Control Systems

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Contents

Abstract—**This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 State-Space Model

1.1 Controllability and Observabiity

1.1.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.

Solution: The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1.1.1.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{1.1.1.2}$$

with parameters listed in Table 1.1.1.

Variable	Size	Description
$\mathbf{u}$	$p \times 1$	input(control) vector
$\mathbf{y}$	$q \times 1$	output vector
$\mathbf{x}$	$n \times 1$	state vector
$\mathbf{A}$	$n \times n$	state or system matrix
$\mathbf{B}$	$n \times p$	input matrix
$\mathbf{C}$	$q \times n$	output matrix
$\mathbf{D}$	$q \times p$	feedthrough matrix

TABLE 1.1.1

1.1.2. Find the transfer function $\mathbf{H}(s)$ for the general system.

Solution: Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}X(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s) \tag{1.1.2.1}$$

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s) + x(0) \tag{1.1.2.2}$$

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{1.1.2.3}$$

and

$$Y(s) = \mathbf{C}X(s) + D\mathbf{I}U(s) \tag{1.1.2.4}$$

Substituting from (1.1.2.3) in the above,

$$Y(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{1.1.2.5}$$

1.1.3. Find $H(s)$ for a SISO (single input single output) system.

Solution:

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + DI \tag{1.1.3.1}$$

1.1.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{1.1.4.1}$$

$$D = 0 \tag{1.1.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.1.4.3}$$

find $\mathbf{A}$ and $\mathbf{C}$ such that the state-space realization is in *controllable canonical form.*

Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{1.1.4.4}$$

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letting

$$\frac{Y(s)}{V(s)} = 1, \tag{1.1.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{1.1.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{1.1.4.7}$$

so equation 0.1.13 can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ s(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{1.1.4.8}$$

So

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{1.1.4.9}$$

$$Y = X_1(s) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} \tag{1.1.4.10}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{1.1.4.11}$$

1.1.5. Obtain **A** and **C** so that the state-space realization in in *observable canonical form*.
Solution: Given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{1.1.5.1}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{1.1.5.2}$$

$$Y(s) \times (s^3 + 3s^2 + 2s + 1) = U(s) \tag{1.1.5.3}$$

$$s^3Y(s) + 3s^2Y(s) + 2sY(s) + Y(s) = U(s) \tag{1.1.5.4}$$

$$s^3Y(s) = U(s) - 3s^2Y(s) - 2sY(s) - Y(s) \tag{1.1.5.5}$$

$$Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{1.1.5.6}$$

let $Y = aU + X_1$
by comparing with equation 1.5.6 we get a=0
and

$$Y = X_1 \tag{1.1.5.7}$$

inverse laplace transform of above equation is

$$y = x_1 \tag{1.1.5.8}$$

so from above equation 1.5.6 and 1.5.7

$$X_1 = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{1.1.5.9}$$

$$sX_1 = -3Y(s) - 2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{1.1.5.10}$$

inverse laplace transform of above equation

$$\dot{x_1} = -3y + x_2 \tag{1.1.5.11}$$

where

$$X_2 = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{1.1.5.12}$$

$$sX_2 = -2Y(s) + s^{-1}(U(s) - Y(s)) \tag{1.1.5.13}$$

inverse laplace transform of above equation

$$\dot{x_2} = -2y + x_3 \tag{1.1.5.14}$$

where

$$X_3 = s^{-1}(U(s) - Y(s)) \tag{1.1.5.15}$$

$$sX_3 = U(s) - Y(s) \tag{1.1.5.16}$$

inverse laplace transform of above equation

$$\dot{x_3} = u - y \tag{1.1.5.17}$$

so we get four equations which are

$$y = x_1 \tag{1.1.5.18}$$

$$\dot{x_1} = -3y + x_2 \tag{1.1.5.19}$$

$$\dot{x_2} = -2y + x_3 \tag{1.1.5.20}$$

$$\dot{x_3} = u - y \tag{1.1.5.21}$$

sub $y = x_1$ in 1.5.19,1.5.20,1.5.21 we get

$$y = x_1 \tag{1.1.5.22}$$

$$\dot{x_1} = -3x_1 + x_2 \tag{1.1.5.23}$$

$$\dot{x_2} = -2x_1 + x_3 \tag{1.1.5.24}$$

$$\dot{x_3} = u - x_1 \tag{1.1.5.25}$$

so above equations can be written as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{1.1.5.26}$$

So

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{1.1.5.27}$$

$$y = x_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{1.1.5.28}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{1.1.5.29}$$

1.1.6. Find the eigenvaues of **A** and the poles of $H(s)$ using a python code.
Solution: The following code

```
codes/ee18btech11004.py
```

gives the necessary values. The roots are the same as the eigenvalues.

