



## Lecture 7: Policy Gradient

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# Outline

- 1 Introduction
- 2 Finite Difference Policy Gradient
- 3 Monte-Carlo Policy Gradient
- 4 Actor-Critic Policy Gradient



## Policy-Based Reinforcement Learning

- In the last lecture we approximated the value or action-value function using parameters  $\theta$ ,

$$V_{\theta}(s) \approx V^{\pi}(s)$$

$$Q_{\theta}(s, a) \approx Q^{\pi}(s, a)$$

- A policy was generated directly from the value function
  - e.g. using  $\epsilon$ -greedy
- In this lecture we will directly parametrise the policy

$$\pi_{\theta}(s, a) = \mathbb{P}[a \mid s, \theta]$$

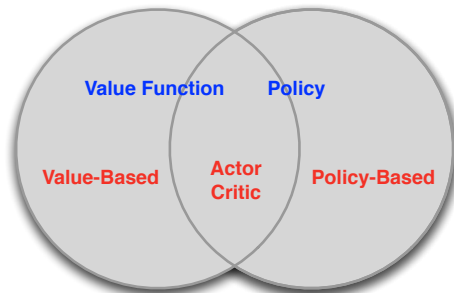


- We will focus again on model-free reinforcement learning

# Value-Based and Policy-Based RL



- Value Based
  - Learnt Value Function
  - Implicit policy (e.g.  $\epsilon$ -greedy)
- Policy Based
  - No Value Function
  - Learnt Policy
- Actor-Critic
  - Learnt Value Function
  - Learnt Policy



# Advantages of Policy-Based RL

## Advantages:

- Better convergence properties
- Effective in high-dimensional or continuous action spaces
- Can learn **stochastic policies**



## Disadvantages:

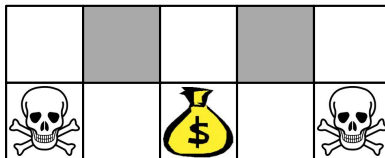
- Typically converge to a local rather than global optimum
- Evaluating a policy is typically inefficient and high variance

# Example: Rock-Paper-Scissors



- Two-player game of rock-paper-scissors
  - Scissors beats paper
  - Rock beats scissors
  - Paper beats rock
- Consider policies for *iterated* rock-paper-scissors
  - A deterministic policy is easily exploited
  - A uniform random policy is optimal (i.e. Nash equilibrium)

# Example: Aliased Gridworld (1)



- The agent cannot differentiate the grey states
- Consider **features** of the following form (for all N, E, S, W)



$$\phi(s, a) = \mathbf{1}(\text{wall to } N, a = \text{move } E)$$

- Compare value-based RL, using an approximate value function

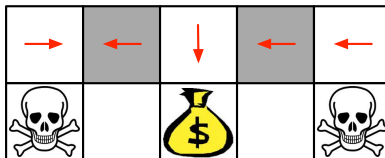
$$Q_{\theta}(s, a) = f(\phi(s, a), \theta)$$



- To policy-based RL, using a parametrised policy

$$\pi_{\theta}(s, a) = g(\phi(s, a), \theta)$$

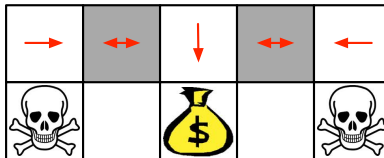
## Example: Aliased Gridworld (2)



- Under aliasing, an optimal deterministic policy will either
  - move W in both grey states (shown by red arrows)
  - move E in both grey states
- Either way, it can get stuck and *never* reach the money
- Value-based RL learns a near-deterministic policy
  - e.g. greedy or  $\epsilon$ -greedy
- So it will traverse the corridor for a long time



## Example: Aliased Gridworld (3)



- An optimal stochastic policy will randomly move E or W in grey states

$$\pi_{\theta}(\text{wall to N and S, move E}) = 0.5$$

$$\pi_{\theta}(\text{wall to N and S, move W}) = 0.5$$

- It will reach the goal state in a few steps with high probability
- Policy-based RL can learn the optimal stochastic policy

