Answer of Problem Set 2

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1 Problem 1. Addictive goods.

1. An approximate map of indifference curves:

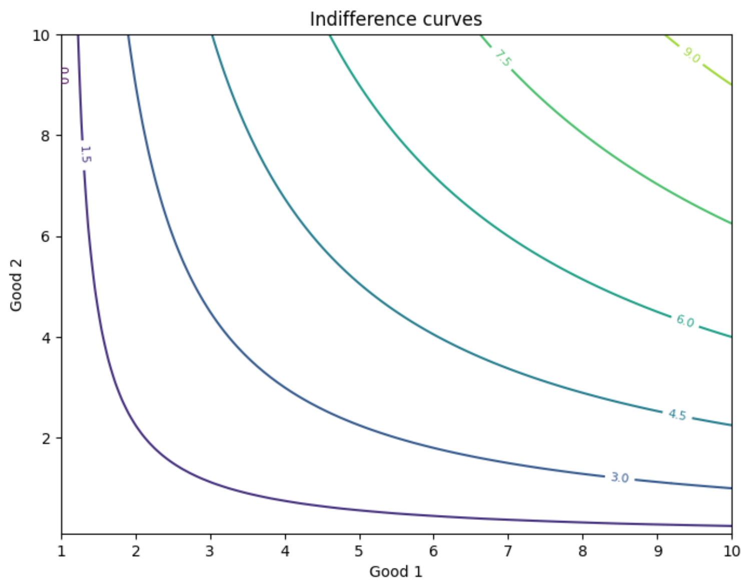

Figure 1.1: indifference Curves

2. The derivative of the utility function with respect to r_1 is:

$$\frac{\partial u}{\partial r_1} = -\alpha(x_1 - r_1)^{\alpha-1} x_2^{\beta}$$

This term is negative because as the reference point r_1 (i.e., addiction level) increases, the utility from consuming a certain quantity of the addictive good x_1 decreases. This is because a higher addiction level reduces the marginal benefit from consuming the good.

3. The marginal utility with respect to x_1 is:

$$\frac{\partial u}{\partial x_1} = \alpha(x_1 - r_1)^{\alpha-1} x_2^{\beta}$$

and the derivative of this marginal utility with respect to r_1 is:

$$\frac{\partial^2 u}{\partial x_1 \partial r_1} = -\alpha(\alpha - 1)(x_1 - r_1)^{\alpha-2} x_2^{\beta}$$

This term is positive because the more accustomed a person becomes to the addictive good (i.e., the higher their addiction level r_1), the more they will want to consume the addictive good (i.e., the higher the marginal utility of x_1).

4. The Lagrangian for this maximization problem is:

$$L = (x_1 - r_1)^\alpha x_2^\beta + \lambda(M - p_1 x_1 - p_2 x_2)$$

5. The first order conditions for this problem are given by the partial derivatives of the Lagrangian with respect to x_1, x_2 and λ:

$$\frac{\partial L}{\partial x_1} = \alpha(x_1 - r_1)^{\alpha-1} x_2^\beta - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = \beta(x_1 - r_1)^\alpha x_2^{\beta-1} - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = M - p_1 x_1 - p_2 x_2 = 0$$

6. Solving these equations gives us the optimal values of x_1 and x_2:

$$x_1^* = \frac{\alpha M + \beta r_1 p_1}{p_1}$$

$$x_2^* = \frac{\beta M - \beta r_1 p_1}{p_2}$$

7. The minimum level of income such that the solution makes sense (i.e., $x_1^* \geq r_1$ and $x_2^* \geq 0$):

$$M \geq r_1 p_1$$

8. The derivative of x_1^* with respect to M is:

$$\frac{dx_1^*}{dM} = \frac{\alpha}{p_1}$$

This result is always positive, meaning that good x_1 is a normal good.

9. The derivative of x_1^* with respect to r_1 is:

$$\frac{dx_1^*}{dr_1} = \beta$$

This result is positive, indicating that as addiction level r_1 increases, the quantity of good x_1 consumed increases. This makes sense because a higher addiction level increases the desire to consume the addictive good.

