Chapter 4-

Part 2 : Advance counting Techniques :-

Recurrence Relations :-

→ The no of bacteria in a colony doubles every hour

→ If a colony begins with five bacteria, how many will be present in n hours ?

→ Because the no of bacteria doubles every hour the relationship $a_n = 2a_{n-1}$ holds whenever n is a positive integer

→ This relationship ($a_n = 2a_{n-1}$) holds together with condition $a_0 = 5$ uniquely determines a_n for all non negative integer n.

→ We can find formula for a_n from this information

→ Hence some counting problems can be solved by finding relationships, called recurrence relations between terms of a sequence.

Recurrence relation :-

→ A recurrence relation for sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of previous terms of sequence namely $a_0, a_1 \ldots a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is non- negative integer.

→ A sequence is called a solution of a recurrence relation if its term satisfy the recurrence relation

Examples:-

1) Let $\{a_n\}$ be sequence of recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4$ and suppose that $a_0 = 3$, $a_1 = 5$, what are a_2, a_3

Given recurrence relation

$$a_n = a_{n-1} - a_{n-2}$$

$a_2 = ?$ $\quad a_2 = a_1 - a_0 = 5 - 3 = 2.$

$a_3 = ?$ $\quad a_3 = a_2 - a_1 = 2 - 5 = -3$

similarly we can find a_4, a_5

∴ sequence we can find a_4, a_5

∴ sequence of recurrence relation solution of recurrence relation $= 3, 5, 2, -3 \ldots\ldots$

2) Determine whether sequence $\{a_n\}$ where $a_n = 3n$ for every non negative integer n is a solution of recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4 \ldots$

Given Recurrence relation

$$a_n = 2a_{n-1} - a_{n-2} \text{ for } n = 2, 3, 4 \ldots$$

$$a_n = 2[3(n-1)] - 3(n-2) \quad [\text{where } a_n = 3n]$$

$$a_n = 6n - 6 - 3n + 6$$

$$\boxed{a_n = 3n}$$

The sequence $\{a_n\}$ where $a_n = 3n$ is a solution of given recurrence relation

$$a_n = 2a_{n-1} - a_{n-2}$$

3) Determine whether sequence $\{a_n\}$ where $a_n = 2^n$ for every non negative integer, n is a solution of recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4 \ldots$ Answer the same question where $a_n = 5$

$a_n = 2^n$

$a_0 = 1 \; a_1 = 2, \; a_2 = 4$

$a_n = 2a_{n-1} - a_{n-2}$

$a_2 = 2a_1 - a_0$

$a_2 = 3$

we got $a_2 = 4$ and with recurrence relation we got $a_2 = 3$

sequence $a_n = 2^n$ does not satisfy recurrence relation

Given $\boxed{a_n = 5}$

$\boxed{a_n = 2a_{n-1} - a_{n-2}}$

for sequence $a_n = 5$, it is that $a_1 = 5$
$a_2 = 5$
$a_3 = 5$
$\vdots$
$a_n = 5$

$a_2 = 2a_1 - a_0 \Rightarrow \boxed{a_2 = 5}$

$a_n = 5$ the sequence (solution of recurrence relation) satisfies given recurrence relation $a_n = 2a_{n-1} - a_{n-2}$

Recursive Algorithm :
It provides solution of a problem of size n in terms of solutions of one or more instance of same problem of smaller size.

Recurrence Relation : when we analyze the complexity of a recursive algorithm, we obtain recurrence relation that expresses the no of operations to solve a problem of size n

Modeling with Recurrence Relation:-

Modeling with Recurrence Relation

1) compound interest - Recurrence Relation
Suppose that a person deposits $10,000 in a savings account at a bank yielding 11% per year with interest compounded anually. How much will be in the account after 30 years.
To solve this problem
Let P_n denote the amount in account after n years
→ The amount in account after n years equals the amount in account after n-1 years plus interest for n^{th} year. We see that recurrence relation

$$P_n = P_{n-1} + 0.11\, P(n-1)$$

$$P_1 = (1.11)\, P_0$$

$$P_2 = (1.11)\, P_1 = (1.11)^2\, P_0$$

$$P_3 = (1.11)^3\, P_0$$

$$P_n = (1.11)^n\, P_0$$

after 30 years $P_{30} = (1.11)^{30} \times 10,000 = \$228,922.97$

Rabbits and fibonacci numbers-

1) Recurrence Relation.

consider this problem which was orginally posed by leonardo pesano, also known as fibonacci in 13th century in his book liberaboii

"A young pair of rabbits (one of each sex) is placed on an island. A pair of rabbit does not breed until they are 2 months old. After they another pair each month as shown find recurrence relation for no of pairs of rabbits on island after n months, assuming that no rabbits ever die.

sol: Let f_n = No of pairs of rabbit after n months

f_n, n = 1, 2, 3

→ Rabbit population can be modeled using recurrence relation

At end of 1st month, the no of pairs of rabbit on island is $f_1 = 1$

This is because this place doesnot breed during the second month $f_2 = 1$

To find no of pair after n months add the no of island the previous month of f_{n-1} and no of newborn pair which equal f_{n-1} because each newborn pair comes form atleast 3 months old.

denotes pair of rab

Reproducing pair (atleast 2 months old)	Young pair (Less than 2 months old)	Month	Reproducing Pair	Young Pair	Total Pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

consequently the sequence $\{f_n\}$ satisfies the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

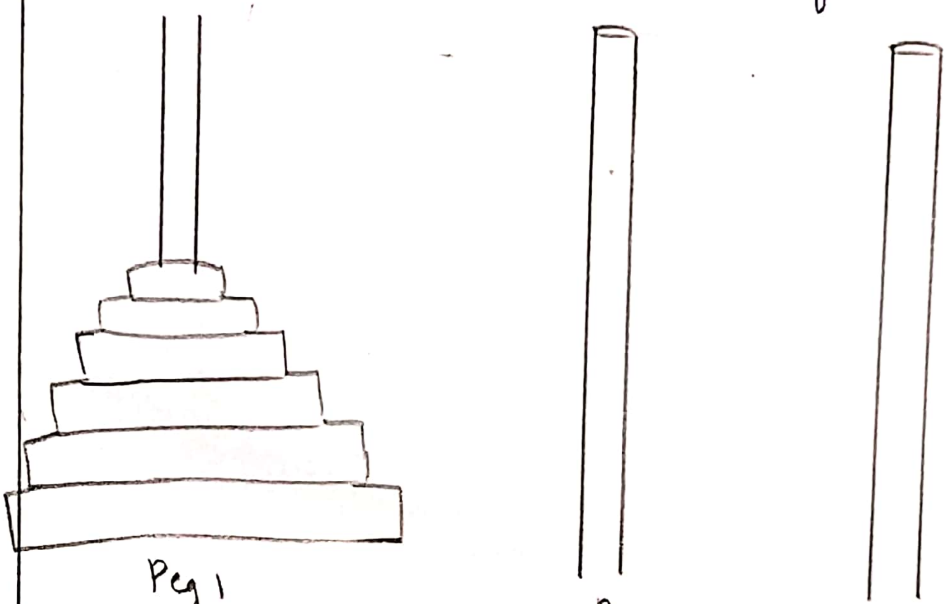
for $n \geq 3$ together with intial conditions $f_1 = 1$ & $f_2 = 1$. This recurrence relation uniquely determine sequence, which is fibonacci sequence

The tower of hanoi:

A popular puzzle of late Nineteenth century invented by french mathematician edward lucas called towers of hanoi consists of three pegs mounted on a board together with disks are placed on first peg in order of size with largest on bottom. The rules of puzzle allow disks to be moved one at a time from one peg to another as longas disk is never

placed on top of smaller disk.
The goal of puzzle is to have all disks one second peg order of size, with largest on bottom.

Let H_n be no of moves needed to solve the tower of hanoi problem with n disks. Setup recurence relation for sequences $\{H_n\}$.



Peg 1 Peg 2 Peg 3.

* Begin with n disks one peg 1.
* we can transfer the top $n-1$ disks, following rulers of puzzle to peg3 using H_{n-1} moves

Peg 1 Peg 2 Peg 3

* Now use one move to transfer the largest disk to the second peg
* we can transfer disks on peg 3 to Peg 2 using H_{n-1} addition al moves, placing them on top of largest disk.

This moves sets up recurrence relation

$H_n = H_{n-1} + 1 + H_{n-1}$

$H_n = 2H_{n-1} + 1$

[$H_1 = 1$ because one disk can be transfered from peg 1 to peg 2 according to rules of puzzle)

We can use iterative approach to solve recurrence relation

$H_n = 2H_{n-1} + 1$

$H_n = 2^2(2H_{n-2} + 1) + 1$

$H_n = 2^2(2H_{n-3} + 1) + 2 + 1$

$H_n = 2^3 H_{n-3} + 2^2 + 2 + 1$

$= 2^{n-1} H_{n-(n-1)} + 2^{n-2} + \ldots 2^2 + 2 + 1$

$H_n = 2^{n-1} H_1 + 2^{n-2} + \ldots + 2^2 + 2 + 1$

$H_n = 2^{n-1} + 2^{n-2} + \ldots + 2^2 + 2 + 1$

$\boxed{H_n = 2^n - 1}$

Find a recurrence relation and give initial conditions for the numbers of bit strings of length n that does not have two consective 0's. How many such bit strings are there of length five

Sol:

a_n - number of bit strings of length n that do not have consecutive 0's

No of strings come under a_0

$a_0 = 0$

0

$a_0 = 0$

Note: There exists no strings of length 0

string come under a_1

$a_1 = 2$

0
1

$a_2 = 3$

strigs come under a_2

01
10
11

$a_3 = 5$

strings came under a_3

010
011
101
110
111

$a_4 = 8$

Strings came under a_4

0101
0110
0111
1010
1011
1100
1110
1111

$a_3 = a_2 + a_1 = 3+2 = 5$
$a_4 = a_3 + a_2 = 5+3 = 8$
With above relations we can conclude

$$a_n = a_{n-1} + a_{n-2} \quad \text{for } n \geq 3$$

Codeward Enumeration :-
A computer system consider a string of decimal digits a valid codeward if
For instance · 1230407869 is valid
whenever 120987045608 is not valid. let
a_n be the number of valid n-digit codewards

Find a recurrence relation for a_n

a_n = No of valid n-digit codewards,
$a_1 = 9$ [From 10 digits (0 to 9) are valid 1-digit codewards]

a_n = Previous valid (n-1) digit codewards by digits (1 to 9) + invalid (n-1 digit codewards appeared by 0'

So we take

$\boxed{a_{n-1} = \text{No of valid } (n-1) - \text{digit codeward}}$

case 1: To make valid (n-1) digit codeward to valid n-digit code word it is append "digit' from 1 to 9 digits Two can be done by 9

∴ $9a_{n-1}$ gives us valid n-digit code — (1)

case 2. we take invalid $(n-1)$ digit strings we extract invalid $(n-1)$ digits from total string of length $(n-1)$ by subtraction valid $(n-1)$ digits

ie 10^{n-1} = No of Total Strings of length $(n-1)$

No of invalid $(n-1)$ digit string = No of Total string of length $(n-1)$ − Valid $(n-1)$ length String

$$= (10^{n-1} - a_{n-1}) \quad - (2)$$

This invalid strings of length $(n-1)$ are made valid n- length strings by appending 'zero' This can be done only one way

