ES3C3 – Planar Structures & Mechanisms, Coursework Assignment

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Input Conditions:

ω = ± 41 rads^{-1} or 0 rads^{-1}

α = ± 6 rads^{-1} or 0 rads^{-2}

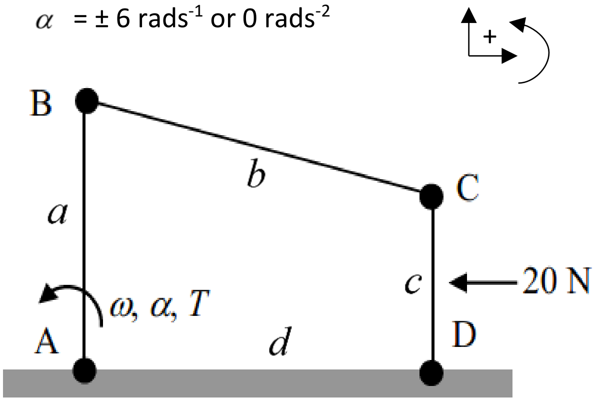


Figure 1: 4 Bar Linkage

Link	AB	BC	CD	DA
Length (m)	0.5	0.7	0.3	0.67
Mass (kg)	0.6	1.0	0.5	-
Moment of Inertia (kgm^2)	0.015	0.04	0.003	-

Table 1: Geometric Data of Mechanism

Introduction

This assignment consists of the analysis of a 4 bar linkage mechanism. A motor provides varying input conditions in the form of a driving torque on link *a*. The dimensions and relevant quantities of the constituent links are shown above, in **Table 1**, along with a diagram of the mechanism. The report consists of 3 phases; kinetic analysis of the mechanism at constant position, kinematic analysis with varying position and lastly analyses and recommendations for construction of the mechanism. Evaluation of the matrix equations associated with phases 1&2 was done computationally using Matlab®, the scripts utilised are attached as appendices at the end of this report along with the m. function files they produced.

Phase 1 – Kinetics at constant position, varying speed and acceleration

The angles of the links to the *x*-axis reference (parallel to link AD) are $\theta_1 = \pi$, $\theta_2 = \pi/2$, $\theta_3 = 5.99$ and $\theta_4 = 3\pi/2$ (all radians). Using non-rotating local axis systems located at the pins, two simultaneous equations for the angular velocities can be expressed as:

$$\begin{aligned} \text{Re}: & -a\omega_2 \sin\theta_2 - b\omega_3 \sin\theta_3 - c\omega_4 \sin\theta_4 = 0 \\ \text{Im}: & \ a\omega_2 \cos\theta_2 + b\omega_3 \cos\theta_3 + c\omega_4 \cos\theta_4 = 0 \end{aligned} \qquad \textbf{(1.1)}$$

Where *a*, *b*, *c* and *d* are the lengths defined in Table 1 and highlighted in Figure 1 above. ω and θ are the angular velocities and positions respectively. Re: , Im: represent the relevant real and imaginary components respectively. The subscripts to each link are as follows: Link $DA_{(1)}$, Link $AB_{(2)}$, Link $BC_{(3)}$, Link $CD_{(4)}$. Substituting the angles θ_1, θ_2, θ_3 and *a*, *b*, *c* as above, into Eqn.(1.1) yields the set of velocities needed for the two simultaneous equations for angular acceleration below **(1.2)**

$$\mathrm{Re}:(-a\omega_2^{\ 2}\cos\theta_2 - a\alpha_2\sin\theta_2) + (-b\omega_3^{\ 2}\cos\theta_3 - b\alpha_3\sin\theta_3) + (-c\omega_4^{\ 2}\cos\theta_4 - c\alpha_4\sin\theta_4) = 0$$

$$\mathrm{Im}:(-a\omega_2^{\ 2}\sin\theta_2 + a\alpha_2\cos\theta_2) + (-b\omega_3^{\ 2}\sin\theta_3 + b\alpha_3\cos\theta_3) + (-c\omega_4^{\ 2}\sin\theta_4 + c\alpha_4\cos\theta_4) = 0$$

