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CSA0672 - Design and Analysis of Algorithm for polynomial problems

1\. If $t1(n) \in O(g1(n))$ and $t2(n) \in O(g2(n))$, prove that $t1(n) + t2(n) \in O(\max\{g1(n), g2(n)\})$

For any four arbitrary real numbers, a_1, b_1, a_2, b_2 such that $a_1 \le b_1$ and $a_2 \le b_2$

we have $a_1 + a_2 \le 2\max\{b_1, b_2\}$

Since $t_1(n) \in O(g_1(n))$, then there exists some constant c_1 and non-negative integer n_1 such that

$$t_1(n) \le c_1 g_1(n) \text{ for all } n \ge n_1$$

Since $t_2(n) \in O(g_2(n))$, then there exists some constant c_2 and non-negative integer n_2 such that

$$t_2(n) \le c_2 g_2(n) \text{ for all } n \ge n_2$$

Let $c_3 = \max\{c_1, c_2\}$ and $n_0 = \max\{n_1, n_2\}$

$$\begin{aligned} t_1(n) + t_2(n) &\le c_1 g_1(n) + c_2 g_2(n) \\ &\le c_3 g_1(n) + c_3 g_2(n) \\ &= c_3\{g_1(n) + g_2(n)\} \\ &\le 2c_3 \max\{g_1(n), g_2(n)\} \end{aligned}$$

Hence, $t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\}$, with constants c and n_0 required by the O definition being $2c_3 = 2\max\{c_1, c_2\}$ and $\max\{n_1, n_2\}$ respectively.

2\. Find the time complexity of the below recurrence equation:

3)

$$T(n) = \begin{cases} 2T\left(\frac{n}{2}\right) + 1 & \text{if } n > 1 \\ 1 & \text{otherwise} \end{cases}$$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$ Masters Theorem

$a = 2$

$b = 2$

$$\log_b a = \log_2 2 = 1$$

$k = 0$

$$1 > 0$$

$$\log_b a > k$$

case i)

$$\theta(n^{\log_b a})$$

$$\theta(n^{1})$$

$$\theta(n)$$

4) $T(n) = \begin{cases} 2T(n-1) & \text{if } n>0 \\ 1 & \text{otherwise.} \end{cases}$

Backward substitution :-

$$T(n) = 2T(n-1) \rightarrow ① \qquad \text{Initial } T(0) = 0$$

$n = n-1$

$$T(n-1) = 2T((n-1)-1)$$

$$T(n-1) = 2T(n-2) \rightarrow ②$$

sub ② in ①

$$T(n) = 2[2T(n-2)]$$

$$T(n) = 2^2 T(n-2) \rightarrow ③$$

$n = n-2$

$$T(n-2) = 2T((n-2)-1)$$

$$T(n-2) = 2T(n-3) \rightarrow ④$$

sub ④ in ③

$$T(n) = 2^2[2T(n-3)]$$

$$T(n) = 2^3 T(n-3) \rightarrow ⑤$$

$n = n-3$

$$T(n-3) = 2T((n-3)-1)$$

$$T(n-3) = 2T(n-4) \rightarrow ⑥$$

sub ⑥ in ⑤

$$T(n) = 2^3[2T(n-4)]$$

$$= 2^4 T(n-4) \rightarrow ⑦$$

$$T(n) = 2^k T(n-k)$$

$$n-k = 0 \Rightarrow n = k$$

if $T(0) = 1$

$$T(n) = 2^k . T(0)$$

$T(n) = 2^k \cdot 1$

$T(n) = 2^k$

$\therefore \quad n = k$

$T(n) = O(2^n)$

5) Big O Notation: show that $f(n) = n^2+3n+5$ is $O(n^2)$

To prove that $f(n) = n^2+3n+5$ is $O(n^2)$ we need to find constants c and n_0 such that $f(n) \leq c \cdot n^2$ for all $n \geq n_0$

$f(n) = n^2+3n+5$

For $n \geq 1$, $n^2 \geq n$ ... so on

$f(n) = n^2+3n+5 \leq n^2+3n^2+5n^2$

$f(n) = n^2+3n+5 \leq 9n^2$ for $n \geq 1$

So, for $c=9$ and $n_0 = 1$.

