Flexible Link Manipulators: Modelling and Control

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Part II

System Modeling (Structurally Flexible Manipulator)

Stress-Strain Relations in the Flexible Manipulator

This section establishes a mathematical relationship between the generalized coordinates, q_i $(for\ i = 2 \dots n)$, of a flexible beam and the strain measurements taken at specific points along the beam.

These generalized coordinates were identified in the preceding section and are important for describing the beam's deformation under various loads.

By linking these coordinates to measurable strain values, we can leverage this relationship for computing the beam's deformation, which is essential for implementing precise control mechanisms in robotics and structural engineering applications.

By expressing the deformation of the beam in terms of its generalized coordinates and relating these to measurable strains, engineers can develop precise control strategies to manipulate and maintain structures under various loads. This approach is foundational in the fields of mechanical and control engineering, particularly within the context of designing flexible and adaptive systems.

When considering a beam with uniform cross-sectional dimensions and homogeneous material properties, the application of a positive bending moment results in the beam experiencing stress.

This stress manifests as compression on one side of the beam's neutral axis and tension on the other.

The magnitude of this stress is proportional to the distance from the neutral axis, reaching its maximum value at the beam's outer surface.

Importantly, the maximum tensile and compressive stresses are equivalent in magnitude but opposite in direction.

This linear stress distribution is a fundamental concept in beam theory, typically visualized in diagrams where y represents the distance from the neutral axis to a point of interest, and c is the maximum distance from the neutral axis to the beam's surface given in Fig. 1

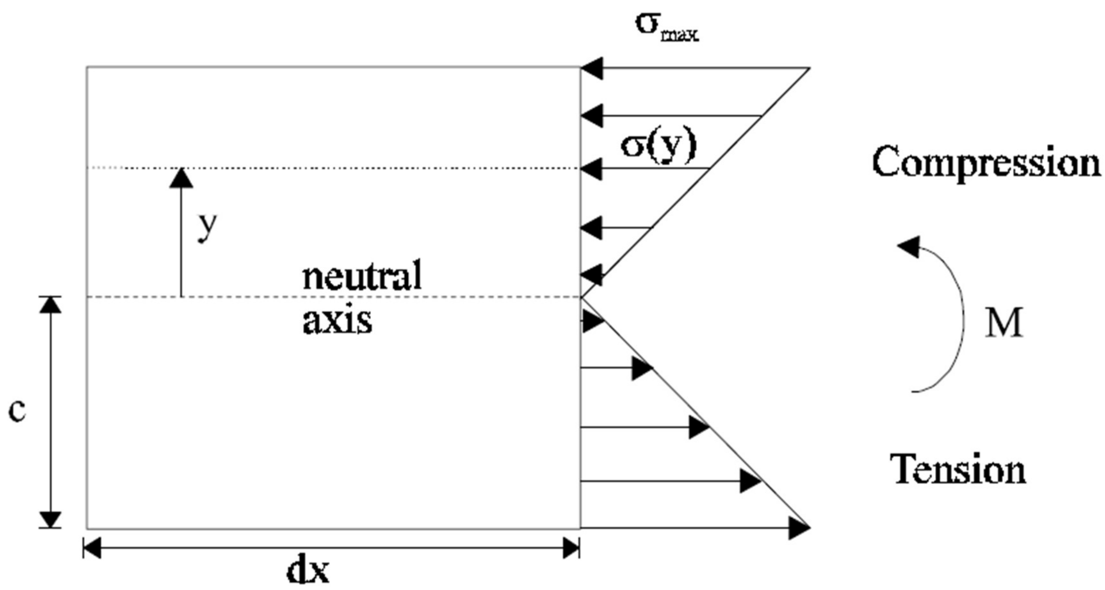

Figure 1: Generic Stress Distribution in a Flexible Beam Segment.

If the maximum stress is denoted by σ_{max} then the stress at any point in the beam is given by:

$$\sigma(y) = -\frac{y}{c}\sigma_{max}$$

This equation correctly relates the stress at a distance y from the neutral axis to the maximum stress σ_{max}, considering c is the distance from the neutral axis to the outermost fiber (where the maximum stress occurs).

The negative sign indicates a linear variation of stress, which changes from tensile to compressive (or vice versa) across the neutral axis. However, in practice, the sign of $\sigma(y)$ is determined by the direction of the bending moment and the orientation of y.

The moment of a force acting over an area of the beam is defined to be

$dM = y.dF = y.\sigma(y).dA$

Integrating dM over the cross-sectional area of the beam to find the total bending moment M

$$M = \int_A^{\cdot} y\,\sigma(y)dA = \int_A^{\cdot} -\frac{y^2}{c}\sigma_{max}dA = -\frac{\sigma_{max}}{c}I$$

$I = \int_A^{\cdot} y^2 dA$ is called the area moment of inertia.

In practice, the minus sign is dropped since the sign of the stress can be found by inspection.

By definition, the strain on the surface of a uniform beam is defined as

$$\epsilon = \frac{change\ in\ len}{original\ lengt}$$

The strain is related to the stress through Young's Modulus, E, as

$\epsilon_{max} = \frac{\sigma_{max}}{E}$, for the maximum stress at the surface, this equation correctly links stress and strain under the assumption of linear elastic behavior of the material.

