Flexible Link Manipulators: Modelling and Control

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Part I

Dynamic Modeling (Structurally Flexible Manipulator)

The actual system dynamics are infinite-dimensional because of the distributed link flexibility. To simplify the problem a conventional method is to approximate the flexible-link system by a system with a finite number of degrees of freedom.

Neglecting second-order effects (rotary inertia, shear deformation, actuator dynamics)

In the domain of space applications, the design paradigm significantly shifts towards prioritizing low-mass structures. This shift is not merely a preference but a critical requirement to achieve escape velocity and ensure missions are accomplished with optimal fuel economy. This necessity introduces a complex interplay between the need for structural rigidity to limit degrees of freedom (DOF) and the desire for structural flexibility to accommodate a wide range of tasks. Furthermore, the design must also consider the prevention of damage to manipulator systems in the event of collisions, alongside the goal of enhancing productivity through high-speed manipulation. Achieving a stable closed-loop system that is lightweight and consumes less energy becomes paramount.

However, the pursuit of flexibility introduces significant challenges in control dynamics, especially when the objective is to maintain precise and stable control over the manipulator's tip. This challenge is compounded by the need to account for deformations resulting from the arms' flexibility within the dynamic equations, complicating both the analysis and the design of control laws.

A particularly daunting task in managing flexible link manipulators arises when attempting to track a specific trajectory of the tip position by applying torques at the actuation ends. Here, we encounter the concept of zero dynamics, which refers to the internal dynamics of the system when the outputs are nullified by specific inputs. A system exhibits non-minimum phase characteristics when these dynamics are unstable, severely limiting the ability of causal controllers to achieve exact asymptotic tracking of desired tip trajectories.

Addressing the issues of flexibility fundamentally requires materials that combine a high modulus of elasticity, low mass density, and high structural damping. These material properties offer a potential solution to the inherent problems presented by flexibility.

From a theoretical standpoint, the dynamic equations governing such systems are infinite-dimensional, described by partial differential equations (PDEs). This complexity arises from the requirement of an infinite number of coordinates to kinematically describe each link. However, for practical control system design, an infinite-dimensional model is untenable due to its complexity and the band-limited nature of sensors and actuators. Consequently, the modeling phase often involves truncating the number of flexible coordinates to manage this complexity.

Nonetheless, the resultant dynamics are characterized by a highly nonlinear and coupled set of differential equations, posing significant challenges in controller design. A critical aspect of this design process involves determining the number and type of sensing points required. Given that the primary control objectives include managing the tip positions and their rates of change, the controller must receive accurate information regarding these parameters. Although tip position data can be acquired through camera vision systems or strain gauge measurements, obtaining direct measurements of tip rates poses additional challenges.

Introduction to the Dynamic Model

The system in question comprises a flexible beam directly connected to a motor.

The objective is to develop a dynamic model that captures the system's behavior under motion.

To achieve this, we leverage the Euler-Lagrange equations.

These equations are foundational in deriving the equations of motion for a wide range of mechanical systems.

They are predicated on the principle of stationary action, utilizing the system's energy (kinetic and potential) to formulate the motion equations.

The Euler-Lagrange equations are expressed as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

where L is the Lagrangian of the system, defined as $L = T - V$, with T representing the kinetic energy and V the potential energy.

The variables q_i are the generalized coordinates, and $\dot{q}_i's$ are their time derivatives.

Q_i represents generalized forces acting on the system.

The generalized coordinates q_i are crucial as they encapsulate the system's configuration with the minimum number of independent variables.

Generalized Coordinates and Flexible Beam Dynamics

For our flexible beam, the generalized coordinates are associated with the beam's strain at specific points.

These points' strain can be quantitatively determined using strain gauges, thus offering a method to indirectly measure the beam's flexible coordinates.

This choice of generalized coordinates is instrumental in simplifying the model while ensuring it adequately describes the system's physical state.