$f(n) \leq c \cdot n^2$ for all $n \geq n_0$

That proves $f(n)$ is $O(n^2)$

6- Big omega notation: prove that $g(n) = n^3+2n^2+4n$ is $\Omega(n^3)$

To prove that $g(n) = n^3+2n^2+4n$ is $\Omega(n^3)$ we need to find constants c and n_0 such that

$g(n) \geq c \cdot n^3$ for all $n \geq n_0$

$g(n) = n^3+2n^2+4n$

For $n \geq 1$,

$g(n) = n^3+2n^2+4n \geq n^3$

Since $2n^2$ and $4n$ are both less than n^3 when $n \geq 1$

So, for $c=1$ and $n_0 = 1$

$g(n) \geq c \cdot n^3$ for all $n \geq n_0$

That proves $g(n)$ is $\Omega(n^3)$

7. Big Theta Notation: Determine whether $h(n) = 4n^2 + 3n$ is $\Theta(n^2)$ or not

1. $h(n) = 4n^2 + 3n$ is $O(n^2)$:

For $n \geq 1$, $h(n) \leq 4n^2 + 3n^2$
(Since $3n$ is less than n^2 when $n \geq 1$)

For This simplifies to $h(n) \leq 7n^2$
for $n \geq 1$

Therefore, $h(n)$ is $O(n^2)$

2. $h(n) = 4n^2 + 3n$ is $\Omega(n^2)$:

For $n \geq 1$, $h(n) \geq 4n^2$
(Since $3n$ is positive)

Therefore $h(n)$ is $\Omega(n^2)$

Since $h(n)$ is both $O(n^2)$ and $\Omega(n^2)$, it is $\Theta(n^2)$

8. Let $f(n) = n^3 - 2n^2 + n$ and $g(n) = -n^2$ show whether $f(n) = \Omega(g(n))$ is true or false and justify your answer.

$n = 1$

$f(1) = 1^3 - 2(1)^2 + 1$
$= 1 - 2 + 1$
$= 0$

$g(1) = \cancel{-}(n)^2$
$= (-1)^2$
$= 1$

$n = 2$

$f(2) = 2^3 - 2(2)^2 + 2$
$= 8 - 8 + 1$
$= 2$

$g(2) = (-2)^2$
$= 4$

$n = 3$

$f(3) = 3^3 - 2(3)^2 + 3$
$= 27 - 18 + 3$
$= 12$

$g(3) = (-3)^2$
$= 9$

$n = 4$

$$f(4) = 4^3 - 2(4)^2 + 4 \qquad g(n) = (-4)^2$$
$$= 64 - 32 + 4 \qquad = 16$$
$$= 32 + 4$$
$$= 36$$

$n = 5$

$$f(5) = 5^3 - 2(5)^2 + 5 \qquad g(n) = (-5)^2$$
$$= 15 - 50 + 5 \qquad = 25$$
$$= 35 + 5$$
$$= 40$$

$$f(n) \geq g(n)$$

so it is best case according to asymptotic notation.

$$f(n) = \Omega(g(n))$$

9. Determine whether $h(n) = n\log n + n$ is in $\Theta(n \log n)$ prove a rigorous proof for your conclusion.

1. <u>upper</u> <u>Bound</u> (O notation):

we need to find c_1 and n_0 such that

$h(n) \leq c_1 \cdot n\log n$ for all $n \geq n_0$

$$h(n) = n\log n + n$$
$$\leq n\log n + n\log n \quad (\text{since } \log n \text{ is increasing})$$
$$= 2n\log n$$

Now, let $c_1 = 2$, then $h(n) \leq 2n\log n$ for all $n \geq 1$

so, $h(n)$ is $O(n\log n)$

2. <u>Lower</u> <u>Bound</u> (Ω notation):

we need to find c_2 and n_0 such that

$h(n) \geq c_2 \cdot n\log n$ for all $n \geq n_0$

$$h(n) = n\log n + n$$
$$\geq \frac{1}{2} \cdot n\log n \ (\text{for } n \geq 2)$$

