

Stat 486 / 886 Survival Analysis

Chapter 4. Comparison of Failure Time Distributions

Professor: Wenyu Jiang

Department of Mathematics and Statistics
Queen's University

Acknowledgement: Profs. Jerry Lawless, David Matthews (Waterloo),
Profs. Bingshu Chen, Paul Peng, Dongsheng Tu (Queen's)

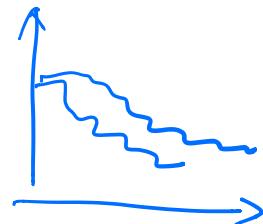
Sections of Chapter 4

1 Parametric Comparison

2 Nonparametric Comparison

- Two-Sample Log Rank Test
- Weighted Log Rank Test

Comparison of Failure Time Distributions



Informal comparisons, for two groups of subjects observed.

- Plot $\hat{S}_{KM}(t)$ for each group, put on the same graph; add confidence intervals if needed.
- Plot and compare $\hat{H}_{NA}(t)$, or other nonparametric estimates, of the groups.

4.1 Parametric Comparison

Aim: Test $H_0 : \underline{S_1(t) = S_2(t)}$, equality of distribution between the 2 groups.

Assumption: S_1, S_2 come from the same parametric family.

Advantage of parametric comparison:

Being able to quantify differences between distributions.

- Log location-scale models.

Assume the failure time T_{ij} has a log location-scale distribution, and $Y_{ij} = \log T_{ij}$, for groups $i = 1, 2$.

Group 1: Log failure times Y_{11}, \dots, Y_{1n_1} .

Group 2: Log failure times Y_{21}, \dots, Y_{2n_2} .

Denote the cumulative distribution functions of the 2 groups by

$$\begin{cases} F_1(y) = G\left(\frac{y-\mu_1}{\sigma_1}\right), \\ F_2(y) = G\left(\frac{y-\mu_2}{\sigma_2}\right). \end{cases}$$

$G(\cdot)$: cdf of the std. dist'n of the loc-scale dist'n family.

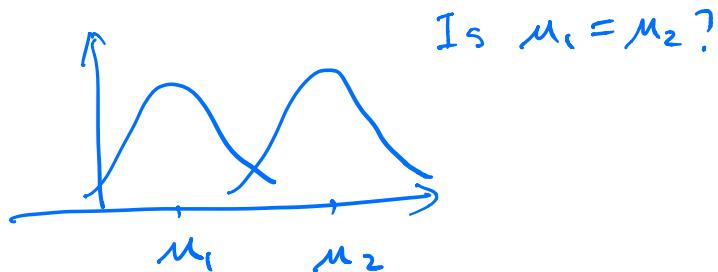
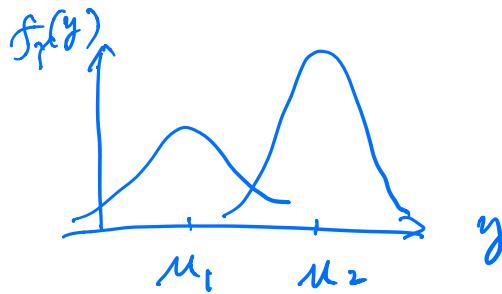
Example: $G(\cdot) : EV(0, 1), T \sim \text{Weibull}.$
 $: N(0, 1), T \sim \log \text{normal}.$

Steps for testing $H_0 : S_1(t) = S_2(t)$.

1. Test $H_0 : \sigma_1 = \sigma_2$.

If rejected, conclude $S_1(t) \neq S_2(t)$, stop.
Otherwise, go to step 2.

2. If $\sigma_1 = \sigma_2$ is not rejected, test $H_0 : \mu_1 = \mu_2$.



Example 4.1.1. Leukemia Remission Times

Leukemia data with two treatment groups (control and drug 6-MP).

Continuation of Examples 1.2.2., 2.5.1.

Assume the failure time of each group has a Weibull distribution. Test if the two groups have the same distribution.

Test equality of distributions based on Weibull failure time models.

- Check on the equality of scale parameters of the 2 groups, $H_0 : \phi_1 = \phi_2$.

Likelihood ratio (LR) statistic $\Lambda \approx \chi^2_1$.