1.1.7. Theoretically, show that eigenvaues of **A** are the poles of $H(s)$. **Solution:**
as we know tthat the characteristic equation is det(sI-A)

$$\mathbf{sI} - \mathbf{A} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix} \tag{1.1.7.1}$$

therfore

$$det(sI - A) = s(s^2 + 3s + 2) + 1(1) = s^3 + 3s^2 + 2s + 1 \tag{1.1.7.2}$$

so from equation 1.6.2 we can see that charcteristic equation is equal to the denominator of the transefer function

1.2 Second Order System

1.2.1. Consider a state-variable model of a system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & -2\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} r \tag{1.2.1.1}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{1.2.1.2}$$

where y is the output, and r is the input. Find the the system transfer function $H(s)$.
Solution: The state space model is given by

$$\dot{X} = AX + BU \tag{1.2.1.3}$$

$$Y = CX + DU \tag{1.2.1.4}$$

From (1.1.2.3) we know that the transfer function for the state space model is:

$$H(s) = C(sI - A)^{-1}B + D \tag{1.2.1.5}$$

$$\implies H(s) = \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s + 2\beta & 1 \\ -\alpha & s \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{s(s + 2\beta) + \alpha} \tag{1.2.1.6}$$

$$= \frac{b_1(s + 2\beta) + b_2}{s^2 + 2s\beta + \alpha} \tag{1.2.1.7}$$

$$\implies H(s) = \frac{b_1 s}{s^2 + 2s\beta + \alpha} + \frac{2b_1\beta + b_2}{s^2 + 2s\beta + \alpha} \tag{1.2.1.8}$$

1.2.2. Find the Damping ratio ζ and the Undamped natural frequency ω_n of the system.
Solution: Generally for a second order system the transfer function is given by

$$H(s) = \frac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2} \tag{1.2.2.1}$$

Now from the transfer function we got we can see that our system is bandpass and lowpass combination but the comaprison of denominator of our transfer function to the general transfer function is still valid.

$$\therefore 2\zeta\omega_n = 2\beta, \tag{1.2.2.2}$$

$$\omega_n^2 = \alpha \tag{1.2.2.3}$$

$$\implies \zeta = \frac{\beta}{\sqrt{\alpha}}, \omega_n = \sqrt{\alpha} \tag{1.2.2.4}$$

1.2.3. What is the significance of ζ and ω_n? Explain through plots. **Solution:**

Damping Ratio	Undamped Natural Frequency
Damping ratio basically indicates the amount of damping present in the overall system.	It's the frequency of oscillation of the system without damping.
ζ >1 $\implies$ Overdamped system ζ = 1 $\implies$ Critically damped system 0 <ζ <1 $\implies$ Underdamped system ζ = 0 $\implies$ Undamped system	Only systems with ζ <1 have a natural frequency ω and only in the case that ζ = 0 will the natural frequency $\omega = \omega_n$, the undamped natural frequency.

Given above is a table explaining about ζ , ω_n and below is a plot explaining what happens if zeta increases. If zeta increases than the magnitude decreases as shown in the Fig. 1.2.3.

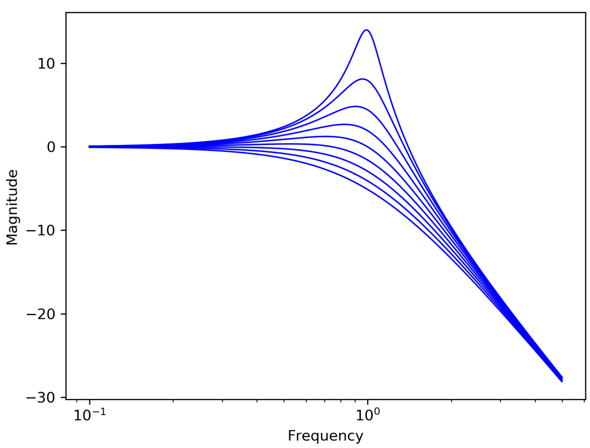

Fig. 1.2.3

1.2.4. How do α and β affect the system performance? Explain through plots.
Solution: From the Fig. 1.2.4 we can say that when β is kept constant and α is increased than so as the magnitude decreases and from the Fig. 1.2.4 we can say that when α is kept constant and β is increased the magnitude is increased.

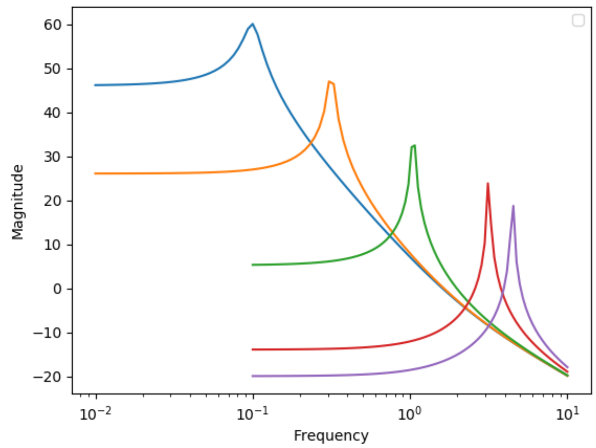

Fig. 1.2.4

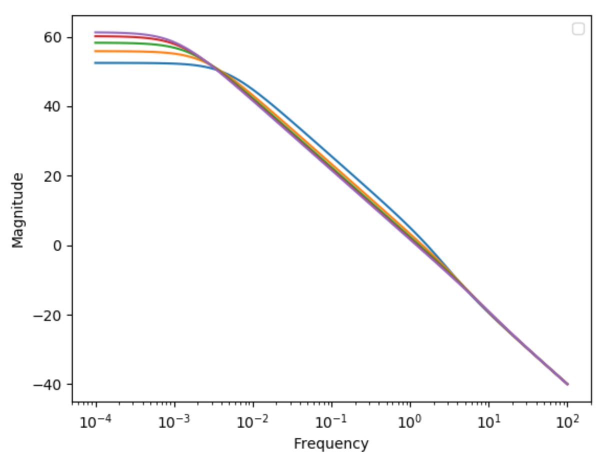

Fig. 1.2.4