If equations are combined, it is possible to relate the strain in the beam with the beam shape as

$$\epsilon(x,t) = c\frac{\partial^2 w(x,t)}{\partial x^2} = c\sum_{i=1}^{n} q_i(t)\frac{d^2\phi_i(x)}{dx^2}$$

This equation appears to relate the curvature of the beam, represented by $\frac{\partial^2 w(x,t)}{\partial x^2}$, to the strain.

However, the direct relationship between strain and curvature for a uniform beam under bending is typically given by:

$$\epsilon(x,t) = -\frac{y}{c}\frac{\partial^2 w(x,t)}{\partial x^2} = -\frac{y}{c}\sum_{i=1}^{n} q_i(t)\frac{d^2\phi_i(x)}{dx^2}$$

The strain $\epsilon(x,t)$ is related to the displacement field $w(x,t)$ and its second derivative, which represents the curvature of the beam.

Expanding upon the process of relating strain measurements to the generalized coordinates of a flexible beam, let's delve into how this can be practically achieved using discrete strain measurements and the principles of matrix algebra and the least squares method.

When strain measurements are made at discrete points along a beam, say at locations x_i for $i = 1, \dots, m$, the relationship between these measurements and the generalized coordinates q_i can be formulated in a given matrix form.

$$\begin{bmatrix} \varepsilon(x_1,t) \\ \varepsilon(x_2,t) \\ \vdots \\ \varepsilon(x_m,t) \end{bmatrix} = \begin{bmatrix} c\left.\frac{d^2\phi_1(x)}{dx^2}\right|_{x=x_1} & c\left.\frac{d^2\phi_2(x)}{dx^2}\right|_{x=x_1} & \cdots & c\left.\frac{d^2\phi_n(x)}{dx^2}\right|_{x=x_1} \\ c\left.\frac{d^2\phi_1(x)}{dx^2}\right|_{x=x_2} & c\left.\frac{d^2\phi_2(x)}{dx^2}\right|_{x=x_2} & \cdots & c\left.\frac{d^2\phi_n(x)}{dx^2}\right|_{x=x_2} \\ \vdots & \vdots & \vdots & \vdots \\ c\left.\frac{d^2\phi_1(x)}{dx^2}\right|_{x=x_m} & c\left.\frac{d^2\phi_2(x)}{dx^2}\right|_{x=x_m} & \cdots & c\left.\frac{d^2\phi_n(x)}{dx^2}\right|_{x=x_m} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix}$$

This is particularly useful for capturing the complex deformation patterns of the beam in terms of a finite set of parameters (q_i) that describe its bending and vibrational modes. The strain at a point x_i is related to the curvature of the beam and thus to the second derivatives of the mode shapes are given.

If these measurements are arranged into vectors and matrices, we get:

$\bar{\varepsilon} = \tilde{\phi}\bar{q}$

where $\bar{\varepsilon}$ is a vector of strain measurements at points x_i, $\tilde{\phi}$ is a matrix whose elements are the second derivatives of the mode shapes evaluated at each measurement point $\phi_j(x_i)$, $\bar{q}$ is a vector of the generalized coordinates.

Matrix Inversion and Least Squares Method

The direct inversion of the matrix $\mathbf{\Phi}$ to solve for $\mathbf{q}$ is only possible if $m = n$ and $\mathbf{\Phi}$ is square and non-singular. However, in practical scenarios, the number of measurement points (m) does not always equal the number of generalized coordinates (n).

- **If** $m > n$ (more measurements than modes), the system is overdetermined. In this case, the matrix $\mathbf{\Phi}$ is not square and cannot be inverted using standard methods. However, the least squares method can be applied to find an approximate solution that minimizes the error between the measured strains and those predicted by the model. This is achieved by minimizing the sum of the squares of the differences between the measured and predicted strains, leading to the least squares solution:
  $\mathbf{q} = (\mathbf{\Phi}^T \mathbf{\Phi})^{-1} \mathbf{\Phi}^T \epsilon$
  This solution not only provides the best-fit values of q_i but also helps in reducing the errors in strain measurements by averaging out the noise over multiple measurements.
- **If** $m < n$ (fewer measurements than modes), the system is underdetermined, and there are infinitely many solutions. Additional constraints or information would be required to uniquely determine q_i.

This approach allows for the practical determination of the generalized coordinates of a flexible beam directly from strain measurements, facilitating the modeling and control of the beam's dynamics. By using the least squares method for $m > n$, errors in the measurements can be effectively minimized, improving the accuracy of the model. This methodology is widely applicable in structural health monitoring, adaptive structures, and robotics, where understanding and controlling the dynamic behavior of flexible elements is crucial.

Derivation of the Euler-Lagrange Equations

The Euler-Lagrange equation for a dynamic system is given by:
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q$$
where $L = T - V$ is the Lagrangian of the system, q represents the generalized coordinates (in this case, q_1 and any coordinates representing the beam's deformation), $\dot{q}$ are their time derivatives, and Q represents the generalized forces acting on the system (here, the motor torque τ).

