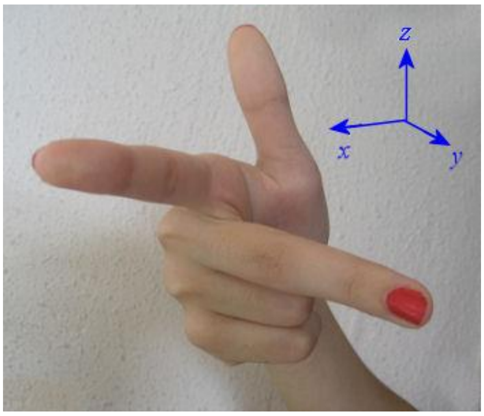
KINEMATICS

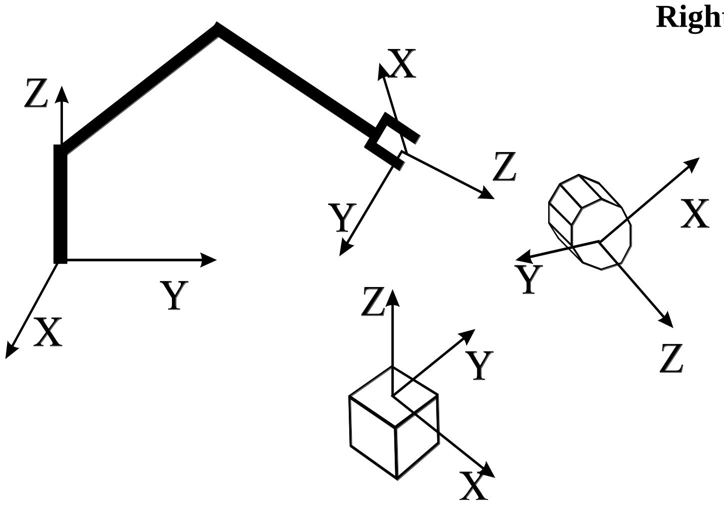
This is the study of motion without regard to the forces that cause it. It refers to all geometrical and time-based properties of motion.

Spatial Descriptions

Around a robot objects are normally defined in terms of a **frame** (i.e. a set of right-handed orthogonal coordinate axes) known as the **origin** or "**world frame**". Usually, this frame coincides with the base of one robot within a work-cell.

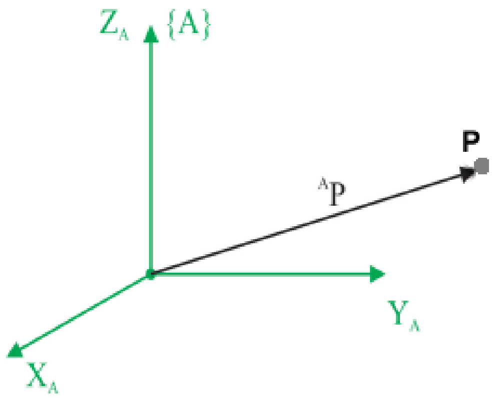


Right-handed coordinate frame



Every object (and the robot tool) within the work-cell is assigned a "conceptual" right-handed frame whose position and orientation is known relative to the origin (see diagram opposite).

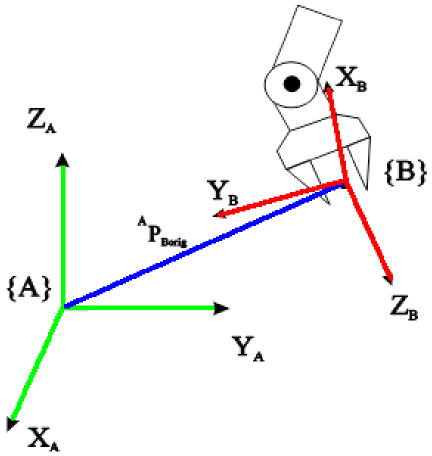
Positions are defined relative to the origin by using 3x1 position vectors.



The diagram opposite shows a frame {A}. The vectors Z_A, Y_A, X_A represent the orthogonal axes associated with this frame. The vector ^AP "points" to a point P whose **position** relative to {A} is known if the values x, y, z of the vector ^AP are known. In general then:

$$^A P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

To find the **orientation** of an object we need to consider the frame attached to the object.



