Summary of Earlier Computations

Notation. We will use the folowing notation:

$$\phi_{ijkl} \triangleq \phi_i \otimes \phi_j \otimes \phi_k \otimes \phi_l,\ \ \phi_{i,p|jk|l,q} \triangleq \partial_p \phi_i \otimes \phi_j \otimes \phi_k \otimes \partial_q \phi_l,$$

$$\phi_{i,pq|jkl} \triangleq \partial^2_{pq} \phi_i \otimes \phi_j \otimes \phi_k \otimes \phi_l \text{ etc.}$$

Recall the following function:

$$\bar{b}(x) \triangleq b(x) + \nabla_x G_1(x, 0).$$

1 Degree 3 case

The coefficient is

$$\frac{1}{48}\phi_{i,i|k|j,j|k} + \frac{1}{48}\phi_{k|i,i|k|j,j} - \frac{1}{48}\phi_{i,i|kk|j,j} - \frac{1}{48}\phi_{k|i,i|j,j|k}.$$

2 Degree 2.5 Case

The quantity $\pi_2 \otimes \pi_2$ (up to order t^3) looks like

$$((\mathrm{I} + \mathrm{II} + \mathrm{III}) \otimes (\mathrm{I} + \mathrm{II} + \mathrm{III}) + \mathrm{V}) \otimes ((\mathrm{I} + \mathrm{II} + \mathrm{III}) \otimes (\mathrm{I} + \mathrm{II} + \mathrm{III}) + \mathrm{V})$$

Note that type I, III-terms have order $\sqrt{t}$ and type II, V terms have order t.

2.1 Choosing V (order structure: $1.5 + 1$ or $1 + 1.5$)

2.1.1 II, I; V

This means the term coming from $(\mathrm{II} \otimes \mathrm{I}) \otimes \mathrm{V}$. Similar convention applies to other cases. The coefficient is

$$-\frac{1}{12}\partial_i \bar{b}^j \phi_{jikk} - \frac{1}{24}\phi_{j,ij|ikk} - \frac{1}{12}\phi_{i,i|j,j|kk} - \frac{1}{24}\phi_{j,j|i|k,i|k} - \frac{1}{24}\phi_{j,j|i|k|k,i}.$$

2.1.2 II, III; V

The coefficient is

$$\frac{1}{48}\phi_{j,j|i|k,i|k} + \frac{1}{48}\phi_{j,j|ik|k,i}.$$

2.1.3 I, II; V

The coefficient is

$$-\frac{1}{24}\phi_{i,i|j,j|kk} - \frac{1}{24}\partial_i \bar{b}^j \phi_{ijkk} - \frac{1}{48}\phi_{i|j,ij|kk} - \frac{1}{48}\phi_{i|j,j|k,i|k} - \frac{1}{48}\phi_{i|j,j|k|k,i}.$$

2.1.4 III, II; V

$$\frac{1}{8}\partial_i\bar{b}^j\phi_{ijkk}+\frac{1}{16}\phi_{i|j,ij|kk}+\frac{1}{24}\phi_{i|j,j|k,i|k}+\frac{1}{24}\phi_{i|j,j|k|k,i}.$$