## Policy Objective Functions



- Goal: given policy  $\pi_\theta(s, a)$  with parameters  $\theta$ , find best  $\theta$
- But how do we measure the quality of a policy  $\pi_\theta$ ?
- In episodic environments we can use the start value

$$J_1(\theta) = V^{\pi_\theta}(s_1) = \mathbb{E}_{\pi_\theta} [v_1]$$



- In continuing environments we can use the **average value**

$$J_{avV}(\theta) = \sum_s d^{\pi_\theta}(s) V^{\pi_\theta}(s)$$

- Or the **average reward per time-step**



$$J_{avR}(\theta) = \sum_s d^{\pi_\theta}(s) \sum_a \pi_\theta(s, a) \mathcal{R}_s^a$$

- where  $d^{\pi_\theta}(s)$  is **stationary distribution** of Markov chain for  $\pi_\theta$

# Policy Optimisation

- Policy based reinforcement learning is an **optimisation** problem
- Find  $\theta$  that maximises  $J(\theta)$
- Some approaches do not use gradient
  - Hill climbing
  - Simplex / amoeba / Nelder Mead
  - Genetic algorithms
- Greater efficiency often possible using gradient
  - Gradient descent
  - Conjugate gradient
  - Quasi-newton
- We focus on **gradient descent**, many extensions possible
- And on methods that exploit sequential structure



# Policy Gradient



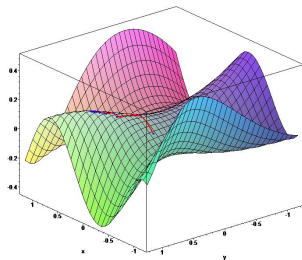
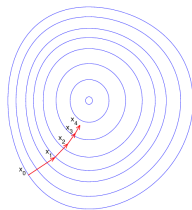
- Let  $J(\theta)$  be any policy objective function
- Policy gradient algorithms search for a *local* maximum in  $J(\theta)$  by ascending the gradient of the policy, w.r.t. parameters  $\theta$

$$\Delta\theta = \alpha \nabla_{\theta} J(\theta)$$

- Where  $\nabla_{\theta} J(\theta)$  is the **policy gradient**

$$\nabla_{\theta} J(\theta) = \begin{pmatrix} \frac{\partial J(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial J(\theta)}{\partial \theta_n} \end{pmatrix}$$

- and  $\alpha$  is a step-size parameter



# Computing Gradients By Finite Differences



- To evaluate policy gradient of  $\pi_{\theta}(s, a)$
- For each dimension  $k \in [1, n]$ 
  - Estimate  $k$ th partial derivative of objective function w.r.t.  $\theta$
  - By perturbing  $\theta$  by small amount  $\epsilon$  in  $k$ th dimension

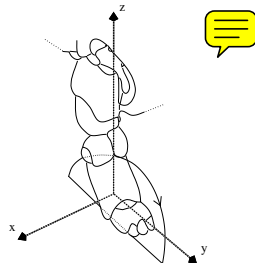


$$\frac{\partial J(\theta)}{\partial \theta_k} \approx \frac{J(\theta + \epsilon u_k) - J(\theta)}{\epsilon}$$

where  $u_k$  is unit vector with 1 in  $k$ th component, 0 elsewhere

- Uses  $n$  evaluations to compute policy gradient in  $n$  dimensions
- Simple, noisy, inefficient - but sometimes effective
- Works for arbitrary policies, even if policy is not differentiable

# Training AIBO to Walk by Finite Difference Policy Gradient



- Goal: learn a fast AIBO walk (useful for Robocup)
- AIBO walk policy is controlled by 12 numbers (elliptical loci)
- Adapt these parameters by finite difference policy gradient
- Evaluate performance of policy by field traversal time

