Appendix B. Computation for $\Phi_3^{(m)}$

The asymptotic expression for $\Phi_3^{(m)}$ in the massless limit is presented here. The arguments may help the reader to understand the estimates for general $\Phi_n^{(m)}$. Recall $\mathfrak{A}_n^{(m)}(t,p) := \underset{0\le t_1<\cdots<t_n\le t}{\int\cdots\int} \bigotimes_{j=1}^n \left(-\frac{\mathcal{M}}{m}e^{-\frac{\mathcal{M}}{m}t_j}p\right)\,\mathrm{d}t_1\ldots\mathrm{d}t_n$ and $\widetilde{\mathbf{\Sigma}}_\sigma := \int_0^\sigma e^{-\mathcal{M}\varsigma}e^{-\mathcal{M}^*\varsigma}\,\mathrm{d}\varsigma$. The proposition below is a special case of Theorem 4.2:

Proposition B.1. *The 3^{rd} level of the expected signature takes the form $\Phi_3^{(m)}(t,p) = \mathfrak{A}_3^{(m)}(t,p) + \mathfrak{B}_3^{(m)}(t,p) + \mathcal{E}_3^{(m)}(t,p)$, where $\mathfrak{B}_3^{(m)}(t,p)$ and $\mathcal{E}_3^{(m)}(t,p)$ are degree-1 homogeneous polynomials in p. Indeed, $\mathfrak{B}_3^{(m)}(t,p) = \mathfrak{A}_1^{(m)}(t,p)\otimes\left\{\mathcal{M}\cdot \fint_0^{\frac{t}{m}}\widetilde{\mathbf{\Sigma}}_\sigma\,\mathrm{d}\sigma - \frac{\mathrm{Id}}{2}\right\}t$ and $\left\|\mathcal{E}_3^{(m)}(t,p)\right\| \le C\left(K,\Lambda,\lambda^{-1},t,d\right)|p|\cdot m$ as $m\to 0^+$.*

Proof of Proposition B.1. From Lemma 4.7 and the subsequent remarks it follows that $\mathfrak{A}_3^{(m)}(t,p)$, $\mathfrak{B}_3^{(m)}(t,p)$, and $\mathcal{E}_3^{(m)}(t,p)$ are polynomials in p of degrees 3, 1, and 1, respectively. Their homogeneity in p holds by the explicit formulae derived below. By the expression (4.5) for Φ_2, the Feynmann–Kac formula (3.2), and the structure of the source term in Eq. (3.1), we find that

$$\begin{aligned}\Phi_3^{(m)}(t,p) = \mathbb{E}^p\Bigg[-\int_0^t \frac{\mathcal{M}}{m}P_s\otimes\Bigg\{\mathfrak{A}_2^{(m)}(t-s,P_s) + \Bigg(\mathcal{M}\cdot\fint_0^{\frac{t-s}{m}}\widetilde{\mathbf{\Sigma}}_\sigma\,\mathrm{d}\sigma - \frac{\mathrm{Id}}{2}\Bigg)(t-s) + \mathcal{E}_2^{(m)}(t-s)\Bigg\}\,\mathrm{d}s\Bigg]\\ + \sum_{j=1}^d e_j\otimes\mathbb{E}^p\left[\int_0^t\left(\partial_{p_j}\mathfrak{A}_2^{(m)}\right)(t-s,P_s)\,\mathrm{d}s\right] + \frac{1}{2}\sum_{j=1}^d e_j\otimes e_j\otimes\mathbb{E}^p\left[\int_0^t\left\{e^{-\frac{\mathcal{M}}{m}(t-s)}-\mathrm{Id}\right\}P_s\,\mathrm{d}s\right].\end{aligned}$$

Let us divide the remaining parts of the proof into five steps below.

Step 1. As $\left\|e^{-\frac{\mathcal{M}}{m}s}\right\| \le Ke^{-\frac{\lambda s}{m}}$, it holds by Lebesgue's dominated convergence theorem that

$$\begin{aligned}\left|\mathbb{E}^p\left[\int_0^t\left\{e^{-\frac{\mathcal{M}}{m}(t-s)}-\mathrm{Id}\right\}P_s\,\mathrm{d}s\right]\right| &= \left|\left(\int_0^t\left\{e^{-\frac{\mathcal{M}}{m}(t-s)}-\mathrm{Id}\right\}e^{-\frac{\mathcal{M}}{m}s}\,\mathrm{d}s\right)\cdot p\right|\\ &= \left|\left\{\int_0^t\left(e^{-\frac{\mathcal{M}}{m}t}-e^{-\frac{\mathcal{M}}{m}s}\right)\mathrm{d}s\right\}\cdot p\right| \le Kd\left(\frac{m}{\lambda}+te^{-\frac{\lambda t}{m}}\right)|p|.\end{aligned}$$