2.1.5 Swapping the two slots

By symmetry, we have

$$C_{V;\mathrm{II,I}}=\sigma(C_{\mathrm{I,II};V})\text{ etc.}$$

where $\sigma:E^{\otimes 4}\to E^{\otimes 4}$ is the linear map induced by

$$\sigma(a\otimes b\otimes c\otimes d)=c\otimes d\otimes a\otimes b.$$

Therefore, we also have coefficients

$$\begin{aligned}
C_{\mathrm{V;II,I}}&=-\frac{1}{12}\partial_i\bar{b}^j\phi_{kkji}-\frac{1}{24}\phi_{kk|j,ij|i}-\frac{1}{12}\phi_{kk|i,i|j,j}-\frac{1}{24}\phi_{k,i|k|j,j|i}-\frac{1}{24}\phi_{k|k,i|j,j|i},\\
C_{\mathrm{V;II,III}}&=\frac{1}{48}\phi_{k,i|k|j,j|i}+\frac{1}{48}\phi_{k|k,i|j,j|i},\\
C_{V;\mathrm{I,II}}&=-\frac{1}{24}\phi_{kk|i,i|j,j}-\frac{1}{24}\partial_i\bar{b}^j\phi_{kkij}-\frac{1}{48}\phi_{kki|j,ij}-\frac{1}{48}\phi_{k,i|ki|j,j}-\frac{1}{48}\phi_{k|k,i|i|j,j},\\
C_{V;\mathrm{III,II}}&=\frac{1}{8}\partial_i\bar{b}^j\phi_{kkij}+\frac{1}{16}\phi_{kki|j,ij}+\frac{1}{24}\phi_{k,i|ki|j,j}+\frac{1}{24}\phi_{k|k,i|i|j,j}.
\end{aligned}$$

2.2 Not choosing V (order structure: $1.5+(0.5+0.5)$ or $(0.5+0.5)+1.5$)

2.2.1 II, I; I, I and I, I; II, I

The coefficient for the case II, I; I, I is

$$\begin{aligned}
C_{\mathrm{II,I;I,I}}=&-\frac{1}{12}\partial_i\bar{b}^j\phi_{jikk}-\frac{55}{432}\partial_i\bar{b}^j\phi_{jkik}-\frac{25}{432}\partial_i\bar{b}^j\phi_{jkki}-\frac{1}{24}\phi_{j,ij|ikk}-\frac{55}{864}\phi_{j,ij|kik}-\frac{25}{864}\phi_{j,ij|kki}\\
&-\frac{1}{12}\phi_{i,i|j,j|kk}-\frac{19}{432}\phi_{i,i|j,k|jk}-\frac{19}{432}\phi_{i,i|j,k|kj}-\frac{35}{288}\phi_{i,i|j|k,k|j}-\frac{7}{144}\phi_{i,i|j|k,j|k}-\frac{7}{144}\phi_{i,i|j|j,k|k}\\
&-\frac{19}{288}\phi_{i,i|jj|k,k}-\frac{1}{24}\phi_{i,i|jk|k,j}-\frac{1}{24}\phi_{i,i|jk|j,k}.
\end{aligned}$$

By symmetry,

$$\begin{aligned}
C_{\mathrm{I,I;II,I}}=&-\frac{1}{12}\partial_i\bar{b}^j\phi_{kkji}-\frac{55}{432}\partial_i\bar{b}^j\phi_{ikjk}-\frac{25}{432}\partial_i\bar{b}^j\phi_{kijk}-\frac{1}{24}\phi_{kk|j,ij|i}-\frac{55}{864}\phi_{ik|j,ij|k}-\frac{25}{864}\phi_{ki|j,ij|k}\\
&-\frac{1}{12}\phi_{kk|i,i|j,j}-\frac{19}{432}\phi_{jk|i,i|j,k}-\frac{19}{432}\phi_{kj|i,i|j,k}-\frac{35}{288}\phi_{k,k|j|i,i|j}-\frac{7}{144}\phi_{k,j|k|i,i|j}-\frac{7}{144}\phi_{j,k|k|i,i|j}\\
&-\frac{19}{288}\phi_{j|k,k|i,i|j}-\frac{1}{24}\phi_{k|k,j|i,i|j}-\frac{1}{24}\phi_{k|j,k|i,i|j}.
\end{aligned}$$