Imagine that the object is the tip of the robot (i.e. the tool). From the diagram opposite, a rotation of the gripper about the Z_B axis will not change the position of the origin of the tool frame {B}. It will, however, cause the robot fingers to rotate (i.e. the Y_B and X_B axes will point in different directions than those shown). Hence an object held in the gripper will also rotate in the same way.

So, finding the **rotation** of frame {B} relative to frame {A}, this will give the **orientation** of frame {B} relative to frame {A}.

Having a generic notation between frames is helpful, and to do we consider e each of the unit vectors of frame {B} (i.e. X_B, Y_B, Z_B). If they are defined relative to frame {A}, the notation will be given as: $^AX_B, {}^AY_B, {}^AZ_B$.

Remember now that any vector **v** can be written in the generic form **v** = x***i*** + y***j*** + z***k***. So applying this to each unit vector $^AX_B, {}^AY_B, {}^AZ_B$, we derive at a 3x3 rotational matrix which describes {B} relative to {A}. In terms of notation we have:

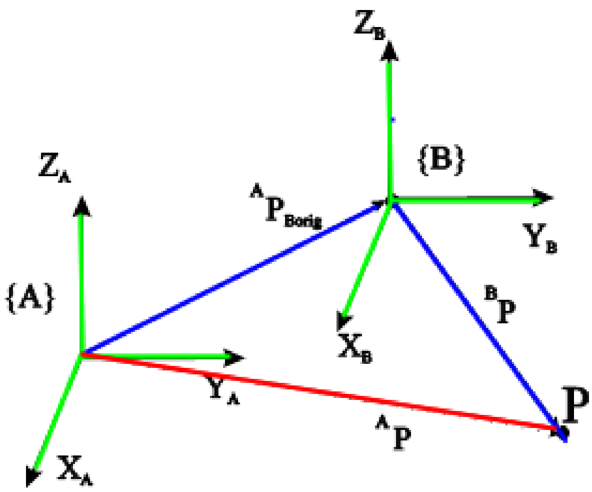
$$^A_BR = \left[{}^AX_B, {}^AY_B, {}^AZ_B\right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

This matrix relationship tells us what rotations we must apply so that, starting from frame {A}, when all is done the orientation will be that of frame {B} (i.e. how we go from {A} to {B})

Translations

In the diagram opposite we define the point P by the vector ^BP. We have also defined the origin of frame {B} relative to {A} through the translation vector $^AP_{Borig}$.

We wish to find the vector ^AP which defines the position of point P relative to frame {A}.



Because both frames have the **same** orientation, a simple vector addition will provide the answer: $^AP = {}^AP_{Borig} + {}^BP$

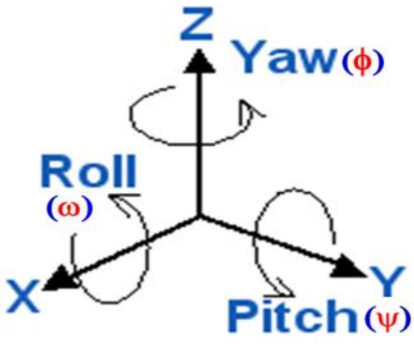
This addition can occur only in the special case of the **<u>same frame orientation.</u>**

Note: All we have done is to evaluate a different description of the point P in space. Nothing else has changed in the system and certainly the point P has not moved.

This is a very important and is central to understanding the way translations work.

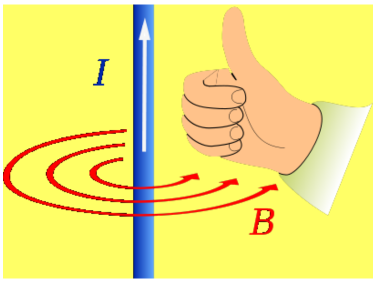
Rotations

On an orthogonal axes system there are 3 rotations that can take place: yaw, ; pitch, ψ ; and roll, ω. They occur about the axes as shown in the diagram.



Rotations can be positive or negative. All the rotations shown in the diagram are positive.

A positive rotation is a clockwise rotation as we look down the axis in the direction the arrow points.

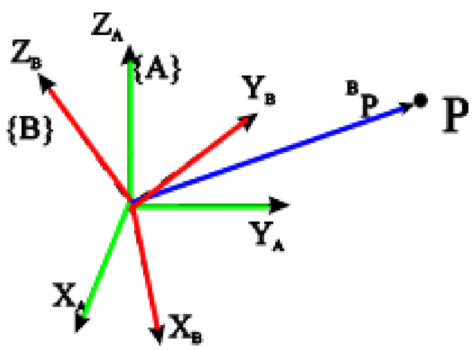


Another way to remember a positive rotation is the right hand rule of current and magnetic flux. With the thumb pointing in the same direction as the current (see figure opposite), the fingers point in the direction of the magnetic flux (clockwise direction looking in the direction of the current flow.)

