Yonghyun Kwon

1. Write $\vec{x}_i$ be i-th data of the Data set D and

x_{ij} be j-th component of $\vec{x}_i$ i.e., $D = [\vec{x}_1, \cdots, \vec{x}_i, \cdots, \vec{x}_n]$

$\vec{x}_i = [x_{i1}, \cdots, x_{ij}, \cdots, x_{id}]$

Then the likelihood of $\vec{\theta}$ is

$$\mathbb{P}(\vec{\theta} \mid D) = \prod_{i=1}^{n} \prod_{j=1}^{d} \theta_j^{x_{ij}} (1-\theta_j)^{1-x_{ij}} = \prod_{j=1}^{d} \theta_j^{\sum_i x_{ij}} (1-\theta_j)^{n-\sum_i x_{ij}}$$

So the log-likelihood is (write $\sum_i x_{ij} = n_j$)

$$LL(\hat{\theta}) = \sum_{j=1}^{d} n_j \log \theta_j + (n - n_j) \log (1-\theta_j) \quad : (\text{concave if } 0 < \theta_j < 1 \ \forall j)$$

Since $\frac{\partial LL(\hat{\theta})}{\theta_j} = \frac{n_j}{\theta_j} - \frac{n-n_j}{1-\theta_j}$, the solution of $\frac{\partial LL(\hat{\theta})}{\partial \theta} = 0$ is

$$\frac{n_j}{\theta_j} = \frac{n-n_j}{1-\theta_j} \quad \forall j = 1, \cdots, d$$

$$\Leftrightarrow n_j - \cancel{n_j \theta_j} = n\theta_j - \cancel{n_j \theta_j} \quad \Leftrightarrow \theta_j = \frac{n_j}{n}$$

(when $n_j = 0$ or $n - n_j = 0$, the mle is also

$\theta_j = 0 = \frac{n_j}{n}$, $\theta_j = 1 = \frac{n_j}{n}$ respectively)

Hence the maximum likelihood estimate for $\vec{\theta}$ is

$$\boxed{\hat{\theta}_i^{ML} = \frac{N_i}{n}} \quad \text{where } N_i = \sum_{j=1}^{n} x_{ji}$$

for $i = 1, \cdots, d$

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$$\hat{p} = \frac{x}{n} = \frac{900}{1000} \qquad x \sim \text{Binomial}(n, p)$$

$$Var(\hat{p}) = \frac{npq}{n^2} = \frac{pq}{n} \qquad \widehat{Var}(\hat{p}) = \frac{\hat{p}(1-\hat{p})}{n} = \frac{1}{1000} \cdot \frac{9}{10} \cdot \frac{1}{10}$$

For large enough sample,

$$\frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \sim N(0,1)$$

So the 95% confidence interval is

$$p \in \left(\hat{p} \pm Z_{0.025} \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right)$$

$$= \left(\frac{9}{10} \pm 1.96 \cdot \sqrt{\frac{9}{10^5}} \right)$$

$$= \boxed{(0.8814, 0.9186)}$$