2.2.2 I, II; I, I and I, I; I, II

The coefficient for the case I, II; I, I is

$$\begin{aligned}
C_{\mathrm{I,II;I,I}}=&-\frac{1}{24}\phi_{i,i|j,j|kk}-\frac{1}{54}\phi_{i,j|k,k|ij}-\frac{1}{54}\phi_{i,j|k,k|ji}-\frac{1}{24}\partial_i\bar{b}^j\phi_{ijkk}-\frac{1}{48}\phi_{i|j,ij|kk}\\
&-\frac{17}{432}\partial_j\bar{b}^k\phi_{ikij}-\frac{17}{864}\phi_{i|k,jk|ij}-\frac{11}{432}\partial_j\bar{b}^k\phi_{ikji}-\frac{11}{864}\phi_{i|k,jk|ji}-\frac{1}{32}\phi_{i|j,j|k,k|i}\\
&-\frac{1}{48}\phi_{i|j,j|k,i|k}-\frac{1}{48}\phi_{i|j,j|i,k|k}-\frac{1}{32}\phi_{i|j,j|i|k,k}-\frac{1}{72}\phi_{i|j,j|k|k,i}-\frac{1}{72}\phi_{i|j,j|k|i,k}.
\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{I,I;I,II}} =& -\frac{1}{24}\phi_{kk|i,i|j,j} - \frac{1}{54}\phi_{ij|i,j|k,k} - \frac{1}{54}\phi_{ji|i,j|k,k} - \frac{1}{24}\partial_i\bar{b}^j\phi_{kkij} - \frac{1}{48}\phi_{kki|j,ij}\\ &- \frac{17}{432}\partial_j\bar{b}^k\phi_{ijik} - \frac{17}{864}\phi_{iji|k,jk} - \frac{11}{432}\partial_j\bar{b}^k\phi_{jiik} - \frac{11}{864}\phi_{jii|k,jk} - \frac{1}{32}\phi_{k,k|ii|j,j}\\ &- \frac{1}{48}\phi_{k,i|ki|j,j} - \frac{1}{48}\phi_{i,k|ki|j,j} - \frac{1}{32}\phi_{i|k,k|i|j,j} - \frac{1}{72}\phi_{k|k,i|i|j,j} - \frac{1}{72}\phi_{k|i,k|i|j,j}.\end{aligned}$$

2.2.3 II, III; I, I and I, I; II, III

The coefficient for the case II, III; I, I is

$$\begin{aligned}C_{\mathrm{II,III;I,I}} =& \frac{13}{144}\phi_{i,i|j|k,k|j} + \frac{1}{36}\phi_{i,i|j|j,k|k} + \frac{1}{36}\phi_{i,i|j|k,j|k} + \frac{1}{36}\phi_{k,k|ij|j,i}\\ &+ \frac{1}{36}\phi_{k,k|ij|i,j} + \frac{5}{144}\phi_{k,k|ii|j,j} + \frac{13}{144}\partial_i\bar{b}^k\phi_{kjij} + \frac{13}{288}\phi_{k,ik|jij}\\ &+ \frac{1}{48}\partial_i\bar{b}^k\phi_{kjji} + \frac{1}{96}\phi_{k,ik|jji} + \frac{7}{144}\phi_{k,k|i,j|ji} + \frac{1}{48}\phi_{k,k|i,j|ij}.\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{I,I;II,III}} =& \frac{13}{144}\phi_{k,k|j|i,i|j} + \frac{1}{36}\phi_{j,k|k|i,i|j} + \frac{1}{36}\phi_{k,j|k|i,i|j} + \frac{1}{36}\phi_{j|j,i|k,k|i}\\ &+ \frac{1}{36}\phi_{j|i,j|k,k|i} + \frac{5}{144}\phi_{i|j,j|k,k|i} + \frac{13}{144}\partial_i\bar{b}^k\phi_{ijkj} + \frac{13}{288}\phi_{ij|k,ik|j}\\ &+ \frac{1}{48}\partial_i\bar{b}^k\phi_{jikj} + \frac{1}{96}\phi_{jik,ik|j} + \frac{7}{144}\phi_{ji|k,k|i,j} + \frac{1}{48}\phi_{ij|k,k|i,j}.\end{aligned}$$