$H_0 : \phi_1 = \phi_2$ Restricted $\mathbb{H}_0 = \{(u_1, u_2, \phi)\}$ 3 dim. --- Model 1

vs. $H_a : \phi_1 \neq \phi_2$. Entire $\mathbb{H} = \{(u_1, \phi_1, u_2, \phi_2)\}$ 4 dim. --- Model 2

Likelihood of Model 2: $L(u_1, \phi_1, u_2, \phi_2) = L(u_1, \phi_1) L(u_2, \phi_2)$

Observed value is $\lambda_{obs} = 2[-64.9 + (-41.7) - (-106.6)] = 0$.

p-value=1. No evidence against $H_0 : \phi_1 = \phi_2$. Model 1.

Log-likelihood for Model 2: $\log L(u_1, \phi_1, u_2, \phi_2) = \log L(u_1, \phi_1) + \log L(u_2, \phi_2)$
 $= l(u_1, \phi_1) + l(u_2, \phi_2)$

Model 1:

$$\frac{L(\mu_1, \mu_2, \phi)}{\text{For Grp 1}} ?$$

$$Y = \mu_1 + \sigma W$$

For Grp 2

$$Y = \mu_2 + \sigma W.$$

By

$$Y = \beta_0 + \beta_1 z + \sigma W;$$

where $z = \begin{cases} 1 & \text{if subj in Grp 1.} \\ 0, & \text{.. .. Grp 2.} \end{cases}$

ctrl.

$$\mu_1 = \beta_0 + \beta_1$$

$$\mu_2 = \beta_0.$$

$$\beta_1 = \mu_1 - \mu_2.$$

$$\textcircled{X} \quad H_0: \beta_1 = 0$$

- Test for equality of location parameters of the 2 groups, $H_0: \mu_1 = \mu_2$.

LR statistic $\Lambda \approx \chi^2_1$.

Restricted: $H_0: \{\mu, \phi\}$, 2-dim
Entire $H: \{\mu_1, \mu_2, \phi\}$, 3-dim

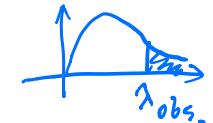
$$\begin{aligned}\Lambda &= 2 [\ell(\hat{\mu}_1, \hat{\mu}_2, \hat{\phi}) - \ell(\tilde{\mu}, \tilde{\phi})] \\ &= 2 [\ell(\hat{\beta}_0, \beta_1, \phi) - \ell(\tilde{\beta}_0, \tilde{\phi})]\end{aligned}$$

Model 1 ; intercept-only model: $Y = \beta_0 + \sigma W$

Observed value is $\lambda_{obs} = 19.65 = 2[(-106.6) - (-116.4)]$.

p-value = 9.3×10^{-6} .

$$p\text{-val} = P(\Lambda > \lambda_{obs})$$



Or test based on Wald statistic. In model 1, $\beta_1 = \mu_2 - \mu_1$.

$$H_0: \mu_1 = \mu_2 \Leftrightarrow H_0: \beta_1 = 0$$

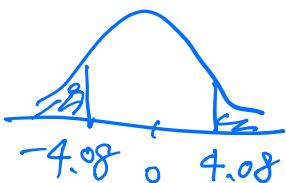
$$Z = \frac{\hat{\beta}_1 - 0}{S.e.(\hat{\beta}_1)} \approx N(0, 1)$$

p-value = 4.5×10^{-5} .