10. The derivative of x_2^* with respect to r_1 is zero, indicating that the quantity of good x_2 consumed does not depend on the addiction level r_1. This result is consistent with the idea that drug addicts can spend virtually all of their income on drugs and none on other goods.

11. Using the envelope theorem, the derivative of the optimal utility with respect to r_1 is:

$$\frac{d}{dr_1} u(x_1^*(p_1, p_2, M, r_1), x_2^*(p_1, p_2, M, r_1), r_1) = -\alpha(x_1^* - r_1)^{\alpha-1} x_2^\beta$$

This indicates that as the addiction level increases, the utility at the optimum decreases, reflecting the decreased satisfaction from consuming the addictive good as addiction grows.

2 Problem 2. Quasi-linear preferences.

1. The Lagrangian for this maximization problem is:

$$L = \phi(x_1) + x_2 + \lambda(M - p_1x_1 - p_2x_2)$$

2. The first order conditions for this problem are given by the partial derivatives of the Lagrangian with respect to x_1, x_2 and λ:

$$\frac{\partial L}{\partial x_1} = \phi'(x_1) - \lambda p_1 = 0$$
$$\frac{\partial L}{\partial x_2} = 1 - \lambda p_2 = 0$$
$$\frac{\partial L}{\partial \lambda} = M - p_1x_1 - p_2x_2 = 0$$

3. From the first order condition for x_2, we see that $\lambda = 1/p_2$. This implies that λ does not depend on p_1 or M, which usually isn't the case. In this context, λ can be interpreted as the marginal utility of wealth. Because the utility function is linear in x_2, the marginal utility of x_2 (which equals 1) is constant, so the marginal utility of wealth is also constant.

4. Plugging the value of λ into the first order condition for x_1, we get:

$$\phi'(x_1) = \frac{p_1}{p_2}$$

This equation implicitly defines x_1^* as a function of the parameters p_1, p_2, M. However, we see that x_1^* does not depend on income M, which means that good 1 is a neutral good.

5. From the first order condition with respect to x_1, taking derivative with respect to p_1, we obtain:

$$\frac{dx_1^*}{dp_1} = -\frac{\phi''(x_1^*)}{\phi''(x_1^*)} < 0$$

This is consistent with our expectation that an increase in the price of a good will decrease its demand (holding everything else constant). The demand for good 1 is solely determined by its own price and does not depend on income.

6. When $u(x_1, x_2) = {x_1}^{\frac{1}{2}} + x_2$, the first order conditions become:

$$\frac{1}{2\sqrt{x_1}} = \frac{p_1}{p_2}$$

Solving these equations, we find:

$$x_1^* = \frac{p_2^2}{4p_1^2}$$
$$x_2^* = M - \frac{p_2}{4p_1}$$

7. x_2^* will be nonnegative as long as $M \geq \frac{p_1}{4p_2}$.

8. The equation of the indifference curve is given by $x_2 = u - \sqrt{x_1}$. For simplicity, let's consider several different values for utility u, say u = 2, 3, 4.

The indifference curves can be graphically represented as follows:

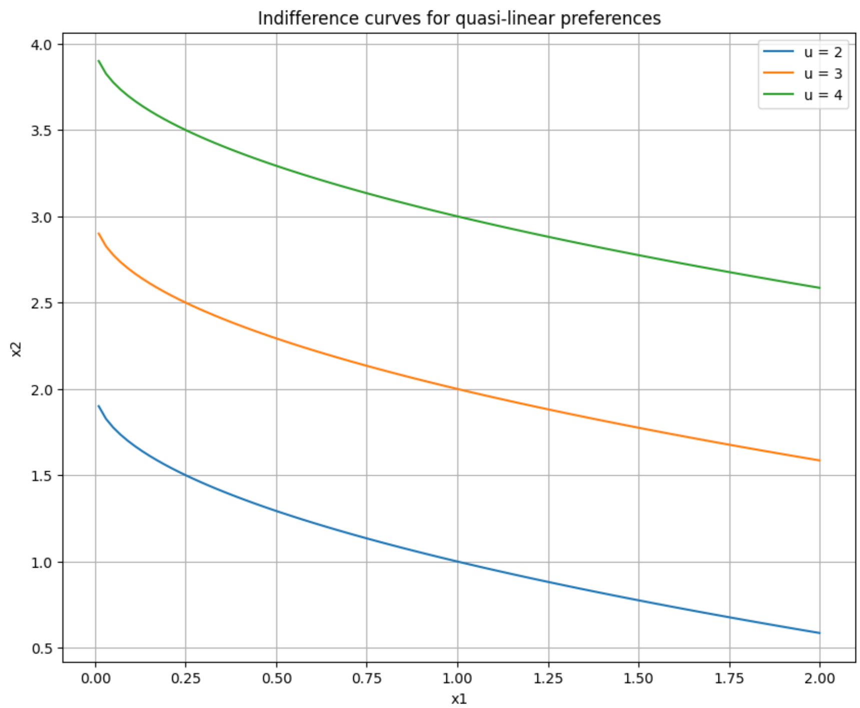

Figure 2.1: The indifference curves

In this graph, the x-axis represents x_1 and the y-axis represents x_2. Different curves represent different levels of utility u. As you can see, the curves are vertical lines for $x_1 = 0$ and flatten as x_1 increases.

This shape of indifference curves is quite unique and contrasts with the more conventional Cobb-Douglas preferences. For Cobb-Douglas preferences, the indifference curves are downward sloping and convex to the origin, indicating a constant rate of trade-off between x_1 and x_2. The marginal rate of substitution between the two goods is not constant but depends on the quantities of both goods.

On the contrary, with these quasi-linear preferences, the marginal rate of substitution is constant and does not depend on the quantity of good 1 consumed. This is shown by the straight line of the indifference curve. The consumer is willing to give up the same amount of x_2 for additional x_1, regardless of how much x_1 they are already consuming. This reflects a constant marginal utility of x_1.

9. The graph is shown below:

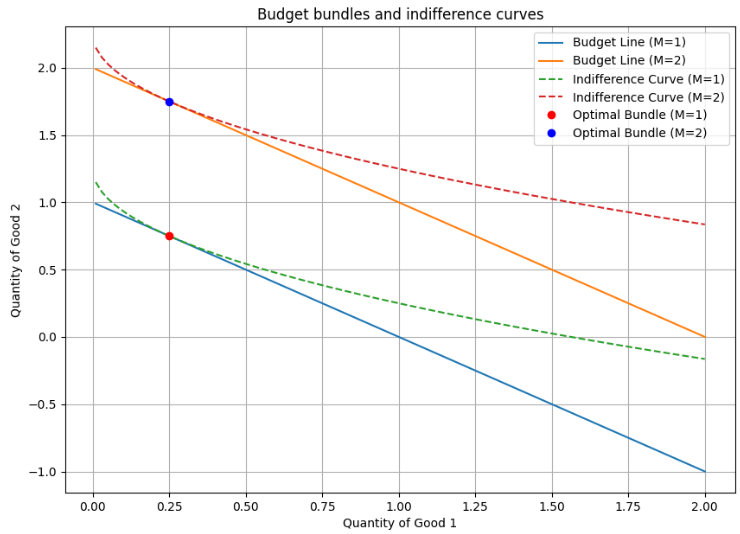

Figure 2.2: Budget bundles and indifference curves

For the given budget lines, the optimal consumption bundles are found by equating the marginal rate of substitution (which is constant at p_1/p_2) to the slope of the budget line. The bundle is at the point where the indifference curve is tangent to the budget line. Since the price ratios are the same in both cases, the optimal quantity of good 1 is the same. The increase in income goes entirely toward good 2, confirming the absence of an income effect for good 1.