Positive rotations about the Z, Y and X axes result in the respective rotational matrices

$$R(z) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R(y) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \quad R(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix}$$

These 3 rotations are commonly referred to as **Yaw**, **Pitch,** and **Roll** respectively.



Consider the two frames {A- in green} and {B – in red}. Which are coincident (i.e. have the same origin) but have different orientations. Consider also the case where point P is known relative to frame {B} (i.e. we know the vector ^{B}P) and also that the rotation of frame {B} is known relative to {A} (i.e. $^{A}_{B}R$ is known). To find the position of P relative to frame {A} (i.e. find ^{A}P) we need to consider the difference in orientation between the two frames {A} and {B}.

First recall that the components of any vector are merely the projections of that vector onto the unit vectors of its frame. Hence the problem is simply to express the vector ${}^{B}P$ from projections onto frame {B}, into projections onto frame {A}. This can be done using vector dot product. Hence:

$${}^{A}p_x = {}^{B}X_A \cdot {}^{B}P \qquad \text{(i.e. the x projection of the } {}^{A}P \text{ vector onto the \{A\} frame)}$$
$${}^{A}p_y = {}^{B}Y_A \cdot {}^{B}P \qquad \text{(i.e. the y projection of the } {}^{A}P \text{ vector onto the \{A\} frame)}$$
$${}^{A}p_z = {}^{B}Z_A \cdot {}^{B}P \qquad \text{(i.e. the z projection of the } {}^{A}P \text{ vector onto the \{A\} frame)}$$

In terms of matrix algebra this can be written in a more compact form as:

$${}^{A}P = \begin{bmatrix} {}^{B}X_A \\ {}^{B}Y_A \\ {}^{B}Z_A \end{bmatrix} . {}^{B}P$$

or by using the notation we used earlier for rotational matrices we can rewrite the above equation as:

$${}^{A}P = {}^{B}_{A}R^{T} . {}^{B}P \qquad ....(1)$$

where ${}^{B}_{A}R^{T}$ is the transposed of the matrix ${}^{B}_{A}R$.

From linear algebra is known that the inverse of a matrix with orthonormal columns is equal to its transposed. Hence:

$${}^{B}_{A}R^{T} = {}^{B}_{A}R^{-1} = {}^{A}_{B}R \qquad ....(2)$$

Equation (2) merely states that if frame {A} is known relative to frame {B}, then frame {B} is known relative to frame {A} through a simple rearrangement (transposition) of the rotational matrix.

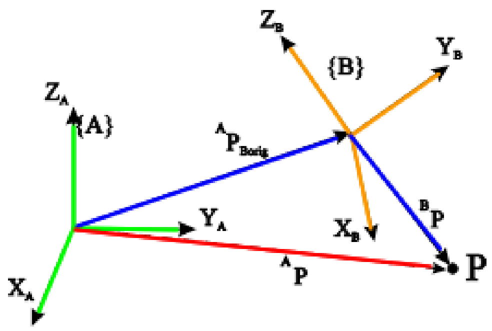
Finally, substitution of (2) into (1) gives:

$${}^{A}P = {}^{A}_{B}R . {}^{B}P \qquad ....(3)$$

The last relationship **merely changes the description** of a point expressed in terms of frame {B} to a description expressed in terms of frame {A}, when the orientation of frame {B} relative to {A} is known.

Combined Rotation and Translation

The problem shown in the diagram opposite is that given ${}^{B}P$ evaluate ${}^{A}P$ if we know ${}^{A}_{B}R$ and ${}^{A}P_{Borig}$.



Clearly we need to combine the individual approaches we have seen so far.