$$p\text{-val} = P(|Z| > |z_{obs}|)$$

$$z_{obs} = 4.08$$

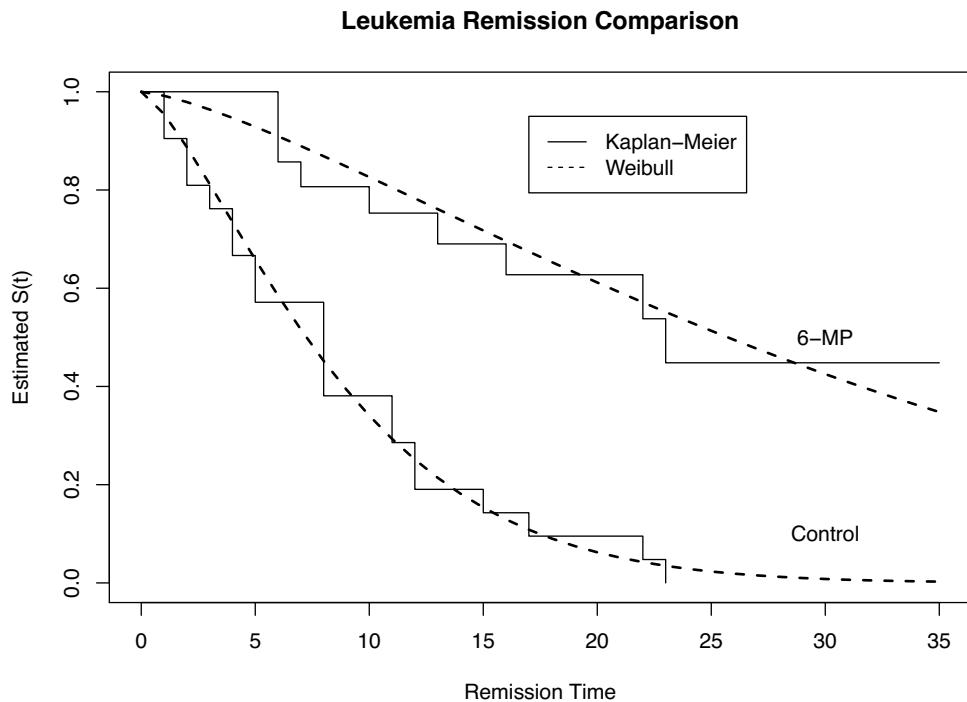
$N(0, 1)$



Very strong evidence against $H_0: \mu_1 = \mu_2$.

Model checking: Are Weibull models good enough for the data?

Plot the estimated survivor function from Weibull fit for each group alone with the corresponding Kaplan-Meier estimate.



Summary and Conclusions:

- Model assumptions

Survival times for the control and drug 6-MP groups follow Weibull distributions. That is, log survival times of the 2 groups have $\text{EV}(\mu_1, \sigma_1)$ and $\text{EV}(\mu_2, \sigma_2)$ distributions respectively.

- Model comparison

We test the equality of survival distributions for control vs. drug 6-MP groups in two steps.

1. Test for the equality of scale parameters ($H_0 : \sigma_1 = \sigma_2$).

A LR test results in a p-value of 1, indicating no difference on the scale parameters between the two models for the two groups.

2. Test for equality of location parameters ($H_0 : \mu_1 = \mu_2$).

The LR test results in a p-value of 9.3×10^{-6} . There is very strong evidence against $H_0 : \mu_1 = \mu_2$. (The Wald test gives the same conclusion.)

- The tests suggest the control and 6-MP groups have different survival (remission) time distributions;
for describing the survival times of the 2 groups, we should consider a Weibull model with $\phi_1 = \phi_2$ ($\sigma_1 = \sigma_2$), but $\mu_1 \neq \mu_2$.
- Model checking
The survivor functions estimated from the above Weibull model are plotted along with the Kaplan-Meier estimates for the 2 groups. The Weibull fit agrees reasonably well with the nonparametric fit. The Weibull model is appropriate for the data.
- Conclusions:
 - ▶ The remission time distributions are significantly different between the patients receiving control treatment and patients receiving drug 6-MP treatment.
 - ▶ A Weibull model with $\phi_1 = \phi_2$, but $\mu_1 \neq \mu_2$ is suitable for describing the survival distributions of the two groups.

```

> library(survival)
> leukdata<-read.table("eg122.txt", header=T)
> # Weibull fit for the data under null hypothesis.
> fitall<-survreg(Surv(time, status) ~ as.factor(grp),
data=leukdata)
> print(summary(fitall))

```

Call:

```

survreg(formula = Surv(time, status) ~ as.factor(grp),
data = leukdata)

```

	Value	Std. Error	z	p
(Intercept)	2.248	0.166	13.55	8.30e-42
as.factor(grp) 1	1.267	0.311	4.08	4.51e-05
Log(scale)	-0.312	0.147	-2.12	3.43e-02