Add (1) & (2) from case 1 & case 2 to get a_n

$$a_n = 9 \times a_{n-1} + (10^{n-1} - a_{n-1})$$

$$\boxed{a_n = 8a_{n-1} + 10^{n-1}}$$

Find a recurrence relation for C_n, the no of ways to paranthesize the product of $n+1$ numbers $x_0, x_1, x_2 \ldots x_n$ to speify the order of multiplication

for example $C_3 = 5$ because there are five ways to paranthesize x_0, x_1, x_2, x_3 to determine the order of multiplication

$$((x_0 \cdot x_1) \cdot x_2) \cdot x_3$$

$$(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3$$

$$(x_0 \cdot x_1) \cdot (x_2 \cdot x_3)$$

$$x_0((x_1 \cdot x_2) \cdot x_3)$$

$$x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$$

The Recurrence relation for $(n+1)$ numbers is $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots C_{n-2} C_1 + C_{n-1} C_0$.

$$\boxed{C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}}$$

initial condition

$C_0 = 1$

$C_1 = 1$

The sequence $\{C_n\}$ is the sequence of catalon numbers

$C_1 = C_0 C_0 = 1$

$C_2 = C_0 C_1 + C_1 C_0 = 2$

$C_3 = C_0 C_2 + C_1 C_1 + C_2 C_0$

$= 2 + 1 + 2$

$= 5$

Recurrence Relation

first order reccurence relation:-

$a_n = c\,a_{n-1} + f(n)$ is called first order reccurence relation.

If $f(n) = 0$ then $a_n = c\,a_{n-1}$ is called 1st order homogeneous reccurence relation.

If $f(n) \neq 0$ then it is called non-homogeneous 1st order reccurence relation.

Solution of the first order reccurence relation is $a_n = c^n a_0$

Solve the reccurence relation $a_{n+1} = 4a_n ; n \geq 0$.
$a_0 = 3$.

Given $\underbrace{a_{n+1} = 4a_n}_{(1)}$ which is a first order homogeneous reccurence relation.

The solution of the above equation is $\underbrace{a_n = c^n a_0}_{(2)}$

take $n = n+1$

$$a_{n+1} = c^{n+1} a_0$$

from eq (1)

$4a_n = c^n \cdot c a_0$

from eq ②

$4 \cancel{c^n} \cancel{a_0} = \cancel{c^n} \cdot c \cdot \cancel{a_0}$

$\boxed{c = 4}$

The solution is $a_n = c^n a_0$

$\boxed{a_n = 4^n \times 3}$

Solve the reccurrence relation $a_n = 7a_{n-1}$ $n \geq 1$

and $a_2 = 98$

Given $a_n = 7a_{n-1}$ — ① which is a first order reccurrence relation.

the Solution of the above equation is $a_n = c^n a_0$ — ②

take $n = n+1$ in eq ②

$a_{n+1} = c^{n+1} a_0$

$a_{n+1} = c^n \cdot c a_0$

in eq ①

take $n = n+1$

$a_{n+1} = 7a_n$

$7a_n = c^n \cdot c \cdot a_0$

from ②

$7 \cancel{c^n} \cancel{a_0} = \cancel{c^n} \cdot \cancel{c} \cdot a_0 \Rightarrow \boxed{c = 7}$

The Solution is $a_n = c^n a_0$

$$\boxed{a_n = 7^n a_0}$$

Given $a_2 = 98$

$$a_2 = 7^2 a_0$$

$$98 = 49 a_0$$

$$a_0 = 98/49$$

$$\boxed{a_0 = 2}$$

$$\therefore \boxed{a_n = 7^n \times 2}$$

If $a_{n+1} = k \cdot a_n$; $n \geq 0$ and $a_3 = \frac{153}{49}$ and $a_5 = \frac{1377}{2401}$ find the value of k.

Given $a_{n+1} = k \cdot a_n$ — ① Which is a first order homogeneous reccurence relation.

The solution of the above equation is

$$a_n = c^n \cdot a_0 \text{ — ②}$$

$$a_{n+1} = c^{n+1} \cdot a_0$$

$$k \cdot a_n = c^n \cdot c \cdot a_0$$

$$k \cdot \cancel{c^n} \cdot a_0 = \cancel{c^n} \cdot c \cdot \cancel{a_0}$$

$$\boxed{c = k}$$

$\therefore a_n = c^n \cdot a_0$ i.e $a_n = k^n \cdot a_0$

but given $a_3 = 153/49$

$$a_3 = k^3 \cdot a_0$$

$$\frac{153}{49} = k^3 \cdot a_0$$

$$a_0 = \frac{153}{49k^3} \quad \text{--- (3)}$$

$$\frac{153}{49 \times \cancel{k^3}} = \frac{1377}{2401 \times \cancel{k^5}^{2}}$$

$$k^2 \times 2401 \times 153 = 1377 \times 49$$

$$k^2 = \frac{1377 \times 49}{2401 \times 153}$$

$$k^2 = 9/49 \Rightarrow \boxed{k = 3/7}$$

Second order linear homogeneous Reccurence Relation

The equation of the 2nd order homogeneous reccurence relation is $C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0$ — (1)

Where C_n, C_{n-1}, C_{n-2} are constants

The solution of the above equation is $a_n = k^n \cdot c$

Substitute these values in eq (1) we get

$$C_n k^n c + C_{n-1} k^{n-1} c + C_{n-2} k^{n-2} \cdot c = 0$$

Divided by $c \cdot k^{n-2}$ on Both sides

we get $\frac{C_n k^n c}{c \cdot k^{n-2}} + \frac{C_{n-1} k^{n-1} c}{c \cdot k^{n-2}} + \frac{C_{n-2} k^{n-2} c}{c \cdot k^{n-2}} = 0$

$C_n k^2 + C_{n-1} k + C_{n-2} = 0$ which is a characteristic equation.

Case-1 :- If the roots of k_1 and k_2 real and different then $a_n = A(k_1)^n + B(k_2)^n$

Case-2 :- If the roots of k_1 and k_2 are real and repeated then $k_1 = k_2 = k$

then $a_n = (A+Bn)k^n$

Case-3 :- If k_1 & k_2 are complex roots where $k_1 = p+iq$, $p-iq$ then

$a_n = r^n (A \cos n\theta + B \sin n\theta)$

$r = \sqrt{p^2+q^2}$, $\theta = \tan^{-1}(q/p)$

(1) Solve the recurrence relation

$a_n + a_{n-1} - 6a_{n-2} = 0 \ \forall n \geq 2$ $a_0 = -1$ $a_1 = 8$

Given $a_n + a_{n-1} - 6a_{n-2} = 0 \rightarrow (1)$ which is a 2^{nd} order homogeneous recurrence relation.

The solution of the above equation is

$a_n = k^n . c$

$k^n . c + k^{n-1} c - 6k^{n-2} . c = 0$

divided by $k^{n-2} . c$ on b.s

we get

$$\frac{k^n \cdot c}{k^{n-2} \cdot c} + \frac{k^{n-1} \cdot c}{k^{n-2} \cdot c} - \frac{6k^{n-2} \cdot c}{k^{n-2} \cdot c}$$

$k^2 + k - 6 = 0$

$k^2 + 3k - 2k - 6 = 0$

$k(k+3) - 2(k+3) = 0$

$\boxed{k_1 = -3}$ $\boxed{k_2 = 2}$

The roots are real and different then

$a_n = A(k_1)^n + B(k_2)^n$

Given $a_0 = -1$

$a_0 = A + B$

$-1 = A + B$

i.e $\boxed{A + B = -1}$

$a_1 = 8$

$a_1 = -3A + 2B$

$\boxed{-3A + 2B = 8}$

$3A + 3B = -3$
$-3A + 2B = 8$

$5B = 5$
$\boxed{B = 1}$

$-3A + 2 = 8$
$-3A = 6$
$\boxed{A = -2}$

$$\boxed{a_n = -2(-3)^n + 1(2)^n}$$

(2) Solve the recurrence relation $a_n - 6a_{n-1} + 9a_{n-2} = 0$; for $n \geq 2$ where $a_0 = 5$, $a_1 = 12$.

Sol :- Given, $a_n - 6a_{n-1} + 9a_{n-2} = 0 \rightarrow (1)$

which is a 2nd order homogeneous recurrence relation.

the solution of the above Equation is

$a_n = k^n . c \longrightarrow (2)$

Substitue eqn (2) in eqn (1) we get

$$k^n c - 6 k^{n-1} c + 9 . k^{n-2} . c = 0$$

divide by $k^{n-2} . c$ on both sides

$$k^2 - 6k + 9 = 0$$

$$k = 3, 3$$

The roots are (Equal) real and repeated

$$k_1 = 3 \quad k_2 = 3$$

The solution of $a_n = (A + Bn) k^n$

$$a_n = (A + Bn) 3^n$$

Given $a_0 = 5$,

$$a_0 = (A + 0) 3^0$$

$$\boxed{A = 5}$$

$a_1 = 12$

$a_1 = (A+B)3$

$12 = (5+B)3$

$12 = 15+3B$ (or)

$12-15 = 3B$

$-3 = 3B$

$\boxed{B=-1}$

$12 = (A+B)3$

$4 = A+B$

$4 = 5+B$

$4-5 = B$

$\boxed{B=-1}$

$\therefore$ the General solution of given eqn is

$a_n = (5-n)3^n$

(3) Solve the recurrence relation

$a_n - 2a_{n-1} + 2a_{n-2} = 0$ and for $n \geq 2$

where $a_0 = 1$, $a_1 = 2$

Sol Given $a_n - 2a_{n-1} + 2a_{n-2} = 0 \rightarrow (1)$

which is a 2nd order homogeneous recurrence relation

The solution of the above Equation is

$a_n = k^n \cdot c \rightarrow (2)$

Sub eqn (2) in eqn (1) we get

$k^n \cdot c - 2k^{n-1}c + 2k^{n-2}c = 0$

divide by $k^{n-2} \cdot c$ on bs

$k^2 - 2k + 2 = 0$

$\cancel{k^2 - 2k - k + 2 = 0}$

$\cancel{k(k-2) - (k-2) = 0}$

$\cancel{k = 1+\sqrt{3}, \ 1-\sqrt{3}}$

$\cancel{k_1 = 1+\sqrt{3} \quad k_2 = 1-\sqrt{3}}$

$k = 1 \pm i$

$k_1 = 1+i \quad k_2 = 1-i$

The roots are complex roots

$k = p \pm iq \qquad p = 1 \quad q = 1$

$a_n = r^n (A \cos n\theta + B \sin n\theta)$

$r = \sqrt{p^2+q^2} = \sqrt{2}$

$\theta = \tan^{-1}(q/p) = \tan^{-1}(1)$

$= 45 \text{ (or) } \pi/4$

$a_n = (\sqrt{2})^n \left(A \cos n\frac{\pi}{4} + B \sin n\frac{\pi}{4}\right)$

Given $a_0 = 1$

$a_0 = (\sqrt{2})^0 \left(A \cos(0)\frac{\pi}{4} + B \sin(0)\frac{\pi}{4}\right)$

$a_0 = 1 \left(A \cos(0) + B \sin(0)\right)$

$1 = 1 \left(A(1) + B(0)\right)$

$\boxed{A = 1}$

$a_1 = 2$

$a_1 = (\sqrt{2})^1 \left(A \cos\frac{\pi}{4} + B \sin\frac{\pi}{4}\right)$

$2 = \sqrt{2} \left(A\left(\frac{1}{\sqrt{2}}\right) + B\left(\frac{1}{\sqrt{2}}\right)\right)$

$2 = \frac{\sqrt{2}}{\sqrt{2}}(A+B)$

$A + B = 2$

$1 + B = 2$

$\boxed{B = 1}$

$\therefore$ The General soln of given eqn is

$a_n = (\sqrt{2})^n \left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]$

Second order non-Homogeneous recurrence relation

The Eqn of the second order non-homogeneous recurrence relation is

$C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = f(n)$

the solution of the above Equation is

$a_n = a_n^{(h)} + a_n^{(p)}$

where $a_n^{(h)}$ is the homogeneous part

$a_n^{(p)}$ is the particular soln.