Hence, this term is a degree-1 homogeneous polynomial in p, which satisfies

$$\frac{1}{2}\left\|\sum_{j=1}^d e_j\otimes e_j\otimes\mathbb{E}^p\left[\int_0^t\left\{e^{-\frac{\mathcal{M}}{m}(t-s)}-\mathrm{Id}\right\}P_s\,\mathrm{d}s\right]\right\| \le \frac{Kd}{2}\left(\frac{m}{\lambda}+te^{-\frac{\lambda t}{m}}\right)|p|. \tag{B.1}$$

Step 2. The term $\mathbb{E}^p\left[-\int_0^t\frac{\mathcal{M}}{m}P_s\otimes\mathcal{E}_2^{(m)}(t-s)\,\mathrm{d}s\right]$ is also easy to control. Note that the error $\mathcal{E}_2^{(m)}$ inherited from the previous level ($n=2$) is independent of p; so this term equals $\int_0^t\left(-\frac{\mathcal{M}}{m}\right)e^{-\frac{\mathcal{M}}{m}s}\otimes\mathcal{E}_2^{(m)}(t-s)\,\mathrm{d}s$. We may use Eq. (4.6) and the simple bound $\int_0^t\left\|\left(-\frac{\mathcal{M}}{m}\right)e^{-\frac{\mathcal{M}}{m}s}\right\|\mathrm{d}s \le \frac{\Lambda Kd}{m}\int_0^t e^{-\frac{\lambda}{m}s}\,\mathrm{d}s \le \frac{\Lambda Kd}{\lambda}$ to estimate that

$$\left\|\mathbb{E}^p\left[-\int_0^t\frac{\mathcal{M}}{m}P_s\otimes\mathcal{E}_2^{(m)}(t-s)\,\mathrm{d}s\right]\right\| \le m\,\frac{\Lambda Kd^2}{\lambda}\left(\frac{K}{\lambda}+\frac{\Lambda K^3d^3}{2\lambda^2}\right)|p|. \tag{B.2}$$

Step 3. We now estimate $\mathbb{E}^p\left[-\int_0^t\frac{\mathcal{M}}{m}P_s\otimes\left(\mathcal{M}\cdot\fint_0^{\frac{t-s}{m}}\widetilde{\mathbf{\Sigma}}_\sigma\,\mathrm{d}\sigma-\frac{\mathrm{Id}}{2}\right)(t-s)\,\mathrm{d}s\right]$. In light of the dominated convergence theorem, this term is equal to $\int_0^t\left(-\frac{\mathcal{M}}{m}\right)e^{-\frac{\mathcal{M}}{m}s}p\otimes\left(\mathcal{M}\cdot\fint_0^{\frac{t-s}{m}}\widetilde{\mathbf{\Sigma}}_\sigma\,\mathrm{d}\sigma-\frac{\mathrm{Id}}{2}\right)(t-$

$s)\,\mathrm{d}s$. Let us write

$$\Xi^{(m)}(T) := \mathcal{M} \cdot \fint_0^T \widetilde{\mathbf{\Sigma}}_\varsigma \,\mathrm{d}\varsigma - \frac{\mathrm{Id}}{2}. \tag{B.3}$$

To control the previous term, first note that

$$\sup_{T>0} \left\| \Xi^{(m)}(T) \right\| \leq \frac{\Lambda K^2 d^2}{2\lambda} + \frac{1}{2}, \tag{B.4}$$

thanks to Eq. (4.4). Hence,

$$\begin{aligned}\left\| \int_0^t \left(-\frac{\mathcal{M}}{m}\right) e^{-\frac{\mathcal{M}}{m}s} p \otimes \Xi^{(m)}(t-s) s \,\mathrm{d}s \right\| &\leq \left(\frac{\Lambda K^2 d^2}{2\lambda} + \frac{1}{2}\right) d^2 |p| \int_0^t \frac{\Lambda}{m} \cdot K e^{-\frac{\lambda}{m}s} s \,\mathrm{d}s \\ &\leq \left(\frac{\Lambda K^2 d^2}{2\lambda} + \frac{1}{2}\right) d^2 |p| \frac{\Lambda K}{m} \int_0^\infty e^{-\frac{\lambda}{m}s} s = \left(\frac{\Lambda K^2 d^2}{2\lambda} + \frac{1}{2}\right) \frac{\Lambda K d^2}{\lambda^2} |p| \cdot m\end{aligned}$$