Scale= 0.732

Weibull distribution

Loglik(model)= -106.6 Loglik(intercept only)= -116.4
 Chisq= 19.65 on 1 degrees of freedom, p= 9.3e-06

n= 42

```
> # Weibull fit for control group.  
> fit1<-survreg(Surv(time,status)~1,data=leukdata,  
subset=grp==0)  
> print(summary(fit1))  
Call:  
survreg(formula = Surv(time, status) ~ 1,  
data = leukdata, subset = grp == 0)  
          Value Std. Error      z       p  
(Intercept) 2.249      0.168 13.40 5.72e-41  
Log(scale)   -0.315      0.174 -1.82 6.94e-02  
Scale= 0.73  
Weibull distribution  
Loglik(model)= -64.9 Loglik(intercept only)= -64.9  
n= 21
```

```
> # Weibull fit for drug 6-MP group.  
> fit2<-survreg(Surv(time,status)~1,data=leukdata,  
subset=grp==1)  
> print(summary(fit2))  
Call:  
survreg(formula = Surv(time, status) ~ 1,  
data = leukdata, subset = grp == 1)  
          Value Std. Error      z       p  
(Intercept) 3.519      0.273 12.87 6.28e-38  
Log(scale)   -0.303      0.278 -1.09 2.77e-01  
Scale= 0.739  
Weibull distribution  
Loglik(model)= -41.7    Loglik(intercept only)= -41.7  
n= 21
```

$$\hat{\beta}_0$$

```
> mu1<-fitall$coeff[1]
> mu1
(Intercept)
 2.248352
> mu2<-fitall$coeff[2]+mu1
> mu2
as.factor(grp) 1
 3.515687
> sig<-fitall$scale
> sig
[1] 0.7321944
>
> # Model checking: plot estimated survival functions
> # by KM method, and Weibull model.
>
> fit.km<-survfit(Surv(time, status) ~ as.factor(grp),
data=leukdata, conf.int=F)
```

```

> # KM plot.
> plot(fit.km, xlab="Remission Time", ylab="Estimated
S(t)", main="Leukemia Remission Comparison")
>
> # Plot estimated survivor functions by Weibull fit.
> t<-0:35
> St1<-exp(-exp((log(t)-mu1)/sig))
> St2<-exp(-exp((log(t)-mu2)/sig))
> text(30, 0.1, "Control")
> text(30, 0.5, "6-MP")
> lines(t, St1, lty=2, lwd=2)
> lines(t, St2, lty=2, lwd=2)
> legend(17, 0.95, c("Kaplan-Meier", "Weibull"), lty=1:2)

```

Recall $W \sim EV(0, 1)$, its
cdf $G(w) = e^{-e^w}$,

$$T \sim \text{Weib.} \Leftrightarrow$$

$$Y = \log T \sim EV(\mu, \sigma).$$

$$\begin{aligned} S_T(t) &= P(T \geq t) \\ &= P(Y \geq \log t) = S_Y(\underline{\log t}) \end{aligned}$$

$$\hat{=} G\left(\frac{\log t - \mu}{\sigma}\right)$$

4.2 Nonparametric Comparison

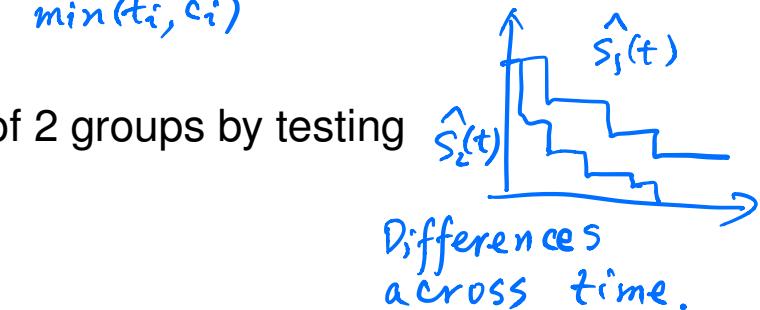
Data : $(x_i, \delta_i, z_i) \stackrel{I(\text{Grp 1})}{\sim} \min(t_i, c_i)$

4.2.1 Two-Sample Log Rank Test

Compare the failure time distributions of 2 groups by testing

$H_0 : S_1(t) = S_2(t)$ for all t , versus

$H_a : S_1(t) \neq S_2(t)$ for some t .