Now, let $c_2 = \frac{1}{2}$, then $h(n) \geq \frac{1}{2} \cdot n \log n$ for all $n \geq 2$. So, $h(n)$ is $\Omega(n \log n)$

3. <u>Combining Bounds</u>:

Since $h(n)$ is both $O(n \log n)$ and $\Omega(n \log n)$, it is also $\Theta(n \log n)$

Thus, $h(n) = n \log n + n$ is in $\Theta(n \log n)$

10. Solve the following recurrence relations and find the order of growth for solutions.

$$T(n) = 4T(n/2) + n^2 \quad , T(1) = 1$$

$$T(n) = aT(n/b) + f(n)$$

$a = 4$
$b = 2$

$$\log_b a = \log_2 4 = 2$$

$$k = 2$$

$$2 = 2$$

$$\log_b a = k$$

Case ii)

$$p > -1 \quad \Theta(n^k \log_n^{p+1})$$

$$\Theta(n^2 \cdot \log_n^{1+1})$$

$$\Theta(n^2 \cdot \log_n^2)$$

$$T(n) = \Theta(n^2 \cdot \log(n))$$

The order of growth for the solution is $n^2 \cdot \log(n)$

2. Demonstrate the Binary search method to search key = 23 from the array

arr [ ] = { 2, 5, 8, 12, 16, 23, 38, 56, 72, 91 }

0	1	2	3	4	5	6	7	8	9
2	5	8	12	16	23	38	56	72	91

low = 0
high = 9

$mid = \frac{low + high}{2}$

$= \frac{0+9}{2} = 4$

a[mid] == key
a[4] == 23
16 != 23
16 < 23
low = mid+1

5	6	7	8	9
23	38	56	72	91

low = 5
high = 9

a[mid] == key

$mid = \frac{5+9}{2}$

$= 7$

a[7] == 23
56 != 23
56 > 23
high = mid - 1

5	6	7
23	38	56

low = 5
high = 7

a[6] == 23

$mid = \frac{5+7}{2} = 6$

38 != 23
38 > 23
high = mid - 1

5
23

low = 5
high = 5

a[5] == key

$mid = \frac{5+5}{2}$

$= 5$

23 == 23

Return the position of the key is 5

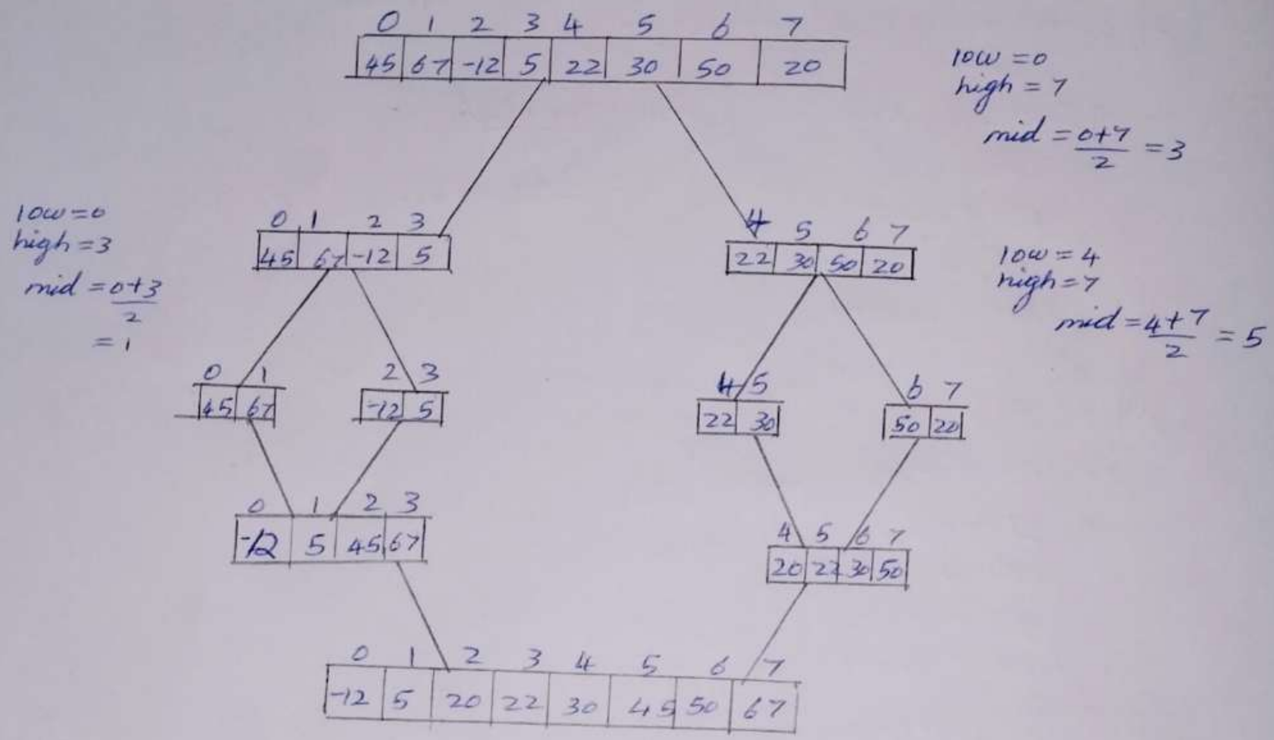
Pseudo code:

```
binary_search (a, n, key):
    low = 0
    high = n-1
    while (low <= high):
        mid = (high + low) / 2
        if a[mid] == key:
            return mid
        a[mid] > key
            high = mid - 1
        a[mid] < key
            low = mid + 1
    return -1      (If not found element)
```