2.2.4 III, II; I, I and I, I; III, II

The coefficient for the case III, II; I, I is

$$\begin{aligned}C_{\mathrm{III,II;I,I}} =& \frac{1}{8}\partial_i\bar{b}^k\phi_{ikjj} + \frac{1}{16}\phi_{i|k,ik|jj} + \frac{11}{144}\partial_j\bar{b}^k\phi_{ikij} + \frac{11}{288}\phi_{i|k,jk|ij} + \frac{1}{16}\partial_j\bar{b}^k\phi_{ikji}\\ &+ \frac{1}{32}\phi_{i|k,jk|ji} + \frac{1}{24}\phi_{i|k,k|j,i|j} + \frac{1}{24}\phi_{i|k,k|i,j|j} + \frac{1}{16}\phi_{i|k,k|j,j|i} + \frac{1}{16}\phi_{i|k,k|i|j,j}\\ &+ \frac{1}{36}\phi_{i|k,k|j|j,i} + \frac{1}{36}\phi_{i|k,k|j|i,j} + \frac{1}{48}\phi_{i,j|k,k|ij} + \frac{5}{144}\phi_{i,j|k,k|ji}.\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{I,I;III,II}} =& \frac{1}{8}\partial_i\bar{b}^k\phi_{jjik} + \frac{1}{16}\phi_{jji|k,ik} + \frac{11}{144}\partial_j\bar{b}^k\phi_{ijik} + \frac{11}{288}\phi_{iji|k,jk} + \frac{1}{16}\partial_j\bar{b}^k\phi_{jiik}\\ &+ \frac{1}{32}\phi_{jii|k,jk} + \frac{1}{24}\phi_{j,i|ji|k,k} + \frac{1}{24}\phi_{i,j|ji|k,k} + \frac{1}{16}\phi_{j,j|ii|k,k} + \frac{1}{16}\phi_{i|j,j|i|k,k}\\ &+ \frac{1}{36}\phi_{j|j,i|i|k,k} + \frac{1}{36}\phi_{j|i,j|i|k,k} + \frac{1}{48}\phi_{ij|i,j|k,k} + \frac{5}{144}\phi_{ji|i,j|k,k}.\end{aligned}$$

2.2.5 I, II; I, III and I, III; I, II

The coefficient for the case I, II; I, III is

$$\begin{aligned}C_{\mathrm{I,II;I,III}} =& \frac{1}{216}\phi_{i,j|k,k|ij} + \frac{1}{216}\phi_{i,j|k,k|ji} + \frac{1}{54}\partial_j\bar{b}^k\phi_{ikij} + \frac{1}{108}\phi_{i|k,jk|ij}\\ &+ \frac{1}{216}\partial_j\bar{b}^k\phi_{ikji} + \frac{1}{432}\phi_{i|k,jk|ji} + \frac{1}{96}\phi_{i|k,k|j,j|i} + \frac{1}{144}\phi_{i|k,k|j|i,j}.\end{aligned}$$

By symmetry,

$$C_{\mathrm{I,III;I,II}} = \frac{1}{216}\phi_{ij|i,j|k,k} + \frac{1}{216}\phi_{ji|i,j|k,k} + \frac{1}{54}\partial_j \bar{b}^k \phi_{ijik} + \frac{1}{108}\phi_{iji|k,jk} + \frac{1}{216}\partial_j \bar{b}^k \phi_{jiik} + \frac{1}{432}\phi_{jii|k,jk} + \frac{1}{96}\phi_{j,j|ii|k,k} + \frac{1}{144}\phi_{j|i,j|i|k,k}.$$