Bernoulli-Euler Beam Theory

To model the beam's behavior, we adopt the Bernoulli-Euler beam theory, focusing on transverse vibrations.

This theory simplifies the beam's dynamic analysis by neglecting torsional and damping effects, which are generally minor for many practical applications.

The beam's displacement at any point x and time t is denoted by $w(x,t)$, representing how far the beam deviates from its original, unstressed position along the x −axis.

Mathematical Representation of Beam Segment Dynamics

Consider a segment of the beam with a small width dx at position x.

This segment experiences a shearing force $V(x)$ and a bending moment $M(x)$.

The differential equations governing the beam's motion, derived from the Bernoulli-Euler beam equation, are given by:

$$EI\frac{\partial^2 w(x,t)}{\partial x^2} = M(x)$$

$$\frac{\partial V}{\partial x} + Q = \rho A\frac{\partial^2 w(x,t)}{\partial t^2}$$

where EI is the flexural rigidity of the beam, Q is the distributed load per unit length (if any), and ρA is the mass per unit length of the beam.

The structure of the flexible beam is given in Fig.1

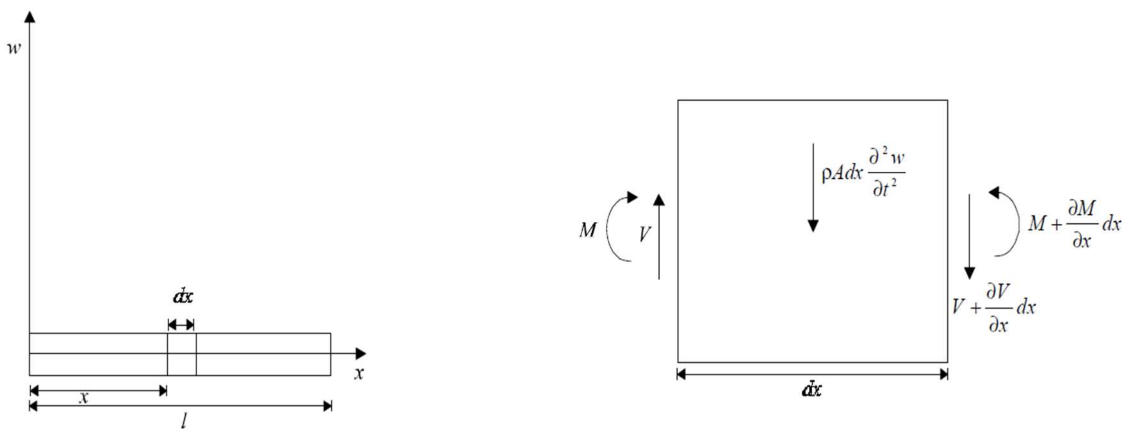

Figure 1: Structure of the Flexible Beam.

The beam's behavior is influenced by forces and moments acting along its length. The primary goal is to develop an equation of motion for the beam, which is essential for designing control systems that can precisely manage its movements.

Force and Moment Analysis

Consider a beam segment of infinitesimal length dx at a position x. The forces and moments acting on this segment can vary along the length of the beam due to applied loads, the beam's weight, or other external factors.

Shearing Force Variation: At the position $x + dx$, the shearing force $V(x + dx)$ can be expressed as:

$$V(x + dx) = V(x) + \frac{\partial V}{\partial x} dx$$

The expression accounts for the incremental change in the shearing force over the small distance dx.

Moment Force Variation: Similarly, the bending moment $M(x + dx)$ at the position $x + dx$ is given by:

$M(x + dx) = M(x) + \frac{\partial M}{\partial x} dx$

Here $\frac{\partial M}{\partial x} dx$ represents the change in the moment over the segment dx.