To apply this to the single flexible link:

1. **Formulate** $L = T - V$, incorporating both the rigid body and flexible beam kinetic energies, and the potential energy due to the beam's deformation.
2. **Compute** $\frac{\partial L}{\partial \dot{q}}$**,** $\frac{\partial L}{\partial q}$**, and** $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$ for each generalized coordinate.
3. **Substitute these derivatives into the Euler-Lagrange equation** to derive the equations of motion for q_1 and any variables representing the beam's deformation.
4. **Relate these equations to the inputs and outputs of the system** (τ, q_1, and $\epsilon(a, t)$) to complete the state-space model.

Unconstrained Motion

To elucidate and derive the model for a single flexible link as described in Fig.2, let's break down the process into a more detailed explanation and step-by-step derivation, using the principles of dynamics and control engineering.

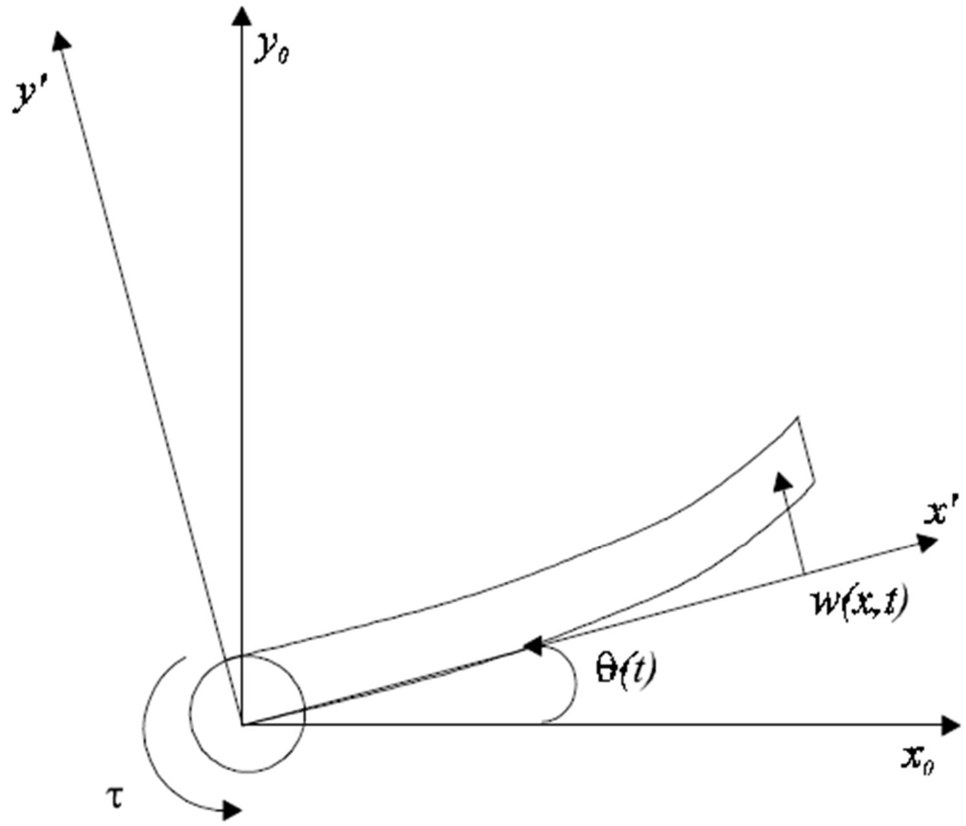

Figure 2: Single flexible link.

The task is to develop a model for a single flexible link that is actuated by a motor but does not interact with the environment. This model aims to capture both the motor's dynamics and the beam's flexibility. The input to this system is the motor torque τ, and the outputs are the motor angle q_1 and the strain at a fixed point on the beam $\varepsilon(a, t)$. The model is derived using Euler-Lagrange equations, which are fundamental in deriving equations of motion for dynamic systems by considering energy contributions.

Kinetic and Potential Energy Contributions

The Euler-Lagrange framework requires expressions for the kinetic (T) and potential (V) energies of the system.

The kinetic energy is divided into two parts: the kinetic energy due to the motor's rotation (T_r), and the kinetic energy due to the beam's elastic deformation.

$T_r = \frac{1}{2} I_h \dot{q}_1^2 + \frac{1}{2} I_b \dot{q}_1^2$

where I_h is the hub inertia of the motor, and $\dot{q}_1$ is the time derivative of the motor angle, indicating its angular velocity, $I_b = \int_0^l x^2 dm$ is the beam inertia.

The flexible beam kinetic energy part accounts for the kinetic energy due to the deformation of the beam. The detailed expression for this component depends on the deformation characteristics of the beam, which would involve integrating over the beam's length and considering its velocity profile due to both translational and rotational movements.

The potential energy primarily comes from the elastic deformation of the beam, which can be modeled based on the beam's material properties and the extent of its deformation. For a linear elastic beam, the potential energy is related to the strain energy stored in the beam due to its bending.