Time complexity:

$O(n \log n)$

13. Apply merge sort and order the list of 8 elements $a = \{45, 67, -12, 5, 22, 30, 50, 20\}$. Set up a recurrence relation for the number of key comparisons made by merge sort.



∴ The sorted list is :

-12, 5, 20, 22, 30, 45, 50, 67

Time complexity:

$O(n \log n)$

Recurrence Relation:

$T(n) = 2T(n/2) + (n-1)$

4. Find the no. of times to perform swapping for selection sort. Also estimate the time complexity. $A = \{12, 7, 5, -2, 18, 6, 13, 4\}$

0	1	2	3	4	5	6	7
12	7	5	-2	18	6	13	4
S			M				

0	1	2	3	4	5	6	7
-2	7	5	12	18	6	13	4
	S						M

0	1	2	3	4	5	6	7
-2	4	5	12	18	6	13	7
		M S M					

0	1	2	3	4	5	6	7
-2	4	5	12	18	6	13	7
			S		M		

0	1	2	3	4	5	6	7
-2	4	5	6	18	12	13	7
				S			M

0	1	2	3	4	5	6	7
-2	4	5	6	7	12	13	18
					M		

0	1	2	3	4	5	6	7
-2	4	5	6	7	12	13	18
						M	

0	1	2	3	4	5	6	7
-2	4	5	6	7	12	13	18
							M

Sorted List:

-2, 4, 5, 6, 7, 12, 13, 18

usually the number swaps required will be n-1. But for this equation there are only 4 swaps.

Time complexity:

$O(n^2)$ for all three cases

15. Find the index of the target value 10 using binary search from the following list of elements. a = {2,4,6,8,10,12,14,16,18,20}