Equilibrium Conditions

Force Equilibrium: By summing forces along the segment, considering the shearing force and inertial force due to acceleration, we have:

$V - \left(V + \frac{\partial V}{\partial x} dx\right) - \rho dx \frac{\partial^2 w}{\partial t^2} = 0$

Simplifying, we obtain the differential equation relating shearing force, mass density, and acceleration:

$\frac{\partial V}{\partial x} + \rho \frac{\partial^2 w}{\partial t^2} = 0$

Moment Equilibrium: The moment equilibrium around one end of the segment yields:

$-V dx + \frac{\partial M}{\partial x} dx = 0$

Introducing the moment-curvature relationship $M = EI \frac{\partial^2 w}{\partial x^2}$, where EI is the flexural rigidity, leads to a crucial insight into how bending moments relate to beam curvature.

Deriving the Bernoulli-Euler Beam Equation

By substituting the $M = EI \frac{\partial^2 w}{\partial x^2}$ into the moment equilibrium condition and integrating the force equilibrium, we can combine these relationships to arrive at the Bernoulli-Euler beam equation.

$\rho A \frac{\partial^2 w}{\partial t^2} + \mathrm{EI} \frac{\partial^4 w}{\partial x^4} = 0$

The equation succinctly captures the dynamics of a vibrating beam, relating its transverse displacement $w(x, t)$ to its material properties (ρ, EI) and the forces acting upon it. The term $\rho A \frac{\partial^2 w}{\partial t^2}$ represents the inertial effects, while $\mathrm{EI} \frac{\partial^4 w}{\partial x^4}$ encapsulates the beam's resistance to bending.

The Bernoulli-Euler beam equation is a foundational component in the analysis and design of flexible structures and robotic manipulators.

It provides a mathematical model that enables engineers to predict the behavior of beams under various loading conditions, which is crucial for ensuring stability, precision, and efficiency in robotic systems.

This detailed analysis and the derived equation highlight the importance of understanding the interplay between physical forces and the structural characteristics of materials.

Understanding Boundary Conditions

Boundary conditions (BCs) are essential for solving differential equations, as they specify the behavior of a physical system at the boundaries of the domain of interest. For the Bernoulli-Euler beam equation, which governs the transverse vibrations of a beam, these conditions relate to the beam's displacement $w(x,t)$ and its derivatives at the ends of the beam ($x = 0$ and $x = l$).

Physical Interpretation of $w(x,t)$ and Its Derivatives

$w(x,t)$: Transverse displacement of the beam at point x and time t.

$\frac{\partial w}{\partial x}$ (First spatial derivative): Slope of the beam, indicating how the angle of the beam changes concerning x.

$\frac{\partial^2 w}{\partial x^2}$ (Second spatial derivative): Bending moment, reflecting the internal moment that causes the beam to bend.

$\frac{\partial^3 w}{\partial x^3}$ (Third spatial derivative): Shearing force, representing the internal force that causes one segment of the beam to slide over an adjacent segment.

Pinned-Free and Clamped-Free Boundary Conditions

In the context of a flexible beam mounted as the last link in a rigid robot, two common sets of boundary conditions are used: pinned-free and clamped-free.

Pinned-free conditions: This setup assumes the beam is attached to a rigid body with a pinned joint, allowing rotation but not translation at the joint. The x −axis of the beam's coordinate system, originating at the pinned end, runs through the beam's center of mass for all t.

Clamped-free conditions: Here, the x −axis remains tangent to the beam at the clamped end for all t, implying the beam cannot rotate or translate at the clamped end.