To find the $a_n^{(p)}$

Case:- 1

Solution of $a_n^{(p)}$

Case :- 1

If $f(n)$ = constant and '1' is not root of the characteristic Equation $a_n = k$

Case :- 2

If $f(n)$ = Constant and '1' is a root of the characteristic Equation is Repeated 'm' times $a_n = n^m . k$

Case :- 3

If $f(n) = ab^n$ and b is not a root of the characteristic Equation then $a_n = kb^n$

Case :- 4

If $f(n) = ab^n$ and b is a root of the characteristic Equation then repeated 'm' times $a_n = n^m [kb^n]$

① Solve the recurrence relation $a_n + 4a_{n-1} + 4a_{n-2} = 8$ and $a_0 = 1$ $a_1 = 2$

Sol :- Given $a_n + 4a_{n-1} + 4a_{n-2} = 8 \rightarrow$ ①
which is 2^{nd} order non-homogeneous recurrence relation

The soln of the above Eqn is

$a_n = a_n^{(h)} + a_n^{(p)}$

To find $a_n^{(h)}$

i.e., we take $a_n + 4a_{n-1} + 4a_{n-2} = 0$

The solution is $a_n = k^n c$

$$k^n c + 4 k^{n-1} c + 4 k^{n-2} c = 0$$

divided by $k^{n-2}c$ on b.s we get

$$k^2 + 4k + 4 = 0$$

$$k = -2, -2$$

the roots are repeated

$$k_1 = -2 \quad k_2 = -2$$

$$a_n = (A + Bn) k^n$$

$$a_n^{(h)} = (A + Bn)(-2)^n$$

To find the $a_n^{(p)}$

Here $f(n) = 8$ = constant

'1' is not root of the characteristic equation

$$a_n = k$$

$$a_{n-1} = k$$

$$a_{n+1} = k$$

$$a_n + 4a_{n-1} + 4a_{n-2} = 8$$

$$k + 4k + 4k = 8$$

$$9k = 8$$

$$\boxed{k = 8/9}$$

$$\therefore \boxed{a_n^{(p)} = \frac{8}{9}}$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = (A + Bn)(-2)^n + \frac{8}{9}$$

Given $a_0 = 1$

$$a_0 = (A + B(0))(-2)^0 + \frac{8}{9}$$

$$1 = (A + B(0)) + \frac{8}{9}$$

$$A + B = 1 - \frac{8}{9}$$

$$A + B = \frac{9-8}{9}$$

$$A + B = \frac{1}{9} \Rightarrow \boxed{A = \frac{1}{9}}$$

$a_1 = 2$

$$a_1 = (A + B)(-2)^1 + \frac{8}{9}$$

$$2 = (A+B)(-2) + \frac{8}{9}$$

$$2 - \frac{8}{9} = (A+B)(-2)$$

$$\frac{18-8}{9} = (A+B)(-2)$$

$$\frac{10}{9} = (A+B)(-2)$$

$$-\frac{10}{18} = A+B$$

$$A+B = -\frac{10}{18}$$

$$B = -\frac{10}{18} - A$$

$$B = -\frac{10}{18} - \frac{1}{9}$$

$$B = \frac{-10-2}{18}$$

$$B = \frac{-12}{18} = \frac{-6}{9}$$

$$\boxed{B = \frac{-2}{3}}$$

$\therefore$ the General solution of given Eqn is

$$a_n = \left(\frac{1}{9} - \frac{2}{3}n\right)(-2)^n + \frac{8}{9}$$

(2) Solve the recurrence relation

$a_n - 3a_{n-1} - 4a_{n-2} = 4^n \quad n \geq 2$

Sol Given $a_n - 3a_{n-1} - 4a_{n-2} = 4^n \rightarrow (1)$

which is 2^{nd} order non-homogeneous recurrence relation.

the solution of the above Equation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

To find $a_n^{(h)}$

$$a_n - 3a_{n-1} - 4a_{n-2} = 0$$

Let the solution of the above Equation is $a_n = k^n c$

$$k^n c - 3k^{n-1}c - 4k^{n-2}c = 0$$

divided by $k^{n-2}c$ on B.S we get

$$k^2 - 3k - 4 = 0$$

$$k = -1, 4$$

$$a_n^{(h)} = A(-1)^n + B(4)^n$$

To find $a_n^{(p)}$

Given $f(n) = 4^n = ab^n$

Here $a = 1$, $b = 4$

$\therefore$ b is a root of the characteristic Equation then

$$a_n = n^m (k b^n)$$

Here $m = 1$ (How many time '4' is there)

$$\left.\begin{aligned} a_n &= n(k4^n) \\ a_{n-1} &= (n-1)k\,4^{n-1} \\ a_{n-2} &= (n-2)k\,4^{n-2} \end{aligned}\right\} \text{Substitute in eqn (1)}$$

we get,

$$nk4^n - 3\left[(n-1)k4^{n-1}\right] - 4\left[(n-2)k4^{n-2}\right] = 4^n$$

$$nk4^n - 3(n-1)k\frac{4^n}{4} - 4(n-2)k\frac{4^n}{4^2} = 4^n$$

$$k\cancel{4^n}\left[n - \frac{3(n-1)}{4} - \frac{\cancel{4}(n-2)}{\cancel{16}\,4}\right] = \cancel{4^n}$$

$$k\left[\frac{4n - 3(n-1) - (n-2)}{4}\right] = 1$$

$$k\left[\frac{\cancel{4n}-\cancel{3n}+3-\cancel{n}+2}{4}\right]=1$$

$$k\left[\frac{5}{4}\right]=1$$

$$\boxed{k=\frac{4}{5}}$$

$$a_n^{(p)}=n^1\left(kb^n\right)$$
$$=n\left(\frac{4}{5}\right)4^n$$
$$=\frac{n4^{n+1}}{5}$$

$$a_n=a_n^{(h)}+a_n^{(p)}$$

$$a_n=A(-1)^n+B(4)^n+\frac{n4^{n+1}}{5}$$

this is the general solution of given Equation

Note :- If $f(n)$ = polynomial of degree r and '1' is not root of the characteristic Equation we can use $a_n=k_r n^r+k_{r-1}n^{r-1}+\dots+k_0$

* If $f(n)$ = polynomial of degree r and '1' is a root of the characteristic Equation then we can use

$$a_n=n^m\left[k_r n^r+k_{r-1}n^{r-1}+\dots+k_0\right]$$

where m denotes no. of times repeated

Solve the recurrence relation a_n
$a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 2$

Given, the recurrence relation is

$$a_{n+2} - 10a_{n+1} + 21a_n = 3n^2 - 1 \rightarrow (1)$$ which is a second order non-homogeneous recurrence relation

the solution of the above Equation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

To find the $a_n^{(h)}$

i.e $a_{n+2} - 10a_{n+1} + 21a_n = 0$

the solution of the above Equation is

$$a_n = k^n . c$$

$$k^{n+2}c - 10k^{n+1}c + 21k^n c = 0$$

Divided by $k^n c$ we get

$$k^2 - 10k + 21$$

$$k = 7, 3$$

$$a_n^{(h)} = A(3)^n + B(7)^n$$

To find the $a_n^{(p)}$

Given $f(n) = 3n^2 - 2$ is a polynomial of degree '2' and '1' is not a root of the characteristic Equation.

r=degree $a_n = k_2 n^2 + k_1 n + k_0$

$a_{n+1} = k_2(n+1)^2 + k_1(n+1) + k_0$

$a_{n+2} = k_2(n+2)^2 + k_1(n+2) + k_0$

these values substitue in Eqn ①

$\Rightarrow \left[k_2(n+2)^2 + k_1(n+2) + k_0\right] - 10\left[k_2(n+1)^2 + k_1(n+1) + k_0\right] + 21\left[k_2 n^2 + k_1 n + k_0\right] = 3n^2 - 2$

$\Rightarrow \left[k_2(n^2+4n+4) + k_1(n+2) + k_0\right] - 10\left[k_2(n^2+1+2n) + k_1(n+1) + k_0\right] + 21k_2n^2 + 21k_1n + 21k_0 = 3n^2 - 2$

$\Rightarrow k_2n^2 + k_2 4n + 4k_2 + k_1 n + 2k_1 + k_0 - 10k_2n^2 - 20k_2n - 10k_2 - 10k_1n - 10k_1 - 10k_0 + 21k_2n^2 + 21k_1n + 21k_0 = 3n^2 - 2$

$\Rightarrow n^2\left[k_2 - 10k_2 + 21k_2\right] + n\left[4k_2 + k_1 - 20k_2 - 10k_1 + 21k_1\right] + 4k_2 + 2k_1 + k_0 - 10k_2 - 10k_1 - 10k_0 + 21k_0 = 3n^2 - 2$

$\Rightarrow n^2\left[12k_2\right] + n\left[-16k_2 + 12k_1\right] - 6k_2 - 8k_1 + 12k_0 = 3n^2 - 2$

$\Rightarrow$ Compairing on both sides

$\cancel{12}^{4}k_2 = \cancel{3}$

$\boxed{k_2 = 1/4}$

$$-16k_2 + 12k_1 = 0$$

$$-\cancel{16}^{4}(\tfrac{1}{\cancel{4}}) + 12k_1 = 0$$

$$-4 + 12k_1 = 0$$

$$^{3}\cancel{12}k_1 = \cancel{4}$$

$$\boxed{k_1 = 1/3}$$

$$-6k_2 - 8k_1 + 12k_0 = -2$$

$$-\cancel{6}^{3}\left(\frac{1}{\cancel{4}_2}\right) - 8\left(\frac{1}{3}\right) + 12k_0 = -2$$

$$-\frac{3}{2} - \frac{8}{3} + 12k_0 = -2$$

$$12k_0 = -2 + \frac{3}{2} + \frac{8}{3}$$

$$12k_0 = \frac{13}{6}$$

$$\boxed{k_0 = \frac{13}{72}}$$

$$a_n^{(p)} = \frac{1}{4}n^2 + \frac{1}{3}n + \frac{13}{72}$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$\boxed{a_n = A(3)^n + B(7)^n + \frac{n^2}{4} + \frac{n}{3} + \frac{13}{72}}$$

this is the general solution of given Equation.

