \quad 0.565].\begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix} = [0.4305 \quad 0.5605]$

i) $\therefore {p_w}^{(2)} = 0.435 = 43.5\%$

ii) $\therefore {p_w}^{(3)} = 0.4305 = 43.05\%$

iii) Let Q=[x y] be the probability vector for which x+y=1

$\therefore QP = Q$

$\therefore [x \quad y].\begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix} = [x \quad y]$

$\Rightarrow 0.6x + 0.3y = x \ , \ 0.4x + 0.7y = y$

$\Rightarrow 0.4x - 0.3y = 0$

$\Rightarrow 0.4x - 0.3(1-x) = 0$

$\Rightarrow 0.4x - 0.3 + 0.3x = 0$

$\Rightarrow 0.7x = 0.3$

$\Rightarrow x = \frac{3}{7}$

$\Rightarrow y = 1 - x \Rightarrow y = 1 - \frac{3}{7} \Rightarrow y = \frac{4}{7}$

$\therefore Q = [p_w \quad p_l] = \begin{bmatrix}\frac{3}{7} & \frac{4}{7}\end{bmatrix}$

10) A Salesman's territory consists of three cities A, B, C. He never sells in the same city on successive days. If he sells in city A then the next day he sells in city B. If he sells in B or C then the next day is twice as likely to sell in city A as than other cities. In long run, how often does he sells in each of the city.

Sol^{n}:

Given a salesman can move to the cities A, B, C with the probabilities as below,

$$P = \begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix}{p_{AA}}^{(1)} & {p_{AB}}^{(1)} & {p_{AC}}^{(1)}\\ {p_{BA}}^{(1)} & {p_{BB}}^{(1)} & {p_{BC}}^{(1)}\\ {p_{CA}}^{(1)} & {p_{CB}}^{(1)} & {p_{CC}}^{(1)}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 2/3 & 0 & 1/3\\ 2/3 & 1/3 & 0\end{bmatrix}$$

Let $Q = [x \quad y \quad z]$ be the probability vector for which x+y+z=1

$\therefore QP = Q$

$\therefore [x \quad y \quad z].\begin{bmatrix}0 & 1 & 0\\ 2/3 & 0 & 1/3\\ 2/3 & 1/3 & 0\end{bmatrix} = [x \quad y \quad z]$

$\Rightarrow \begin{bmatrix}\frac{2y}{3} + \frac{2z}{3} & x + \frac{z}{3} & \frac{y}{3}\end{bmatrix} = [x \quad y \quad z]$

$\Rightarrow \frac{2y}{3} + \frac{2z}{3} = x \ , \ x + \frac{z}{3} = y \ , \ \frac{y}{3} = z$

$\Rightarrow 3x - 2y - 2z = 0 \ , \ 3x - 3y + z = 0$

$\Rightarrow 3x - 2y - 2(1 - x - y) = 0 \ , \ 3x - 3y + (1 - x - y) = 0$

$\Rightarrow 3x - 2y - 2 + 2x + 2y = 0 \ , \ 3x - 3y + 1 - x - y = 0$

$\Rightarrow 5x = 2 \ , \ 2x - 4y = -1$

$\Rightarrow x = \frac{2}{5}$

$\Rightarrow 4y = \frac{9}{5} \Rightarrow y = \frac{9}{20}$

$\Rightarrow z = 1 - x - y \Rightarrow z = 1 - \frac{2}{5} - \frac{9}{20} \Rightarrow z = \frac{3}{20}$

$\therefore Q = [x \quad y \quad z] = \begin{bmatrix}\frac{2}{5} & \frac{9}{20} & \frac{3}{20}\end{bmatrix}$

Thus, the salesman in the long run sells,

$\frac{2}{5}\ in\ city\ A = 40\%$, $\frac{9}{20}\ in\ city\ B = 45\%$, $\frac{3}{20}\ in\ city\ C = 15\%$

11) Every year, a man trades his car for a new car. If he has a Maruthi, he trades it for an Ambassador. If he has an Ambassador, he trades it for Santro. However if he had a Santro, he is just as likely to trade it for a Maruthi or an Ambassador. In 2000 he bought his first car which was a Santro. Find the probability that he has,

i) 2002 Santro

ii) 2002 Maruthi

iii) 2003 Ambassador

iv) 2003 Santro

Sol^n:

Given a man trades his car for a new car with the probabilities as below,

$$P = \begin{array}{c} M \\ A \\ S \end{array}\begin{bmatrix} p_{MM}{}^{(1)} & p_{MA}{}^{(1)} & p_{MS}{}^{(1)} \\ p_{AM}{}^{(1)} & p_{AA}{}^{(1)} & p_{AS}{}^{(1)} \\ p_{SM}{}^{(1)} & p_{SA}{}^{(1)} & p_{SS}{}^{(1)} \end{bmatrix} = \begin{array}{c} M \\ A \\ S \end{array}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Also given, he has bought his first car in 2000 was Santro.

$\therefore$ The initial probability vector $p^{(0)} = [p_M^{(0)} \quad p_A^{(0)} \quad p_S^{(0)}] = [0 \quad 0 \quad 1]$

$$\therefore P^2 = P.P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}.\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$\Rightarrow P^3 = P^2.P = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}.\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\therefore p^{(2)} = p^{(0)}P^2 = [0 \quad 0 \quad 1].\begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} = [0 \quad 1/2 \quad 1/2] = [p_M^{(2)} \quad p_A^{(2)} \quad p_S^{(2)}]$$

$$\therefore p^{(3)} = p^{(0)}P^3 = [0 \quad 0 \quad 1].\begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = [1/4 \quad 1/4 \quad 1/2] = [p_M^{(3)} \quad p_A^{(3)} \quad p_S^{(3)}]$$

i) $\therefore$ The probability to have a Santro car in the year 2002, $p_S^{(2)} = 1/2 = 50\%$

ii) $\therefore$ The probability to have a Maruthi car in the year 2002, $p_M^{(2)} = 0 = 0\%$

iii) $\therefore$ The probability to have an Ambassador car in the year 2003, $p_A^{(3)} = 1/4 = 25\%$

iv) $\therefore$ The probability to have a Santro car in the year 2003, $p_S^{(3)} = 1/2 = 50\%$

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