The flexible beam reference frames are represented in Fig.2

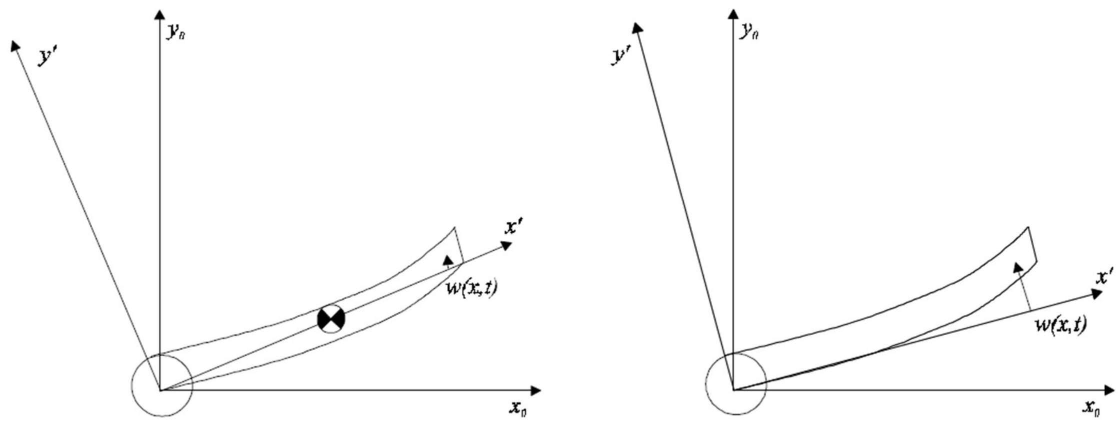

Figure 2: Flexible Beam Reference Frames.

Mathematical Formulation of Clamped-Free Boundary Conditions

Given the decision to focus on clamped-free BCs, the conditions can be mathematically expressed as follows for the beam clamped at $x = 0$ and free at $x = l$:

1. **At the clamped end ($x = 0$):**
   - The beam's transverse displacement is zero: $w(0, t) = 0$.
   - The slope of the beam is zero: $\left.\frac{\partial w}{\partial x}\right|_{x=0} = 0$.
2. **At the free end ($x = l$):**
   - The bending moment is zero: $\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=l} = 0$, indicating no internal moment at the free end.
   - The shearing force is zero: $\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=l} = 0$, indicating no internal shear at the free end.

These boundary conditions set the stage for solving the Bernoulli-Euler beam equation, allowing for the determination of the beam's dynamic response under various loading conditions. Specifically, they ensure that at the clamped end, the beam is immovable and has no initial slope, reflecting the constraints of being rigidly fixed. At the free end, the absence of bending moment and shearing force reflects the physical reality that no external forces or moments are applied at this boundary.

This framework is pivotal in analyzing and designing control systems for robotic manipulators with flexible elements. It facilitates the prediction of how the flexible beam will behave under specific conditions, which is crucial for ensuring precise control and operation of robotic systems.

To thoroughly explain and derive the solution to the Bernoulli-Euler beam equation, let's proceed with a step-by-step analysis, incorporating the concept of separation of variables to tackle this partial differential equation (PDE). This approach is fundamental in mechanical engineering and control systems, especially when dealing with the dynamics of flexible structures such as beams in robotics.

Assumption of Separable Solutions

The key assumption here is that the solution of the Bernoulli-Euler equation can be decomposed into a product of two functions, one depending only on spatial variables (x) and the other only on time (t):

$$w(x,t) = \phi(x)q(t)$$

This assumption is based on the principle that the beam's displacement at any point and time can be represented as the product of a spatial function, describing the shape of the beam's deflection, and a temporal function, describing how this deflection varies over time.

Rewriting the Bernoulli-Euler Equation

Given the Bernoulli-Euler beam equation in its standard form,

$$\rho A \frac{\partial^2 w}{\partial t^2} + \mathrm{EI} \frac{\partial^4 w}{\partial x^4} = 0$$

and substituting the assumed solution $w(x,t) = \phi(x)q(t)$,

we separate the variable, leading to:

$$EIq(t)\frac{d^4\phi(x)}{dx^4} = -\rho A\phi(x)\frac{d^2q(t)}{dt^2}$$

$$\frac{EI}{\rho A}\frac{1}{\phi(x)}\frac{d^4\phi(x)}{dx^4} = -\frac{1}{q(t)}\frac{d^2q(t)}{dt^2} = \omega^2$$

where ω is a separation constant that is equal on both sides of the equation, ensuring that each side is constant and thus facilitating the separation of variables.

