# Answer of Problem Set 3

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July 26, 2023

# 1 Problem 1. Expenditure Minimization.

1. The partial derivative of the utility function with respect to $x_1$ is given by:

$$\frac{\partial u}{\partial x_1} = \frac{1}{2}(x_1x_2x_3)^{-\frac{1}{2}} \cdot x_2x_3 = \frac{x_2x_3}{2\sqrt{x_1x_2x_3}}$$
$$\frac{\partial^2 u}{\partial x_1^2} = -\frac{1}{4}(x_1x_2x_3)^{-\frac{3}{2}} \cdot x_2^2x_3^2 = -\frac{x_2^2x_3^2}{4(x_1x_2x_3)^{3/2}}$$

Therefore, the utility function is increasing in $x_1$ (since the first derivative is positive), and is concave in $x_1$ (since the second derivative is negative).

2. The expenditure minimization problem is given by:

$$\min_x p \cdot \mathbf{x} \quad \text{s.t.} \quad u(\mathbf{x}) = \bar{u}$$

3. The Lagrangian function for this problem is:

$$\mathcal{L}(\mathbf{x}, \lambda) = p \cdot x + \lambda(\bar{u} - u(\mathbf{x}))$$

4. The first order conditions are given by:

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \lambda \frac{\partial u}{\partial x_i} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \bar{u} - u(x) = 0$$

5. compute the partial derivatives of the utility function:

$$p_1 - \lambda \frac{x_2x_3}{2\sqrt{x_1x_2x_3}} = 0$$
$$p_2 - \lambda \frac{x_1x_3}{2\sqrt{x_1x_2x_3}} = 0$$
$$p_3 - \lambda \frac{x_1x_2}{2\sqrt{x_1x_2x_3}} = 0$$

Next, set the partial derivatives equal to each other and solve for the relationship between $x_1, x_2, x_3$:

$$\frac{p_1}{p_2} = \frac{x_2x_3}{x_1x_3} \Rightarrow x_1 = \frac{p_2}{p_1}x_2$$
$$\frac{p_1}{p_3} = \frac{x_2x_3}{x_1x_2} \Rightarrow x_3 = \frac{p_2}{p_3}x_2$$

Substitute the relationship between $x_1, x_2, x_3$ into the utility function:

$$\begin{aligned}
\bar{u} &= \sqrt{x_1 x_2 x_3} \\
&= \sqrt{\left(\frac{p_2}{p_1} x_2\right) \cdot x_2 \cdot \left(\frac{p_2}{p_3} x_2\right)} \\
&= \sqrt{\left(\frac{p_2^2}{p_1 p_3}\right) x_2^3} \\
&= \frac{p_2}{\sqrt{p_1 p_3}} x_2^{3/2}
\end{aligned}$$

Finally, solve for $x_2$ to get $h_2$, and use $h_2$ to find $h_1$ and $h_3$:

$$\begin{aligned}
h_1 &= \left(\frac{\bar{u}\sqrt{p_2 p_3}}{p_1}\right)^{2/3} \\
h_2 &= \left(\frac{\bar{u}\sqrt{p_1 p_3}}{p_2}\right)^{2/3} \\
h_3 &= \left(\frac{\bar{u}\sqrt{p_1 p_2}}{p_3}\right)^{2/3}
\end{aligned}$$

6. Compute the derivatives and determine whether goods are substitutes or complements:

$$\begin{aligned}
\frac{\partial h_1(p; \bar{u})}{\partial p_2} &= \frac{1}{3}\left(\frac{\bar{u}\sqrt{p_3}}{p_1 p_2}\right)^{2/3} \geq 0 \\
\frac{\partial h_1(p; \bar{u})}{\partial p_3} &= \frac{1}{3}\left(\frac{\bar{u}\sqrt{p_2}}{p_1 p_3}\right)^{2/3} \geq 0
\end{aligned}$$

   Here we see that $h_1$ increases with $p_2$ and $p_3$, so $x_1$ and $x_2$ are net substitutes while $x_1$ and $x_3$ are net substitutes.

