Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Second order System

1.1 Damping

1.1.1. List the different kinds of damping for a second order system defined by

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2} \tag{1.1.1.1}$$

where ω is the natural frequency and ζ is the damping factor.
Solution: The details are available in Table 1.1.1

Damping Ratio	**Damping Type**
$\zeta > 1$	Overdamped
$\zeta = 1$	Critically Damped
$0 < \zeta < 1$	Underdamped
$\zeta = 0$	Undamped

TABLE 1.1.1

1.1.2. Classify the following second-order systems according to damping.

a) $H(s) = \frac{15}{s^2+5s+15}$
b) $H(s) = \frac{25}{s^2+10s+25}$
c) $H(s) = \frac{35}{s^2+18s+35}$

Solution: For

$$H(s) = \frac{25}{s^2 + 10s + 25}, \tag{1.1.2.1}$$

$$\omega^2 = 25, 2\zeta\omega = 10 \tag{1.1.2.2}$$

$$\implies \omega = 1, \zeta = 1 \tag{1.1.2.3}$$

and the system is critically damped. Similarly, the damping factors for other systems in Problem 1.1.2 are calculated and listed in Table 1.1.2

$H(s)$	ω	ζ	**Damping Type**
$\frac{35}{s^2+18s+35}$	$\sqrt{35}$	$\sqrt{\frac{81}{35}} > 1$	Overdamped
$\frac{25}{s^2+10s+25}$	5	1	Critically Damped
$\frac{15}{s^2+5s+15}$	$\sqrt{15}$	$\sqrt{\frac{5}{12}} < 1$	Underdamped

TABLE 1.1.2

1.1.3. By choosing an appropriate input, illustrate the effect of damping using a Python code to sketch the response.

2 State-Space Model

2.1 Controllability and Observabiity

2.1.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.
Solution: The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{2.1.1.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{2.1.1.2}$$

with parameters listed in Table 2.1.1.

2.1.2. Find the transfer function $\mathbf{H}(s)$ for the general system.

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Variable	Size	Description
u	$p \times 1$	input(control) vector
y	$q \times 1$	output vector
x	$n \times 1$	state vector
A	$n \times n$	state or system matrix
B	$n \times p$	input matrix
C	$q \times n$	output matrix
D	$q \times p$	feedthrough matrix

TABLE 2.1.1

Solution: Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}X(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s) \tag{2.1.2.1}$$

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s) + x(0) \tag{2.1.2.2}$$

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{2.1.2.3}$$

and

$$Y(s) = \mathbf{C}X(s) + D\mathbf{I}U(s) \tag{2.1.2.4}$$

Substituting from (2.1.2.3) in the above,

$$Y(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{2.1.2.5}$$

2.1.3. Find $H(s)$ for a SISO (single input single output) system.
Solution:

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + DI \tag{2.1.3.1}$$

2.1.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{2.1.4.1}$$

$$D = 0 \tag{2.1.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.1.4.3}$$

find **A** and **C** such that the state-space realization is in *controllable canonical form.*

Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{2.1.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{2.1.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{2.1.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{2.1.4.7}$$

so equation 0.1.13 can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ s(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{2.1.4.8}$$

So

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{2.1.4.9}$$

$$Y = X_1(s) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} \tag{2.1.4.10}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{2.1.4.11}$$

2.1.5. Obtain **A** and **C** so that the state-space realization in in *observable canonical form.*
Solution: Given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{2.1.5.1}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{2.1.5.2}$$

$$Y(s) \times (s^3 + 3s^2 + 2s + 1) = U(s) \tag{2.1.5.3}$$

$$s^3Y(s) + 3s^2Y(s) + 2sY(s) + Y(s) = U(s) \tag{2.1.5.4}$$

$$s^3Y(s) = U(s) - 3s^2Y(s) - 2sY(s) - Y(s) \tag{2.1.5.5}$$

$$Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{2.1.5.6}$$

let $Y = aU + X_1$
by comparing with equation 1.5.6 we get a=0 and

$$Y = X_1 \tag{2.1.5.7}$$

inverse laplace transform of above equation is

$$y = x_1 \tag{2.1.5.8}$$

so from above equation 1.5.6 and 1.5.7

$$X_1 = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{2.1.5.9}$$

$$sX_1 = -3Y(s) - 2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{2.1.5.10}$$