solve the recurrence relation $a_n - 2a_{n-1} + a_{n-2} = 5n$

Given, the recurrence relation is

$a_n - 2a_{n-1} + a_{n-2} = 5n \rightarrow (1)$ which is a second order non-homogeneous recurrence relation.

the solution of the above Equation is

$a_n = a_n^{(h)} + a_n^{(p)}$

To find the $a_n^{(h)}$

i.e $a_n - 2a_{n-1} + a_{n-2} = 0$

the solution of the above Equation

$a_n = k^n . c$

$k^n . c - 2k^{n-1} c + k^{n-2} c = 0$

Divided by $k^{n-2} c$ we get

$k^2 - 2k + 1 = 0$

$k^2 - k - k + 1 = 0$

$k(k-1) - 1(k-1) = 0$

$k = 1, 1$

$a_n(h) = (A + Bn) 1^n$

To find the $a_n(p)$

Given $f(n) = 5n$ is a polynomial of degree '1' and '1' is a root of the characteristic Equation

$a_n = n^2 [k_1 n^1 + k_0]$

$$a_{n-1} = (n-1)^2 [k(n-1) + k_0]$$

$$a_{n-2} = (n-2)^2 [k(n-2) + k_0]$$

$$\Rightarrow n^2[k_1 n + k_0] - 2[(n-1)^2[k(n-1) + k_0]]$$
$$+ (n-2)^2 [k(n-2) + k_0] = 5n$$

$$\Rightarrow n^2 k_1 n + n^2 k_0 - 2[(n^2 + 1 - 2n)[kn - k + k_0]]$$
$$+ (n^2 + 4 - 4n)[kn - 2k + k_0] = 5n$$

$$\Rightarrow n^2 k_1 n + n^2 k_0 - 2[kn^3 - kn^2 + n^2 k_0 + kn - k + k_0$$
$$- 2kn^2 + 2kn - 2nk_0] + [n^3 k - 2n^2 k + n^2 k_0$$
$$+ 4kn - 8k + 8kn - 4kn^2 + 8kn - 4nk_0]$$
$$= 5n$$

Solution of the recurrence relation by the method of generating function (1)

Note:- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$

$(1-2x)^{-1} = 1 + 2x + 4x^2 + \cdots = \sum_{n=0}^{\infty} (2x)^n$

First order recurrence relation by the method of generating function.

Let, $a_n = c a_{n-1} + \phi(n)$

(or)

$a_{n+1} = c a_n + \phi(n)$

which is called first order recurrence relation.

By the method of generating function

$$f(x) = \frac{a_0 + x g(x)}{1 - cx}$$

where $g(x) = \sum_{n=0}^{\infty} \phi(n) x^n$

Let $a_n = c a_{n-1}$ (or) $a_{n+1} = c a_n$ are called first order homogeneous recurrence relation. Then $f(x) = \dfrac{a_0}{1-cx}$

The solution a_n is the quotient of x^n in the generating function.

① Solve the recurrence relation $a_n - 4a_{n-1} = 0$ $n \geq 1$ $a_0 = 1$ by the method of generating function

Sol Given the Equation $a_n - 4a_{n-1} = 0$ which is a first order homogeneous we have the form of the first order homogeneous recurrence relation is

$$a_n = C a_{n-1}$$

$$\therefore \boxed{C = 4}$$

$$f(x) = \frac{a_0}{1 - Cx} = \frac{1}{1 - 4x}$$

$$f(x) = (1 - 4x)^{-1} = 1 + 4x + 16x^2 + \cdots$$

$$(1-2x)^{-1} = 1 + 2x + 4x^2 + \cdots = \sum_{n=0}^{\infty} (2x)^n$$

$$= \sum_{n=0}^{\infty} (4x)^n$$

$$= \sum_{n=0}^{\infty} 4^n x^n$$

$$\therefore a_n = 4^n$$

② Solve the recurrence relation $a_n - a_{n-1} = 0$ for $n \geq 1$ $a_0 = 6$.

Given the Equation $a_n - a_{n-1} = 0$ which is a first order homogeneous we have the form of the first order homogeneous recurrence relation is

$$a_n = C a_{n-1}$$

$$a_n = a_{n-1}$$

$$\therefore \boxed{C = 1}$$

by the method of generating function

$$f(x) = \frac{a_0}{1-cx} = \frac{6}{1-x}$$

$(1-x)^{-1} = 1+x+x^2+x^3\ldots$

$$= 6(1-x)^{-1}$$

$$= 6\left[1+x+x^2+\ldots\right]$$

$$= 6\sum_{n=0}^{\infty} x^n$$

$$= 6 \times (1)^n$$

(3) Solve the recurrence relation by the method of generating function $a_{n+1} - a_n = 3^n$ $n \geq 0$ and $a_0 = 1$

Sol:- Given $a_{n+1} - a_n = 3^n$ which is a first order non-homogeneous recurrence relation.

The form of the non-homogeneous first order recurrence relation is

$$a_{n+1} - a_n = 3^n$$

$$a_{n+1} - C.a_n = \phi(n)$$

$\therefore \boxed{C = 1}$ $\boxed{\phi(n) = 3^n}$

by the method of generating function

$$f(x) = \frac{a_0 + x\,g(x)}{1-Cx}$$

$$g(x) = \sum_{n=0}^{\infty} \phi(n)\, x^n$$

$$g(x) = \sum_{n=0}^{\infty} 3^n . x^n$$

$$= \sum_{n=0}^{\infty} (3x)^n$$
$$= 1 + 3x + 9x^2 + \dots$$
$$g(x) = (1-3x)^{-1}$$

$$f(x) = \frac{a_0 + x\,g(x)}{1 - Cx}$$
$$= \frac{1 + x(1-3x)^{-1}}{1-x}$$
$$= \frac{1 + \frac{x}{1-3x}}{1-x}$$
$$= \frac{\frac{1-3x+x}{1-3x}}{1-x}$$
$$\therefore f(x) = \frac{1-2x}{(1-3x)(1-x)}$$

Now, $\frac{1-2x}{(1-x)(1-3x)} = \frac{A}{1-x} + \frac{B}{1-3x}$

$$\frac{1-2x}{\cancel{(1-x)(1-3x)}} = \frac{A(1-3x) + B(1-x)}{\cancel{(1-x)(1-3x)}}$$

$$1 - 2x = A(1-3x) + B(1-x)$$

if $x = 0$
$1 = A(1-0) + B(1-0)$
$1 = A + B$
$\therefore A + B = 1 \longrightarrow (1)$

if $x = 1$
$-1 = A(-2) + B(0)$
$-2A = -1$

$\therefore \boxed{A = 1/2}$ sub in (1)

$A + B = 1$

$B = 1 - A$

$B = 1 - 1/2$

$\boxed{B = 1/2}$

$$\frac{1-2x}{(1-x)(1-3x)} = \frac{1}{2} \cdot \frac{1}{(1-x)} + \frac{1}{2} \frac{1}{(1-3x)}$$

$$= \frac{1}{2}\left[(1-x)^{-1} + (1-3x)^{-1}\right]$$

$$= \frac{1}{2}\left[(1+x+x^2+\dots) + (1+3x+9x^2+\dots)\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n\right]$$

$$= \frac{1}{2}\left[(1)^n + (3)^n\right] //$$

Solution of the second order recurrence relation by generating function method

Let $a_n + Aa_{n-1} + Ba_{n-2} = f(n);\ n \geq 2$ → (1)

$a_{n+2} + Aa_{n+1} + Ba_n = \phi(n);$ → (2)

eqn (1) & (2) are called second order non-homogeneous recurrence relation.

By the method of generating function

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x + x^2 g(x)}{1 + Ax + Bx^2}$$

if it is a homogeneous

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x}{1 + Ax + Bx^2} \qquad g(x) = \sum \phi(n) x^n$$

(1) Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ $\forall\ n \geq 2$ $a_0 = -1$ $a_1 = 8$ by the method of generating function.

Given, recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0$ which is the second order homogeneous recurrence relation

here $A = 1$ $B = -6$ $a_0 = -1$, $a_1 = 8$

by the method of generating function

$$f(x) = \frac{a_0 + (a_1 + a_0 A)x}{1 + Ax + Bx^2}$$

$$f(x) = \frac{-1 + (8 - 1)x}{1 + x - 6x^2}$$

$$= \frac{-1 + 7x}{1 + x - 6x^2}$$

$$f(x) = \frac{7x - 1}{-6x^2 + x + 1} = \frac{7x - 1}{-(6x^2 - x - 1)}$$

$$-\frac{7x-1}{(2x-1)(3x+1)} = \frac{7x-1}{(1-2x)(-1-3x)}$$

$$\cancel{\frac{7x-1}{(1-2x)(-1-3x)}} = \cancel{\frac{A}{(1-2x)}}$$

$$-\frac{7x-1}{(2x-1)(3x+1)} = \frac{A}{(2x-1)} + \frac{B}{(3x+1)}$$

$$-7x - 1 = A(3x+1) + B(2x-1)$$

Let $x=0$

$1 = A - B \rightarrow (1)$

Let $x=1$

$-6 = 4A+B \rightarrow (2)$

$$A - \cancel{B} = 1$$
$$4A + \cancel{B} = -6$$
$$5A = -5$$

$\boxed{A = -1}$ sub in eqn (1)

$$-1 - B = 1$$
$$-B = 2$$
$$\boxed{B = -2}$$

$$f(x) = \frac{-1}{2x-1} - \frac{2}{3x+1}$$
$$= \frac{1}{1-2x} - \frac{2}{1+3x}$$
$$= (1-2x)^{-1} - 2(1+3x)^{-1}$$
$$= [1+2x+4x^2+ \ldots] - 2[1-3x+9x^2 \ldots]$$
$$= \sum_{n=0}^{\infty} (2x)^n - 2 \sum_{n=0}^{\infty} (3x)^n (-1)^n$$

$$a_n = 2^n - 2 \cdot 3^n (-1)^n$$

(2) Solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$ $n \geq 0$ $a_0 = 1$, $a_1 = 2$

Sol Given, the recurrence relation is $a_{n+2} - 2a_{n+1} + a_n = 2^n$ which is 2^{nd} order non-homogeneous recurrence relation.