Solution to the Temporal Part

Solving for $q(t)$ with respect to the temporal derivative gives us a second-order differential equation, which has a well-known solution in terms of harmonic functions:

$$q(t) = A\cos(\omega t) + B\sin(\omega t)$$

where A and B are constants determined by the initial conditions of the system.

Solution to the Spatial Part

Solving for $\phi(x)$ with respect to the spatial derivative involves finding the roots of the characteristic equation derived from the spatial part of the separation:

$$\phi(x) = C_1\sin(kx) + C_2\cos(kx) + C_3\sinh(kx) + C_4\cosh(kx)$$

where $k = \sqrt{\frac{\rho A}{EI}}\omega$ is the wave number, related to the physical and geometric properties of the beam as well as the separation constant ω.

The solution to the Bernoulli-Euler beam equation, as derived through the separation of variables, illustrates how the displacement of a beam under transverse vibrations can be modeled as the product of spatial and temporal functions. This approach is instrumental in understanding the dynamic behavior of flexible structures in robotics, allowing for the prediction and control of their movements under various conditions. The harmonic nature of the temporal solution reflects the oscillatory behavior of the beam, while the spatial solution encompasses both trigonometric and hyperbolic functions, capturing the complex deflection shapes possible under different boundary conditions. This detailed analysis serves as a foundation for designing control strategies and predicting the performance of robotic manipulators equipped with flexible links.

Continuing from where we left off, to apply the clamped-free boundary conditions to the spatial part of the solution $\phi(x)$ and further analyze the resulting implications for the Bernoulli-Euler beam equation solution, let's delve deeper into the mathematical formulation and its practical implications.

Clamped-Free Boundary Conditions Applied to $\phi(x)$

Given the clamped-free boundary conditions, which are now exclusively related to $\phi(x)$ since they do not involve time t, we can write:

At the clamped end ($x = 0$):

- $\phi(0) = 0$ (The beam has zero displacement at the clamped end).
- $\left.\frac{d\phi}{dx}\right|_{x=0} = 0$ (The slope of the beam is zero at the clamped end).

At the free end ($x = l$):

- $\left.\frac{d^2\phi}{dx^2}\right|_{x=l} = 0$ (The bending moment is zero).
- $\left.\frac{d^3\phi}{dx^3}\right|_{x=l} = 0$ (The shearing force is zero).

Using these boundary conditions and denoting the derivative concerning x, then the following matrix can be formed.

$$\begin{bmatrix} \phi(0) \\ \dot{\phi}(0) \\ \ddot{\phi}(l) \\ \dddot{\phi}(l) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ -\sin(kl) & -\cos(kl) & \sinh(kl) & \cosh(kl) \\ -\cos(kl) & \sin(kl) & \cosh(kl) & \sinh(kl) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The obtain a solution, the $4x4$ matrix must have zero determinant.

This leads to:

Matrix Formulation and Determinant Condition

To satisfy these boundary conditions, a matrix representation of the system of equations derived from the spatial solution $\phi(x)$ can be set up. This matrix relates the coefficients of the solution to the boundary conditions. For a non-trivial solution to exist (i.e., for $\phi(x)$ not to be identically zero), the determinant of this matrix must be zero. This requirement leads to the transcendental equation that must be satisfied by k, the wave number:
$\cos(kl)\cosh(kl) = -1$

Solution to the Spatial Part with Boundary Conditions

The solutions for k that satisfy this equation, denoted as k_i, lead to a family of spatial functions $\phi_i(x)$, each corresponding to a different mode of vibration. Incorporating the boundary conditions into the spatial solutions yields:
$\phi_i(x) = C_3(\sin(k_i x) - \sinh(k_i x)) + C_3\left(-\frac{\sin(k_i L)+\sinh(k_i L)}{\cos(k_i L)+\cosh(k_i L)}\right)(\cos(k_i x) - \cosh(k_i x))$