inverse laplace transform of above equation

$$\dot{x_1} = -3y + x_2 \tag{2.1.5.11}$$

where

$$X_2 = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{2.1.5.12}$$

$$sX_2 = -2Y(s) + s^{-1}(U(s) - Y(s)) \tag{2.1.5.13}$$

inverse laplace transform of above equation

$$\dot{x_2} = -2y + x_3 \tag{2.1.5.14}$$

where

$$X_3 = s^{-1}(U(s) - Y(s)) \tag{2.1.5.15}$$

$$sX_3 = U(s) - Y(s) \tag{2.1.5.16}$$

inverse laplace transform of above equation

$$\dot{x_3} = u - y \tag{2.1.5.17}$$

so we get four equations which are

$$y = x_1 \tag{2.1.5.18}$$

$$\dot{x_1} = -3y + x_2 \tag{2.1.5.19}$$

$$\dot{x_2} = -2y + x_3 \tag{2.1.5.20}$$

$$\dot{x_3} = u - y \tag{2.1.5.21}$$

sub $y = x_1$ in 1.5.19,1.5.20,1.5.21 we get

$$y = x_1 \tag{2.1.5.22}$$

$$\dot{x_1} = -3x_1 + x_2 \tag{2.1.5.23}$$

$$\dot{x_2} = -2x_1 + x_3 \tag{2.1.5.24}$$

$$\dot{x_3} = u - x_1 \tag{2.1.5.25}$$

so above equations can be written as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{2.1.5.26}$$

So

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{2.1.5.27}$$

$$y = x_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{2.1.5.28}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{2.1.5.29}$$

2.1.6. Find the eigenvaues of **A** and the poles of $H(s)$ using a python code.
Solution: The following code

```
codes/ee18btech11004.py
```

gives the necessary values. The roots are the same as the eigenvalues.

2.1.7. Theoretically, show that eigenvaues of **A** are the poles of $H(s)$. **Solution:**
as we know tthat the characteristic equation is det(sI-A)

$$\mathbf{sI} - \mathbf{A} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix} \tag{2.1.7.1}$$

therfore

$$det(sI - A) = s(s^2 + 3s + 2) + 1(1) = s^3 + 3s^2 + 2s + 1 \tag{2.1.7.2}$$

so from equation 1.6.2 we can see that charcteristic equation is equal to the denominator of the transefer function

2.2 *Second Order System*

2.2.1. Consider a state-variable model of a system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & -2\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} r \tag{2.2.1.1}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.2.1.2}$$

where y is the output, and r is the input.

2.2.2. List the various state matrices in (2.2.1.1)
Solution: The state space model is given by

$$\dot{X} = AX + BU \tag{2.2.2.1}$$

$$Y = CX + DU \tag{2.2.2.2}$$

Comparing this with (2.2.1.1) we get the state matrices as,

$$A = \begin{pmatrix} 0 & 1 \\ -\alpha & -2\beta \end{pmatrix} \tag{2.2.2.3}$$

$$B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{2.2.2.4}$$

$$C = \begin{pmatrix} 1 & 0 \end{pmatrix} \tag{2.2.2.5}$$

$$D = 0 \tag{2.2.2.6}$$

2.2.3. Find the the system transfer function $H(s)$.
Solution: From (2.1.1.1) and , (2.1.3.1), the transfer function for the state space model is

$$H(s) = C(sI - A)^{-1}B + D \tag{2.2.3.1}$$

$$= \frac{\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} s + 2\beta & 1 \\ -\alpha & s \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{s(s + 2\beta) + \alpha} \tag{2.2.3.2}$$

$$= \frac{b_1(s + 2\beta) + b_2}{s^2 + 2s\beta + \alpha} \tag{2.2.3.3}$$

$$\implies H(s) = \frac{b_1 s}{s^2 + 2s\beta + \alpha} + \frac{2b_1\beta + b_2}{s^2 + 2s\beta + \alpha} \tag{2.2.3.4}$$

2.2.4. Find the Damping ratio ζ and the Undamped natural frequency ω_n of the system.
Solution: Generally for a second order system the transfer function is given by 1.1.1.1

$$H(s) = \frac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2} \tag{2.2.4.1}$$

Comparing the denominator of the above with (2.2.3.4),

$$2\zeta\omega_n = 2\beta, \tag{2.2.4.2}$$

$$\omega_n^2 = \alpha \tag{2.2.4.3}$$

$$\implies \zeta = \frac{\beta}{\sqrt{\alpha}}, \omega_n = \sqrt{\alpha} \tag{2.2.4.4}$$

2.2.5. Using Table 1.1.1, explain how the damping conditions depend upon α and β.
Solution: Using Table 1.1.1, and equation (2.2.4.4) we can say the following things about system,

$$\beta > \sqrt{\alpha} \implies Overdamped \tag{2.2.5.1}$$

$$\beta = \sqrt{\alpha} \implies CriticallyDamped \tag{2.2.5.2}$$

$$0 < \beta < \sqrt{\alpha} \implies Underdamped \tag{2.2.5.3}$$

$$\beta = 0 \implies Undamped \tag{2.2.5.4}$$