#SAT based Bounded Model Checking

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- Background
- BMC for MCs
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Background

- **Markov Decision Process (MDP)**

  A Markov decision process is a 4-tuple (S, A, P_a, R_a)

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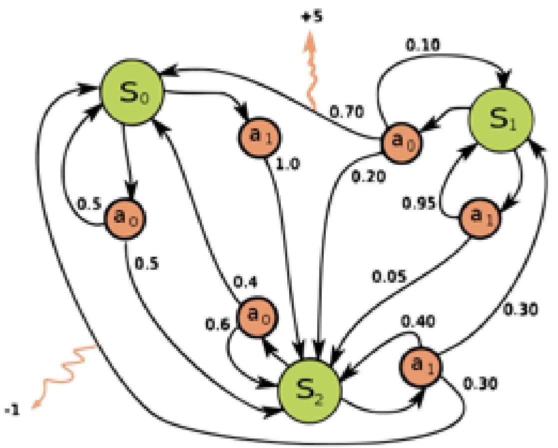

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- *S* is a finite set of states
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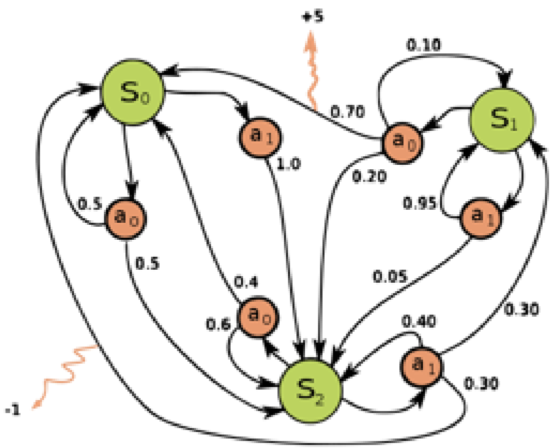

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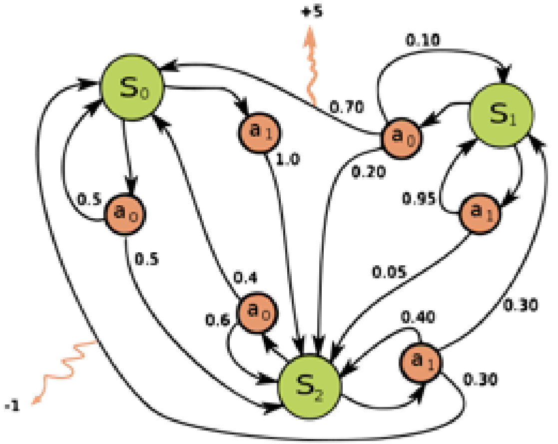

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Equivalently, a k-step policy specifies the action to be taken at each state for each step made by the agent

A k-step policy induces a Markov Chain with |*S*|*(*k+1*) number of states

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No of satisfying assignments of $(a \vee b)$ is 3.
For simplicity, denoted as:

$$\#(a \vee b) = 3$$

Similarly,

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Input : Input variables denote the possible paths that a run of the MC may take
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$$F = (s_{(0,0)} \wedge (\bigwedge_{j=1}^{|S|-1} \overline{s_{(j,0)}})) \wedge (\bigwedge_{i=0}^{k-1} \bigwedge_{j=0}^{|S|-1} PE_\pi(s_{(j,i)})) \wedge (\bigvee_{i=0}^{k} s_{(|S|-1,i)})$$

where, $PE_\pi(s_{(j,i)})$ encodes the probability distribution for state no *j* at step *i+1*

Encoding Probabilities

- **Biased Coins**

The fraction of assignments that satisfy a given Boolean formula is the probability that the formula encodes

Unbiased Coin: (p=0.5)$\equiv c$ $\quad \frac{\#(c)}{2} = 0.5$

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For any rational number *p/q* : $\Pr(x < p \mid x < q)$ = $\dfrac{\Pr(x < p)}{\Pr(x < q)} = \dfrac{\frac{p}{2^n}}{\frac{q}{2^n}} = \dfrac{p}{q}$