2.2.6 II, I; I, III and I, III; II, I

The coefficient for the case II, I; I, III is

$$C_{\mathrm{II,I;I,III}} = \frac{1}{108}\partial_j \bar{b}^k \phi_{kiij} + \frac{1}{216}\phi_{k,jk|iij} + \frac{17}{216}\partial_j \bar{b}^k \phi_{kiji} + \frac{17}{432}\phi_{k,jk|iji} + \frac{7}{432}\phi_{k,k|i,j|ji} + \frac{7}{432}\phi_{k,k|i,j|ij} + \frac{7}{96}\phi_{k,k|i|j,j|i} + \frac{5}{144}\phi_{k,k|ij|i,j}.$$

By symmetry,

$$C_{\mathrm{I,III;II,I}} = \frac{1}{108}\partial_j \bar{b}^k \phi_{ijki} + \frac{1}{216}\phi_{ij|k,jk|i} + \frac{17}{216}\partial_j \bar{b}^k \phi_{jiki} + \frac{17}{432}\phi_{ji|k,jk|i} + \frac{7}{432}\phi_{ji|k,k|i,j} + \frac{7}{432}\phi_{ij|k,k|i,j} + \frac{7}{96}\phi_{j,j|i|k,k|i} + \frac{5}{144}\phi_{j|i,j|k,k|i}.$$

2.2.7 III, II; I, III and I, III; III, II

The coefficient for the case III, II; I, III is

$$C_{\mathrm{III,II;I,III}} = -\frac{1}{36}\partial_j \bar{b}^k \phi_{ikij} - \frac{1}{72}\phi_{i|k,jk|ij} - \frac{1}{72}\partial_j \bar{b}^k \phi_{ikji} - \frac{1}{144}\phi_{i|k,jk|ji}.$$

By symmetry,

$$C_{\mathrm{I,III;III,II}} = -\frac{1}{36}\partial_j \bar{b}^k \phi_{ijik} - \frac{1}{72}\phi_{iji|k,jk} - \frac{1}{72}\partial_j \bar{b}^k \phi_{jiik} - \frac{1}{144}\phi_{jii|k,jk}.$$

2.2.8 II, III; I, III and I, III; II, III

The coefficient for the case II, III; I, III is

$$C_{\mathrm{II,III;I,III}} = -\frac{5}{72}\partial_j \bar{b}^k \phi_{kiji} - \frac{5}{144}\phi_{k,jk|iji} - \frac{1}{36}\phi_{k,k|i,j|ji} - \frac{1}{16}\phi_{k,k|i|j,j|i} - \frac{1}{36}\phi_{k,k|ij|i,j}.$$

By symmetry,

$$C_{\mathrm{I,III;II,III}} = -\frac{5}{72}\partial_j \bar{b}^k \phi_{jiki} - \frac{5}{144}\phi_{ji|k,jk|i} - \frac{1}{36}\phi_{ji|k,k|i,j} - \frac{1}{16}\phi_{j,j|i|k,k|i} - \frac{1}{36}\phi_{j|i,j|k,k|i}.$$

2.2.9 I, II; III, I and III, I; I, II

The coefficient for the case I, II; III, I is

$$C_{\mathrm{I,II;III,I}} = \frac{1}{12}\phi_{i,i|k,k|jj} + \frac{7}{216}\phi_{i,j|k,k|ij} + \frac{7}{216}\phi_{i,j|k,k|ji} + \frac{1}{12}\partial_i \bar{b}^k \phi_{ikjj} + \frac{1}{24}\phi_{i|k,ik|jj} + \frac{13}{216}\partial_j \bar{b}^k \phi_{ikij} + \frac{13}{432}\phi_{i|k,jk|ij} + \frac{5}{108}\partial_j \bar{b}^k \phi_{ikji} + \frac{5}{216}\phi_{i|k,jk|ji} + \frac{5}{96}\phi_{i|k,k|i|j,j} + \frac{5}{144}\phi_{i|k,k|j|j,i} + \frac{5}{144}\phi_{i|k,k|j|i,j} + \frac{1}{24}\phi_{i|k,k|j,i|j} + \frac{1}{48}\phi_{i|k,k|i,j|j}.$$