The position of a beam segment in the $x_0 - y_0$, the frame is given by transforming coordinates from the beam's local frame $x' - y'$, to the global frame. This transformation accounts for the rotation q_1 of the beam due to the motor's movement and the beam's deformation $w(x,t)$ (transverse displacement of the bema as a function of position along the beam's length (x) and time (t). The position vector $P(x)$ is given as:

$$P(x) = \begin{bmatrix} \cos(q_1)x - s\quad(q_1)w(x,t) \\ \sin(q_1)x + \cos(q_1)w(x,t) \end{bmatrix}$$

The kinetic energy (dT_f) of an infinitesimal segment of the beam (dm) is given by the expression:

$$dT_f = \frac{1}{2}\dot{P}^T\dot{P}dm$$

Here, $\dot{P}$ represents the derivative of the position vector of the beam segment for time, indicating its velocity. The position vector P incorporates both the translation and rotational movements of the beam segment. For a segment at position x along the beam's length and undergoing a transverse displacement $w(x,t)$, its velocity is influenced by both the motion due to the motor angle (q_1) and the deformation of the beam itself.

$$T_f = \frac{1}{2}\int_0^l \dot{P}^T\dot{P}dm = \frac{1}{2}\int_0^l (x^2 + w^2)\dot{q}_1^2 dm + \frac{1}{2}\int_0^l \dot{w}^2 dm + \frac{1}{2}\int_0^l \dot{w}x\dot{q}_1 dm$$

If it is assumed that the beam is only subjected to small deflections then it is possible to simplify by nothing that $w << x$. This gives the following expression for the kinetic energy of the flexible beam.

$$T_f \cong \frac{1}{2}\int_0^l (x^2\dot{q}_1^2)dm + \frac{1}{2}\int_0^l \dot{w}^2 dm + \frac{1}{2}\int_0^l \dot{w}x\dot{q}_1 dm$$

This approximation acknowledges that the contribution of w to the kinetic energy through $x^2\dot{q}_1^2$ is negligible compared to its direct contribution via $\dot{w}^2$ and the interaction term $\dot{w}x\dot{q}_1$.

Finally, the total kinetic energy of the single flexible link is

T=$T_f + T_r = \frac{1}{2}I_h\dot{q}_1^2 + \frac{1}{2}I_b\dot{q}_1^2 + \frac{1}{2}\int_0^l \dot{w}^2 dm + \frac{1}{2}\int_0^l \dot{w}x\dot{q}_1 dm$

Assuming that the plane in which the link rotates is perpendicular to the gravity vector, then the potential energy due to gravity remains constant and can be ignored.

Considering the beam's deformation, the potential energy (V) due to the elastic deformation is

$V = \frac{1}{2}\int_0^l EI\left(\frac{\partial^2 w}{\partial x^2}\right)dx$

This expression encapsulates the elastic potential energy stored within the beam due to bending, where EI is the flexural rigidity of the beam, and $\frac{\partial^2 w}{\partial x^2}$ is the curvature of the beam due to bending.

With this expression for the energy of the system, the Lagrangian is,

$L = T - V$

This framework allows us to generate dynamic equations that describe the time evolution of the system's states, which in this context are the motor angle (q_1) and the deformation of the beam (w).

Derivation Process

1. **Formulate Kinetic and Potential Energies:** Based on the system's physical characteristics, write down expressions for T and V.
2. **Construct the Lagrangian:** Subtract V from T to obtain L.
3. **Apply Euler-Lagrange Equation:** Use the equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$ for each generalized coordinate (q_1 and w) to derive the equations of motion.

This approach will yield a set of differential equations that capture the dynamics of the single flexible link, considering both its rigid body motion and flexible behavior. The equations of motion can then be used to analyze the system's response to inputs (e.g., motor torque) and design control strategies to achieve desired outcomes (e.g., specific angles or positions).

Assumed Mode Approximation for Beam Deformation

The deformation $w(x,t)$ of the beam is represented by a series expansion in terms of mode shapes $\phi_k(x)$ and time-varying generalized coordinates $q_k(t)$, specifically for the flexible modes of the beam

$$w(x,t) = \sum_{k=2}^{n} \phi_k(x) q_k(t)$$

Here q_2 corresponds to the first mode of oscillation of the beam, as q_1 is reserved for describing the rigid body motion due to the motor's rotation.

For a beam with clamped-free boundary conditions, the differential equation governing the mode shapes is:

$$\frac{d^2\phi_i(x)}{dx^2} = -k_i^2 \phi_i(x), \text{ for } i = 2, \dots, n$$

This represents a simplified model where k_i are constants related to the natural frequencies of the beam's modes.

Also, recall that the frequency of oscillation ω_i for each mode is related to the beam's physical properties and the mode number i:

$$\omega_i = k_i^2 \sqrt{\frac{EI}{\rho A}} \quad \text{for } i = 2, \dots, n$$

Replacing dm with ρdx; (integrating to the beam's density and length, we define:

$$\gamma_i = \int_0^l x\phi_i(x)dx, \text{ for } i: 2, \dots, n$$

This integration gives a measure of the contribution of each mode shape to the beam's kinetic energy and dynamics.