To proceed, we estimate by the triangle inequality and change of variables:

$$\begin{aligned}\left\| \int_0^t \left(-\frac{\mathcal{M}}{m}\right) e^{-\frac{\mathcal{M}}{m}s} p \otimes \Xi^{(m)}(t-s) t \,\mathrm{d}s \right\| &\leq \left\| \int_0^t \left(-\frac{\mathcal{M}}{m}\right) e^{-\frac{\mathcal{M}}{m}s} p \otimes \Xi^{(m)}(t) \,\mathrm{d}s \right\| \cdot t \\ &+ \left\| \int_0^{\frac{t}{m}} (-\mathcal{M})\, e^{-\mathcal{M}\sigma} p \otimes \left\{ \Xi^{(m)}(t) - \Xi^{(m)}(t - m\sigma) \right\} \,\mathrm{d}\sigma \right\| \cdot t =: J_1 + J_2.\end{aligned}$$

For the second term J_2, we utilise Lemma 4.6 and Eq. (B.4) for $\mathbf{\Sigma}$ to bound

$$\left\| \Xi^{(m)}(t) - \Xi^{(m)}(t - m\sigma) \right\| \leq \left(\frac{\Lambda K^2 d^2}{2\lambda} + \frac{1}{2}\right) \frac{2m\sigma}{t} \leq \frac{2\Lambda K^2 d^2}{\lambda} \cdot \frac{m\sigma}{t}. \tag{B.5}$$

Hence

$$|J_2| \leq \frac{2\Lambda K^2 d^2}{\lambda} \frac{2m}{t} \cdot \Lambda K d^2 |p| \int_0^{\frac{t}{m}} e^{-\lambda\sigma} \sigma \,\mathrm{d}\sigma \cdot t \leq m \cdot \frac{4\Lambda^2 K^3 d^4}{\lambda^3} |p|.$$

This together with J_1 justifies the limit

$$\lim_{m \to 0^+} \left\{ \int_0^t \left(-\frac{\mathcal{M}}{m}\right) e^{-\frac{\mathcal{M}}{m}s} p \otimes \Xi^{(m)}(t-s) t \,\mathrm{d}s \right\} = t \left\{ \lim_{m \to 0^+} \left[\int_0^t \left(-\frac{\mathcal{M}}{m}\right) e^{-\frac{\mathcal{M}}{m}s} p \,\mathrm{d}s \right] \otimes \Xi^{(m)}(t) \right\}.$$

Furthermore, recall from §4.3.1 that $\int_0^t \left(-\frac{\mathcal{M}}{m}\right) e^{-\frac{\mathcal{M}}{m}s} p \,\mathrm{d}s = \left(-\mathrm{Id} + e^{-\frac{\mathcal{M}}{m}t}\right) p = \mathfrak{A}_1^{(m)}(p,t)$.

Putting together the above arguments in this step, we arrive at the bound

$$\begin{aligned}&\mathbb{E}^p \left[-\int_0^t \frac{\mathcal{M}}{m} P_s \otimes \left(\mathcal{M} \cdot \fint_0^{\frac{t}{m}} \widetilde{\mathbf{\Sigma}}_\sigma \,\mathrm{d}\sigma - \frac{\mathrm{Id}}{2} \right) (t-s) \,\mathrm{d}s \right] \\ &\qquad = \mathfrak{A}_1^{(m)}(p,t) \otimes \left(\mathcal{M} \cdot \fint_0^{\frac{t}{m}} \widetilde{\mathbf{\Sigma}}_\sigma \,\mathrm{d}\sigma - \frac{\mathrm{Id}}{2} \right) t + \mathcal{R}(K, m, t, \Lambda, \lambda, p),\end{aligned} \tag{B.6}$$

where $\left\| \mathcal{R}(K, m, t, \Lambda, \lambda, p) \right\| \leq m \cdot \frac{6\Lambda^2 K^3 d^4}{\lambda^3} |p|$. (Recall that $0 < \lambda < \Lambda < \infty$.)

Step 4. Next we turn to $\sum_{j=1}^d e_j \otimes \mathbb{E}^p \left[\int_0^t \left(\partial_{p_j} \mathfrak{A}_2^{(m)} \right) (P_s, t-s) \,\mathrm{d}s \right]$, the only term involving derivatives in p. It is a degree-1 polynomial in p. We argue below that this term $\lesssim \mathcal{O}(m)$.