By symmetry,

$$\begin{aligned}C_{\mathrm{III,I;I,II}} =&\frac{1}{12}\phi_{jj|i,i|k,k}+\frac{7}{216}\phi_{ij|i,j|k,k}+\frac{7}{216}\phi_{ji|i,j|k,k}+\frac{1}{12}\partial_i\bar{b}^k\phi_{jjik}+\frac{1}{24}\phi_{jji|k,ik}\\&+\frac{13}{216}\partial_j\bar{b}^k\phi_{ijik}+\frac{13}{432}\phi_{iji|k,jk}+\frac{5}{108}\partial_j\bar{b}^k\phi_{jiik}+\frac{5}{216}\phi_{jii|k,jk}+\frac{5}{96}\phi_{i|j,j|i|k,k}\\&+\frac{5}{144}\phi_{j|j,i|i|k,k}+\frac{5}{144}\phi_{j|i,j|i|k,k}+\frac{1}{24}\phi_{j,i|ji|k,k}+\frac{1}{48}\phi_{i,j|ji|k,k}.\end{aligned}$$

2.2.10 II, I; III, I and III, I; II, I

The coefficient for the case II, I; III, I is

$$\begin{aligned}C_{\mathrm{II,I;III,I}} =&\frac{1}{6}\partial_i\bar{b}^k\phi_{kijj}+\frac{1}{12}\phi_{k,ik|ijj}+\frac{13}{216}\partial_j\bar{b}^k\phi_{kiij}+\frac{13}{432}\phi_{k,jk|iij}+\frac{19}{108}\partial_j\bar{b}^k\phi_{kiji}\\&+\frac{19}{216}\phi_{k,jk|iji}+\frac{1}{6}\phi_{k,k|i,i|jj}+\frac{31}{432}\phi_{k,k|i,j|ij}+\frac{31}{432}\phi_{k,k|i,j|ji}+\frac{11}{96}\phi_{k,k|ii|j,j}\\&+\frac{13}{144}\phi_{k,k|ij|j,i}+\frac{13}{144}\phi_{k,k|ij|i,j}+\frac{1}{12}\phi_{k,k|i|j,i|j}+\frac{1}{16}\phi_{k,k|i|i,j|j}.\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{III,I;II,I}} =&\frac{1}{6}\partial_i\bar{b}^k\phi_{jjki}+\frac{1}{12}\phi_{jj|k,ik|i}+\frac{13}{216}\partial_j\bar{b}^k\phi_{ijki}+\frac{13}{432}\phi_{ij|k,jk|i}+\frac{19}{108}\partial_j\bar{b}^k\phi_{jiki}\\&+\frac{19}{216}\phi_{ji|k,jk|i}+\frac{1}{6}\phi_{jj|k,k|i,i}+\frac{31}{432}\phi_{ij|k,k|i,j}+\frac{31}{432}\phi_{ji|k,k|i,j}+\frac{11}{96}\phi_{i|j,j|k,k|i}\\&+\frac{13}{144}\phi_{j|j,i|k,k|i}+\frac{13}{144}\phi_{j|i,j|k,k|i}+\frac{1}{12}\phi_{j,i|j|k,k|i}+\frac{1}{16}\phi_{i,j|j|k,k|i}.\end{aligned}$$