Simplification of the Lagrangian

Using the above definitions and the orthogonality condition, the Lagrangian can be simplified to

$$L = \frac{1}{2}(I_h + I_b)\dot{q}_1^2 + \frac{\rho}{2}\sum_{k=2}^{n} \dot{q}_k^2(t) + \rho\dot{q}_1 \sum_{k=2}^{n} \dot{q}_k(t)\gamma_k - \frac{\rho}{2}\sum_{k=2}^{n} \omega_k^2 q_k^2(t)$$

Using the Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i \text{ , for } i = 1, \dots, n$$

τ_i is the generalized torque, gives the dynamic equations of the single flexible link

$$\ddot{q}_1 = \frac{\tau + \sum_{k=2}^{n} \rho\omega_k^2\gamma_k q_k}{I_h}$$

$$\ddot{q}_k = \frac{-\tau\gamma_k}{I_h} - q_k^2\omega_k^2\left[1 + \frac{\rho\gamma_k^2}{I_h}\right] - \sum_{j\neq k; j=2}^{n} \frac{\rho q_j^2\omega_j^2\gamma_j\gamma_k}{I_h}, \quad \text{for } k = 2, \dots, n$$

These dynamic equations encapsulate the motion of the single flexible link, including the rigid body rotation induced by the motor torque and the flexible deformations characterized by the mode shapes and frequencies. The generalized torque reflects the external inputs to the system, which, for $i = 1$, is the motor torque, $i = 2, \dots, n$, represents the effective torque due to the beam's flexibility.

This detailed derivation presents a comprehensive model for the dynamics of a single flexible link, capturing both rigid and flexible behaviors through the Euler-Lagrange formalism. Such models are crucial for designing control systems in robotics, particularly for applications involving flexible structures where precision and responsiveness are paramount. The analytical approach enables engineers to predict system behavior under various conditions and to devise strategies for control and stabilization.

For the experimental setup with the assumption that the flexible beam exhibits only one significant mode of oscillation, simplifying the analysis to $n = 2$ streamlines the model to focus on this primary mode. This simplification leads to a specific state space representation, detailing the dynamics of both the motor angle (q_1) and the mode associated with the flexible deformation (q_2)

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{\rho\omega_2^2\gamma_2}{I_h} & 0 \\ 0 & 0 & -\omega_2^2\left(1+\dfrac{\rho\gamma_2^2}{I_h}\right) & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{I_h} \\ \frac{-\gamma_2}{I_h} \end{bmatrix} \tau$$

$$\begin{bmatrix} q_1 \\ \varepsilon(a) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & c\left.\dfrac{d^2\phi_2(x)}{dx^2}\right|_{x=a} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tau$$

where the strain gauge is mounted at position a, measured from the clamped end, on the flexible beam.

1. **For $\ddot{q}_1$:**
   $\ddot{q}_1 = \frac{\tau + \rho\omega_2^2\gamma_2 q_2}{I_h}$
2. **For $\ddot{q}_2$:**
   $\ddot{q}_2 = -\frac{\tau\gamma_2}{I_h} - q_2^2\omega_2^2\left(1 + \frac{\rho\gamma_2^2}{I_h}\right)$

These equations represent the second derivatives of q_1 and q_2, indicating the accelerations for both states within the system. Here, τ represents the external torque applied, ρ is a constant related to the system's parameters, ω_2 is the natural frequency of the second mode, γ_2 is a damping factor associated with the second mode, I_h is the moment of inertia for q_1, and q_2 is the position (or angular displacement) of the second mode.

The term $\frac{\rho\omega_2^2\gamma_2}{I_h}$ in the third row reflects the coupling effect of the beam's oscillation mode on the motor's angular acceleration, whereas the term $-\omega_2^2\left(1+\frac{\rho\gamma_2^2}{I_h}\right)$ in the fourth row captures the natural dynamics of the beam's oscillation and its interaction with the motor dynamics through γ_2.

This model encapsulates the dynamics of the system, including the interaction between the motor's rotation and the beam's first mode of oscillation, influenced by the input torque.

The outputs were chosen because they are the actual measured values on the apparatus. One of the more common things to control in flexible robotics is the tip position. This quantity is quite important in manipulation tasks and can be easily calculated from the two measured quantities of the above matrix equation. Denoting the tip position by, y, it is possible to write

$$y = lq_1 + w(l,t)$$

which with only one significant flexible mode gives the output

$$y = lq_1 + \phi_2(l)\left(c\left.\frac{d^2\phi_2}{dx^2}\right|_{x=a}\right)^{-1}\varepsilon(a)$$

This expression links the measurable quantities to the tip position, facilitating control over the beam's end effector.

The choice of output influences the passivity of the system's transfer function, which is crucial for ensuring stability under the selected control technique. This modified output $y_r = lq_1 - w(l,t)$ leads to a system that, when considering the transfer function from torque to y_r, exhibits a well-defined relative degree and passivity for small hub inertias. This adjustment ensured that the control system could be designed to guarantee stability, especially important in applications requiring precise manipulation and interaction with environments.

Simplifying the model to focus on the primary mode of oscillation allows for a clearer understanding of the system's dynamics and how they can be manipulated through input torque. The careful selection of outputs and consideration of the system's passivity are crucial in designing effective control strategies. This approach demonstrates the intricacies of modeling and controlling flexible robotic systems, highlighting the balance between theoretical models and practical considerations in engineering design and application.