Here $A=-2,\ B=1,\ a_0=1\quad a_1=2\quad \phi(n)=2^n$

by the method of generating function

$$f(x)=\frac{a_0+(a_1+a_0A)x+x^2g(x)}{1+Ax+Bx^2}$$

$$=\frac{1+(2-2)x+x^2g(x)}{1-2x+x^2}$$

$$=\frac{1+x^2g(x)}{x^2-2x+1}$$

$$g(x)=\Sigma\phi(n)x^n$$
$$=\Sigma 2^n x^n$$
$$=\Sigma(2x)^n$$
$$=1+2x+4x^2+\dots$$
$$g(x)=(1-2x)^{-1}$$

$$=\frac{1+x^2\left[(1-2x)^{-1}\right]}{x^2-2x+1}$$

$$=\frac{1+\frac{x^2}{1-2x}}{x^2-2x+1}$$

$$=\frac{\frac{1-2x+x^2}{1-2x}}{x^2-2x+1}$$

$$=\frac{\cancel{1-2x+x^2}}{(1-2x)\cancel{(x^2-2x+1)}}$$

$$=\frac{1}{1-2x}$$

$$=(1-2x)^{-1}$$

$$=1+2x+4x^2+\dots$$

$$=\sum_{n=0}^{\infty}(2x)^n$$

$$f(x)=\sum_{n=0}^{\infty}2^n x^n$$

$$\boxed{a_n=2^n}$$

③ If $f(x) = (1-x)^{-1}$ find the sequence by the method of generating function? By definition of

Sol generating function.

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

Given $f(x) = (1-x)^{-1}$

$$= 1 + x + x^2 + x^3 + \dots$$

$a_0 = 1 \quad a_1 = 1 \quad a_2 = 1$

$\therefore$ The sequence $1, 1, 1, \dots$

Note :-

$$(1+x)^n = 1 + nx + \frac{n!}{(n-2)!\,2!} x^2 + \dots$$

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!} x^2 + \frac{n(n+1)(n+2)}{3!} x^3 + \dots$$

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \dots$$

④ Find the sequence generated by the following function $f(x) = (3+x)^3$

Sol :- Given $f(x) = (3+x)^3$

$$= 3^3 (1 + x/3)^3$$

we known that $(1+x)^n = 1 + nx + \frac{n!}{(n-2)!\,2!} x^2 + \dots$

$$= 3^3 \left[1 + 3(x/3) + \frac{3!}{(3-2)!\,2!} (x/3)^2 + \dots\right]$$

$$= 3^3 \left[1 + x + \frac{\cancel{3} x^2}{\cancel{9}_3} + \dots\right]$$

$$= 3^3 + 3^3 x + 3^3 \frac{x^2}{3} + \dots$$

$= 3^3 + 3^3 x + 3^2 x^2 + \cdots$

$= [27 + 27x + 9x^2 + \cdots]$

$\therefore 27, 27, 9, \cdots$

(5) Find the sequence generated by following function $f(x) = 2x^2(1-x)^{-1}$

Sol: Given $f(x) = 2x^2(1-x)^{-1}$

$= 2x^2[1 + x + x^2 + \cdots]$

$= [2x^2 + 2x^3 + 2x^4 + \cdots]$

$= 0 + 0x + 2x^2 + 2x^3 + 2x^4 + \cdots$

$\therefore 0, 0, 2, 2, 2, \cdots$

(6) Find the sequence generated by the following function $f(x) = \frac{1}{1-x} + 2x^3$

Sol: Given $f(x) = \frac{1}{1-x} + 2x^3$

$f(x) = (1-x)^{-1} + 2x^3$

$= [1 + x + x^2 + x^3 + \cdots] + 2x^3$

$= 1 + x + x^2 + 3x^3 + \cdots$

$\therefore 1, 1, 1, 3, \cdots$

(7) Find the sequence generated by the following function $f(x) = 3x^3 + e^{2x}$

Sol: Given $f(x) = 3x^3 + e^{2x}$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$f(x) = 3x^3 + 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$

$= 3x^3 + 1 + 2x + \frac{4x^2}{2} + \frac{\cancel{4}8x^3}{\cancel{6}3} + \cdots$

$= 1 + 2x + 2x^2 + 3x^3 + \frac{4x^3}{3} + \cdots$

$= 1 + 2x + 2x^2 + \frac{9x^3 + 4x^3}{3} + \cdots$

$= 1 + 2x + 2x^2 + \frac{13}{3}x^3 + \cdots$

$\therefore 1, 2, 2, 13/3 \cdots$

(8) Find the generating function from the following sequence 1, 2, 3, 4.

Sol :- we know that,

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \rightarrow (1)$

Given sequence is 1, 2, 3, 4

$a_0 = 1 \quad a_1 = 2 \quad a_2 = 3, \quad a_3 = 4$ Sub in (1)

$f(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$

$\therefore f(x) = (1-x)^{-2}$

(9) Find the generating function from the following sequence 1, -2, 3, -4

Sol we know that

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \rightarrow (1)$

Given sequence is 1, -2, 3, -4

$a_0 = 1 \quad a_1 = -2 \quad a_2 = 3 \quad a_3 = -4$ Sub in (1)

$f(x) = 1 - 2x + 3x^2 - 4x^3 + \cdots$

$\therefore f(x) = (1+x)^{-2}$

(10) Find the generating function for the following sequence 1, 1, 0, 1, 1, 1

Sol :- we know that

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \rightarrow (1)$

Given sequence is 1, 1, 0, 1, 1, 1

$a_0 = 1 \quad a_1 = 1 \quad a_2 = 0 \quad a_3 = 1 \quad a_4 = 1$ Sub in (1)

$f(x) = 1 + 1x + 0x^2 + 1x^3 + 1x^4 + 1x^5 + \ldots$

$= 1 + x + x^3 + x^4 + x^5 \ldots$

$= 1 + x + x^2 + x^3 + x^4 + x^5 - x^2$

$\therefore f(x) = (1-x)^{-1} - x^2$

Method - 2

Solve the recurrence relation by the method of generating function

$a_n = a_{n-1} + n \; ; \; n \geq 1 \quad a_0 = 1$

Sol :- Given, the recurrence relation is

$a_n = a_{n-1} + n \rightarrow (1)$

we have $g(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$

$= \sum_{n=0}^{\infty} a_n x^n$

Multiply x^n on both sides for eqn (1)

$a_n x^n = a_{n-1} x^n + n x^n$

$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} n x^n$

$\sum_{n=1}^{\infty} a_n x^n = \underbrace{x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}}_{\text{Multiply and divided by } x} + \sum_{n=1}^{\infty} n x^n \rightarrow (2)$

Take first term

$$\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 - a^0$$

$$= g(x) - a_0 \longrightarrow (3)$$

Consider the second term

$$= x \sum_{n=1}^{\infty} a_{n-1} x^{n-1}$$

$$= x [a_0 + a_1 x + a_2 x^2 + \dots]$$

$$= x [g(x)] \longrightarrow (4)$$

substitute in (3), (4) in (2)

$$g(x) - a_0 = x g(x) + x + 2x^2 + 3x^3 + \dots$$

$$g(x) - 1 = x g(x) + x [1 + 2x + 3x^2 + \dots]$$

$$g(x) - 1 = x g(x) + x [(1-x)^{-2}]$$

$$g(x) - x g(x) = 1 + \frac{x}{(1-x)^2}$$

$$g(x)[1-x] = 1 + \frac{x}{(1-x)^2} \Rightarrow g(x)[1-x] = 1 + \frac{x}{(1-x)^2}$$

$$g(x)[1-x] = \frac{(1-x)^2 + x}{(1-x)^2}$$

$$g(x)(1-x) = \frac{1 - 2x + x^2 + x}{(1-x)^2}$$

$$g(x)(1-x) = \frac{1 - x + x^2}{(1-x)^2}$$

$$\therefore \boxed{g(x) = \frac{1 - x + x^2}{(1-x)^3}}$$

$$g(x) = \frac{1}{1-x} + \frac{x}{(1-x)^3}$$

$$g(x) = (1-x)^{-1} + x(1-x)^{-3}$$

$$= [1+x+x^2+\dots] + x[1+3x+6x^2+\dots]$$

$$(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\dots$$

$$= [1+x+x^2+\dots] + [x+3x^2+6x^3+\dots]$$

$$= \sum_{n=0}^{\infty} x^n + \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n$$

Solve the recurrence relation

$a_n - 2a_{n-1} - 3a_{n-2} = 0$; $n \geq 2$ $a_0 = 3$, $a_1 = 1$

~~Sol~~ :-

By the method of generating function

Sol :- Given, the recurrence relation is

$$a_n - 2a_{n-1} - 3a_{n-2} = 0 \longrightarrow (1)$$

$$\cancel{a_n = 2a_{n-1} + 3a_{n-2}}$$

Multiply x^n on both sides to eqn (1)

$$a_n x^n - 2a_{n-1} x^n - 3a_{n-2} x^n = 0$$

$$\sum_{n=2}^{\infty} a_n x^n - 2x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - 3x^2 \sum_{n=2}^{\infty} a_{n-1} x^{n-2} \rightarrow (2)$$

we have $g(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

Take first term in eqn (2)

$$\sum_{n=2}^{\infty} a_n x^n$$

$$= a_2 x^2 + a_3 x^3 + \cdots$$

$$= [a_0 + a_1 x + a_2 x^2 + a_3 x^3] - a_0 - a_1 x$$

$$= g(x) - a_0 - a_1 x \longrightarrow (3)$$

Consider Second term in eqn (2)

$\cancel{\sum_{n=2}^{\infty}}$ we can ~~rearrange~~ 2nd term as

$$\cancel{\sum_{n=2}^{\infty}} \quad \sum_{n=2}^{\infty} a_{n-1} x^{n-1}$$

$$= a_1 x + a_2 x^2 + \cdots$$

$$= a_0 + a_1 x + a_2 x^2 - a_0$$

$$= g(x) - a_0 \longrightarrow (4)$$

Consider third term in eqn (2)

$$\sum_{n=2}^{\infty} a_{n-2} x^{n-2} = a_0 + a_1 x + a_2 x^2$$

$$= g(x) \longrightarrow (5)$$

Substitute (3), (4) (5) in (2)

$$\Rightarrow g(x) - a_0 - a_1 x - 2x[g(x) - a_0] - 3x^2 g(x) = 0$$

$$\Rightarrow g(x) - 3 - x - 2x g(x) + 6x - 3x^2 g(x) = 0$$

$$\Rightarrow g(x)[1 - 2x - 3x^2] + 5x - 3 = 0$$

$$\Rightarrow g(x)[1 - 2x - 3x^2] = 3 - 5x$$

$$\Rightarrow g(x) = \frac{3-5x}{1-2x-3x^2}$$

$$g(x) = \frac{-[5x-3]}{-[3x^2+2x-1]}$$

$$g(x) = \frac{5x-3}{3x^2+2x-1}$$

$$= \frac{5x-3}{(x+1)(3x-1)}$$

$$\frac{5x-3}{(x+1)(3x-1)} = \frac{A}{1+x} + \frac{B}{3x-1}$$

$$5x-3 = A(3x-1) + B(x+1)$$

$x = -1$

$$-8 = A(-4)$$

$$\boxed{A = 2}$$

$x = 0$

$$-3 = -A + B$$

$$-3 = -2 + B$$

$$\boxed{B = -1}$$

$$\frac{2}{1+x} + \frac{-1}{3x-1}$$

$$= \frac{2}{1+x} + \frac{1}{1-3x}$$

$= 2(1+x)^{-1} + (1-3x)^{-1}$

$= 2[1+x+x^2+\dots] + [1+3x+9x^2+\dots]$

$= 2\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} (3x)^n$

$= 2(1)^n + 3^n$

prove that $1+3x+6x^2+10x^3+\dots = 1/(1-x)^3$

Sol Let $A = 1+3x+6x^2+10x^3+\dots$ $\rightarrow$ eqn (1)