2.2.11 III, II; III, I and III, I; III, II

The coefficient for the case III, II; III, I is

$$\begin{aligned}C_{\mathrm{III,II;III,I}} =&-\frac{1}{4}\partial_i\bar{b}^k\phi_{ikjj}-\frac{1}{8}\phi_{i|k,ik|jj}-\frac{1}{8}\partial_j\bar{b}^k\phi_{ikij}-\frac{1}{16}\phi_{i|k,jk|ij}-\frac{1}{9}\partial_j\bar{b}^k\phi_{ikji}\\&-\frac{1}{18}\phi_{i|k,jk|ji}-\frac{5}{72}\phi_{i|k,k|j|j,i}-\frac{5}{48}\phi_{i|k,k|i|j,j}-\frac{5}{72}\phi_{i|k,k|j|i,j}\\&-\frac{1}{12}\phi_{i|k,k|j,i|j}-\frac{1}{24}\phi_{i|k,k|i,j|j}-\frac{1}{18}\phi_{i,j|k,k|ji}-\frac{1}{24}\phi_{i,j|k,k|ij}.\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{III,I;III,II}} =&-\frac{1}{4}\partial_i\bar{b}^k\phi_{jjik}-\frac{1}{8}\phi_{jji|k,ik}-\frac{1}{8}\partial_j\bar{b}^k\phi_{ijik}-\frac{1}{16}\phi_{iji|k,jk}-\frac{1}{9}\partial_j\bar{b}^k\phi_{jiik}\\&-\frac{1}{18}\phi_{jii|k,jk}-\frac{5}{72}\phi_{j|j,i|i|k,k}-\frac{5}{48}\phi_{i|j,j|i|k,k}-\frac{5}{72}\phi_{j|i,j|i|k,k}\\&-\frac{1}{12}\phi_{j,i|ji|k,k}-\frac{1}{24}\phi_{i,j|ji|k,k}-\frac{1}{18}\phi_{ji|i,j|k,k}-\frac{1}{24}\phi_{ij|i,j|k,k}.\end{aligned}$$

2.2.12 II, III; III, I and III, I; II, III

The coefficient for the case II, III; III, I is

$$\begin{aligned}C_{\mathrm{II,III;III,I}} =&-\frac{1}{16}\phi_{k,k|ii|j,j}-\frac{1}{18}\phi_{k,k|ij|j,i}-\frac{1}{18}\phi_{k,k|ij|i,j}-\frac{1}{24}\phi_{k,k|i|j,i|j}\\&-\frac{1}{18}\phi_{k,jk|iji}-\frac{1}{9}\partial_j\bar{b}^k\phi_{kiji}-\frac{5}{72}\phi_{k,k|i,j|ji}-\frac{1}{24}\partial_j\bar{b}^k\phi_{kiij}\\&-\frac{1}{48}\phi_{k,jk|iij}-\frac{1}{24}\phi_{k,k|i,j|ij}-\frac{1}{24}\phi_{k,k|i|i,j|j}.\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{III,I;II,III}} =& -\frac{1}{16}\phi_{i|j,j|k,k|i} - \frac{1}{18}\phi_{j|j,i|k,k|i} - \frac{1}{18}\phi_{j|i,j|k,k|i} - \frac{1}{24}\phi_{j,i|j|k,k|i}\\ &- \frac{1}{18}\phi_{ji|k,jk|i} - \frac{1}{9}\partial_j\bar{b}^k\phi_{jiki} - \frac{5}{72}\phi_{ji|k,k|i,j} - \frac{1}{24}\partial_j\bar{b}^k\phi_{ijki}\\ &- \frac{1}{48}\phi_{ij|k,jk|i} - \frac{1}{24}\phi_{ij|k,k|i,j} - \frac{1}{24}\phi_{i,j|j|k,k|i}.\end{aligned}$$