0	1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18	20

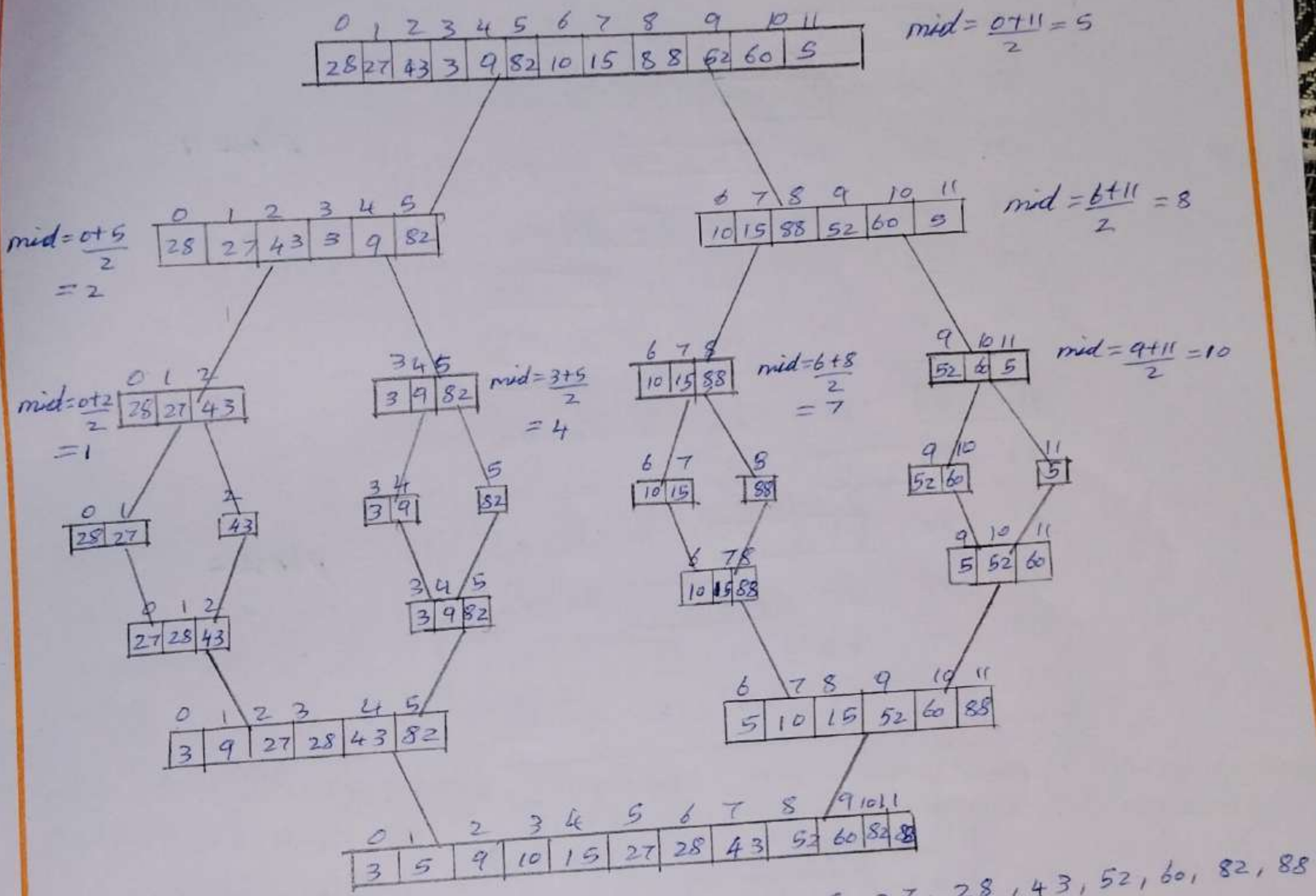
low = 0
high = 9
mid = $\frac{0+9}{2}$
= 4

a [mid] = = key
a [4] = = 10
10 = = 10
Return the position of the key is 4.

Pseudocode:

```
binary_search ( arr , size of array, key )
  low = 0
  high = size -1
  while (low <= high)
   mid = (high+low)/2
   if a [mid] = = key
     return mid
     a [mid] > key
       high = mid -1
     a [mid] < key
       low = mid +1
     return -1   ( if not found element)
```

Solve the elements using merge sort divide & conquer strategy.
A = {28, 27, 43, 3, 9, 82, 10, 15, 88, 52, 60, 5} & analyze the time complexity.



∴ The sorted list is: 3, 5, 9, 10, 15, 27, 28, 43, 52, 60, 82, 88

Pseudo code:

```
position (low, high)
  if l<h.
   mid = h+l/2
   position ( l, mid)
   position (mid+1, h)
   po merge ( l, mid, h)
  end if
```

$T(n) = 2T(n/2) + n^{-1}$

$a = 2 \quad k = 1$
$b = 2$

$\log_b a = \log_2 2 = 1$

$\log_b a = k$

∴ case ii)

$p > -1$

$\therefore O(n^k \log^{p+1} n)$

Time complexity:

$O(n \log n)$

17. Sort the array 64, 34, 25, 12, 22, 11, 90 using Bubble sort. What is the time complexity of selection sort in Best, average, worst case?