Thus we first express ${}^{B}P$ in terms of the orientation of frame {A} and then we add the translation vector ${}^{A}P_{Borig}$ i.e. the origin of the frame {B} relative to frame {A}. Hence:

$$ {}^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{Borig} \qquad \text{......(4)}$$

Equation (4) can be written more compactly through the use of **Homogeneous transforms**. The latter is a construction which enables the use of simple matrix operations to obtain the results of equation (4) in one step. It arises out of the fact that equation (4) can be written as:

$$\left[\begin{array}{c} {}^{A}P \\ \hdashline 1 \end{array}\right] = \left[\begin{array}{c:c} {}^{A}_{B}R & {}^{A}P_{Borig} \\ \hdashline 0 \quad 0 \quad 0 & 1 \end{array}\right]\left[\begin{array}{c} {}^{B}P \\ \hdashline 1 \end{array}\right] \qquad \text{...(5)}$$

The following points are worth mentioning regarding this representation:

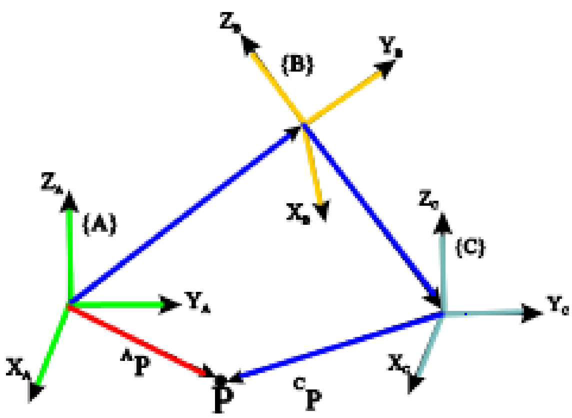
1. The 3x1 vectors ${}^{A}P$ and ${}^{B}P$ have been augmented by a 1 to become 4x1 vectors.
2. A row of [0 0 0 1] is added to change the combined part $[{}^{A}_{B}R, {}^{A}P_{Borig}]$ so it changes from a 3x4 matrix to 4x4 **square** matrix. This is important since most functions on matrices, particularly inversions, require square matrices.
3. The addition of the 1 and the 0s in this way has no effect on the values of the evaluated matrices and vectors.

Due to this result we can now rewrite equation (5) as:

$$ {}^{A}P = {}^{A}_{B}T \cdot {}^{B}P \qquad \text{.......(6)}$$

where ${}^{A}P$ and ${}^{B}P$ are the new augmented vectors, and ${}^{A}_{B}T$ is the 4x4 matrix above.

Compound Transformations



The diagram opposite indicates a case where we know the position of P relative to frame {C} and we require the position of P relative to frame {A} (i.e. we need to find ${}^{A}P$.

Working in the same way as before and using now homogeneous transforms, we can transform ${}^{C}P$ into ${}^{B}P$ according to:

$$ {}^{B}P = {}^{B}_{C}T \cdot {}^{C}P \qquad \text{......(7)}$$

We can also transform ${}^{B}P$ into ${}^{A}P$ according to :

$$ {}^{A}P = {}^{A}_{B}T \cdot {}^{B}P \qquad \text{.......(8)}$$

Combining (7) and (8) we have:

$$ {}^{A}P = {}^{A}_{B}T \cdot {}^{B}_{C}T \cdot {}^{C}P \qquad \text{or} $$

$$ {}^{A}P = {}^{A}_{C}T \cdot {}^{C}P \qquad ....(9) $$

In terms of our notation so far, ${}^{A}_{C}T$ is given by the 4x4 matrix:

$$ {}^{A}_{C}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R\,{}^{B}_{C}R & & {}^{A}_{B}R\,{}^{B}P_{Corig} + {}^{A}P_{Borig} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] $$

Inverse Transformations

It is often necessary to obtain inverse transformations (i.e. given that a frame {B} is known relative to frame {A}, we may require a description of frame {A} relative to {B}). In such cases we need to evaluate inverse transformations. One way of doing this is to compute the inverse of a 4x4 homogeneous transform. Doing this however does not exploit the properties of the transforms, and is computationally not simple.

Suppose we know ${}^{A}_{B}T$ and we require to find ${}^{B}_{A}T$. To do so we must compute ${}^{B}_{A}R$ and ${}^{B}P_{Aorig}$ from ${}^{A}_{B}R$ and ${}^{A}P_{Borig}$.

In general we have: ${}^{A}P = {}^{A}_{B}R \cdot {}^{B}P + {}^{A}P_{Borig}$. This is a general equation for any vector known with respect to frame {B} and expressed in terms of frame {A}. We can use it to find the vector of the point P which coincides with the origin of frame {B} relative to frame {B}. In other words find the origin of frame {B} relative to itself.