To make our estimates transparent, let us write all tensors component-wise. By definition of $\mathfrak{A}_2^{(m)}$,

$$\left[\partial_{p_j} \mathfrak{A}_2^{(m)} \right]^{\alpha\beta} (t,p) = \iint_{0 \leq t_1 < t_2 \leq t} \sum_{\delta=1}^d \left(-\mathcal{M} e^{-\mathcal{M} t_1} \right)_j^\alpha \left(-\mathcal{M} e^{-\mathcal{M} t_2} \right)_\delta^\beta p^\delta \mathrm{d}t_1 \mathrm{d}t_2$$

$$+\iint_{0\le t_1<t_2\le t}\sum_{\gamma=1}^{d}\left(-\mathcal{M}e^{-\mathcal{M}t_1}\right)^{\alpha}_{\gamma}p^{\gamma}\left(-\mathcal{M}e^{-\mathcal{M}t_2}\right)^{\beta}_{j}\mathrm{d}t_1\mathrm{d}t_2.$$

Evaluating this tensor at $(t-s,P_s)$, integrating over $s\in[0,t]$, and taking the expectation $\mathbb{E}^p$, we obtain that

$$\left\|\mathbb{E}^p\left[\int_0^t\left(\partial_{p_j}\mathfrak{A}_2^{(m)}\right)(t-s,P_s)\,\mathrm{d}s\right]\right\|\le\underbrace{\int_0^t\iint_{0\le t_1<t_2\le\frac{t-s}{m}}\left\|\left(-\mathcal{M}e^{-\mathcal{M}t_1}\right)^{\alpha}_{j}\left(-\mathcal{M}e^{-\mathcal{M}t_2}e^{-\frac{\mathcal{M}}{m}s}p\right)^{\beta}\right\|\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}s}_{=:\mathcal{K}}+[\cdots].$$

Here $[\cdots]$ is the term obtained from $\mathcal{K}$ by switching t_1 and t_2 in the integrand (thus satisfying the same estimate as for $\mathcal{K}$). On the other hand,

$$\mathcal{K}\le\Lambda^2K^3d^4|p|\int_0^t e^{-\frac{\lambda s}{m}}\iint_{0\le t_1<t_2\le\frac{t-s}{m}}e^{-\lambda(t_1+t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}s\le\frac{\Lambda^2K^3d^4|p|}{2\lambda^2}\int_0^t e^{-\frac{\lambda s}{m}}\,\mathrm{d}s\le\frac{\Lambda^2K^3d^4|p|}{2\lambda^3}\cdot m.$$

The second line follows from the symmetrisation trick (Lemma 4.10). Hence,

$$\left\|\mathbb{E}^p\left[\int_0^t\left(\partial_{p_j}\mathfrak{A}_2^{(m)}\right)(t-s,P_s)\,\mathrm{d}s\right]\right\|\le\frac{\Lambda^2K^3d^4|p|}{\lambda^3}\cdot m. \tag{B.7}$$

Step 5. What is left is the hardest term $\mathbb{E}^p\left[-\int_0^t\frac{\mathcal{M}}{m}P_s\otimes\mathfrak{A}_2^{(m)}(t-s,P_s)\,\mathrm{d}s\right]$. The estimate for this term with $\mathfrak{A}_n^{(m)}$ in lieu of $\mathfrak{A}_2^{(m)}$ for general n only requires a little extra work, so we shall treat for general n at one strike. Indeed, as a special case of Eq. (4.22), one has that

$$\left\|\mathbb{E}^p\left[-\int_0^t\frac{\mathcal{M}}{m}P_s\otimes\mathfrak{A}_2^{(m)}(t-s,P_s)\,\mathrm{d}s\right]-\mathfrak{A}_3^{(m)}(t,p)\right\|\le\frac{\Lambda^3e^{-\lambda}|p|}{2\lambda^4}\cdot m. \tag{B.8}$$

Estimates in Eqs. (B.1), (B.2), (B.6), (B.7), and (B.8) together complete the proof. □

Siran Li: School of Mathematical Sciences and CMA-Shanghai, Shanghai Jiao Tong University, No. 6 Natural Sciences Building, 800 Dongchuan Road, Minhang, Shanghai, China (200240)

Email address: siran.li@sjtu.edu.cn

H. Ni: Department of Mathematics, University College, Room 603, 25 Gordon St, London WC1H 0AY, London, UK; and the Alan Turing Institute, 96 Euston Rd, London NW1 2DB, UK

Email address: h.ni@ucl.ac.uk

Q. Zhu: New York University Shanghai, 567 West Yangsi Road, Pudong New District, Shanghai, China (200126); current address: Center for Computational Science and Engineering, Massachusetts Institute of Technology, 77 Massachusetts Ave Cambridge, MA 02139

Email address: qianyu_z@mit.edu