Intuition about log rank statistic:

Let $t_1 < t_2 < \dots < t_K$ be the distinct failure times for the pooled sample.

t_j

At time t_j	# failures	# did not fail	# at risk
Group 1	d_{1j}	$n_{1j} - d_{1j}$	n_{1j}
Group 2	d_{2j}	$n_{2j} - d_{2j}$	n_{2j}
	d_j	$n_j - d_j$	n_j

Let D_{ij} be the number of failures in group i at time t_j .

Under H_0 that the 2 groups have the same distribution, D_{ij} can be modeled by a hypergeometric distribution, with

$$P(D_{ij} = d_{ij}) = \frac{\binom{d_j}{d_{1j}} \binom{n_j - d_j}{n_{1j} - d_{1j}}}{\binom{n_j}{n_{1j}}}.$$

The log rank statistic takes the form

$$U = \sum_{j=1}^K (d_{1j} - e_{1j}), \text{ with } e_{1j} = E(D_{1j}) = n_{1j} \left(\frac{d_j}{n_j} \right).$$

observed # of failures in Grp 1.

expected # of failures in Grp 1.

prob. of failure in both grps. under H_0 .

Its variance is estimated by

$$V = \sum_{j=1}^K v_{1j}, \text{ with } v_{1j} = \text{Var}(D_{1j}) = n_{1j} \left(\frac{d_j}{n_j} \right) \left(1 - \frac{d_j}{n_j} \right) \left(\frac{n_j - d_j}{n_j - 1} \right)$$

The log rank test is based on the result that, under H_0 ,

$$W^2 = \frac{U^2}{V} \approx \chi^2_1, \text{ or } W = \frac{U}{\sqrt{V}} \approx N(0, 1).$$

Example:

Observed failure times: Grp 1: 3, 5, 7, 9*, 19
Grp 2: 12, 19, 20, 20*, 33*

Distinct failure time (pooled sample):

3, 5, 7, 12, 19, 20
 $t_1 < t_2 < \dots < t_6$

At $t_5 = 19$

	# failure	# did not fail	# at risk
Grp 1	1	0	1
Grp 2	1	3	4
	2	3	5

$$d_{51} = 1, \quad e_{51} = 1 \times \left(\frac{2}{5}\right) = \frac{2}{5}.$$

Extension to 3-sample comparison.

$$H_0: S_1(t) = S_2(t) = S_3(t).$$

$$\text{Log rank stat: } U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \rightarrow \begin{array}{l} \text{for Grp 1} \\ \text{for Grp 2.} \end{array}$$

V is a 2×2 matrix.

$$W^2 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T V^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \sim \chi^2_2.$$

4.2.2 Weighted Log Rank Test

$(x_i, \delta_i), i=1, \dots, n.$

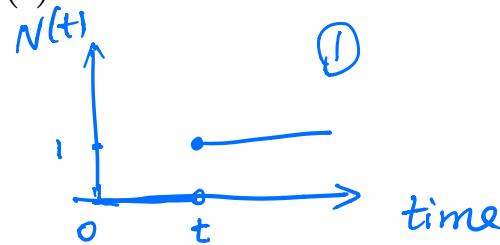
- Counting process notation for describing lifetime

- More general.
- More convenient for theoretical development of methods.
- More modern.

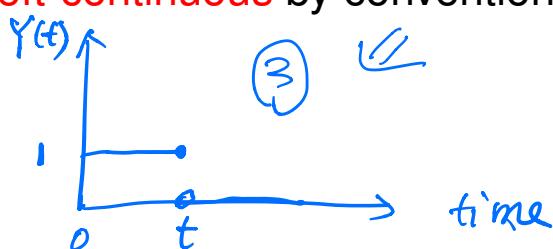
Describe the survival time of 1 subject:

$N(t)$ = number of failures up to time t . Right-continuous by convention.