64 34 25 12 22 11 90
34 64 25 12 22 11 90
34 25 64 12 22 11 90
34 25 12 64 22 11 90 — phase 1
34 25 12 22 64 11 90
34 25 12 22 11 64 90
34 25 12 22 11 64 90

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34 25 12 22 11 64 90
25 34 12 22 11 64 90
25 12 34 22 11 64 90 — phase 2
25 12 22 34 11 64 90
25 12 22 11 34 64 90
25 12 22 11 34 64 90

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25 12 22 11 34 64 90
12 25 22 11 34 64 90
12 22 25 11 34 64 90 — phase 3
12 22 11 25 34 64 90
12 22 11 25 34 64 90

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12 22 11 25 34 64 90
12 22 11 25 34 64 90
12 11 22 25 34 64 90 — phase 4
12 11 22 25 34 64 90

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12 11 22 25 34 64 90
11 12 22 25 34 64 90 — phase 5
11 12 22 25 34 64 90

---

11 12 22 25 34 64 90
11 12 22 25 34 64 90 — phase 6

Time complexity:
$O(n)^2$

Best case - $O(n^2)$
worst case - $O(n^2)$
Average case - $O(n^2)$

Sort the array 64, 25, 12, 22, 11, 90 using selection sort. What is time complexity.

64	25	12	22	11	90
s				m	

11	25	12	22	64	90
	s	M			

11	12	25	22	64	90
		s	M		

11	12	22	25	64	90
			s M		

11	12	22	25	64	90
				s M	

11	12	22	25	64	90
					s M

The sorted list is : 11, 12, 22, 25, 64, 90

Time complexity - $O(n^2)$

Best case - $O(n^2)$

worst case - $O(n^2)$

Average case - $O(n^2)$

9. Solve the following using Insertion sort using Brute-force approach [38, 27, 43, 3, 9, 82, 10, 15, 88, 52, 60, 5] & analyse the complexity.