Rotating Flexible Beam with Moving Particle

Assumptions and Scenario Breakdown

The text describes a dynamic system consisting of a flexible beam and a particle moving along this beam. The key assumptions and elements of the system are as follows:

1. **Flexible Beam Characteristics:**
   - **Mass density (ρ):** This refers to the mass per unit length of the beam.
   - **Cross-sectional area (A):** The area of the beam's cross-section, which, along with the mass density, determines the beam's mass distribution.
   - **Young's modulus (E):** A measure of the stiffness of the beam's material.
   - **Length (L):** The total length of the beam.
   - **Moment of inertia (I):** This refers to the beam's cross-sectional area moment of inertia, which affects its resistance to bending.
2. **Beam Dynamics:**
   - The beam can rotate in the vertical plane, experiencing arbitrary rotations and rotation rates, under the influence of gravity and external torque (τ) at its base. The base is rigidly attached to a rotating reference frame, adding complexity to the dynamic analysis due to the non-inertial effects.
3. **Particle Dynamics:**
   - A particle of mass (m) moves along the beam with arbitrary displacement (u) and velocity ($\dot{u}$) relative to the beam, adding to the system's dynamic interaction.
4. **Elastic Deformations:**
   - Small elastic deformations of the beam are denoted by δ, indicating that linear elasticity theories can be applied for simplification.

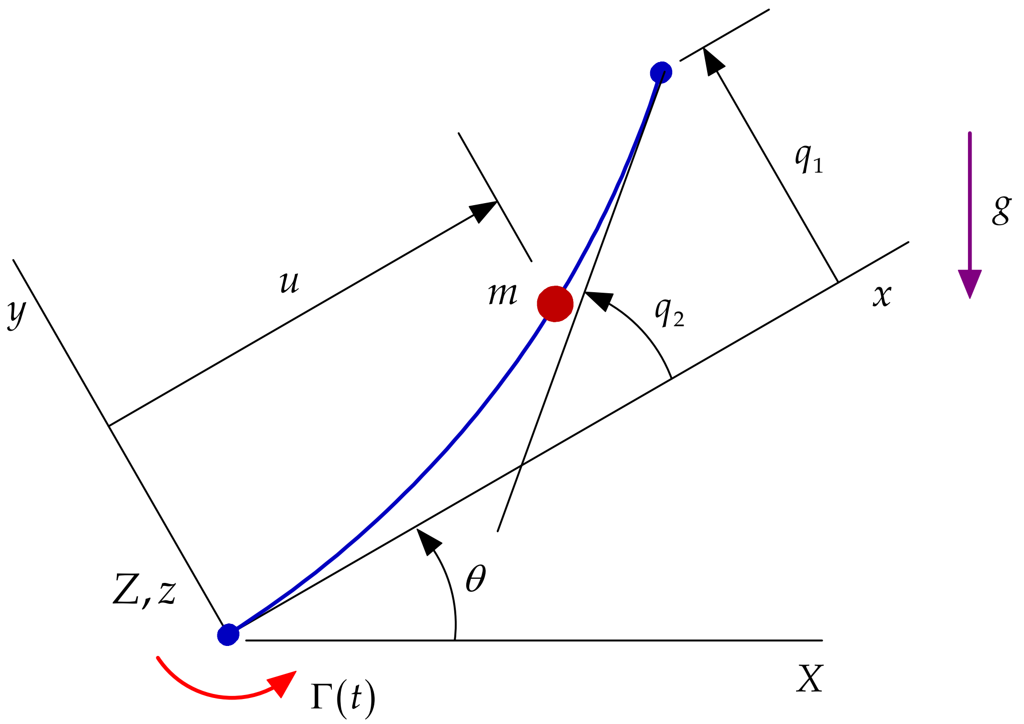

Figure 3: Rotating Flexible Beam with Moving Particle.

Mathematical Modeling

To derive the symbolic representations and mathematical models for this system, we'll proceed with the following steps:

Beam Dynamics Model

1. **Euler-Bernoulli Beam Theory:**
   - We start with the Euler-Bernoulli beam theory for small deformations, which provides a relation between the bending moment (M) and the curvature ($\frac{d^2\delta}{dx^2}$) of the beam as:
     $M = EI\frac{d^2\delta}{dx^2}$
   - For a rotating beam, the bending moment will also include contributions from centrifugal forces due to rotation and the particle's mass.
2. **Rotational Dynamics:**
   - The rotation of the beam introduces Coriolis and centrifugal forces, which must be accounted for in the model. The external torque (τ) and the gravitational force will also influence the beam's rotational motion.

Particle Dynamics Model

1. **Particle Motion Relative to the Beam:**
   - The motion of the particle along the beam introduces additional forces and moments. The kinetic energy of the particle can be expressed as a function of its displacement (u) and velocity ($\dot{u}$) relative to the beam.

System Equations

1. **Lagrangian Mechanics:**
   - We can use Lagrangian mechanics to formulate the equations of motion for the entire system. The Lagrangian (L) is defined as the difference between the kinetic (T) and potential energy (V) of the system:
     $L = T - V$
   - Applying the Euler-Lagrange equation for each degree of freedom (δ for the beam and u for the particle) gives us:
     $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$
     where q_i represents the generalized coordinates (δ, u) and Q_i represents the non-conservative forces or moments (like external torque).

Deriving Symbolic Representations

To derive the specific symbolic representations, we follow these steps:

Define the Kinetic and Potential Energy:

For the beam and particle, consider rotational and translational motions, gravitational potential energy, and the energy due to elastic deformations.

Apply the Euler-Lagrange Equation:

For each degree of freedom, derive the equations of motion considering the contributions from external forces and moments.