# AIBO Walk Policies



- Before training
- During training
- After training

# Score Function

- We now compute the policy gradient analytically
- Assume policy  $\pi_\theta$  is differentiable whenever it is non-zero
- and we know the gradient  $\nabla_\theta \pi_\theta(s, a)$
- Likelihood ratios exploit the following identity

$$\begin{aligned}\nabla_\theta \pi_\theta(s, a) &= \pi_\theta(s, a) \frac{\nabla_\theta \pi_\theta(s, a)}{\pi_\theta(s, a)} \\ &= \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)\end{aligned}$$



- The **score function** is  $\nabla_\theta \log \pi_\theta(s, a)$



## Softmax Policy



- We will use a softmax policy as a running example
- Weight actions using linear combination of features  $\phi(s, a)^\top \theta$
- Probability of action is proportional to exponentiated weight

$$\pi_\theta(s, a) \propto e^{\phi(s, a)^\top \theta}$$

- The score function is

$$\nabla_\theta \log \pi_\theta(s, a) = \phi(s, a) - \mathbb{E}_{\pi_\theta} [\phi(s, \cdot)]$$

## Gaussian Policy

- In continuous action spaces, a Gaussian policy is natural
- Mean is a linear combination of state features  $\mu(s) = \phi(s)^\top \theta$
- Variance may be fixed  $\sigma^2$ , or can also be parametrised
- Policy is Gaussian,  $a \sim \mathcal{N}(\mu(s), \sigma^2)$
- The score function is

$$\nabla_{\theta} \log \pi_{\theta}(s, a) = \frac{(a - \mu(s))\phi(s)}{\sigma^2}$$

# One-Step MDPs

- Consider a simple class of **one-step** MDPs
  - Starting in state  $s \sim d(s)$
  - Terminating after one time-step with reward  $r = \mathcal{R}_{s,a}$
- Use likelihood ratios to compute the policy gradient



$$J(\theta) = \mathbb{E}_{\pi_\theta} [r]$$



$$= \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s, a) \mathcal{R}_{s,a}$$

$$\nabla_\theta J(\theta) = \sum_{s \in \mathcal{S}} d(s) \sum_{a \in \mathcal{A}} \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a) \mathcal{R}_{s,a}$$

$$= \mathbb{E}_{\pi_\theta} [\nabla_\theta \log \pi_\theta(s, a) r]$$



# Policy Gradient Theorem

- The policy gradient theorem generalises the likelihood ratio approach to multi-step MDPs
- Replaces instantaneous reward  $r$  with long-term value  $Q^\pi(s, a)$
- Policy gradient theorem applies to start state objective, average reward and average value objective

## Theorem

*For any differentiable policy  $\pi_\theta(s, a)$ ,  
for any of the policy objective functions  $J = J_1, J_{avR}$ , or  $\frac{1}{1-\gamma} J_{avV}$ ,  
the policy gradient is*

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta} [\nabla_\theta \log \pi_\theta(s, a) Q^{\pi_\theta}(s, a)]$$

# Monte-Carlo Policy Gradient (REINFORCE)

- Update parameters by stochastic gradient ascent
- Using policy gradient theorem
- Using return  $v_t$  as an unbiased sample of  $Q^{\pi_\theta}(s_t, a_t)$

$$\Delta\theta_t = \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$$

## function REINFORCE

Initialise  $\theta$  arbitrarily

**for** each episode  $\{s_1, a_1, r_2, \dots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$  **do**

**for**  $t = 1$  to  $T - 1$  **do**

$\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) v_t$

**end for**

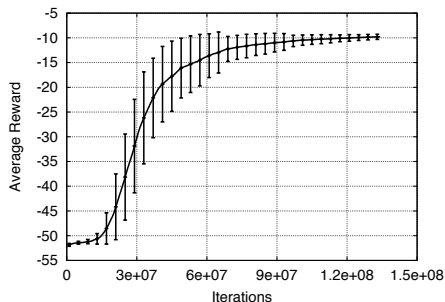
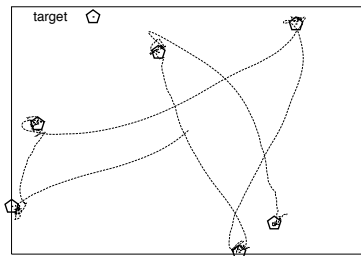
**end for**

