Physics of
`oseiskar.github.io/black-hole`

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Ray paths and gravitational lensing

As usual in raytracing, each sample (pixel) in the image corresponds to a path starting from the position $\boldsymbol{p} = (x_0, y_0, z_0)$ of the camera, initially travelling to a direction $\boldsymbol{d}_o \in \mathbb{R}^3$ specified by the camera coordinate system. The path represents photons arriving to the camera from that direction. Instead of travelling in straight lines until hitting an object, the paths solve the photonic geodesic equation in the Schwarzschild space-time, which in certain units and spherical coordinates (t, r, θ, ϕ) takes the form

$$\mathrm{d}s^2 = \left(1 - \frac{1}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{1}{r}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 - \sin^2\theta \cdot \mathrm{d}\phi^2) = 0.$$

Without loss of generality, one can fix $\theta = \frac{\pi}{2}$ and, even though not very obviously (see, e.g., [3] and [2]), this gives the ordinary differential equation

$$u''(\phi) = -u(\phi)\left(1 - \frac{3}{2}u^2(\phi)\right), \tag{1}$$

where $u := 1/r$.

To solve this in the raytracer, I use the initial conditions

$$u(0) = \frac{1}{\|\boldsymbol{p}\|}, \qquad u'(0) = \frac{-\boldsymbol{d}_0 \cdot \boldsymbol{n}}{u(0)\ \boldsymbol{d}_0 \cdot \boldsymbol{t}},$$

where the normal and tangential components of $\boldsymbol{d}_0$ are

$$\boldsymbol{n} = \frac{\boldsymbol{p}}{\|\boldsymbol{p}\|} \qquad \text{and} \qquad \boldsymbol{t} = \frac{(\boldsymbol{n} \times \boldsymbol{d}_0) \times \boldsymbol{n}}{\|(\boldsymbol{n} \times \boldsymbol{d}_0) \times \boldsymbol{n}\|}.$$

Equation (1) is integrated using the Leapfrog method until either a maximum number of steps N is reached or $u_j + \Delta u_j < 0$, which signals that the ray escapes at an angle $\phi \in (\phi_j, \phi_j + \Delta\phi_j)$.

To compute light travel times, i.e., the Schwarzschild time coordinate t, along the path, the numerical integration is amended with

$$t'(\phi) = -\frac{\sqrt{(u')^2 + u^2(1-u)}}{u^2(1-u)}.$$

In order to do the integration in a visually accurate way, the step size $\Delta\phi_j$ is varied using an *ad hoc* formula, in addition to using the classical approximation $\mathrm{d}t^2 \approx \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ at large distances r from the black hole.

When the observer is moving with velocity $\boldsymbol{v}$ w.r.t. the Schwarzschild coordinate system, the ray directions $\boldsymbol{d}$ in the coordinate system of the observer (the camera coordinate system) are transformed by *relativistic aberration*, which is corresponds to the Lorentz velocity transformation

$$\boldsymbol{d}_0 = \frac{1}{1 + v d_\parallel}\left((d_\parallel + v)\boldsymbol{v}^0 + \frac{1}{\gamma}\boldsymbol{d}_\perp\right)$$

where $\boldsymbol{v} = \boldsymbol{v}^0 v$, $d_\parallel = \boldsymbol{v}^0 \cdot \boldsymbol{d}$, $\boldsymbol{d}_\perp = \boldsymbol{d} - \boldsymbol{d}_\parallel$ and

$$\gamma = 1/\sqrt{1 - v^2}. \tag{2}$$

Circular orbits

The observer (unless stationary) and the planet both move on circular orbits with angular velocities given by [2]

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{1}{r\sqrt{2(r-1)}}. \tag{3}$$

On such orbits, the time dilation of the observer relative to the planet is computed by updating the t-coordinate as

$$\Delta t = \sqrt{\frac{(\Delta t_0)^2(1 - v^2)}{1 - \frac{1}{r}}},$$

where Δt_0 is the elapsed wall clock time between consecutive animation frames and $v = r\mathrm{d}\phi/\mathrm{d}t$.

When computing intersection between the planet and light paths, the motion of the planet has to be taken into account. Within the ODE integration steps, a ray-sphere intersection model can be used after mapping the ray to the moving coordinate system of the planet using a Lorentz velocity transformation.

Doppler shift and beaming

The *Doppler factor* for light arriving to the observer from direction $\boldsymbol{d}$, emitted by a source moving at velocity $\boldsymbol{v}$ is [1]

$$\delta = \gamma(1 + \boldsymbol{d} \cdot \boldsymbol{v}), \tag{4}$$

where γ is the Lorentz factor (2). The Doppler factor for a ray path connecting a moving light source and a moving observer can be computed as

$$\delta = \delta_o \delta_s = \gamma_o \gamma_s (1 - \boldsymbol{d}_o \cdot \boldsymbol{v}_o)(1 + \boldsymbol{d}_s \cdot \boldsymbol{v}_s),$$

where $\boldsymbol{d}_o$ and $\boldsymbol{d}_s$ are the initial and final (at intersection with the light source) photon directions on the ray path. The velocities $\boldsymbol{v}_o$ and $\boldsymbol{v}_s$ of the observer and source can be determined by (3). All of these are given in the "stationary" Schwarzschild coordinate system. For the accretion disk, the velocity $\boldsymbol{v}_s$ is computed assuming the disk consists of glowing matter moving in circular orbits determined by (3).

The wavelengths of the emitted and observed light are related by $\lambda_o = \delta\lambda_s$ and intensities by $I(\lambda_o) = \delta^{-3} I(\lambda_s)$ [1]. The perceived (RGB) colors corresponding to any fixed spectrum I can be precomputed in a one-dimensional lookup table (texture) as a function of the Doppler factor. This is done by integrating the shifted spectra with respect to the standard CIE color sensitivity measures [4].

References

[1] Jeremy Goodman, Topics in High-Energy Astrophysics `http://www.astro.princeton.edu/~jeremy/heap.pdf`

[2] Riccardo Antonelli, Visualizing a Black Hole `http://spiro.fisica.unipd.it/~antonell/schwarzschild/`

[3] Wikipedia, Schwarzschild Geodesics `https://en.wikipedia.org/wiki/Schwarzschild_geodesics`

[4] Munsell Color Science Laboratory, CIE data `http://www.cis.rit.edu/research/mcsl2/online/cie.php`

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