Linearize the Equations:

Assuming small deformations allows us to linearize the equations around the equilibrium position, simplifying the analysis.

This approach leads to a set of differential equations that describe the dynamics of the beam and the particle under the given assumptions. The solution of these equations requires numerical methods, especially for complex boundary conditions and non-linear dynamics introduced by arbitrary rotations and velocities.

The detailed analysis and derivation process outlined above provides a comprehensive mathematical model for understanding the dynamics of a flexible beam and a particle moving along it in a rotating reference frame.

This model incorporates the complexities of real-world applications in robotics and mechanical engineering, such as non-linear dynamics, material properties, and the influence of external forces and torques.

The equations of motion have been obtained using Lagrange's method and a finite element approximation of the beam deformation.

The equations were formulated using the Symbolic Toolbox of Matlab.

The equations of motion are given as:

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{u} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ u \\ \theta \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{Bmatrix}$$

where

$$m_{11} = \frac{13}{35}\rho AL + m\left(9\frac{u^4}{L^4} - 12\frac{u^5}{L^5} + 4\frac{u^6}{L^6}\right)$$

$$m_{12} = m_{21} = -\frac{11}{210}\rho AL^2 + m\left(-3\frac{u^4}{L^3} + 5\frac{u^5}{L^4} - 2\frac{u^6}{L^5}\right)$$

$$m_{13} = m_{31} = 0$$

$$m_{14} = m_{41} = \frac{7}{20}\rho AL^2 + m\left(3\frac{u^3}{L^2} - 2\frac{u^4}{L^3}\right)$$

$$m_{22} = \frac{1}{105}\rho AL^3 + m\left(\frac{u^4}{L^2} - 2\frac{u^5}{L^3} + \frac{u^6}{L^4}\right)$$

$$m_{23} = m_{32} = 0$$

$$m_{24} = m_{42} = -\frac{1}{20}\rho AL^3 + m\left(-\frac{u^3}{L} + \frac{u^4}{L^2}\right)$$

$$m_{33} = m$$

$$m_{34} = m_{43} = m\left[\frac{u^2}{L} + \left(-3\frac{u^2}{L^2} + 2\frac{u^3}{L^3}\right)q_1 - \frac{u^3}{L^2}q_2\right]$$

$$m_{44} = \frac{1}{3}\rho AL^4 + \frac{\rho AL}{105}\left(39q_1^2 - 11Lq_1q_2 + L^2q_2^2\right)$$
$$+m\begin{bmatrix} u^2 + \left(9\frac{u^4}{L^4} - 12\frac{u^5}{L^5} + 4\frac{u^6}{L^6}\right)q_1^2 \\ + \left(-6\frac{u^4}{L^3} + 10\frac{u^5}{L^4} - 4\frac{u^6}{L^5}\right)q_1q_2 \\ + \left(\frac{u^4}{L^2} - 2\frac{u^5}{L^3} + \frac{u^6}{L^4}\right)q_2^2 \end{bmatrix}$$

$$k_{11} = 12\frac{EI_z}{L^3}$$

$$k_{12} = k_{21} = -6\frac{EI_z}{L^2}$$

$$k_{22} = 4\frac{EI_z}{L}$$

$$k_{13} = k_{31} = k_{14} = k_{41} = k_{23} = k_{32} = k_{24} = k_{42} = k_{34} = k_{43} = k_{33} = k_{44} = 0$$

$$r_1 = -\frac{\rho AL}{210}\dot{\theta}^2\left(-78q_1 + 11Lq_2\right)$$
$$-m\left\{\begin{aligned} &\dot{u}\begin{bmatrix} \left(36\frac{u^3}{L^4} - 60\frac{u^4}{L^5} + 24\frac{u^5}{L^6}\right)\dot{q}_1 \\ + \left(-12\frac{u^3}{L^3} + 25\frac{u^4}{L^4} - 12\frac{u^5}{L^5}\right)\dot{q}_2 \\ + \left(12\frac{u^2}{L^2} - 10\frac{u^3}{L^3}\right)\dot{\theta} \end{bmatrix} \\ &+\dot{\theta}^2\begin{bmatrix} \left(-9\frac{u^4}{L^4} + 12\frac{u^5}{L^5} - 4\frac{u^6}{L^6}\right)q_1 \\ + \left(3\frac{u^4}{L^3} - 5\frac{u^5}{L^4} + 2\frac{u^6}{L^5}\right)q_2 \end{bmatrix} \end{aligned}\right\}$$
$$-g\cos\theta\left[\frac{\rho AL}{2} + m\left(3\frac{u^2}{L^2} - 2\frac{u^3}{L^3}\right)\right]$$