**return**  $\theta$

**end function**



# Puck World Example



- Continuous actions exert small force on puck
- Puck is rewarded for getting close to target
- Target location is reset every 30 seconds
- Policy is trained using variant of Monte-Carlo policy gradient

## Reducing Variance Using a Critic



- Monte-Carlo policy gradient still has high variance
- We use a **critic** to estimate the action-value function,

$$Q_w(s, a) \approx Q^{\pi_\theta}(s, a)$$



- Actor-critic algorithms maintain two sets of parameters

**Critic** Updates action-value function parameters **w**

**Actor** Updates policy parameters  **$\theta$** , in direction suggested by critic

- Actor-critic algorithms follow an *approximate* policy gradient

$$\nabla_\theta J(\theta) \approx \mathbb{E}_{\pi_\theta} [\nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)]$$

$$\Delta \theta = \alpha \nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)$$



# Estimating the Action-Value Function

- The critic is solving a familiar problem: policy evaluation
- How good is policy  $\pi_\theta$  for current parameters  $\theta$ ?
- This problem was explored in previous two lectures, e.g.
  - Monte-Carlo policy evaluation
  - Temporal-Difference learning
  - $TD(\lambda)$
- Could also use e.g. least-squares policy evaluation



# Action-Value Actor-Critic

- Simple actor-critic algorithm based on action-value critic
- Using linear value fn approx.  $Q_w(s, a) = \phi(s, a)^\top w$

Critic Updates  $w$  by linear TD(0)

Actor Updates  $\theta$  by policy gradient



**function** QAC

    Initialise  $s, \theta$

    Sample  $a \sim \pi_\theta$

**for** each step **do**

        Sample reward  $r = \mathcal{R}_s^a$ ; sample transition  $s' \sim \mathcal{P}_{s, \cdot}^a$ .

        Sample action  $a' \sim \pi_\theta(s', a')$

$\delta = r + \gamma Q_w(s', a') - Q_w(s, a)$

$\theta = \theta + \alpha \nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)$

$w \leftarrow w + \beta \delta \phi(s, a)$

$a \leftarrow a', s \leftarrow s'$

**end for**

**end function**

# Bias in Actor-Critic Algorithms

- Approximating the policy gradient introduces bias
- A biased policy gradient may not find the right solution
  - e.g. if  $Q_w(s, a)$  uses aliased features, can we solve gridworld example?
- Luckily, if we choose value function approximation carefully
- Then we can avoid introducing any bias
- i.e. We can still follow the *exact* policy gradient

# Compatible Function Approximation

## Theorem (Compatible Function Approximation Theorem)

*If the following two conditions are satisfied:*

- 1 Value function approximator is *compatible* to the policy

$$\nabla_w Q_w(s, a) = \nabla_\theta \log \pi_\theta(s, a)$$

- 2 Value function parameters  $w$  minimise the mean-squared error

$$\varepsilon = \mathbb{E}_{\pi_\theta} [(Q^{\pi_\theta}(s, a) - Q_w(s, a))^2]$$

*Then the policy gradient is exact,*

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta} [\nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)]$$

# Proof of Compatible Function Approximation Theorem

If  $w$  is chosen to minimise mean-squared error, gradient of  $\varepsilon$  w.r.t.  $w$  must be zero,

$$\nabla_w \varepsilon = 0$$

$$\mathbb{E}_{\pi_\theta} [(Q^\theta(s, a) - Q_w(s, a)) \nabla_w Q_w(s, a)] = 0$$

$$\mathbb{E}_{\pi_\theta} [(Q^\theta(s, a) - Q_w(s, a)) \nabla_\theta \log \pi_\theta(s, a)] = 0$$

$$\mathbb{E}_{\pi_\theta} [Q^\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)] = \mathbb{E}_{\pi_\theta} [Q_w(s, a) \nabla_\theta \log \pi_\theta(s, a)]$$