Temporal Solution Component

Each spatial mode $\phi_i(x)$ is associated with a temporal mode $q_i(t)$, which describes how the amplitude of the mode varies with time:
$q_i(t) = A\cos(\omega_i t) + B\sin(\omega_i t)$

Overall Solution and Mode Summation

The overall solution to the Bernoulli-Euler beam equation is a superposition of these modes:
$w(x,t) = \sum_{i=2}^{\infty} \phi_i(x) q_i(t)$

This summation starts from $i = 2$ because the first mode ($q_1(t)$) is reserved for representing the motor joint angle, highlighting the integration of the flexible beam's dynamics into the robotic system.

Orthogonality and Modal Analysis

The spatial functions $\phi_i(x)$, or mode shape functions, are orthogonal, meaning that the integral of the product of any two different mode shapes over the length of the beam equals zero. This property is crucial for simplifying the analysis of the system and for the modal decomposition of the beam's response.

$$\int_0^l \phi_i(x)\phi_j(x)dx = \begin{cases} 0, & \text{for } i \neq j \\ m_i, & \text{for i = j} \end{cases}$$

where m_i is the square of the magnitude of the mode shape function.

Practical Considerations and Model Simplification

In practice, while theoretically, an infinite number of modes exists, only a finite number, typically $n = 2$ or 3, are considered due to the diminishing contribution of higher-order modes to the beam's overall displacement $w(x,t)$. This approximation simplifies the analysis and is often sufficient for engineering applications, where the focus is on the most significant modes of vibration that impact the system's performance.

The first five modal frequencies w.r.t wave numbers are given in Fig.3

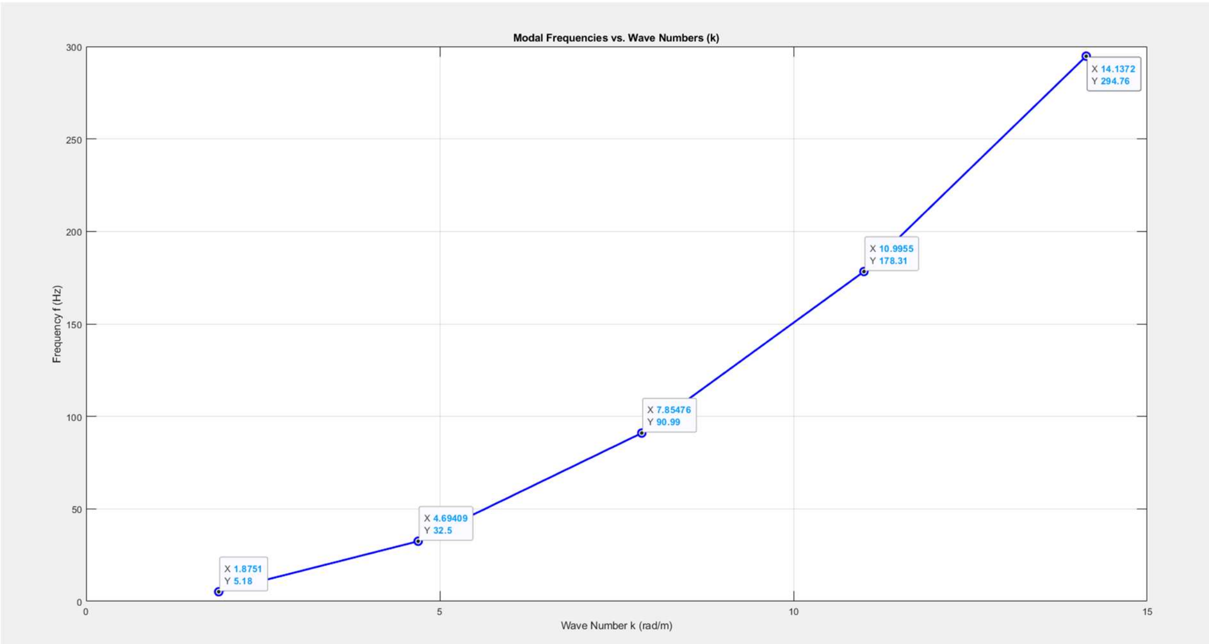

Figure 3: Modal frequencies vs. Wave numbers.