Multiply 'x' to eqn (1) on both sides

$Ax = x+3x^2+6x^3+10x^4+\dots \rightarrow$ (2)

eqn (1) - (2)

$(A-Ax) = (1-0)+(3x-x)+(6x^2-3x^2)+\dots$

$(A-Ax) = 1+2x+3x^2+\dots$

$A(1-x) = (1+2x+3x^2+\dots)$

$A = \dfrac{(1+2x+3x^2+\dots)}{(1-x)}$

$A = (1+2x+3x^2+\dots)(1-x)^{-1}$

$A = (1+2x+3x^2+\dots)(1+x+x^2+\dots)$

~~$A = (1+2x^2+3$~~

$A = (1-x)^{-2}(1-x)^{-1}$

$A = (1-x)^{-3}$

$$A = \frac{1}{(1-x)^3}$$

Solve $1+4x+9x^2+16x^3+\cdots = \frac{1+x}{(1-x)^3}$

Let $A = 1+4x+9x^2+16x^3+\cdots \rightarrow (1)$

Multiply by x to eqn (1)

$Ax = x+4x^2+9x^3+16x^4+\cdots \longrightarrow (2)$

eqn (1) − (2), we get

$(A - Ax) = (1-0)+(4x-x)+(9x^2-4x^2)$
$+(16x^3-9x^3)+\cdots$

$A(1-x) = 1+3x+5x^2+7x^3+\cdots$
$\rightarrow (3)$

Multiply by x to eqn (3) on B.S

$Ax - Ax^2 = x+3x^2+5x^3+7x^4+\cdots \longrightarrow (4)$

~~A(~~

eqn (3) − (4), we get

$A - Ax - Ax + Ax^2 = (1-0)+(3x-x)+(5x^2-3x^2)$
$+(7x^3-5x^3)+\cdots$

$$A - 2Ax + Ax^2 = 1 + 2x + 2x^2 + 2x^3 + \cdots$$

$$A(1-x)^2 = 1 + 2x(1 + x + x^2 + \cdots)$$

$$A(1-x)^2 = 1 + 2x(1-x)^{-1}$$

$$A(1-x)^2 = 1 + \frac{2x}{(1-x)}$$

$$A(1-x)^2 = \frac{1 - x + 2x}{(1-x)}$$

$$A(1-x)^2 = \frac{(1+x)}{(1-x)}$$

$$\boxed{A = \frac{(1+x)}{(1-x)^3}}$$

Find ~~the~~ ~~sequence~~ generating function of the sequence $0^2, 1^2, 2^2, 3^2, 4^2 \cdots$

Given $a_0 = 0^2, a_1 = 1^2, a_2 = 2^2, a_3 = 3^2$
$a_4 = 4^2 \cdots$

we know that,

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= 0 + x + 4x^2 + 9x^3 + \cdots$$

$$= x + 4x^2 + 9x^3 + \cdots$$

$$= x\left[1 + 4x + 9x^2 + \cdots\right]$$

$$= x\left[(1+x)/(1-x)^3\right]$$

Principle of Inclusion and Exclusion

Consider, a ~~find~~ finite set 's' containing 'p' no. of Elements. Here, number p is called the order, size, or the cardiality of set 's'. It is denoted by $n(s)$ or $|s|$

If A and B are two sets then principle of Inclusion and exclusion is given by

$$|A \cup B| = |A| + |B| - |A \cap B|$$

If A and B are disjoint sets then we have $|A \cup B| = |A| + |B|$

$\because A \cap B = \phi$

If A, B and C are three sets then the principle of inclusion and exclusion is given by

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Problem

1) In a class 52 students. 30 are studing C++, 18 are studying pascal and 13 are studying both languages. How many in this class are studing atleast one of this languages. How many are studying neither of these languages

Sol: Let A is denoted by studying C++ language

B is denoted by studying pascal

U is denoted by universal set

Given : $|A| = 30$

$|B| = 18$

$|A \cap B| = 13$

By the principle of inclusion & Exclusion is given by.

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 30 + 18 - 13$

$|A \cup B| = 35$

$|\overline{A \cup B}| = U - |A \cup B|$

$= 52 - 35$

$= 17$

(2) In a sample of 100 logic chips 23 have a defective D 26 have a defective D_2, 30 have a defective D_3, 7 have defects D_1 & D_2, 8 have defects D_1 & D_3, 10 have defects D_2 & D_3 and 3 have all the 3 defectives.

(i) Find the no. of chips having atleast one defect

(ii) No defect

Sol Let U denoted the set of all chips having the defects D_1, D_2 & D_3 respectively

$|U| = 100$ $\quad$ $|D_3| = 30$ $\quad$ $|D_2 \cap D_3| = 10$

$|D_1| = 23$ $\quad$ $|D_1 \cap D_2| = 7$ $\quad$ $|D_1 \cap D_2 \cap D_3| = 3$

$|D_2| = \cancel{30}\ 26$ $\quad$ $|D_1 \cap D_3| = 8$

by the principle of inclusion and Exclusion is given by

$$|D_1 \cup D_2 \cup D_3| = |D_1| + |D_2| + |D_3| - |D_1 \cap D_2| - |D_1 \cap D_3| - |D_2 \cap D_3| + |D_1 \cap D_2 \cap D_3|$$

$$= 23 + 26 + 30 - 7 - 8 - 10 + 3$$

$$= 57$$

$$|\overline{D_1 \cup D_2 \cup D_3}| = U - |D_1 \cup D_2 \cup D_3|$$

$$= 100 - 57$$

$$= 43$$

(3) A Survey of 500 TV viewers of a sports channel produced the following information 285 watched cricket, 195 watched hockey, 115 watched football, 45 Cricket and football, 70 Cricket and hockey, 50 watch hockey & football 50 do not watch any of the 3 kinds of game.

(i) How many viewers in Survey watch all 3 kinds of game.

(ii) How many viewers exactly watch one of the sports.

Sol :- Let A is denoted as watching Cricket

B is denoted as hockey

C is denoted as football

Now $|U| = 500$, $|A| = 285$ $|B| = 195$

$|C| = 115$

$|A \cap B| = 70$, $|A \cap C| = 45$ $|B \cap C| = 50$

$|\overline{A \cup B \cup C}| = 50$

by the principle of inclusion and Exclusion is given by

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap C| - |A \cap B|$
$- |B \cap C| + |A \cap B \cap C|$

we have $|\overline{A \cup B \cup C}| = |U| - |A \cup B \cup C|$

$50 = 500 - |A \cup B \cup C|$

$|A \cup B \cup C| = 450$

$450 = 285 + 195 + 115 - 45 - 70$
$- 50 + |A \cap B \cap C|$

$450 = 430 + |A \cap B \cap C|$

$\therefore |A \cap B \cap C| = 20$

(ii) Let A_1 denote the set of viewers only watch Cricket

B_1 denote the set of viewers who watch only hockey

C_1 denote the set of viewers who watch only football

$|A_1| = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 285 - 70 - 45 + 20$

$|A_1| = 190$

$|B_1| = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$

$= 195 - 70 - 50 + 20$

$|B_1| = 95$

$|C_1| = |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 115 - 45 - 50 + 20$

$= 40$

∴ The viewers who watch exactly ~~of~~ one of the sports equal to

$= |A_1| + |B_1| + |C_1|$

$= 190 + 95 + 40$

$= 325$

(4) Determine the no. of positive integer ≤ 100 which are divisible by 3 (or) 7.

Sol Let $S = \{1, 2, 3, \ldots, 100\}$

$|S| = 100$

Let A_1 & A_2 be the subset of S whose Elements are divisibly by 3 & 7 respectively.

we have to find A_1 (or) A_2

$|A_1 \cup A_2|$

By the ~~prici~~ principle of inclusion and Exclusion is given by

$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

$|A_1|$ = no. of Elements in S that are divisible by 3

$= 100/3$

$= 33$

$|A_2|$ = no. of elements in S that are divisible by 7

$= 100/7$

$= 14$

$|A_1 \cap A_2|$ = no. of elements in S that are divisible by both 3 & 7

$= \frac{100}{21} = 4$

$|A_1 \cup A_2| = 33 + 14 - 4$

$\therefore |A_1 \cup A_2| = 43$

① A Box 'A' Contains 4 red balls and 5 black balls. Box 'B' Contains 5 red balls and 4 Black balls. Box 'C' contains 6 red balls and 3 black balls one box is choosen at random and two balls drawn they happened to be red and black what is the probability that they are Coming from Box 1, Box 2, Box 3.

Sol Let E_1, E_2 and E_3 denote of drawing a Box 'A', Box 'B' and Box 'C'.

Let event A denote drawing a red and black balls

$$P(E_1) = \frac{1}{3}$$

$$P(E_2) = \frac{1}{3}$$

$$P(E_3) = \frac{1}{3}$$

$$P(A/E_1) = \frac{4C_1 \times 5C_1}{9C_2} = \frac{4 \times 5}{36} = \frac{20}{36}$$

$$P(A/E_2) = \frac{5C_1 \times 4C_1}{9C_2} = \frac{5 \times 4}{36} = \frac{20}{36}$$

$$P(A/E_3) = \frac{6C_1 \times 3C_1}{9C_2} = \frac{6 \times 3}{36} = \frac{18}{36}$$

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\left(\frac{1}{3}\right)\left(\frac{20}{36}\right)}{\left(\frac{1}{3}\right)\left(\frac{20}{36}\right) + \left(\frac{1}{3}\right)\left(\frac{20}{36}\right) + \left(\frac{1}{3}\right)\left(\frac{18}{36}\right)}$$

$$= \frac{\frac{20}{108}}{\frac{20}{108} + \frac{20}{108} + \frac{18}{108}}$$

$$= \frac{20/\cancel{108}}{\frac{20+20+18}{\cancel{108}}} = \frac{20}{58}$$

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\left(\frac{1}{3}\right)\left(\frac{20}{36}\right)}{(1/3)(20/36) + (1/3)(20/36) + (1/3)(18/36)}$$

$$= \frac{20/108}{\frac{20}{108} + \frac{20}{108} + \frac{18}{108}}$$

$$= \frac{20}{58}$$

$$P(E_3/A) = \frac{P(E_3)\,P(A/E_3)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{18/\cancel{108}}{58/\cancel{108}} = \frac{18}{58}$$

$= P(E_1/A) + P(E_2/A) + P(E_3/A)$

$= \frac{20}{58} + \frac{20}{58} + \frac{18}{58}$

$= \frac{58}{58}$

$= 1$

(2) In a bolt's factory machines A, B, C manufactured 20%, 30%, and 50% of the total of these output and 6%, 3% and 2% defective. A bolt is drawn at random and found to be defective find the probability is that it is manufacture from

(1) Machine A

(2) Machine B

(3) Machine C

Sol Let, E_1 denote ~~of~~ that the bolts are manufactured by machine 'A'.