Where α is the angular acceleration of each link. The solution of Eqn.(1.2) completes the kinematic analysis. Kinetic analysis initially treats each link as a free body. The dynamic equations for link BC (link 3) are formulated by resolving the member forces horizontally and vertically along with their resulting moments at both pins:

$$\begin{aligned} &F_{23x} + F_{43x} = m_3 a_{3x} \\ &F_{23y} + F_{43y} = m_3 a_{3y} \qquad \textbf{(1.3)} \\ &-R_{23y}F_{23x} + R_{23x}F_{23y} - R_{43y}F_{43x} + R_{43x}F_{43y} = I_3\alpha_3 \end{aligned}$$

Each value of R_{ij} is either the horizontal or vertical component of the reaction forces at each pin joint. Values for F_{ij} indicate the horizontal and vertical components of the external forces on these pins from the other links (subscripts $_{2,3,4}$). I_3 and m_3 are the moment of inertia and mass of the link BC, respectively. α_3 and a_3 represent the angular accelerations and linear accelerations of link 3, respectively. Other links generate similar sets with slight variations according to the conditions on the particular link.

Using Newton's laws to balance all the resulting force components at each pin, the dynamic equations for each link can be combined into the vector-matrix kinetic model: $[\mathbf{G}]\mathbf{F} = \mathbf{R}$ **(1.4).** [**G**] is the matrix representing the geometrical arrangement of the mechanism, **F** is a vector of forces at the pins and **R** a vector representing the inertial effects. Their full forms for the mechanism are on the next page.

$$
\begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-R_{12y} & R_{12x} & R_{32y} & -R_{32x} & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -R_{23y} & R_{23x} & R_{43y} & -R_{43x} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -R_{34y} & R_{34x} & R_{14y} & -R_{14x} & 0
\end{bmatrix}
\begin{pmatrix} F_{12x} \\ F_{12y} \\ F_{23x} \\ F_{23y} \\ F_{34x} \\ F_{34y} \\ F_{41x} \\ F_{41y} \\ T_{in} \end{pmatrix}
=
\begin{pmatrix} m_2 a_{2x} \\ m_2 a_{2y} \\ I_2 \alpha_2 \\ m_3 a_{3x} \\ m_3 a_{3y} \\ I_3 \alpha_3 \\ m_4 a_{4x} - F_{cx} \\ m_4 a_{4y} \\ I_4 \alpha_4 \end{pmatrix}
$$

All the terms follow the pattern indicated by Eqn.(1.3) except for the addition of the external force F_{cx} which acts horizontally at the midpoint of link *c* with a magnitude of 20N. Applying the combinations of angular speeds and accelerations specified, the solution of Eqn.(1.4), computed using Matlab®, generates the data in **Table 1** overleaf. The forces on the pin are found by solving this set of equations. The reaction forces, as elements of the geometric arrangement matrix, were taken to be functions of the horizontal or vertical components of the vector lengths of each link in the mechanism. In combination with the known quantities for the masses and moments of inertia of each link, the force vector **F** was found as the solution of the system of linear equations.

Table 1: Force exerted on pin C by link CD.

ω (rads^{-1})	α (rads^{-2})	F_{Cx} (N)	F_{Cy} (N)
+41	6	-16.9260	628.0862
0	6	9.5250	-2.3926
-41	6	-16.9260	628.0862
+41	0	-16.4510	627.4973
0	0	10.0000	-2.9814
-41	0	-16.4510	627.4973

With these modes of operation of the mechanism, the following observations can be made:

- The peak forces acting vertically on pin C are more than an order of magnitude greater than any of the other forces and always act upwards in the positive y direction
- Angular accelerations in the range 0-6rads^{-2} have little effect on the resultant forces compared to that of varying the magnitudes and directions of the driving angular velocity
- The forces are identical in both direction and magnitude for equal but opposite velocity

Phase 2 – Kinetics at varying positions with fixed speed

A positional analysis is now needed for each specified orientation of link AB. This starts by setting up simultaneous equations that equate to zero the sums of the *x*- and *y*-components of the link (vector) lengths

$$\begin{aligned} &\mathrm{Re}:(a\cos\theta_2 + b\cos\theta_3 + c\cos\theta_4 + d\cos\theta_1) = 0 \\ &\mathrm{Im}:(a\sin\theta_2 + b\sin\theta_3 + c\sin\theta_4 + d\sin\theta_1) = 0 \end{aligned} \quad \textbf{(2.1)}$$