The beam properties are given in Table 1.

Table 1: The properties of the selected aluminum beam.

Property	Symbol	Value	Units
Length of the beam	L	1	meters (m)
Side length of square cross-section	a	6.35e-3	meters (m)
Moment of inertia	I	$a^4/12$	meters^4 (m^4)
Cross-sectional area	A	a^2	meters^2 (m^2)
Density of aluminum	ρ	2700	kilograms per cubic meter (kg/m^3)
Young's modulus	E	69e9	pascals (Pa)

To evaluate the first five modal frequencies (k_i, ω_i in radians per second, and f_i in Hz) of a beam analytically, we can use the well-known solution to the Euler-Bernoulli beam equation for a clamped-free boundary condition. The mode shapes and frequencies are determined by solving the characteristic equation derived from the boundary conditions. For a clamped-free beam, the characteristic equation involves transcendental functions, making the roots non-trivial to find analytically. However, we can approximate these roots numerically using MATLAB.

Given:

- EI as the flexural rigidity of the beam,
- ρA as the mass per unit length (ρ is the density and A is the cross-sectional area),
- L as the length of the beam,

the natural frequencies (ω_n) can be found from the roots of the characteristic equation, and they relate to k as $\omega_n^2 = \left(\frac{k_n^2 EI}{\rho A}\right)$. The frequencies in Hz can be found by $f_n = \frac{\omega_n}{2\pi}$

Fundamental Texts and References:

Timoshenko, S. P., & Young, D. H. (1961). "Vibration Problems in Engineering." Wiley.

This classic book offers a comprehensive treatment of vibration problems, including detailed discussions on beam vibrations, which is fundamental for understanding Euler-Bernoulli beam theory and its applications.

Meirovitch, L. (2001). "Fundamentals of Vibrations." McGraw-Hill Education.

Meirovitch provides an in-depth exploration of vibration theory, covering both free and forced vibrations, which is essential for analyzing the modal frequencies and responses of beams under various loading conditions.

Reddy, J. N. (2006). "Theory and Analysis of Elastic Plates and Shells." CRC Press.

While focused on plates and shells, Reddy's book offers valuable insights into the elastic behavior of structural elements, including a thorough treatment of the mathematical foundations relevant to beam analysis.

Rao, S. S. (2007). "Vibration of Continuous Systems." Wiley.

Rao's work is particularly useful for understanding the continuous nature of beam dynamics and provides methods for analyzing vibrations, including the use of numerical techniques such as the bvp4c method discussed.

Inman, D. J. (2013). "Engineering Vibration." Pearson.

This textbook introduces engineering vibration in a more accessible manner, including discussions on modal analysis and the physical interpretation of beam vibrations, making it suitable for undergraduate and graduate levels.

Additional Resources:

MATLAB & Simulink Documentation. MathWorks.

The official MATLAB documentation is invaluable for understanding the specific functions (bvp4c, fsolve, etc.) used in numerical analysis and simulation of beam dynamics. Available online: MathWorks Documentation.

Hodges, D. H., & Pierce, G. A. (2002). "Introduction to Structural Dynamics and Aeroelasticity." Cambridge University Press.

This book offers insights into the dynamic behavior of structural elements in aerospace engineering, including beams, and discusses analytical and numerical methods for their analysis.

These references collectively cover the theoretical, analytical, and numerical aspects of beam dynamics, offering a solid foundation for academic research, study, and application in engineering disciplines.