38 27 43 3 9 82 10 15 88 52 60 5

27 38 43 3 9 82 10 15 88 52 60 5

27 38 43 3 9 82 10 15 88 52 60 5

27 38 3 43 9 82 10 15 88 52 60 5

27 3 38 43 9 82 10 15 88 52 60 5

3 27 38 43 9 82 10 15 88 52 60 5

3 27 38 9 43 82 10 15 88 52 60 5

3 27 9 38 43 82 10 15 88 52 60 5

3 9 27 38 43 82 10 15 88 52 60 5

3 9 27 38 43 82 10 15 88 52 60 5

3 9 27 38 43 82 10 15 88 52 60 5

3 9 27. 38 43 10 82 15 88 52 60 5

3 9 27 38 10 43 82 15 88 52 60 5

3 9 27 10 38 43 82 15 88 52 60 5

3 9 10 27 38 43 82 15 88 52 60 5

3 9 10 27 38 43

3 9 10 27 38 $\boxed{43 \quad 15}$ 82 88 52 60 5

3 9 10 27 $\boxed{38 \quad 15}$ 43 82 88 52 60 5

3 9 10 $\boxed{27 \quad 15}$ 38 43 82 88 52 60 5

3 9 $\boxed{10 \quad 15}$ 27 38 43 82 88 52 60 5

3 9 10 15 27 38 43 $\boxed{82 \quad 88}$ 52 60 5

3 9 10 15 27 38 43 82 $\boxed{88 \quad 52}$ 60 5

3 9 10 15 27 38 43 $\boxed{82 \quad 52}$ 88 60 5

3 9 10 15 27 38 $\boxed{43 \quad 52}$ 82 88 60 5

3 9 10 15 27 38 43 52 82 $\boxed{88 \quad 60}$ 5

3 9 10 15 27 38 43 52 $\boxed{82 \quad 60}$ 88 5

3 9 10 15 27 38 43 $\boxed{52 \quad 60}$ 82 88 5

3 9 10 15 27 38 43 52 60 82 $\boxed{88 \quad 5}$

3 9 10 15 27 38 43 52 60 $\boxed{82 \quad 5}$ 88

3 9 10 15 27 38 43 52 $\boxed{60 \quad 5}$ 82 88

3 9 10 15 27 38 43 $\boxed{52 \quad 5}$ 60 82 88

3 9 10 15 27 38 $\boxed{43 \quad 5}$ 52 60 82 88

3 9 10 15 27 $\boxed{38 \quad 5}$ 43 52 60 82 88

3 9 10 15 $\boxed{27 \quad 5}$ 38 43 52 60 82 88

3 9 10 $\boxed{15 \quad 5}$ 27 38 43 52 60 82 88

3 9 $\boxed{10 \quad 5}$ 15 27 38 43 52 60 82 88

3 $\boxed{9 \quad 5}$ 10 15 27 38 43 52 60 82 88

$\boxed{3 \quad 5}$ 9 10 15 27 38 43 52 60 82 88

3 5 9 10 15 27 38 43 52 60 82 88

Pseudo code:

```
Insertion_sort (a, size of a):
  for i in range (2, size):
      key = a[i]
       j = i-1
      while j>0 and a[j]>key
          a[j+1]= a[j]
           j-= 1
          a [i+1] = key
      return a
```

Time complexity:

$T(n) = O(n^2)$

Given an array of [4, -2, 5, 3, 10, -5, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9] integers. Sort the following elements using insertion sort using Brute Force Approach strategy analysis complexity of the algorithm.

4 -2 5 3 10 -5 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 4 5 3 10 -5 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 4 5 3 10 -5 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 4 3 5 10 -5 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 3 4 5 10 -5 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 3 4 5 10 -5 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 3 4 5 -5 10 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 3 4 -5 5 10 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 3 -5 4 5 10 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-2 -5 3 4 5 10 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 3 4 5 10 2 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 3 4 5 2 10 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 3 4 2 5 10 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 3 2 4 5 10 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 2 3 4 5 10 8 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 2 3 4 5 8 10 -3 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 2 3 4 5 8 -3 10 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 2 3 4 5 -3 8 10 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 2 3 4 -3 5 8 10 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 2 3 -3 4 5 8 10 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 2 -3 3 4 5 8 10 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -2 -3 2 3 4 5 8 10 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 8 10 6 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 8 6 10 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 6 8 10 7 -4 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 6 8 7 10 -4 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 6 7 8 10 -4 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 6 7 8 -4 10 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 6 7 -4 8 10 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 6 -4 7 8 10 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 5 -4 6 7 8 10 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 4 -4 5 6 7 8 10 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 3 -4 4 5 6 7 8 10 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 2 -4 3 4 5 6 7 8 10 1 9 -1 0 -6 -8 11 -9
-5 -3 -2 -4 2 3 4 5 6 7 8 10 1 9 -1 0 -6 -8 11 -9
-5 -3

-5 $\boxed{-3\ -4}$ -2 2 3 4 5 6 7 8 10 1 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 3 4 5 6 7 8 10 1 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 3 4 5 6 7 8 $\boxed{10\ 1}$ 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 3 4 5 6 7 $\boxed{8\ 1}$ 10 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 3 4 5 6 $\boxed{7\ 1}$ 8 10 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 3 4 5 $\boxed{6\ 1}$ 7 8 10 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 3 4 $\boxed{5\ 1}$ 6 7 8 10 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 3 $\boxed{4\ 1}$ 5 6 7 8 10 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 2 $\boxed{3\ 1}$ 4 5 6 7 8 10 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 $\boxed{2\ 1}$ 3 4 5 6 7 8 10 9 -1 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 4 5 6 7 8 $\boxed{10\ 9}$ -1 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 4 5 6 7 8 9 $\boxed{10\ -1}$ 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 4 5 6 7 8 $\boxed{9\ -1}$ 10 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 4 5 6 7 $\boxed{8\ -1}$ 9 10 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 4 5 6 $\boxed{7\ -1}$ 8 9 10 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 4 5 $\boxed{6\ -1}$ 7 8 9 10 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 4 $\boxed{5\ -1}$ 6 7 8 9 10 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 3 $\boxed{4\ -1}$ 5 6 7 8 9 10 0 -6 -8 11 -9