Where all lengths and angles are as previously defined. When squared and added together, these two equations eliminate the variable θ_4. Rearranging the resulting equation and rewriting the $\cos\theta_3$ and $\sin\theta_3$ expressions in terms of $\tan\theta_3$ gives Freudenstein's equation. The form shown below has been derived with the use of a quadratic equation in *t,* an arbitrary parameter

Solving $(A-B)t^2 + Ct + (A+B) = 0$ gives $t = \dfrac{-C \pm \sqrt{C^2 - 4(A^2 - B^2)}}{2(A-B)} = \tan\left(\dfrac{\theta_3}{2}\right)$

where A, B and C represent terms of the Freudenstein equation in relation to the vector components of the resultant forces on each member

$$\begin{aligned} A &= a^2 + b^2 - c^2 + d^2 - 2ad\cos\theta_2 \\ B &= 2b(a\cos\theta_2 - d) \\ C &= 4ab\sin\theta_2 \end{aligned} \quad \textbf{(2.2)}$$

These equations can then be used to solve for θ_4 as shown below (**2.3)**. θ_3 may then be found by computing the four-quadrant inverse tangent of the resulting components of the Freudenstein expression. The use of this method in Matlab® ensures that only the real component is returned.

$$\theta_3 = 2\arctan\left(\frac{-C \pm \sqrt{C^2 - 4(A^2 - B^2)}}{2(A-B)}\right),\ \theta_4 = \arcsin\left(\frac{a\sin\theta_2 + b\sin\theta_3}{-c}\right) \quad \textbf{(2.3)}$$

All the positional information is now available, so the kinematic and kinetic analysis can proceed by exactly the same process as in Phase 1, solving via Eqn.(1.1) to Eqn.(1.4). Results for the specified positions of link AB are summarized overleaf in **Table 2** and were computed using Matlab®.

Table 2: Orientation, torque and forces and constant ω = 41 rads^{-1}

θ_2 (°)	θ_3 (°)	θ_4 (°)	T_{in} (Nm)	Pin B (N)		Pin D (N)	
				F_{23x}	F_{23y}	F_{41x}	F_{41y}
45	-28.28	-175.81	-2290.50	3204.20	-3274.40	6547.30	930.90
60	-20.57	-141.43	-270.80	203.40	-731.10	1095.60	977.60
80	-17.01	-106.53	-26.75	-37.48	-520.70	250.14	903.19
90	-16.60	-90.00	49.99	-99.98	-493.17	5.31	977.71
100	-17.10	-72.83	216.90	-361.00	-450.80	-340.40	1274.30
120	N/A	N/A	N/A	N/A	N/A	N/A	N/A

From table 2 above, the following observations have been made:

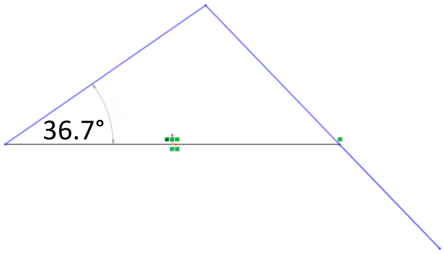
- An increase in driving angle, θ_2 , reduces the angles of the other links 3 and 4.
- The force F_{23y} consistently decreases with increased driving angle
- The mechanism cannot extend to a driving angle of 120°

Phase 3 – Recommendations to the designer and discussion of failure modes

Based on the results and discussions above, it is recommended that the design team consider seriously the following issues before proceeding with this design:

- The specified geometry of the mechanism limits the driving angle to the range $36.7° \leq \theta_2 \leq 116.6°$. Beyond these points the mechanism folds back on itself, this is important to consider with regards to the intended range of motion of the mechanism.
- With constant angular driving velocity, the geometry of the mechanism causes link *c* to accelerate drastically, applying forces to its associated pins of the order 10^6-10^7N. This occurs within about 2 ° of both the minimum and maximum driving angles yet with a particular peak towards the maximum. This is demonstrated in **Figure 4** overleaf. It would be wise to limit the range of the mechanism, so as to avoid these immense peak forces, to say $38° \leq \theta_2 \leq 115°$.
- Considering the point above, if the mechanism were to not be limited and was to experience the full peak forces possible – the stresses experienced by the mechanism would exceed those withstand-able by some of the strongest engineering materials currently known.

