Solution of the Problem 'Magic Square'

Remarks:

- Any magic square with size NxN consists of the numbers from 1 to N^2 so the total sum of the numbers is:

  Total Sum = $N^2*(N^2+1)/2$.

- The sum of each row and column must be the same. Because of there are N rows and N columns. The sum of the numbers in each row and column is;

  Sum of a row or column = $(N^2*(N^2+1)/2)/N = N*(N^2+1)/2$

- There are at most tree numbers in a row and two numbers in a column are missing. It means there can be at most 20 numbers missing.

- The position of he missing numbers are given in the problem description. We can determine the set of the values of the missing numbers.

- There is always a solution.

- If there is a single missing number in a row (or column), it ca be calculated directly.

Algorithm:

There can be 20 numbers missing so we cannot generate all the permutations of them, and try one by one until we find the solution.

```
x1 x2 x3 22 18
20 11 7  3  24
21 x4 x5 x6  5
2  23 19 15  6
8  4  25 x7 x8
```

Here is the magic square which is given in the problem. Instead of the zeros, I used x1, x2,… x8 to represent the missing numbers. The missing numbers are 1, 9, 10, 12, 13, 14, 16, 17

In that configuration, x1 and x8 can be calculated directly. After finding the x8, x7 and x6 can be calculated. x1=14, x8=12, x7 = 16, and x6=9. There are four more unknown numbers we must find out, that is, x2, x3, x4, and x5. To calculate those numbers, we must know at least one of them. We chose one number out of four (1, 10, 13, 17) for a missing position (let's say x2) and check if it leads us the solution. We continue to choose a number until we get the solution.

How can we generalize this solution for any kind of configuration?

Think about the biggest block of missing positions; six unknowns together.

```
x1 x2 x3
x4 x5 x6
```

In this configuration, at the beginning, we must chose two numbers (x1 and x6 for instance) check if they can lead us the solution.

For six unknowns the time complexity will be arrangement of 2 out of 6, that is (6!/(6-2)!*2!)*2! = 6!/4!. In a 10x10 magic square, we can have at most six of such blocks. So the total complexity is the arrangement 6 of 18 (not 20 because the last two numbers must be stand alone). One can prove that this is the worst case for a 10x10 magic square three missing numbers in a row and two missing numbers in a column.

Think about another configuration; 4 missing numbers together.

x1	x2
x3	x4

For such a block we need to start only with one number to find out other three. In a 10x10 magic square we may have 5 such blocks, and we must chose 5 numbers out of 20 to complete the magic square.

The problem became how to calculate at least how many numbers we need to complete the rest of the magic square and to find the positions of those numbers. The answer is the solution of linear equation systems. Let’s turn back to our 5x5 magic square with 8 missing numbers.

x1	x2	x3	22	18
20	11	7	3	24
21	x4	x5	x6	5
2	23	19	15	6
8	4	25	x7	x8

Because we know the sum of each row and diagonal, we can obtain a system of linear equations with m equations and k unknowns from this magic square: There may be m=k, m>k or m<k.

The equations we can obtain from the magic square above are:

$$\left.\begin{aligned} x1 + x2 + x3 &= 25\\ X4 + x5 + x6 &= 39\\ X7 + x8 &= 28 \end{aligned}\right\}\quad \text{from the rows.}$$

$$\left.\begin{aligned} X1 &= 14\\ X2 + x4 &= 27\\ X3 + x5 &= 14\\ X6 + x7 &= 25\\ X8 &= 12 \end{aligned}\right\}\quad \text{from the columns.}$$

$$\left.\begin{aligned} X1 + x5 + x8 &=39\\ X5 &= 13 \end{aligned}\right\}\quad \text{from the columns}$$

Solving this equation system with Gaussian elimination method will give us a set of the unknowns we need to complete the rest of the magic square. After applying the elementary row and column operations you will have such a matrix below. The numbers on the diagonal (dark shaded) are different than zero, the numbers on the light shaded are can be zero or different than zero. In that case, 4 numbers must be selected at the beginning xa, xb, xc and xc).

x1	x3	x3	.	.	.	xa	xb	xc	xn	Result
0										
0	0									
0	0	0								
0	0	0	0							
0	0	0	0	0						
0	0	0	0	0	0					
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Question: Is the solution unique?