-5 -4 -3 -2 1 2 $\boxed{3\ -1}$ 4 5 6 7 8 9 10 0 -6 -8 11 -9

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-5 -4 -3 -2 -1 1 2 3 4 5 6 7 8 $\boxed{9\ 0}$ 10 -6 -8 11 -9

-5 -4 -3 -2 -1 1 2 3 4 5 6 7 $\boxed{8\ 0}$ 9 10 -6 -8 11 -9

-5 -4 -3 -2 -1 1 2 3 4 5 6 $\boxed{7\ 0}$ 8 9 10 -6 -8 11 -9

-5 -4 -3 -2 -1 1 2 3 4 5 $\boxed{6\ 0}$ 7 8 9 10 -6 -8 11 -9

-5 -4 -3 -2 -1 1 2 3 4 $\boxed{5\ 0}$ 6 7 8 9 10 -6 -8 11 -9

-5 -4 -3 -2 -1 1 2 3 $\boxed{4\ 0}$ 5 6 7 8 9 10 -6 -8 11 -9

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$\boxed{-6 \quad -8}$ -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 $\boxed{10 \quad 11}$ -9

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-8 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 $\boxed{10 \quad -9}$ 11

-8 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 $\boxed{9 \quad -9}$ 10 11

-8 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 $\boxed{8 \quad -9}$ 9 10 11

-8 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 $\boxed{7 \quad -9}$ 8 9 10 11

-8 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 $\boxed{6 \quad -9}$ 7 8 9 10 11

-8 -6 -5 -4 -3 -2 -1 0 1 2 3 4 $\boxed{5 \quad -9}$ 6 7 8 9 10 11

-8 -6 -5 -4 -3 -2 -1 0 1 2 3 $\boxed{4 \quad -9}$ 5 6 7 8 9 10 11

-8 -6 -5 -4 -3 -2 -1 0 1 2 $\boxed{3 \quad -9}$ 4 5 6 7 8 9 10 11

-8 -6 -5 -4 -3 -2 -1 0 1 $\boxed{2 \quad -9}$ 3 4 5 6 7 8 9 10 11

-8 -6 -5 -4 -3 -2 -1 0 $\boxed{1 \quad -9}$ 2 3 4 5 6 7 8 9 10 11

-8 -6 -5 -4 -3 -2 -1 $\boxed{0 \quad -9}$ 1 2 3 4 5 6 7 8 9 10 11

-8 -6 -5 -4 -3 -2 $\boxed{-1 \quad -9}$ 0 1 2 3 4 5 6 7 8 9 10 11

-8 -6 -5 -4 -3 $\boxed{-2 \quad -9}$ -1 0 1 2 3 4 5 6 7 8 9 10 11

-8 -6 -5 -4 $\boxed{-3 \quad -9}$ -2 -1 0 1 2 3 4 5 6 7 8 9 10 11

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$\boxed{-8 \quad -9}$ -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11

-9 -8 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11

Pseudo code:

```
Insertion_sort (a, size of a):
  for i in range (2, size):
    key = a[i]
    j = j-1
    while j>0 and a[j]>key
      a[j+1] = a[j]
        j -= 1
      a[i+1] = key
      return a
```

Time complexity:

$T(n) = O(n^2)$

11. Given an array of [4, -2, 5, 3, 10, -5, 2, 8, -3, 6, 7, -4, 1, 9, -1, 0, -6, -8, 11, -9] integers find the maximum and minimum product that can be obtained by multiplying two integers from the array.

Sorted array:

[-9, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

The two largest numbers are 10, 11
The two smallest numbers are -9, -8
The largest positive numbers is 11
The largest negative number is -9

Maximum product:

$10 \times 11 = 110$
$-9 \times -8 = 72$

The maximum product is 110

Minimum product:

$11 \times -9 = -99$
$-9 \times -8 = 72$

The minimum product is -99