2.2.13 I, II; III, III and III, III; I, II

The coefficient for the case I, II; III, III is

$$\begin{aligned}C_{\mathrm{I,II;III,III}} =& -\frac{1}{54}\phi_{i,j|k,k|ji} - \frac{1}{54}\phi_{i,j|k,k|ij} - \frac{5}{108}\partial_j\bar{b}^k\phi_{ikij} - \frac{5}{216}\phi_{i|k,jk|ij}\\ &- \frac{1}{54}\partial_j\bar{b}^k\phi_{ikji} - \frac{1}{108}\phi_{i|k,jk|ji} - \frac{1}{36}\phi_{i|k,k|j|i,j}.\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{III,III;I,II}} =& -\frac{1}{54}\phi_{ji|i,j|k,k} - \frac{1}{54}\phi_{ij|i,j|k,k} - \frac{5}{108}\partial_j\bar{b}^k\phi_{ijik} - \frac{5}{216}\phi_{iji|k,jk}\\ &- \frac{1}{54}\partial_j\bar{b}^k\phi_{jiik} - \frac{1}{108}\phi_{jii|k,jk} - \frac{1}{36}\phi_{j|i,j|i|k,k}.\end{aligned}$$

2.2.14 II, I; III, III and III, III; II, I

The coefficient for the case II, I; III, III is

$$\begin{aligned}C_{\mathrm{II,I;III,III}} =& -\frac{4}{27}\partial_j\bar{b}^k\phi_{kiji} - \frac{4}{54}\phi_{k,jk|iji} - \frac{1}{27}\partial_j\bar{b}^k\phi_{kiij} - \frac{1}{54}\phi_{k,jk|iij}\\ &- \frac{19}{432}\phi_{k,k|i,j|ij} - \frac{19}{432}\phi_{k,k|i,j|ji} - \frac{7}{72}\phi_{k,k|ij|i,j}.\end{aligned}$$

By symmetry,

$$\begin{aligned}C_{\mathrm{III,III;II,I}} =& -\frac{4}{27}\partial_j\bar{b}^k\phi_{jiki} - \frac{4}{54}\phi_{ji|k,jk|i} - \frac{1}{27}\partial_j\bar{b}^k\phi_{ijki} - \frac{1}{54}\phi_{ij|k,jk|i}\\ &- \frac{19}{432}\phi_{ij|k,k|i,j|} - \frac{19}{432}\phi_{ji|k,k|i,j|} - \frac{7}{72}\phi_{j|i,j|k,k|i}.\end{aligned}$$

2.2.15 II, III; III, III and III, III; II, III

The coefficient for the case II, III; III, III is

$$C_{\mathrm{II,III;III,III}} = \frac{1}{9}\partial_j\bar{b}^k\phi_{kiji} + \frac{1}{18}\phi_{k,jk|iji} + \frac{5}{72}\phi_{k,k|i,j|ji} + \frac{5}{72}\phi_{k,k|ij|i,j}.$$

By symmetry,

$$C_{\mathrm{III,III;II,III}} = \frac{1}{9}\partial_j\bar{b}^k\phi_{jiki} + \frac{1}{18}\phi_{ji|k,jk|i} + \frac{5}{72}\phi_{ji|k,k|i,j} + \frac{5}{72}\phi_{j|i,j|k,k|i}.$$

2.2.16 III, II; III, III and III, III; III, II

The coefficient for the case III, II; III, III is

$$\begin{aligned} C_{\text{III,III;III,II}} =& \frac{1}{12}\partial_j \bar{b}^k \phi_{ikij} + \frac{1}{24}\phi_{i|k,jk|ij} + \frac{1}{18}\partial_j \bar{b}^k \phi_{ikji} + \frac{1}{36}\phi_{i|k,jk|ji} \\ &+ \frac{1}{18}\phi_{i,j|k,k|ji} + \frac{1}{18}\phi_{i|k,k|j|i,j}. \end{aligned}$$

By symmetry,

$$\begin{aligned} C_{\text{III,III;III,II}} =& \frac{1}{12}\partial_j \bar{b}^k \phi_{ijik} + \frac{1}{24}\phi_{iji|k,jk} + \frac{1}{18}\partial_j \bar{b}^k \phi_{jiik} + \frac{1}{36}\phi_{jii|k,jk} \\ &+ \frac{1}{18}\phi_{ji|i,j|k,k} + \frac{1}{18}\phi_{j|i,j|i|k,k}. \end{aligned}$$