1. (a) Bayes decision rule for minimum-error-rate classification (two categories)

; Decide ω_1 if $\mathbb{P}(\omega_1 | x) > \mathbb{P}(\omega_2 | x)$

$$\Leftrightarrow \frac{\mathbb{P}(x|\omega_1)\cdot\mathbb{P}(\omega_1)}{\cancel{\mathbb{P}(x)}} > \frac{\mathbb{P}(x|\omega_2)\cdot\mathbb{P}(\omega_2)}{\cancel{\mathbb{P}(x)}}$$

$$\Leftrightarrow \frac{1}{\cancel{\sqrt{2\pi\cdot 1}}}\exp\left[-\frac{1}{2}(x-1)^2\right]\cdot\cancel{\frac{1}{3}} > \frac{1}{\cancel{\sqrt{2\pi\cdot 1}}}\exp\left[-\frac{1}{2}(x-3)^2\right]\cdot\frac{2}{\cancel{3}}$$

Since log-ftn is increasing function, without flipping the inequality sign,

$$\Leftrightarrow \ln\exp\left[-\frac{1}{2}(x-1)^2\right] > \ln\left(\exp\left[-\frac{1}{2}(x-3)^2\right]\cdot 2\right)$$

$$\Leftrightarrow -\frac{1}{2}(x-1)^2 > -\frac{1}{2}(x-3)^2 + \ln 2$$

$$\Leftrightarrow \cancel{-\frac{1}{2}x^2} + x - \frac{1}{2} > \cancel{-\frac{1}{2}x^2} + 3x - \frac{9}{2} + \ln 2$$

$$\Leftrightarrow 2x < 4 - \ln 2 \quad \Leftrightarrow x < 2 - \frac{1}{2}\ln 2$$

$$\boxed{\therefore \text{Decide } \begin{cases} \omega_1 & \text{if } x < 2 - \frac{1}{2}\ln 2 \\ \omega_2 & \text{if } x \geq 2 - \frac{1}{2}\ln 2 \end{cases}}$$

1. (b) Bayes decision rule for minimum risk classification:

Decide ω_1 if $\sum_{j=1}^{2} \lambda(\alpha_1|\omega_j)\, p(\omega_j|x) < \sum_{j=1}^{2} \lambda(\alpha_2|\omega_j)\, p(\omega_j|x)$

$\Leftrightarrow \quad 2p(\omega_2|x) < p(\omega_1|x)$

$\Leftrightarrow \quad 2\cdot p(x|\omega_2)\cdot p(\omega_2) < p(x|\omega_1)\, p(\omega_1)$

$\Leftrightarrow \quad 2\cdot \frac{1}{\cancel{\sqrt{2\pi}}} \exp\left[-\frac{1}{2}(x-3)^2\right]\cdot \frac{2}{\cancel{3}} < \frac{1}{\cancel{\sqrt{2\pi}}} \exp\left[-\frac{1}{2}(x-1)^2\right]\cdot \frac{1}{\cancel{3}}$

Similarly, take log on both sides and get

$\Leftrightarrow \quad -\frac{1}{2}(x-3)^2 + \log 4 < -\frac{1}{2}(x-1)^2$

$\Leftrightarrow \quad 3x - \frac{9}{2} + \log 4 < x - \frac{1}{2}$

$\Leftrightarrow \quad 2x < 4 - \log 4 \quad \Leftrightarrow \quad x < 2 - \log 2$

$$\boxed{\therefore \text{Decide} \begin{cases} \omega_1 & \text{if} \quad x < 2 - \log 2 \\ \omega_2 & \text{if} \quad x \geq 2 - \log 2 \end{cases}}$$

2. The minimum probability rate is achieved by deciding a category with greater posterior probability.

$\Rightarrow$ set the discriminant function as

$$g_j(\vec{x}) = \ln\left[\mathbb{P}(\vec{x}\mid w_j)\cdot \mathbb{P}(w_j)\right]$$

($\because$ log-ftn is increasing ftn)

Note that

$$\left.\begin{array}{l}\mathbb{P}(x_i=1\mid w_j) = P_{ij}\\ \mathbb{P}(x_i=0\mid w_j) = 1-P_{ij}\end{array}\right] \quad (\because \mathbb{P}(x_i=0 \text{ or } x_i=1)=1)$$

So $\mathbb{P}(x_i\mid w_j)$ can be written as

$$p(x_i\mid w_j) = P_{ij}^{\,x_i}(1-P_{ij})^{1-x_i}$$

Hence,

$$\begin{aligned} g_j(\vec{x}) &= \ln P(\vec{x}\mid w_j) + \ln P(w_j)\\ &= \ln \prod_{i=1}^{d} p(x_i\mid w_j) + \ln p(w_j)\\ &= \sum_{i=1}^{d} \ln p(x_i\mid w_j) + \ln p(w_j)\\ &= \sum_{i=1}^{d}\left[x_i \ln P_{ij} + (1-x_i)\ln(1-P_{ij})\right] + \ln p(w_j)\\ &= \sum_{i=1}^{d}\left[\left(\ln\frac{P_{ij}}{1-P_{ij}}\right)x_i\right] + \sum_{i=1}^{d}\ln(1-P_{ij}) + \ln p(w_j)\end{aligned}$$

Therefore, c_{ij} and b_j can be expressed as

$$\boxed{c_{ij} = \ln\frac{P_{ij}}{1-P_{ij}} \qquad b_j = \sum_{i=1}^{d}\ln(1-P_{ij}) + \ln p(w_j)}$$