E_2 denote of that the bolts are manufactured by machine 'B'

E_3 denote of that the bolts are manufactured by machine 'C'

$P(E_1) = \frac{20}{100}$

$P(E_2) = \frac{30}{100}$

$P(E_3) = \frac{50}{100}$

Let an event 'A' is drawing of defective bolt

$$P(A/E_1) = \frac{6}{100}$$

$$P(A/E_2) = \frac{3}{100}$$

$$P(A/E_3) = \frac{2}{100}$$

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\left(\frac{20}{100}\right)\left(\frac{6}{100}\right)}{\left(\frac{20}{100}\right)\left(\frac{6}{100}\right) + \left(\frac{30}{100}\right)\left(\frac{3}{100}\right) + \left(\frac{50}{100}\right)\left(\frac{2}{100}\right)}$$

$$= \frac{\frac{120}{10000}}{\frac{120}{10000} + \frac{90}{10000} + \frac{100}{10000}}$$

$$= \frac{120/10000}{\frac{310}{10000}} = \frac{12}{31}$$

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

$$= \frac{\left(\frac{30}{100}\right)\left(\frac{3}{100}\right)}{\left(\frac{20}{100}\right)\left(\frac{6}{100}\right) + \left(\frac{30}{100}\right)\left(\frac{3}{100}\right) + \left(\frac{50}{100}\right)\left(\frac{2}{100}\right)}$$

$$= \frac{\frac{90}{10000}}{\frac{310}{10000}} = \frac{9}{31}$$

$$P(E_3/A) = \frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\left(\frac{50}{100}\right)\left(\frac{2}{100}\right)}{\left(\frac{20}{100}\right)\left(\frac{6}{100}\right) + \left(\frac{30}{100}\right)\left(\frac{3}{100}\right) + \left(\frac{50}{100}\right)\left(\frac{2}{100}\right)}$$

$$= \frac{\frac{100}{10000}}{\frac{310}{10000}} = \frac{10}{31}$$

$\therefore P(E_1/A) + P(E_2/A) + P(E_3/A)$

$$= \frac{12}{31} + \frac{9}{31} + \frac{10}{31}$$

$$= \frac{31}{31}$$

$$= 1$$

(3) Suppose 5 men out of 100 and 25 women out of 10000 are colour blind. A colour blind person is choosen at random. What is the probability of the person being a male. Assume that male and female to be in equal numbers.

Sol: Let M denotes an event of men, w is denote of an event of women. A colour blind person is denoted by A.

$P(M) = \frac{1}{2}$

$P(w) = \frac{1}{2}$

$$P(A/M) = \frac{5}{100} = 0.05$$

$$P(A/W) = \frac{25}{10000} = 0.0025$$

$$P(M/A) = \frac{P(M)\,P(A/M)}{P(M)\,P(A/M) + P(W)\,P(A/W)}$$

$$= \frac{\left(\frac{1}{2}\right)\left(\frac{5}{100}\right)}{\left(\frac{1}{2}\right)\left(\frac{5}{100}\right) + \left(\frac{1}{2}\right)\left(\frac{25}{10000}\right)}$$

$$= \frac{\frac{5}{200}}{\frac{5}{200} + \frac{25}{20000}}$$

$$= \frac{\frac{5}{200}}{0.05 + 0.0025}$$

$$= \frac{0.05}{0.05 + 0.0025} = 0.95$$

Random variable :-

A real number is assigned to the outcomes of a random experiment is called random variable.

eg :- tossing of a coin, throwing of a die

there are 2 types of random variable

i) Discrete random variable

ii) Continuous random variable

Discrete random variable :-

A random variables takes finite values (or) integer values is called discrete random variable

eg :- No. of rooms in a building.

No. of std's in a class.

Continuous random variable :-

A random variable's which takes the infinite value's in a given interval.

Eg :- age, weight, height etc... and also temperature.

Probability function :

Let x be a discrete random variable which takes the values $x_1, x_2 ---- x_n$ & corresponding probabilities $p(x_1), P(x_2) --- p(x_n)$ then the set $\{x_i : p(x_i)\}$ is called probability function it satisfies the following conditions

(1) $p(x_i) > 0$

(2) $\sum p(x_i) = 1$

probability distribution function :-

Let X be a discrete random variable with probability function $p(x)$ then the pdf (or) Commulative distribution function of random variable X is given by $\boxed{F(x) = P(X \leq x)}$

$$= \sum P(X = x)$$

Eg:- If a coin is tossed 2 times & random variable denotes no. of heads obtained. what is probability function of x & pdf of x.

Sol: The coin is tossed 2 times the Sample $S = \{HH, HT, TH, TT\}$

Let X denotes no. of heads

X	0	1	2
p(x)	1/4	2/4	1/4

pdf is given by $p(X \leq x)$

$$= p(X \leq 0)$$

$$= \frac{1}{4}$$

$$f(1) = p(X \leq 1) = p(X=0) + p(X=1)$$

$$= \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$$

$$f(2) = p(X \leq 2) = p(X=0) + p(X=1) + p(X=2)$$

$$= \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$$

Mean & variance of the discrete random Variable

Let x be a discrete random variable which takes the values $x_1, x_2 \ldots x_n$ & Corresponding probabilities $p(x_1), P(x_2) \ldots p(x_n)$ then the mean / Avg of random variable x is given by

$$\boxed{E(x) = x_1 p(x_1) + x_2 p(x_2) + \ldots x_n P(x_n)}$$

$$\boxed{\therefore E(x) = \sum x p(x)}$$

Mean Square:

Let x be a discrete random variable which takes the values $p(x_1), p(x_2) \ldots p(x_n)$ then the mean square of the random variable x is given by

$$\boxed{E(x^2) = \sum x^2 p(x)}$$

Variance of DRV:-

If x be a DRV with mean $E(x)$ & mean square $E(x^2)$ then variance of x is given by

$$\boxed{\sigma^2 = E(x^2) - [E(x)]^2}$$

Problems :-

① A discrete random variable X is defined as Sum of the no's on faces when 2 dic's are thrown. Find the mean of X

Sol Let random variable X denotes sum of the 2 no's when 2 dic's are thrown i.e random variable X takes the values

X	2	3	4	5	6	7	8	9	10	11	12
P(x)	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$$E(x) = \sum x\, p(x)$$

$$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36}$$

$$= \frac{252}{36} = 7$$

$$\boxed{\therefore E(X) = 7}$$

② Find mean & variance of the following probability function.

X	8	12	16	20	24
P(x)	1/8	1/6	3/8	1/4	1/12

$$\text{Mean} = E(x) = \sum x\, p(x)$$

$$= \frac{8}{8} + \frac{12}{6} + \frac{48}{8} + \frac{20}{4} + \frac{24}{12}$$

$$= 1 + 2 + 6 + 5 + 2$$

$\boxed{E(x) = 16}$

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \sum x^2 p(x)$$

$$= \frac{64}{8} + \frac{144}{6} + \frac{768}{8} + \frac{400}{4} + \frac{576}{12}$$

$$= 8 + 24 + 96 + 100 + 48$$

$$= 276$$

$$= 276 - [16]^2$$

$$= 276 - 256$$

$$\boxed{V(x) = 20}$$

Addition theorem of Probability

If A and B are not disjoint events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are disjoint events then

$P(A \cup B) = P(A) + P(B)$

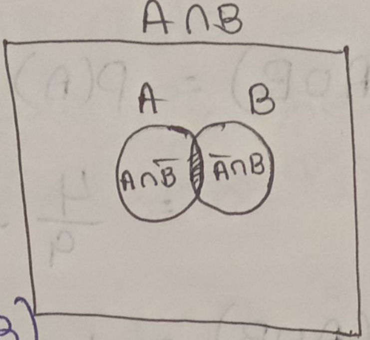
Proof :-

Let sample S contains two events A and B.

Consider, venn diagram

Consider,

$A \cup B$



$A \cup B = A \cup (\bar{A} \cap B)$

$p(A \cup B) = P(A) + P(\bar{A} \cap B)$

since A, $\bar{A} \cap B$ are mutually exclusive events

$$P(A \cup B) = P(A) + P(\bar{A} \cap B) + P(A \cap B) - P(A \cap B)$$

$$= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B)$$

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

① A Box Contains four (4) red and 5 black balls one ball is drawn at random. what is the probability it is either Red or Black.

Let A be the Event of drawing a Red ball

B be the Event of drawing a Black ball.

$$P(A) = \frac{4C_1}{9C_1} = \frac{4}{9}$$

$$P(B) = \frac{5C_1}{9C_1} = \frac{5}{9}$$

$P(A \cap B) = 0$ [only one ball can be drawn so $P(A \cap B)$ becomes 0]

Since A and B are mutually Exclusive events

By addition thereom of Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{4}{9} + \frac{5}{9} - 0$$

$$P(A \cup B) = 1$$

(2) From a pack of 52 Cards 1 Card is drawn at random what is the probability that it is Either a spade or a diamond.

Let A be the Event of drawing a spade Card

$$P(A) = \frac{13C_1}{52C_1} = \frac{13}{52} = \frac{1}{4}$$

Let B be the Event of drawing a diamond Card $P(B) = \frac{13C_1}{52C_1} = \frac{13}{52} = \frac{1}{4}$

$$P(A \cap B) = 0$$

By Addition theorem of probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{1}{4} + \frac{1}{4} - 0$$

$$= \frac{2}{4} - 0$$

$$= \frac{1}{2}$$

$P(A \cup B) = 0.5$ [50% Chances to get a spade or diamond]

③ From a pack of 52 Cards one Card is drawn at random what is the probability that it is Either a spade or jack.

Sol In a 52 Pack of Cards there are 13 spades and 4 jacks

Let A be the Event of getting a spade

$P(A) = \frac{13C_1}{52C_1} = \frac{13}{52} = \frac{1}{4}$

Let B be the Event of getting a jack

$P(B) = \frac{4C_1}{52C_1} = \frac{4}{52} = \frac{1}{13}$

$P(A \cap B) = \frac{1C_1}{52C_1}$ [Because, in spades one jack card is present

$= \frac{1}{52}$

By addition theorem of probability

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$

$= \frac{16}{52}$

$= \frac{4}{13}$

$P(A \cup B) = 0.307$ [30% chance]

Multiplication theorem of Probability

If A and B are two dependent Events then

$$\boxed{P(A\cap B) = P(A).\,P(B/A)}$$

If A and B are two independent Events then $\boxed{P(A\cap B) = P(A).\,P(B)}$

A Box Contains 4 Red and 5 black balls two balls are drawn one after the another with and without replacement. what is the probability that first ball is a red and second ball is black.

MTP

Sol :- Let A be the Event of getting a Red Ball

B be the Event of getting a black ball

without replacement

$$P(A) = \frac{4C_1}{9C_1} = \frac{4}{9}$$

$$P(B/A) = \frac{5C_1}{8C_1} = \frac{5}{8}$$

Since the drawn red ball is not included in the box. and also A and B are dependent events.