Figures 2 and 3 (left to right) – Mechanism at position of minimum driving angle and maximum angle, respectively.



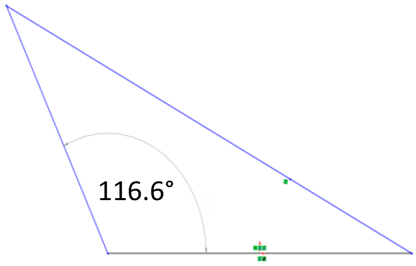
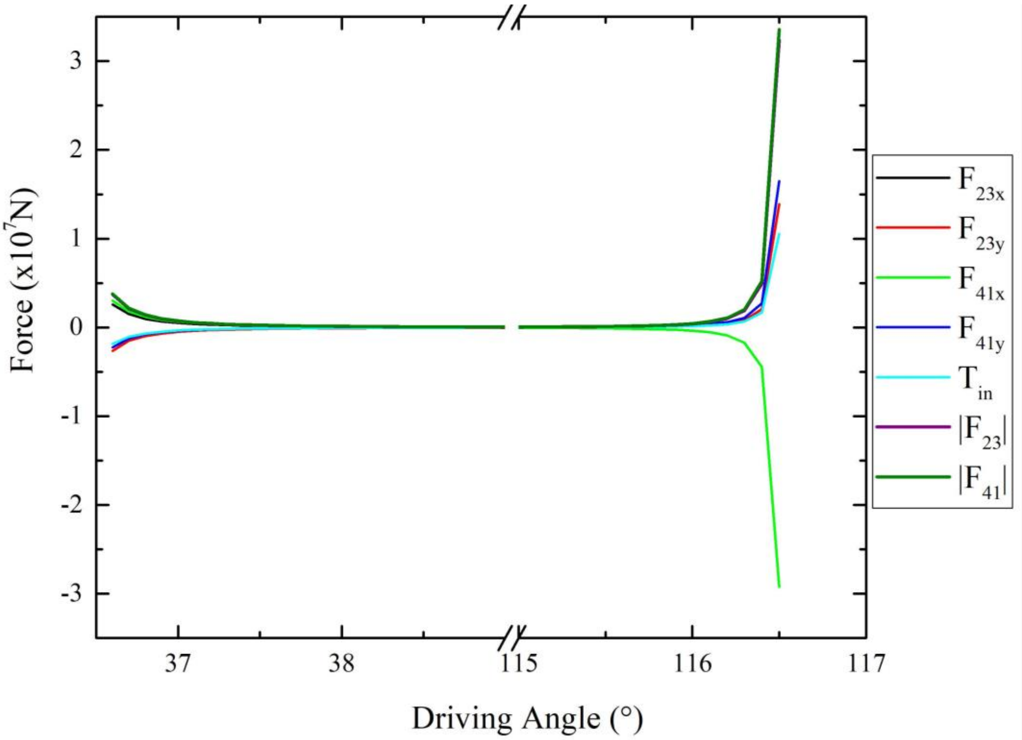


Figure 4 – Plot of Force Components vs Driving Angle (θ_2), interpolated from Matlab® data



Stress Analyses (Pure Tensile Load)

Assumed Link Cross Sectional Area, **A** (m^2) = $4x10^{-4}$, Peak Force in Mechanism, **F** (N) = ~ $3x10^7$

Resulting Stress **σ** = $75x10^9$ GPa   due to $\boldsymbol{\sigma = F/A}$

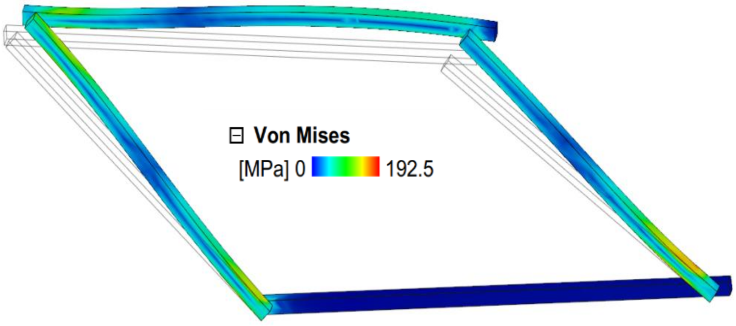
Tensile Yield Strength of Engineering Materials (Mild Steel) = 215MPa, (Grade 4 Titanium) = 552MPa, (Carbon Kevlar) = 3.69 GPa

