EE2227 Control Systems
Presentation-1

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Consider a state-variable model of a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & -2\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where y is the output, and r is the input. The damping ratio ζ and the undamped natural frequency ω_n (rad/sec) of the system are given by:

(A) $\zeta = \frac{\beta}{\sqrt{\alpha}}, \omega_n = \sqrt{\alpha}$

(B) $\zeta = \sqrt{\alpha}, \omega_n = \frac{\beta}{\sqrt{\alpha}}$

(C) $\zeta = \frac{\alpha}{\sqrt{\beta}}, \omega_n = \sqrt{\beta}$

(D) $\zeta = \sqrt{\beta}, \omega_n = \sqrt{\alpha}$

Solution - Transformation of State Equations to a Single Differential Equation

The state equations $\dot{x}$ =Ax+Br for a linear second-order system with a single input are a pair of coupled first-order differential equations in the two state variables: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} r.$

Or

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1 r.$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2 r.$$

The state-space system representation may be transformed into a single differential equation in either of the two state-variables.

Transformation of State Equations to a Single Differential Equation

Taking the Laplace transform of the state equations

$$(sI - A)X(s) = BR(s)$$

$$X(s) = (sI - A)^{-1}BR(s)$$

$$X(s) = \frac{1}{det[sI - A]} \begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} R(s)$$

$$det[sI - A]X(s) = \begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} R(s)$$

Transformation of State Equations to a Single Differential Equation

from this we can write

$$\frac{d^2x_1}{dt^2}-(a_{11}+a_{22})\frac{dx_1}{dt}+(a_{11}a_{22}-a_{12}a_{21})x_1 = b_1\frac{dr}{dt}+(a_{12}b_2-a_{22}b_1)r.$$

and

$$\frac{d^2x_2}{dt^2}-(a_{11}+a_{22})\frac{dx_2}{dt}+(a_{11}a_{22}-a_{12}a_{21})x_2 = b_2\frac{dr}{dt}+(a_{21}b_1-a_{11}b_2)r.$$

which can be written in terms of the two parameters ω_n and ζ as follows:

Transformation of State Equations to a Single Differential Equation

$$\frac{d^2x_1}{dt^2} + 2\zeta\omega_n\frac{dx_1}{dt} + \omega_n^2 x_1 = b_1\frac{dr}{dt} + (a_{12}b_2 - a_{22}b_1)r.$$

and

$$\frac{d^2x_2}{dt^2} + 2\zeta\omega_n\frac{dx_2}{dt} + \omega_n^2 x_1 = b_2\frac{dr}{dt} + (a_{21}b_1 - a_{11}b_2)r.$$

where ζ is the system(dimensionless) damping ratio and the undamped natural frequency with units of radian/second is ω_n.

By comparing above equations we get that:

$$\omega_n = \sqrt{a_{11}a_{22} - a_{12}a_{21}}$$

and

$$\zeta = -\frac{(a_{11} + a_{22})}{2\omega_n} = \frac{-(a_{11} + a_{22})}{2\sqrt{a_{11}a_{22} - a_{12}a_{21}}}$$

Transformation of State Equations to a Single Differential Equation

Now putting the give values in the variables we get,

$$\zeta = \frac{-1(0-2\beta)}{2\sqrt{0(-2\beta)-1(-\alpha)}} = \frac{\beta}{\sqrt{\alpha}}$$

So the answer is Option(A).

$$\omega_n = \sqrt{0(-2\beta)-1(-\alpha)} = \sqrt{\alpha}$$