Hence $\qquad {}^{B}P_{Borig} = {}^{B}_{A}R\,{}^{A}P_{Borig} + {}^{B}P_{Aorig}$

But of course the LHS of the above equation is 0. Hence the equation reduces to:

$$ {}^{B}P_{Aorig} = -\,{}^{B}_{A}R\,{}^{A}P_{Borig} \qquad ......(10) $$

and since $\qquad {}^{B}_{A}R = {}^{A}_{B}R^{T} \qquad ....(11),$

we finally have that: $\qquad {}^{B}P_{Aorig} = -\,{}^{A}_{B}R^{T} \cdot {}^{A}P_{Borig} \qquad ....(12)$

In accordance with previous results we can now write:

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{B}_{A}R & & {}^{B}P_{Aorig} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] $$

or, in view of (11) and (12)

$$ {}^{B}_{A}T = \left[\begin{array}{ccc|c} & {}^{A}_{B}R^{T} & & -{}^{A}_{B}R^{T} \cdot {}^{A}P_{Borig} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \qquad ....(13) $$

Examples of Rotations and Transformations

1. A frame {B} is rotated with respect to frame {A} first about the x-axis (roll) and then about the y-axis (pitch), through angles ω and ψ respectively. Find the matrix which describes frame {B} relative to frame {A} (i.e. the ${}^{A}_{B}R$ matrix).

   Solution

$$ {}^{A}_{B}R(\omega, \psi) = R(\psi)R(\omega) \qquad \text{.........(1)} $$

Note the order of multiplication of the above matrices (it is typical of rotations about fixed frames, discussed later).

According to the earlier equations we have that:

$$ {}^{A}_{B}R(\omega,\psi) = \begin{bmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{bmatrix} $$

$$ = \begin{bmatrix} \cos\psi & \sin\psi\sin\omega & \sin\psi\cos\omega \\ 0 & \cos\omega & -\sin\psi \\ -\sin\psi & \cos\psi\sin\omega & \cos\psi\cos\omega \end{bmatrix} $$

2. A frame {B} has been rotated relative to frame {A} about the Z axis through an angle $\varphi = 30^0$. Given that ${}^{B}P = [0, 2, 0]^T$ evaluate ${}^{A}P$.

   There is clearly no frame translation, hence ${}^{A}P_{Borig} = [0, 0, 0]^T$.

   $\cos\varphi = c\varphi = \cos 30^0 = 0.866$ and $\sin\varphi = s\varphi = \sin 30^0 = 0.5$

Hence
$$ R(Z) = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

As there are no more rotations to consider it follows that ${}^{A}_{B}R = R(Z)$ and we can use this matrix to construct ${}^{A}_{B}T$ and hence evaluate ${}^{A}P$ as follows:

$$ {}^{A}P = {}^{A}_{B}T.{}^{B}P $$

It should be remembered that since ${}^{A}_{B}T$ is a 4x4 matrix, in the above equation both ${}^{A}P$ and ${}^{B}P$ will be in the augmented form (i.e. with an additional row of 1s), giving:

$$ {}^{A}P = \left[\begin{array}{ccc:c} 0.866 & -0.5 & 0 & 0 \\ 0.5 & 0.866 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \left[\begin{array}{c} -1 \\ 1.732 \\ 0 \\ \hdashline 1 \end{array}\right] $$

or $\qquad {}^{A}P = [-1, 1.732, 0]^T$

3. A frame {B} is rotated relative to a frame {A} about the Z axis by $\varphi = 30^0$ and translated by 10 units along X_A and 5 units along Y_A. Find ${}^{A}P$ if ${}^{B}P = [3, 7, 0]^T$.

Clearly,

$${}^{A}P_{Borig} = [10, 5, 0]^T, \quad \text{and} \quad {}^{A}_{B}R = R(Z) = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence:

$${}^{A}P = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 10 \\ 0.5 & 0.866 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 3 \\ 7 \\ 0 \\ 1 \end{bmatrix} = \left[\begin{array}{c} 9.098 \\ 12.562 \\ 0 \\ \hline 1 \end{array}\right]$$

$$\text{or} \qquad {}^{A}P = [9.098, 12.562, 0]^T$$

4. A frame {B} is rotated relative to frame{A} about the Z axis by $\varphi = 30^0$ and translated by 4 units along X_A and 3 units along Y_A. Find ${}^{B}_{A}T$.
It is known that:
$c\varphi = \cos 30^0 = 0.866$, $s\varphi = \sin 30^0 = 0.5$, $\cos\psi = \cos\omega = 0$, $\sin\psi = \sin\omega = 1$.