7. The expenditure function is given by the sum of prices times the quantities:

$$e(\mathbf{p}; \bar{u}) = p_1 h_1(\mathbf{p}; \bar{u}) + p_2 h_2(\mathbf{p}; \bar{u}) + p_3 h_3(\mathbf{p}; \bar{u})$$

8. First, substitute the Hicksian demand functions into the expenditure function:

$$\begin{aligned}
e(p; \bar{u}) &= p_1 h_1(p; \bar{u}) + p_2 h_2(p; \bar{u}) + p_3 h_3(p; \bar{u}) \\
&= 3\left(\bar{u}\sqrt{p_1 p_2 p_3}\right)^{2/3}
\end{aligned}$$

   Then substitute $\bar{u}$ with $v(\mathbf{p}; M)$ and set the expenditure equal to $M$:

$$M = 3\left(v(\mathbf{p}; M)\sqrt{p_1 p_2 p_3}\right)^{2/3}$$

   Rearranging to isolate $v(\mathbf{p}; M)$ and considering that the terms under each cube root in the equation are the same, we get:

$$v(\mathbf{p}; M) = \frac{M\sqrt{3M}}{9\sqrt{p_1 p_2 p_3}}$$

9. Derive the Marshallian demand functions:

$$x_1^*(\mathbf{p}; M) = -\frac{\frac{\partial v(\mathbf{p};M)}{\partial p_1}}{\frac{\partial v(\mathbf{p};M)}{\partial M}} = \frac{M}{3p_1}$$
$$x_2^*(\mathbf{p}; M) = -\frac{\frac{\partial v(\mathbf{p};M)}{\partial p_2}}{\frac{\partial v(\mathbf{p};M)}{\partial M}} = \frac{M}{3p_2}$$
$$x_3^*(\mathbf{p}; M) = -\frac{\frac{\partial v(\mathbf{p};M)}{\partial p_3}}{\frac{\partial v(\mathbf{p};M)}{\partial M}} = \frac{M}{3p_3}$$

10. Compute these derivatives:

$$\frac{\partial x_1^*(\mathbf{p}; M)}{\partial M} = \frac{1}{3p_1} > 0$$
$$\frac{\partial x_2^*(\mathbf{p}; M)}{\partial M} = \frac{1}{3p_2} > 0$$
$$\frac{\partial x_3^*(\mathbf{p}; M)}{\partial M} = \frac{1}{3p_3} > 0$$

As $\frac{\partial x_i^*}{\partial M}$ are all positive, goods $x_1, x_2, x_3$ are normal goods.

11. In general, economic theory suggests that the quantity demanded for a good decreases as its price increases, all other things being equal. This is known as the law of demand. This implies that the derivatives $\partial x_i(p; M)/\partial p_i$ should be negative.

## 2 Problem 2. Expenditure minimization: tricky cases.

1. (a) In this case, the utility of the consumer is determined by the smaller of the two quantities $x_1$ and $x_2$. Therefore, for the consumer to achieve a certain level of utility u, it is necessary that $x_1 = x_2 = \bar{u}$ Therefore, $h_1 = h_2 = \bar{u}$

   (b) In this situation, $h_1$ and $h_2$ are not dependent on $p_1$ and $p_2$. They only depend on the level of utility u that the consumer wants to achieve. Hence, the partial derivatives of $h_1$ and $h_2$ with respect to $p_1$ and $p_2$ are zero. The substitution effect of a change in price is zero because the consumer will always consume equal amounts of both goods.

   The graph is shown below:

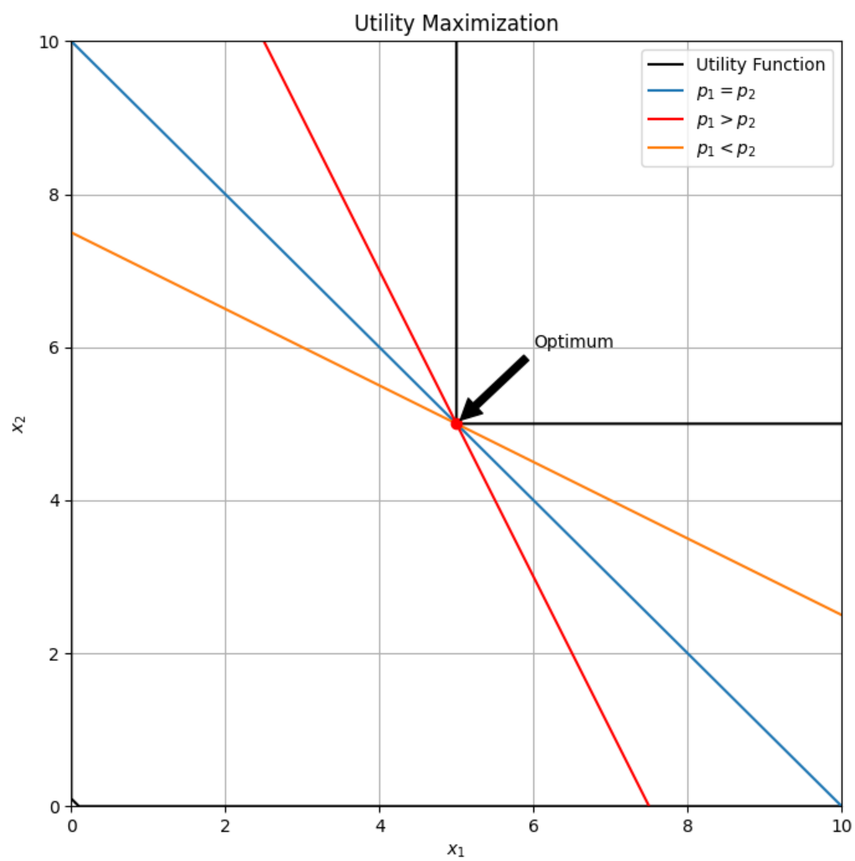


Figure 2.1: The graph of $h_1$ and $h_2$

2. In this case, the utility of the consumer is determined by the sum of the squares of $x_1$ and $x_2$. The consumer will try to distribute his expenditure in such a way that he maximizes his utility. Therefore, the consumer will consume more of the good which has lower price per unit. This is because, for the same amount of money, he can consume more of the good with lower price, and hence increase his utility. Therefore, the Hicksian compensated demand functions $h_1$ and $h_2$ will depend on the relative prices $p_1$ and $p_2$. The graph is shown below:

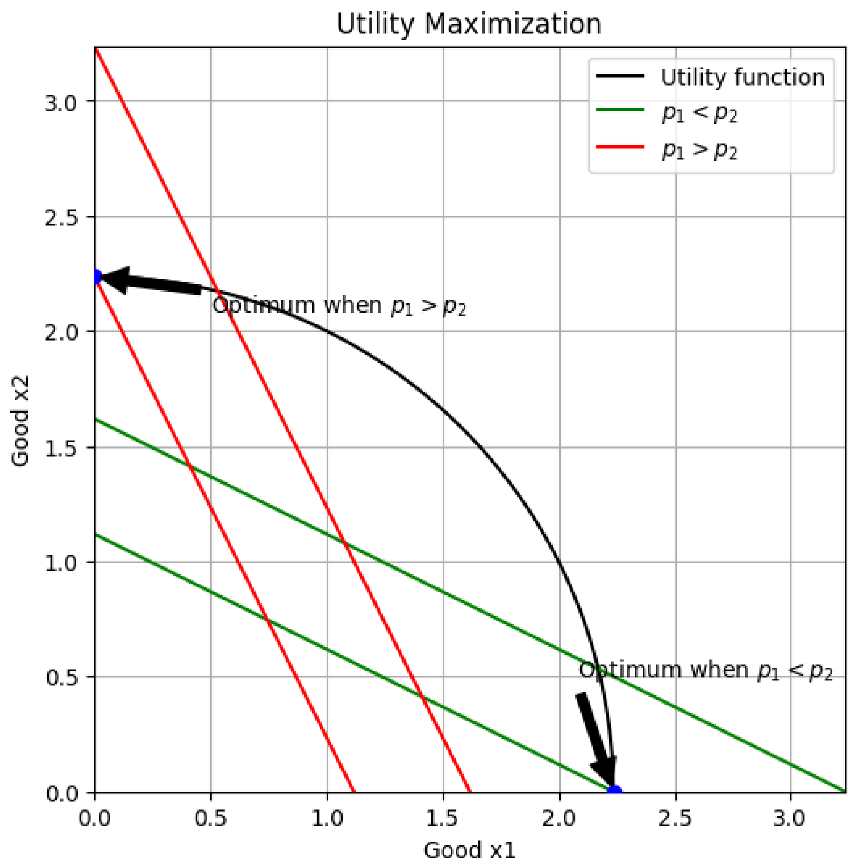


Figure 2.2: The graph of $h_1$ and $h_2$