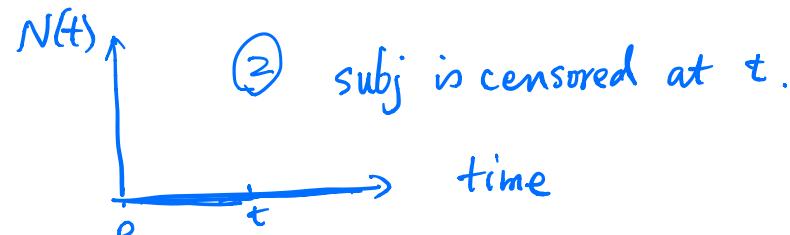
$Y(t)$ = number at risk at time t . Left-continuous by convention.



The subj failed at t .



① & ③: failure
② & ③: censoring



② subj is censored at t .

For subjects $l = 1, \dots, n$ under study, at time t , describe their lifetime by failure processes $N_1(t), \dots, N_n(t)$, and at risk processes $Y_1(t), \dots, Y_n(t)$.

Further define

$$N_{\cdot}(t) = \sum_{l=1}^n N_l(t) : \# \text{ of failures up to } t \text{ among all subj.}$$

$$Y_{\cdot}(t) = \sum_{l=1}^n Y_l(t) : \# \text{ at risk at } t \quad , \quad , \quad ,$$

$$dN_{\cdot}(t) = \sum_{l=1}^n dN_l(t) : \# \text{ of failures at } t.$$

Recall that in general, if T is continuous,

$$H(t) = \int_0^t h(u) du = \underbrace{\int_0^t dH(u)}_{\text{if } T \text{ is cont.}}$$

$$\text{and } \hat{H}_{NA}(t) = \sum_{j: a_j < t} \hat{h}_j = \sum_{j: a_j < t} \frac{d_j}{n_j}$$

Using the counting process notation

$$\hat{H}_{NA}(t) = \int_0^t d\hat{H}(u) = \int_0^t \frac{dN_{\cdot}(t)}{Y_{\cdot}(t)}.$$

- Idea for weighted log rank test

Group 1: subjects $l = 1, \dots, n_1$.

Counting processes: $N_{11}(t), \dots, N_{1n_1}(t)$; and $Y_{11}(t), \dots, Y_{1n_1}(t)$.

Group 2: subjects $l = 1, \dots, n_2$.

Counting processes: $N_{21}(t), \dots, N_{2n_2}(t)$; and $Y_{21}(t), \dots, Y_{2n_2}(t)$.

Test for $H_0 : S_1(t) = S_2(t)$ through a weighted comparison of \hat{S}_1 and \hat{S}_2 .
or equivalently \hat{H}_1 and \hat{H}_2

$$(H(t) = -\log S(t)).$$

The weighted log rank test consider the statistic

$$U = \int_0^\infty w(t) \frac{Y_{1\cdot}(t) Y_{2\cdot}(t)}{Y_{\cdot\cdot}(t)} \left[d\hat{H}_1(t) - d\hat{H}_2(t) \right], \quad (1)$$

where $w(t)$ is the pre-specified weight,

$$Y_{1\cdot}(t) = \sum_{l=1}^{n_1} Y_{1l}(t) : \# \text{ at risk at } t \text{ in grp 1}$$

$$Y_{2\cdot}(t) = \sum_{l=1}^{n_2} Y_{2l}(t) : \dots \dots \dots \dots \text{ grp 2}$$

$$Y_{\cdot\cdot}(t) = Y_{1\cdot}(t) + Y_{2\cdot}(t) = \dots \dots \dots \dots \text{ in both grp's.}$$

$\hat{H}_i(t)$ is the NA estimate for group i , $i = 1, 2$;

for group i , $d\hat{H}_i(t) = \frac{dN_{i\cdot}(t)}{Y_{i\cdot}(t)}$, estimated hazard at t . $(= \frac{d_j}{n_j})$
at t_j .

$$N_{i\cdot}(t) = \sum_{l=1}^{n_i} N_{il}(t) = \text{failure process of grp } i.$$

$dN_{i\cdot}(t)$ = # of failures at t in grp i .

Denote $dN_{i\cdot}(t)$ at time t_j by $di\cdot(t_j)$, $i = 1, 2$.