$${}^{A}_{B}T = \left[\begin{array}{ccc|c} 0.866 & -0.5 & 0 & 4 \\ 0.5 & 0.866 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

To evaluate ${}^{B}_{A}T$ we need to invert the above matrix. Following the rule we learned on the inversion of **homogeneous** transforms we need to evaluate the following result:

$${}^{B}_{A}T = \left[\begin{array}{c|c} {}^{A}_{B}R^T & -{}^{A}_{B}R^T \cdot {}^{A}P_{Borig} \\ \hline 0, \; 0, \; 0 & 1 \end{array}\right]$$

Hence it can easily be verified that: $${}^{B}_{A}T = \left[\begin{array}{ccc|c} 0.866 & 0.5 & 0 & -4.964 \\ -0.5 & 0.866 & 0 & -0.598 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

MORE ON ROTATIONS

There are several ways to describe rotations. Typically rotations can be considered to occur relative to **stationary** or **fixed** frames, or relative to **non-stationary** or **moving** frames. In the first case the overall result is obtained through pre-multiplication of the individual rotational matrices, whilst in the second case the overall result is obtained through post-multiplication.

Roll, Pitch & Yaw about Fixed Axes

One method of describing the orientation of frame {B} relative to frame {A} is as follows:

1. Start with the frame coincident with a known reference frame {A}.
2. First rotate {B} about X_A by an angle ω (roll).
3. Then rotate {B} about Y_A by an angle ψ (pitch).
4. Finally rotate about Z_A by an angle φ (yaw).

The result is the following rotational matrix:

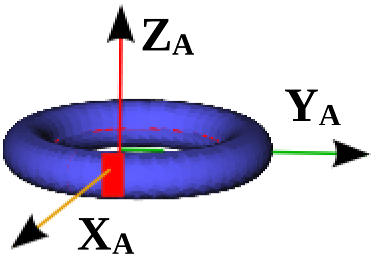
$$ {}^{A}_{B}R(\omega, \psi, \varphi) = R(Z, \varphi)R(Y, \psi)R(X, \omega) $$

$$
= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\psi & 0 & s\psi \\ 0 & 1 & 0 \\ -s\psi & 0 & c\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\omega & -s\omega \\ 0 & s\omega & c\omega \end{bmatrix}
$$

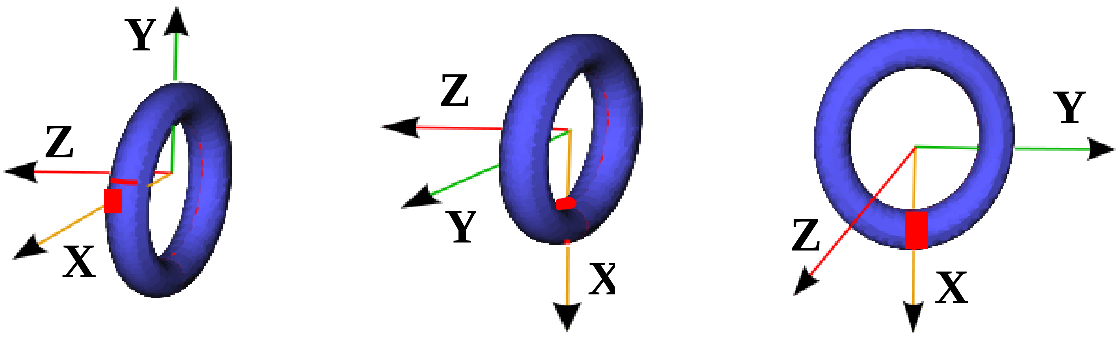
$$
= \begin{bmatrix} c\phi c\psi & c\phi s\psi s\omega - s\phi c\omega & c\phi s\psi c\omega + s\phi s\omega \\ s\phi c\psi & s\phi s\psi s\omega + c\phi c\omega & s\phi s\psi c\omega - c\phi s\omega \\ -s\psi & c\psi s\omega & c\psi c\omega \end{bmatrix} \qquad ....(14)
$$

Note: the order of multiplication. It is **pre-multiplication** and is typical of rotations occurring about fixed (non-moving) frames. **Also note that equation above is correct <u>only</u> for rotations performed in the given order (roll, pitch, yaw).**

Here is what happens graphically. The tyre-like object has a red band painted in the front (but not in the back) as seen. The start position is given opposite. All rotations are +ve 90^0 occurring one-at-a-time) **<u>about the X_A, Y_A and Z_A axes of the axes in the original position shown opposite</u>**.