Considering the results of the above rudimentary stress analysis, the mechanism with its full range of motion is infeasible and would undergo failure before completion of 1 cycle of operation. If one was to consider a reduced angle range such as that presented by the increments in **Table 2,** $45° \leq \theta_2 \leq 100°$ then the forces and stresses would be significantly reduced.

Stress with peak forces in range 45°≤θ_2≤100° (F_{max} = 6547N)

σ (new) = 16MPa = well within material limits

Figure 5 – Von Mises FEA Simulation for θ_2= 100° and constant ω=41rads^{-1}



The results of both the FEA and tabulated data clearly indicate that a reduced range of motion is necessary to allow for feasible operation of the mechanism in order to prevent immediate tensile overload. Even the higher Von Mises results (Peak of 193MPa) fall comfortably within the yield limit of mild steel provided the range is reduced to 45°≤θ_2≤100°.

Further possible modes of failure

Fatigue: As with any mechanism operating cyclically, fatigue is a major consideration with regards to potential failure. Considering the materials selected, mild steel has a fatigue limit of 205MPa, Titanium=425MPa and Kevlar=1.5-2.7GPa. These are all above the peak stresses experienced. Consequently, fatigue is neglected as a possible cause of failure for this mechanism.

Buckling: With Euler's formula, the likelihood of buckling can be assessed considering the longest link BC in mild steel. This can be considered to be the worst case scenario for the mechanism:

$F = \dfrac{n\pi^2 EI}{L^2}$ where n = 1 for two pinned ends and I = $\dfrac{bd^3}{12} = 13\times10^{-9}\,m^4$

F (critical) therefore = 55kN. Considering the restricted range of motion, Buckling is unlikely.

Bending: This is assumed with a peak force of 6547N acting at a distance of 0.15m from pin D (F_{41x}). Modelled as a simply supported beam, the maximum deflection is calculated as follows:

$$y_{\max} = \frac{1}{48}\frac{WL^3}{EI} = 6.75\times10^{-4}\,m$$

This may be problematic depending on the intended use of the mechanism, worth considering as the designer. With all the above in mind, there are many factors to consider to avoid failure of the mechanism and these should all be paid attention to when designing this linkage for actual use.

Appendix: Matlab® code used to aid evaluation

Code for fbassign.m – establishes parameters and input conditions

```
%Four-bar linkage for ES3C3 Assignment
%
% Run this code by typing
%      fbassign

% Enter your student ID:
student_ID = 15154864;
%-------------------------
% set up basic parameters

% Get parameters for defined link 2
th2deg = input('Enter drive angle (degrees) ');
th2 = th2deg*pi/180;
om2 = input('angular speed omega for link 2 (rad/s)  ');
al2 = input('angular acceleration alpha for link 2 (rad/s^2)  ');

% define link lengths (units)
a = 0.5;
b = 0.7;
c = 0.3;
d = sqrt(0.7*0.7-0.2*0.2);

% define active link inertias
m2 = 0.6;
m3 = 1;
m4 = 0.5;

I2 = 0.015;
I3 = 0.04;
I4 = 0.003;

% define external force on link 4
f4x = -20;
%------------------------
% run analysis using other m-files
%fbpos;     %position established (reporting theta3)
%fbkinemat;     %velocities & acc'ns established
x=fbforce(a,b,c,d,m2,m3,m4,I2,I3,I4,f4x,th2,om2,al2)    %forces
established in vector F.
%------------------------
% add code to display results:
```