So  $Q_w(s, a)$  can be substituted directly into the policy gradient,

$$\nabla_\theta J(\theta) = \mathbb{E}_{\pi_\theta} [\nabla_\theta \log \pi_\theta(s, a) Q_w(s, a)]$$

## Reducing Variance Using a Baseline

- We subtract a baseline function  $B(s)$  from the policy gradient
- This can reduce variance, without changing expectation

$$\begin{aligned}\mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) B(s)] &= \sum_{s \in \mathcal{S}} d^{\pi_{\theta}}(s) \sum_a \nabla_{\theta} \pi_{\theta}(s, a) B(s) \\ &= \sum_{s \in \mathcal{S}} d^{\pi_{\theta}}(s) B(s) \nabla_{\theta} \sum_{a \in \mathcal{A}} \pi_{\theta}(s, a) \\ &= 0\end{aligned}$$

- A good baseline is the state value function  $B(s) = V^{\pi_{\theta}}(s)$
- So we can rewrite the policy gradient using the **advantage function**  $A^{\pi_{\theta}}(s, a)$

$$\begin{aligned}A^{\pi_{\theta}}(s, a) &= Q^{\pi_{\theta}}(s, a) - V^{\pi_{\theta}}(s) \\ \nabla_{\theta} J(\theta) &= \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) A^{\pi_{\theta}}(s, a)]\end{aligned}$$

# Estimating the Advantage Function (1)

- The advantage function can significantly reduce variance of policy gradient
- So the critic should really estimate the advantage function
- For example, by estimating *both*  $V^{\pi_\theta}(s)$  and  $Q^{\pi_\theta}(s, a)$
- Using two function approximators and two parameter vectors,

$$\begin{aligned}V_v(s) &\approx V^{\pi_\theta}(s) \\ Q_w(s, a) &\approx Q^{\pi_\theta}(s, a) \\ A(s, a) &= Q_w(s, a) - V_v(s)\end{aligned}$$

- And updating *both* value functions by e.g. TD learning

## Estimating the Advantage Function (2)

- For the true value function  $V^{\pi_{\theta}}(s)$ , the TD error  $\delta^{\pi_{\theta}}$

$$\delta^{\pi_{\theta}} = r + \gamma V^{\pi_{\theta}}(s') - V^{\pi_{\theta}}(s)$$



- is an unbiased estimate of the advantage function

$$\begin{aligned}\mathbb{E}_{\pi_{\theta}} [\delta^{\pi_{\theta}} | s, a] &= \mathbb{E}_{\pi_{\theta}} [r + \gamma V^{\pi_{\theta}}(s') | s, a] - V^{\pi_{\theta}}(s) \\ &= Q^{\pi_{\theta}}(s, a) - V^{\pi_{\theta}}(s) \\ &= A^{\pi_{\theta}}(s, a)\end{aligned}$$



- So we can use the TD error to compute the policy gradient

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) \delta^{\pi_{\theta}}]$$

- In practice we can use an approximate TD error

$$\delta_v = r + \gamma V_v(s') - V_v(s)$$

- This approach only requires one set of critic parameters  $v$

# Critics at Different Time-Scales

- Critic can estimate value function  $V_\theta(s)$  from many targets at different time-scales From last lecture...

- For MC, the target is the return  $v_t$

$$\Delta\theta = \alpha(v_t - V_\theta(s))\phi(s)$$

- For TD(0), the target is the TD target  $r + \gamma V(s')$

$$\Delta\theta = \alpha(r + \gamma V(s') - V_\theta(s))\phi(s)$$

- For forward-view TD( $\lambda$ ), the target is the  $\lambda$ -return  $v_t^\lambda$

$$\Delta\theta = \alpha(v_t^\lambda - V_\theta(s))\phi(s)$$

- For backward-view TD( $\lambda$ ), we use eligibility traces

$$\delta_t = r_{t+1} + \gamma V(s_{t+1}) - V(s_t)$$

$$e_t = \gamma\lambda e_{t-1} + \phi(s_t)$$

$$\Delta\theta = \alpha\delta_t e_t$$



# Actors at Different Time-Scales

- The policy gradient can also be estimated at many time-scales

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) A^{\pi_{\theta}}(s, a)]$$

- Monte-Carlo policy gradient uses error from complete return

$$\Delta\theta = \alpha(v_t - V_v(s_t)) \nabla_{\theta} \log \pi_{\theta}(s_t, a_t)$$

