Answer of Problem Set 1

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1 Problem 1. Univariate unconstrained maximization.

1. The first order condition for this problem is given by setting the derivative of the function with respect to x equal to zero:

$$\frac{df}{dx} = -2(x - x_0)\exp(-(x - x_0)^2) = 0$$

2. Solving the above equation gives us the value of x that maximizes $f(x; x_0)$:

$$x^* = x_0$$

3. The second order condition is given by the second derivative of the function with respect to x:

$$\frac{d^2 f}{dx^2} = -2\exp(-(x - x_0)^2) + 4(x - x_0)^2 \exp(-(x - x_0)^2)$$

If we substitute $x^* = x_0$ into this equation, we get:

$$\left.\frac{d^2 f}{dx^2}\right|_{x=x^*} = -2$$

This is a negative value, which implies that $x^* = x_0$ is a maximum, as expected.

4. As $x^* = x_0$, we have $\frac{dx^*}{dx_0} = 1$

5. If we plug in $x^* = x_0$ into the function f, and then take the derivative with respect to x_0, we get:

$$\frac{df}{dx_0} = \frac{d}{dx_0}\exp(0) = 0$$

By the envelope theorem, the derivative of the value function with respect to x_0 is also 0, which is consistent with the above result.

6. As shown above, the second derivative of f with respect to x is:

$$\frac{d^2 f}{dx^2} = -2\exp(-(x - x_0)^2) + 4(x - x_0)^2 \exp(-(x - x_0)^2)$$

As this function is not always negative (for example, when $x = x_0$ it equals -2, but when $x \neq x_0$ it could be positive), it indicates that the function f is not concave in x.

2 Problem 2. Multivariate unconstrained maximization.

1. The first order conditions for this problem with respect to x and y are given by setting the derivatives of the function equal to zero:
$$\frac{\partial f}{\partial x} = 2ax - 1 = 0$$
$$\frac{\partial f}{\partial y} = 2by - 1 = 0$$

2. Solving these equations gives us the values of x and y that maximize $f(x; y; a; b)$:
$$x^* = \frac{1}{2a} \text{ , } y^* = \frac{1}{2b}$$

3. The second order conditions are given by the second derivatives of the function with respect to x and y:
$$\mathbf{H} = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$$
And its determinant is:
$$\det(\mathbf{H}) = 4ab$$
The point (x^*, y^*) is a maximum if the determinant of the Hessian matrix is positive and the second order partial derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are negative. Therefore, a and b need to be negative for (x^*, y^*) to be a maximum.

4. To compute $\frac{dy^*}{da}$, we use the implicit function theorem, which gives us:
$$\frac{dy^*}{da} = -\frac{\frac{\partial^2 f}{\partial y \partial a}}{\frac{\partial^2 f}{\partial y^2}} = 0$$
So $\frac{dy^*}{da} = 0$. The result is the same as we compute it directly using the solution that we obtained in point 2

5. If we plug in $x^*(a; b)$ and $y^*(a; b)$ into f, and then take the derivative with respect to a, we get:
$$\frac{\partial f}{\partial a} = \frac{\partial}{\partial a}(a(x^*)^2 - x^* + b(y^*)^2 - y^*) = (x^*)^2$$
By the envelope theorem, the derivative of the value function with respect to a is $(x^*)^2$.

   We get the same result! The second method is faster.

6. The function f is concave in x and y when a and b are negative, as that would ensure the second partial derivatives $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial y^2}$ are negative, which is the condition for concavity. The function f is convex in x and y when a and b are positive.

3 Problem 3. Multivariate constrained maximization.

1. The Lagrangian function is a method to solve optimization problems with constraints. In this case, the Lagrangian is:
$$\mathcal{L}(x, y, \lambda) = x^\alpha y^\beta + \lambda(M - p_x x - p_y y)$$
where λ is the Lagrange multiplier.

2. The first order conditions are given by the partial derivatives of the Lagrangian with respect to x, y and λ:
$$\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^\beta - \lambda p_x = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = \beta x^\alpha y^{\beta-1} - \lambda p_y = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = M - p_x x - p_y y = 0$$

3. Solving the above system of equations gives us:
$$x^* = \frac{\alpha M}{p_x(\alpha + \beta)}$$
$$y^* = \frac{\beta M}{p_y(\alpha + \beta)}$$

4. The solutions for x^* and y^* satisfy the constraints $x > 0$ and $y > 0$ as long as p_x, p_y and M are all positive. Also, we must have $\alpha M > 0$ and $\beta M > 0$ which is true given the condition $0 < \alpha, \beta < 1$ and $M > 0$.

5. The derivative of x^* with respect to p_x is negative:
$$\frac{dx^*}{dp_x} = -\frac{\alpha M}{p_x^2(\alpha + \beta)} < 0$$
This means that as the price of good x increases, the quantity consumed decreases, which is intuitive.

6. The derivative of x^* with respect to p_y is:
$$\frac{dx^*}{dp_y} = 0$$
This implies that the quantity of good x consumed is not affected by the price of good y, reflecting the assumption of no substitution between goods in the Cobb-Douglas utility function.

7. The derivative of x^* with respect to M is:
$$\frac{dx^*}{dM} = \frac{\alpha}{p_x(\alpha + \beta)} > 0$$
This implies that as the total income M increases, the quantity of good x consumed increases, which also makes intuitive sense.

8. To calculate $\frac{\partial u(x^*(p_x,p_y,M), y^*(p_x,p_y,M))}{\partial p_x}$, we use the envelope theorem, which gives us:
$$\frac{\partial u(x^*(p_x, p_y, M), y^*(p_x, p_y, M))}{\partial p_x} = -\lambda^* x^* = -\frac{M}{p_x + p_y} x^*$$
This means that as the price of good x increases, utility at the optimum decreases. This result is not surprising because an increase in price without an increase in income leads to lower consumption and hence lower utility.

9. Similarly, we use the envelope theorem to calculate $\frac{\partial u(x^*(p_x,p_y,M),y^*(p_x,p_y,M))}{\partial M}$:

$$\frac{\partial u(x^*(p_x,p_y,M),y^*(p_x,p_y,M))}{\partial M} = \lambda^* = \frac{1}{p_x+p_y} > 0$$

This means that as the total income M increases, utility at the optimum increases. This result is also not surprising because an increase in income allows for higher consumption of goods, leading to higher utility.