$$
\begin{aligned}
r_2 = & -\frac{\rho A L^2}{210}\dot{\theta}^2\left(11q_1 - 2Lq_2\right) \\
& -m\left\{\begin{aligned} & \dot{u}\left[\begin{aligned} & \left(-12\frac{u^3}{L^3}+25\frac{u^4}{L^4}-12\frac{u^5}{L^5}\right)\dot{q}_1 \\ & +\left(4\frac{u^3}{L^2}-10\frac{u^4}{L^3}+6\frac{u^5}{L^4}\right)\dot{q}_2 \\ & +\left(-4\frac{u^2}{L}+5\frac{u^3}{L^2}\right)\dot{\theta} \end{aligned}\right] \\ & +\dot{\theta}^2\left[\begin{aligned} & \left(3\frac{u^4}{L^3}-5\frac{u^5}{L^4}+2\frac{u^6}{L^5}\right)q_1 \\ & +\left(-\frac{u^4}{L^2}+2\frac{u^5}{L^3}-\frac{u^6}{L^4}\right)q_2 \end{aligned}\right] \end{aligned}\right\} \\
& -g\cos\theta\left[-\frac{\rho A L^2}{12}+m\left(-\frac{u^2}{L}+\frac{u^3}{L^2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
r_3 = & -m\left\{\begin{aligned} & \left(-18\frac{u^3}{L^4}+30\frac{u^4}{L^5}-12\frac{u^5}{L^6}\right)\dot{q}_1^2 \\ & +\left(12\frac{u^3}{L^3}-25\frac{u^4}{L^4}+12\frac{u^5}{L^5}\right)\dot{q}_1\dot{q}_2 \\ & +\left(-2\frac{u^3}{L^2}+5\frac{u^4}{L^3}-3\frac{u^5}{L^4}\right)\dot{q}_2^2+\dot{\theta}\left[\begin{aligned} & \left(-12\frac{u^2}{L^2}+10\frac{u^3}{L^3}\right)\dot{q}_1 \\ & +\left(4\frac{u^2}{L}-5\frac{u^3}{L^2}\right)\dot{q}_2 \end{aligned}\right] \\ & +\dot{\theta}^2\left[\begin{aligned} & -u+\left(-18\frac{u^3}{L^4}+30\frac{u^4}{L^5}-12\frac{u^5}{L^6}\right)q_1^2 \\ & +\left(12\frac{u^3}{L^3}-25\frac{u^4}{L^4}+12\frac{u^5}{L^5}\right)q_1q_2 \\ & +\left(-2\frac{u^3}{L^2}+5\frac{u^4}{L^3}-3\frac{u^5}{L^4}\right)q_2^2 \end{aligned}\right] \end{aligned}\right\} \\
& -mg\left\{\sin\theta+\cos\theta\left[\left(6\frac{u}{L^2}-6\frac{u^2}{L^3}\right)q_1+\left(-2\frac{u}{L}+3\frac{u^2}{L^2}\right)q_2\right]\right\}
\end{aligned}
$$

$$r_4 = \frac{\rho AL}{105}\dot{\theta}\left[\left(78q_1 - 11Lq_2\right)\dot{q}_1 + \left(-11Lq_1 + 2L^2q_2\right)\dot{q}_2\right]$$

$$-m\left\{\begin{array}{l} \dot{u}^2\left[\begin{array}{l}\left(-6\frac{u}{L^2}+6\frac{u^2}{L^3}\right)q_1 \\ +\left(2\frac{u}{L}+3\frac{u^2}{L^2}\right)q_2\end{array}\right] \\ +\dot{u}\left[\begin{array}{l}\left(6\frac{u^2}{L^2}-6\frac{u^3}{L^3}\right)\dot{q}_1 \\ +\left(-2\frac{u^2}{L}+3\frac{u^3}{L^2}\right)\dot{q}_2\end{array}\right] \\ +2\dot{u}\dot{\theta}\left[\begin{array}{l}u+\left(18\frac{u^3}{L^4}-30\frac{u^4}{L^5}+12\frac{u^5}{L^6}\right)q_1^2 \\ +\left(-12\frac{u^3}{L^3}+25\frac{u^4}{L^4}-12\frac{u^5}{L^5}\right)q_1q_2 \\ +\left(2\frac{u^3}{L^2}-5\frac{u^4}{L^3}+3\frac{u^5}{L^4}\right)q_2^2\end{array}\right] \\ +2\dot{\theta}\dot{q}_1\left[\begin{array}{l}\left(9\frac{u^4}{L^4}-12\frac{u^5}{L^5}+4\frac{u^6}{L^6}\right)q_1 \\ +\left(-3\frac{u^4}{L^3}+5\frac{u^5}{L^4}-2\frac{u^6}{L^5}\right)q_2\end{array}\right] \\ +2\dot{\theta}\dot{q}_2\left[\begin{array}{l}\left(-3\frac{u^4}{L^3}+5\frac{u^5}{L^4}-2\frac{u^6}{L^5}\right)q_1 \\ \left(\frac{u^4}{L^2}-2\frac{u^5}{L^3}+\frac{u^6}{L^4}\right)q_2\end{array}\right]\end{array}\right\}$$

$$-\frac{\rho AL}{12}g\left[6L\cos\theta + \left(-6q_1 + Lq_2\right)\sin\theta\right]$$

$$-mg\left\{u\cos\theta + \left[\left(-3\frac{u^2}{L^2}+2\frac{u^3}{L^3}\right)q_1 + \left(\frac{u^2}{L}-\frac{u^3}{L^2}\right)q_2\right]\sin\theta\right\}$$

Important Note:

a) All terms that contain second-order elastic deformation terms of the form q_iq_j are much smaller than other terms.

b) Terms involving $u, \dot{u}$ are generally not small.