Code for fbpos.m – conducts positional analysis

```
% four-bar link - position analysis
% All link lengths are known, and input angle drive to link 2
% Uses link lengths a,b,c,d; link 2 is . . . .

% set up length components for link 2 (strict cyclic angles)
function [ax,ay,bx,by,cx,cy,th3,th4]=fbpos(th2,a,b,c,d)
```

```
    ax = a*cos(th2);
    ay = a*sin(th2);

    % use 'Freudenstein eqn' in form on ES3C3 slide 6 in
    % https://www2.warwick.ac.uk/es3c3/resources/es3c3_slides_2_2016-
2017---10nov.pdf

    A = a*a + b*b - c*c + d*d -2*ax*d;
    B = 2*b*(ax-d);
    C = 4*ay*b;

    % evaluate quadratic for tan half-angle
    t = roots([(A-B) C (A+B)]);
    th3 = 2*atan(t(2));
    th3deg = th3*180/pi
    %NB th3 is not quadrant-ambiguous here because t is for a half-angle

    % calculate length components for link 3
    % use the t formulae to avoid quadrant ambiguity from sin, cos.
    bx = b *(1-t(2)^2)/(1+t(2)^2);
    by = b * 2*t(2)/(1+t(2)^2);

    % now find length components for link 4 by direct use of
    % closure eqns
    cx = -ax-bx+d;
    cy = -ay-by;

    % evaluate link 4 angle,
    % working from cx, cy always gets quadrant right
    th4 = atan2(cy, cx);
    th4deg = th4*180/pi
end
```

Code for fbkinemat.m – conducts kinematic analysis

```
% Four-bar link - kinematic analysis
% Matrix form of component closure equations for velocities
function
[a2x,a2y,a3x,a3y,al3,a4x,a4y,al4]=fbkinemat(ay,ax,by,bx,cy,cx,om2,al2)
CO = [-by -cy; bx cx];
V2 = om2*[ay -ax]';
OM = CO\V2;
om3 = OM(1);
om4 = OM(2);
% Component closure equations for accelerations
A2 = al2*[ay -ax]' + om2^2*[ax ay]' + ...
        om3^2*[bx by]' + om4^2*[cx cy]';
AL = CO\A2;
al3 = AL(1);  al4 = AL(2);
 % Now compute key accelerations at link centres
A2G = al2*[-ay ax]'/2 - om2^2*[ax ay]'/2;
a2x = A2G(1); a2y = A2G(2);
 % For link 3 add relative values to that of pin2-3
A3G = al3*[-by bx]'/2 - om3^2*[bx by]'/2 + 2*A2G;
a3x = A3G(1); a3y = A3G(2);
```

```
 % For link 4, go backwards from fixed pin4-1,
% so flipping all signs
A4G = al4*[cy -cx]'/2 + om4^2*[cx cy]'/2;
a4x = A4G(1); a4y = A4G(2);
end
```

Code for fbforce.m – solves force equation set

```
% Four-bar force analysis
% Using variables:
%  active link lengths a,b,c
%  active link masses m2, m3, m4
%  active link inertias I2, I3, I4
%  link length components (strict cyclic angle forms)
%        ax, ay;   bx, by;   cx, cy
%  link accelerations
%        a2x, a2y, al2;  a3x, a3y, al3;  a4x, a4y, al4
%  external force on link 4 = f4x
% Set up 9x9 geometry matrix
function F=fbforce(a,b,c,d,m2,m3,m4,I2,I3,I4,f4x,th2,om2,al2)
    %y=[ay,ax,by,bx,cy,cx,th3,th4];
    [ax,ay,bx,by,cx,cy,th3,th4]=fbpos(th2,a,b,c,d);

[a2x,a2y,a3x,a3y,al3,a4x,a4y,al4]=fbkinemat(ay,ax,by,bx,cy,cx,om2,al2);
    G =[1      0    -1      0     0      0     0   0     0
        0      1     0     -1     0      0     0   0     0
        ay/2 -ax/2 ay/2 -ax/2 0      0     0   0     1
        0      0     1      0    -1      0     0   0     0
        0      0     0      1     0     -1     0   0     0
        0      0     by/2 -bx/2 by/2 -bx/2 0   0     0
        0      0     0      0     1      0    -1   0     0
        0      0     0      0     0      1     0  -1     0
        0      0     0      0     cy/2  -cx/2    cy/2 -cx/2 0];
 % Set up inertia force vector (rhs vector)
    R = [m2*a2x m2*a2y I2*al2, ...
            m3*a3x m3*a3y I3*al3,...
                m4*a4x-f4x m4*a4y I4*al4]';
 % Solve force system equations
    F=G\R;
  end
```