At time t_j	# failures	# did not fail	# at risk
Group 1	$d_{1\cdot}(t_j)$		$Y_{1\cdot}(t_j)$
Group 2	$d_{2\cdot}(t_j)$		$Y_{2\cdot}(t_j)$
	$d_{\cdot\cdot}(t_j)$		$Y_{\cdot\cdot}(t_j)$

$$U = \sum_{j=1}^K w(t_j) \frac{Y_{1\cdot}(t_j) Y_{2\cdot}(t_j)}{Y_{\cdot\cdot}(t_j)} \left[\frac{d_{1\cdot}(t_j)}{Y_{1\cdot}(t_j)} - \frac{d_{2\cdot}(t_j)}{Y_{2\cdot}(t_j)} \right]$$

$$d_{2\cdot}(t_j) = d_{\cdot\cdot}(t_j) - d_{1\cdot}(t_j).$$

Under H_0 , $P(\text{failure in both grps}) = \frac{d_{\cdot\cdot}(t_j)}{Y_{\cdot\cdot}(t_j)}$

The weighted log rank statistic has the form

$$U = \sum_{j=1}^K w(t_j) \left[d_{1\cdot}(t_j) - \frac{Y_{1\cdot}(t_j)}{Y_{\cdot\cdot}(t_j)} d_{\cdot\cdot}(t_j) \right] \quad (2)$$

↑
expect # failure
in grp 1

Some results for U , the weighted log rank statistic (2).

- $E(U) = 0$.
- The variance of U can be estimated by

$$V = \sum_{j=1}^K [w(t_j)]^2 Y_{1\cdot}(t_j) \frac{d_{..}(t_j)}{Y_{..}(t_j)} \left[1 - \frac{d_{..}(t_j)}{Y_{..}(t_j)} \right] \frac{Y_{2\cdot}(t_j)}{Y_{..}(t_j) - 1}.$$

The weighted log rank test is based on the result that, under H_0 ,

$$W^2 = \frac{U^2}{V} \approx \chi_1^2, \text{ or } W = \frac{U}{\sqrt{V}} \approx N(0, 1).$$

Remarks:

- The log rank test and the Wilcoxon test are the most popular among the weighted log rank tests.
- Taking $w(t_j) = 1$ in (2) gives the log rank statistic; taking $w(t_j) = \hat{S}(t)$ gives the Wilcoxon statistic for testing H_0 .
- Advantage of nonparametric methods: No distributional assumptions.

Example 4.2.1 Leukemia Remission Time

Leukemia data with two treatment groups (control and drug 6-MP).

Continuation of Examples 1.2.2., 2.5.1, 4.1.1.

Test equality of distributions through nonparametric methods.

Methods:

Log rank test and Wilcoxon test for $H_0 : S_1(t) = S_2(t)$,

(group 1 = those given control treatment, group 2 = those given drug 6-MP).

Results and conclusions:

- Log rank test and Wilcoxon test both give very small p-values (4×10^{-5} and 1×10^{-4}).
- There are very strong evidences against the null hypothesis.
- The distributions of remission time are significantly different between the drug 6-MP group and the control group.

```

> library(survival)
> leuk<-read.table("eg122.txt", header=T)
> # Log rank test.
> logrk<-survdiff(Surv(time, status) ~ as.factor(grp),
data=leuk)
> logrk
Call:
survdiff(formula = Surv(time, status) ~ as.factor(grp),
data = leuk)

          N Observed Expected (O-E)^2/E (O-E)^2/V
as.factor(grp)=0 21        21     10.7      9.77    16.8
as.factor(grp)=1 21         9     19.3      5.46    16.8

  Chisq= 16.8 on 1 degrees of freedom, p= 4e-05
>

```

```

> # Wilcoxon test.
> wilcx<-survdiff(Surv(time, status) ~ as.factor(grp),
data=leuk, rho=1)
> wilcx
Call:
survdiff(formula = Surv(time, status) ~ as.factor(grp),
data = leuk, rho = 1)

              N Observed Expected (O-E)^2/E (O-E)^2/V
as.factor(grp)=0 21      14.55     7.68      6.16    14.5
as.factor(grp)=1 21       5.12    12.00      3.94    14.5
Chisq= 14.5 on 1 degrees of freedom, p= 1e-04

```

Notes on survdiff() in R