- Actor-critic policy gradient uses the one-step TD error

$$\Delta\theta = \alpha(r + \gamma V_v(s_{t+1}) - V_v(s_t)) \nabla_{\theta} \log \pi_{\theta}(s_t, a_t)$$

# Policy Gradient with Eligibility Traces

- Just like forward-view TD( $\lambda$ ), we can mix over time-scales

$$\Delta\theta = \alpha(v_t^\lambda - V_v(s_t))\nabla_\theta \log \pi_\theta(s_t, a_t)$$

- where  $v_t^\lambda - V_v(s_t)$  is a biased estimate of advantage fn
- Like backward-view TD( $\lambda$ ), we can also use eligibility traces
  - By equivalence with TD( $\lambda$ ), substituting  $\phi(s) = \nabla_\theta \log \pi_\theta(s, a)$

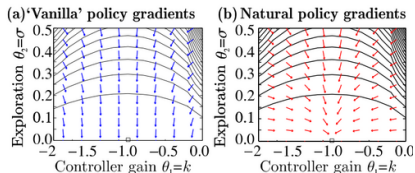
$$\begin{aligned}\delta &= r_{t+1} + \gamma V_v(s_{t+1}) - V_v(s_t) \\ e_{t+1} &= \lambda e_t + \nabla_\theta \log \pi_\theta(s, a) \\ \Delta\theta &= \alpha \delta e_t\end{aligned}$$

- This update can be applied online, to incomplete sequences

# Alternative Policy Gradient Directions

- Gradient ascent algorithms can follow *any* ascent direction
- A good ascent direction can significantly speed convergence
- Also, a policy can often be reparametrised without changing action probabilities
- For example, increasing score of all actions in a softmax policy
- The vanilla gradient is sensitive to these reparametrisations

# Natural Policy Gradient



- The **natural policy gradient** is parametrisation independent
- It finds ascent direction that is closest to vanilla gradient, when changing policy by a small, fixed amount

$$\nabla_{\theta}^{nat} \pi_{\theta}(s, a) = G_{\theta}^{-1} \nabla_{\theta} \pi_{\theta}(s, a)$$

- where  $G_{\theta}$  is the Fisher information matrix

$$G_{\theta} = \mathbb{E}_{\pi_{\theta}} \left[ \nabla_{\theta} \log \pi_{\theta}(s, a) \nabla_{\theta} \log \pi_{\theta}(s, a)^T \right]$$

# Natural Actor-Critic

- Using compatible function approximation,

$$\nabla_w A_w(s, a) = \nabla_\theta \log \pi_\theta(s, a)$$

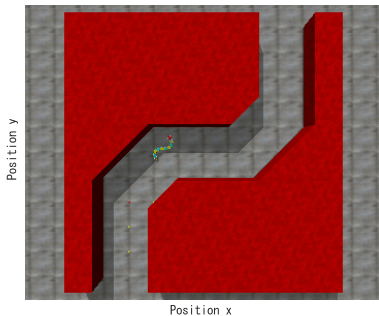
- So the natural policy gradient simplifies,

$$\begin{aligned}\nabla_\theta J(\theta) &= \mathbb{E}_{\pi_\theta} [\nabla_\theta \log \pi_\theta(s, a) A^{\pi_\theta}(s, a)] \\ &= \mathbb{E}_{\pi_\theta} \left[ \nabla_\theta \log \pi_\theta(s, a) \nabla_\theta \log \pi_\theta(s, a)^T w \right] \\ &= G_\theta w\end{aligned}$$