By Multiplication theorem of probability

$P(A \cap B) = P(A)\, P(B/A)$

$= 4/9 \times 5/8$

$= \frac{20}{72} = 0.277$

with replacement

$P(A) = \frac{4C_1}{9C_1} = \frac{4}{9}$

$P(B) = \frac{5C_1}{9C_1} = \frac{5}{9}$

Since A and B independent events

By Multiplication theorem of probability

$P(A \cap B) = P(A).\, P(B)$

$= 4/9 \times 5/9$

$= \frac{20}{81} = 0.24$

Topic 3 Divide and Conquer Algorithms & Recurrence Relations

→ Many recursive algorithms take a problem with a given input and divide it into one or more smaller problems.

→ This reduction is successively applied until the solutions of the smaller problem can be found quickly.

→ These procedures of dividing.. and solving smaller problems is called Divide and Conquer Algorithms (DAC- Dividing-and-Conquer Algorithms).

→ DAC-Algorithms divide a problem into one or more instances of the same problem of smaller size and they conquer the problem by using the solutions of the smaller problems to find a solution of the original problem, perhaps with some additional work.

→ Let us see how recurrence relation can be used to analyze the computational complexity of DAC-Algorithms.

→ Recurrence relations of DAC-Algorithms are used to estimate the number of operations used to solve the problem.

DAC Recurrence Relations

→ Suppose that a recursive algorithm divides a problem of size n into "a" subproblems, where each subproblem is of size "n/b".

→ Also, suppose that a total of $g(n)$ extra operations are required in the Conquer step of the algorithm to Combine the solutions of the subproblem into a solution of the original problem.

→ Then, if $f(n)$ represents the number of operations required to solve the problem of size n, it follows that f satisfies the recurrence relation

$$\boxed{f(n) = a \cdot f(n/b) + g(n)}$$

→ This is called a DAC-Recurrence Relation.

→ Algorithms using DAC approach are

i) Binary Search.
ii) Finding the Max. & Min. of a Sequence
iii) Merge Sort
iv) Fast multiplication of Integers
v) Fast matrix Multiplication
vi) Closest-Pair problem.

Binary Search

→ It reduces the search for an element in a search sequence of size n to the binary search for this element in a search sequence of size $n/2$ when n is even

→ Two comparisons are needed to implement this reduction (One to determine which half of the list to use and the other to determine whether any terms of the list remain).

→ Hence, if $f(n)$ is the number of comparisons required to search for an element in a search sequence of size n, then

$$f(n) = f(n/2) + 2$$

when n is even.

Finding the Max & Min. of Sequence

→ Let $f(n)$ be the total number of comparisons needed to find the max. and min. elements of the sequence with n elements.

→ The problem of size n can be reduced into two problems of size $n/2$, when n is even, using two comparisions, one to compare the maxima of the sequences and the other to compare the minima of the two sequences.

→ This gives the recurrence relations

$$\boxed{f(n) = 2f(n/2) + 2}$$

<u>Merge Sort</u>

→ This algorithm splits a list to be sorted with n items, where n is even, in to two lists with $n/2$ elements each and uses fewer than n comparisions to merge the two sorted $n/2$ list items.

→ $M(n)$ – No. of comparisons by merge sort to sort n-elements.

→ $\boxed{M(n) = 2M(n/2) + n}$

Fast Multiplication of Integers

→ This algorithm proceeds by splitting each of two 2n-bit integers into two blocks, each with n bits.

→ $a = (a_{2n-1} a_{2n-2} \cdots a_2 a_1 a_0)$ → 2n-bit integers
$b = (b_{2n-1} b_{2n-2} \cdots b_2 b_1 b_0)$ → 2n-bit integers

→ A_0, A_1, B_0, B_1 are blocks which stores n bits.

→ $A_0 = (a_{n-1} a_{n-2} \cdots a_2 a_1 a_0)$ | $B_0 = (b_{n-1} b_{n-2} \cdots b_1 b_0)$
$A_1 = (a_{2n-1} a_{2n-2} \cdots a_{n+2} a_{n+1} a_n)$ | $B_1 = (b_{2n-1} b_{2n-2} \cdots b_{n+1} b_n)$

→ Let
$$a = 2^n A_1 + A_0 \qquad b = 2^n B_1 + B_0$$

→ $$ab = 2^n A_1 2^n B_1 + 2^n A_1 B_0 + 2^n A_0 B_1 + A_0 B_0$$
$$= 2^{2n} A_1 B_1 + 2^n A_1 B_0 + 2^n A_0 B_1 + A_0 B_0$$

Add & Subtract $2^n A_1 B_1$ & $2^n A_0 B_0$ terms

$$ab = 2^{2n} A_1 B_1 + 2^n A_1 B_0 + 2^n A_0 B_1 + A_0 B_0 + 2^n A_1 B_1 - 2^n A_1 B_1 + 2^n A_0 B_0 - 2^n A_0 B_0$$

$$ab = \left[2^{2n} + 2^{n}\right] A_1 B_1 + 2^{n}\left[A_1 B_0 + A_0 B_1 - A_1 B_1 - A_0 B_0\right]$$
$$+ \left[2^{n} + 1\right] A_0 B_0$$

$$ab = \left[2^{2n} + 2^{n}\right] A_1 B_1 + 2^{n}\left[(A_1 - A_0)(B_0 - B_1)\right] + \left[2^{n} + 1\right] A_0 B_0$$

Multiplication of above two 2n-bit integers (ab) involves 3 multiplications of n-bit integers, together with additions, subtractions and shifts.

→ f(n) is total number of bit operations needed to multiply two n-bit integers

then

$$f(2n) = 3 f(n) + Cn$$

→ 3 multiplications of n-bit integers are carried out using 3f(n)-bit operations.

→ Each of the additions, Subtraction and Shifts uses a constant multiple of n-bit operations. and Cn represents the total number of bit operation used by these operations.

Fast Matrix Multiplication

- Multiplying two $n \times n$ matrices requires n^3 multiplications and $n^2(n-1)$ additions.
- Consequently, this $n \times n$ multiplication of matrices requires $O(n^3)$ operations
- Surprisingly, there are more efficient DAC algorithms for multiplying two $n \times n$ matrices.
- Such an algorithm, invented by V. Strassen in 1969, reduces the multiplication of two $n \times n$ matrices, when n is even, to Seven multiplication of two $(n/2) \times (n/2)$ matrices and 15 additions of $(n/2) \times (n/2)$ matrices.
- Hence, $f(n)$ = Total no. of operations in Strassen's matrix multiplication algorithm
- $\therefore \boxed{f(n) = 7 f(n/2) + 15(n^2/4)}$

when n is even.

Note:

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a \cdot f(n/b) + c$$

when n is divisible by b,

$a \geq 1$

$b > 1$

c is positive real number

then

$$f(n) = \begin{cases} O(n^{\log_b a}) & \text{if } a > 1 \\ O(\log n) & \text{if } a = 1 \end{cases}$$

$n^{\log_b a}$

$n^{\log_2 5}$

$5^{\log_2 n}$

Furthermore, when $n = b^k$, where k is a positive integer.

$$f(n) = C_1 \cdot n^{\log_b a} + C_2$$

$n = 2^k$

$\log_2 n = k$

$n = b^k$

where

$$C_1 = f(1) + \frac{c}{(a-1)}$$

$$C_2 = \frac{-c}{(a-1)}$$

Example Let $f(n) = 5f(n/2) + 3$ & $f(1) = 7$

find $f(2^k)$, where k is a positive integer also estimate $f(n)$ if f is an increasing function.

Sol: Given $a = 5$, $b = 2$; $c = 3$; $n = 2^k \Rightarrow k = \log_2 n$

as $a > 1$

$$f(n) = \left[f(1) + \frac{3}{(5-1)}\right] \cdot O(n^{\log_2 5}) - \frac{3}{(5-1)}$$

$$f(n) = \left[7+\frac{3}{4}\right] O\left(5^{\log_2 n}\right) - \frac{3}{4}$$

w.k.T $\boxed{k = \log_2 n}$ $\left[\because n^{\log_b a} = a^{\log_b n}\right.$

$$\boxed{f(n) = \left(\frac{31}{4}\right) O(5^k) - \frac{3}{4}}$$

Estimate the number of Comparisions used by

Binary Search

$f(n) = f(n/2) + 2$

Here $a = 1$
$b = 2$

$$\boxed{f(n) = c_1 O(\log n) + c_2}$$

Finding Max. & Min element in a Sequence

$f(n) = 2f(\frac{n}{2}) + 2$

Here $a = 2$
$b = 2$

$f(n) = c_1 O(n^{\log_b a}) + c_2$

$f(n) = c_1 O(n^{\log_2 2}) + c_2$

$$\boxed{f(n) = c_1 O(n) + c_2}$$

Master's Theorem

Let f be an increasing function that satisfies the recurrence relation

$$f(n) = a \cdot f\left(\frac{n}{b}\right) + cn^d$$

when $n = b^k$; $a \geq 1$; $b > 1$ c & d are real numbers with c positive and d nonnegative then

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

Complexity of Merge Sort

$M(n) = 2M(n/2) + n$

Compare with

$f(n) = af(\frac{n}{b}) + cn^d$

$\therefore a = 2;\ b = 2;\ c = 1;\ d = 1$

Check for a & b^d relationship

if $\boxed{2 = 2^1 \Rightarrow a = b^d}$ then

$f(n) = O(n^d \log n)$ given $d = 1$

$\therefore \boxed{f(n) = O(n \log n)}$

Estimate the number of bit operations needed to multiply two n-bit integers using fast multiplication algorithm.

Sol: $f(n) = 3f(n/2) + Cn$

$a = 3;\ b = 2;\ c = C;\ d = 1$

If $\boxed{3 > 2^1 \Rightarrow a > b^d}$ then

$f(n) = O(n^{\log_b a})$

$f(n) = O(n^{\log_2 3})$ $\boxed{\log_2 3 \sim 1.6}$

$\boxed{f(n) = O(n^{1.6})}$

Note: Conventional algorithm complexity is $O(n^2)$.

Estimate the number of multiplication and additions required to multiply the two $n \times n$ matrices using the matrix multiplication algorithm

Sol: $f(n) = 7f\left(\frac{n}{2}\right) + \frac{15n^2}{4}$

given $a = 7$; $b = 2$; $c = \frac{15}{4}$; $d = 2$

If $\boxed{7 > 2^2 \Rightarrow a > b^d}$ then

$f(n) = O(n^{\log_b a})$

$f(n) = O(n^{\log_2 7})$

$\boxed{f(n) = O(n^{2.8})}$

$\boxed{\log_2 7 \sim 2.8}$

Note: Conventional algorithm complexity is $O(n^3)$.

The Closest Pair problem

- The Recurrence relation is

$f(n) = 2f(n/2) + 7n$

given $a = 2$; $b = 2$; $c = 7$; $d = 1$

If $\boxed{2 = 2^1 \Rightarrow a = b^d}$ then

$f(n) = O(n^d \log n)$

$\therefore \boxed{f(n) = O(n \log n)}$

Scanned with CamScanner