That is, only allow *q* inputs to be "valid" and *p* of them to be accepting

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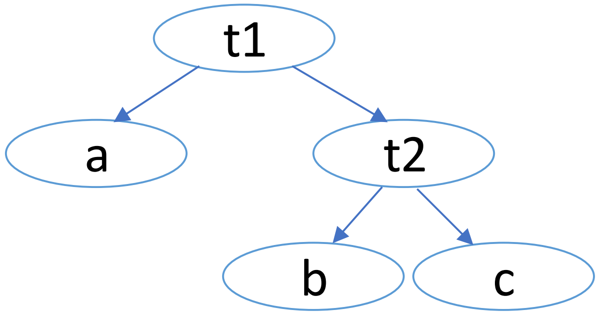

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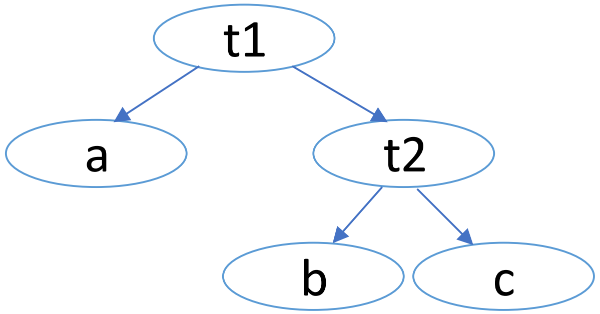

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$$F = (a \leftrightarrow (t_1 \wedge \overline{c_1})) \wedge (t_2 \leftrightarrow (t_1 \wedge c_1)) \wedge (b \leftrightarrow (t_2 \wedge \overline{c_2})) \wedge (c \leftrightarrow (t_2 \wedge c_2))$$

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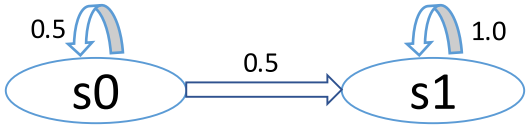
- **Example**

|*S*|=2, *k*=2

$$\Pr(s_0, s_0) = 0.5 \equiv 0.1 \quad \Pr(s_0, s_1) = 0.5 \equiv 0.1 \quad \Pr(s_1, s_1) = 1.0 \equiv 1.0$$



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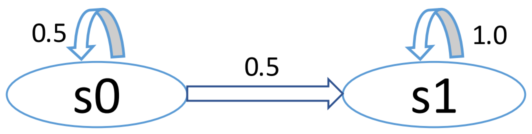

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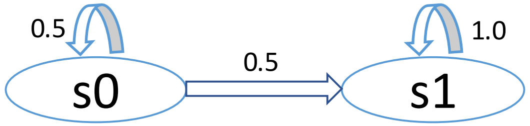

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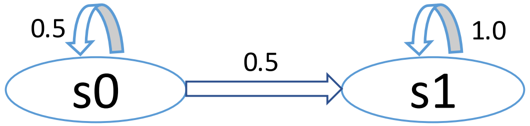

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$$F = c_1 \vee c_2$$

$$\frac{\#(F)}{2^2} = \frac{3}{4}$$

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Problem: Does there exist a k-step Policy which ensures that the Reachability Probability of a target state from a given initial state within k steps exceeds a certain threshold?

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$$\bigwedge_{h=0}^{|A|-1} \left(PE(s_{(j,i,h)}) \leftrightarrow a_{(j,i,h)}\right)$$

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Essentially the **Max#SAT** problem

Policy Iteration is a more scalable approach

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  - Precompute and generate encoding for each of the $|S|*|A|$ Probability distributions
  - Given a policy, stitch the Boolean formula on the fly
  - Formula generation is the bottleneck

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  Policy Iteration primarily only changes a single action choice => only a few clauses are altered

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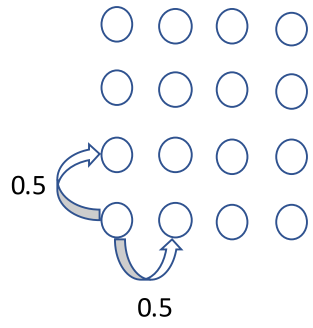
Experimental Results

PC Specs: Intel i5, 8GB RAM

	States in k-MDP	Variable Count in Formula	Clauses in Formula
2-state MDP (k=10)	22	37	78
Finite Reachability (k=10)	44	81	131
Grid MDP (5x5)	200	312	508
Grid MDP optimized (5x5)	200	24	126

	Storm (DRN input)	#SAT with approxMC3
5x5	0.012s	0.023s
15x15	0.027s	0.114
100x100	>1min	7.213s



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  Eg: **Simplified Grid Problem**

  The formula can be as simple as a circuit which adds k-bits