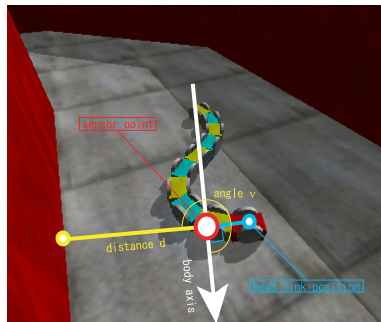
$$\nabla_\theta^{\text{nat}} J(\theta) = w$$

- i.e. update actor parameters in direction of critic parameters

# Natural Actor Critic in Snake Domain

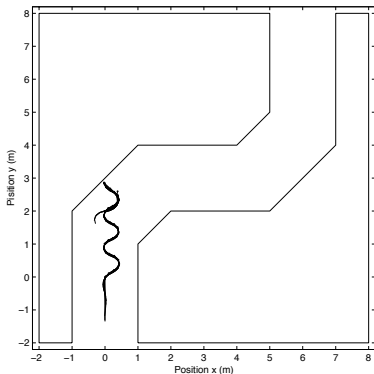


(a) Crank course

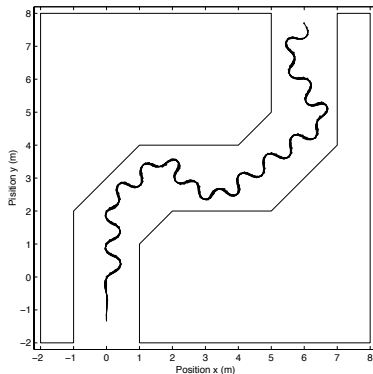


(b) Sensor setting

# Natural Actor Critic in Snake Domain (2)

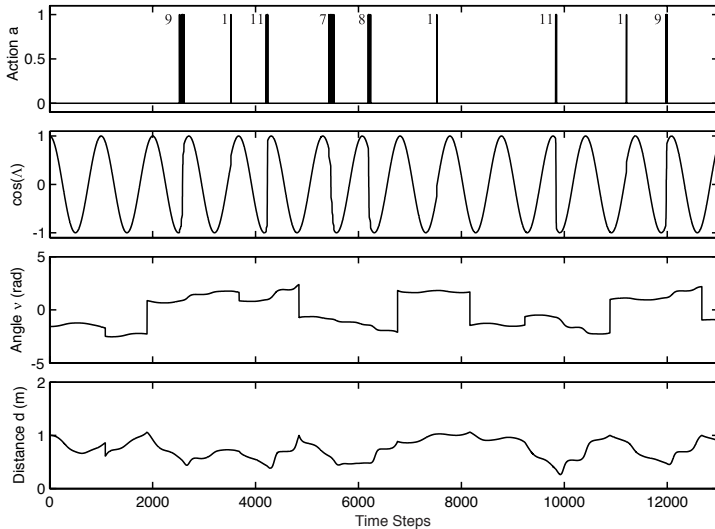


(a) Before learning



(b) After learning

# Natural Actor Critic in Snake Domain (3)





# Summary of Policy Gradient Algorithms

- The **policy gradient** has many equivalent forms

$\nabla_{\theta} J(\theta) = \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) \mathbf{v}_t]$	REINFORCE
$= \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) \mathbf{Q}^w(s, a)]$	Q Actor-Critic
$= \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) \mathbf{A}^w(s, a)]$	Advantage Actor-Critic
$= \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) \delta]$	TD Actor-Critic
$= \mathbb{E}_{\pi_{\theta}} [\nabla_{\theta} \log \pi_{\theta}(s, a) \delta \mathbf{e}]$	TD( $\lambda$ ) Actor-Critic
$G_{\theta}^{-1} \nabla_{\theta} J(\theta) = \mathbf{w}$	Natural Actor-Critic

- Each leads a stochastic gradient ascent algorithm
- Critic uses **policy evaluation** (e.g. MC or TD learning) to estimate  $Q^{\pi}(s, a)$ ,  $A^{\pi}(s, a)$  or  $V^{\pi}(s)$