The results are shown below:



Note: The frame shown in the 3 diagrams is the object's own frame not the one about which rotations are taking place.

Z-Y-X Euler Angles

Another possible description of a frame {B} is as follows:

1. Start with the frame coincident to a known frame {A}.

2. First rotate {B} about Z_B by an angle φ (yaw).

3. Then rotate {B} about the new Y'_B by an angle ψ (pitch).

4. Finally rotate about the new X''_B by an angle ω (roll).

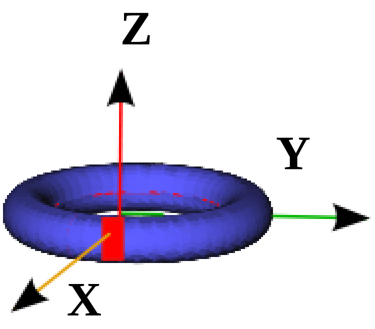
Because the rotations are about a rotating (moving) system of axis, the overall rotational matrix ${}^{A}_{B}R\,(\varphi, \psi, \omega)$ is given as:

$$
{}^{A}_{B}R\,(\varphi, \psi, \omega) = R(Z, \quad \varphi)R(Y', \quad \psi)R(X'', \quad \omega)
$$

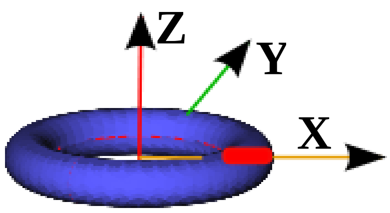
$$
= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\psi & 0 & s\psi \\ 0 & 1 & 0 \\ -s\psi & 0 & c\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\omega & -s\omega \\ 0 & s\omega & c\omega \end{bmatrix}
$$

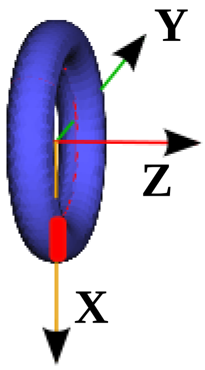
$$
= \begin{bmatrix} c\phi c\psi & c\phi s\psi s\omega - s\phi c\omega & c\phi s\psi c\omega + s\phi s\omega \\ s\phi c\psi & s\phi s\psi s\omega + c\phi c\omega & s\phi s\psi c\omega - c\phi s\omega \\ -s\psi & c\psi s\omega & c\psi c\omega \end{bmatrix} \qquad \text{....(15)}
$$

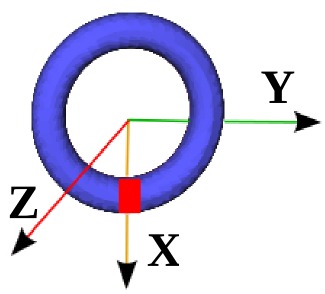
Note the order of multiplication. It is **post-multiplication** and it is the reverse of the previous case examined. <u>Note that the result is the same as equation (14) because the rotations here were performed in a different order than in the previous case.</u>



Here is what happens graphically. The tyre-like object has a red band painted in the front (but not in the back). The start position is given opposite. All rotations are +ve 90^0 occurring one-at-a-time) about the X, Y, and Z axes of the object as they appear after each rotation. The results are:







Z-Y-Z Euler Angle

Another possible description of a frame {B} relative to {A} is the following:

1. Start with the frame coincident to a known frame {A}.
2. First rotate {B} about Z_B by an angle φ (yaw).
3. Then rotate {B} about Y'_B by an angle ψ (pitch).
4. Finally rotate about Z''_B by an angle ω (roll).

$$ {}^A_B R(\varphi, \quad \psi, \quad \omega) = R(Z, \varphi)R(Y', \psi)R(Z'', \omega)$$

$$= \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\psi & 0 & s\psi \\ 0 & 1 & 0 \\ -s\psi & 0 & c\psi \end{bmatrix} \begin{bmatrix} c\omega & -s\omega & 0 \\ s\omega & c\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\phi c\psi c\omega - s\phi s\omega & -c\phi c\psi s\omega - s\phi c\omega & c\phi s\psi \\ s\phi c\psi c\omega + c\phi s\omega & -s\phi c\psi s\omega + c\phi c\omega & s\phi s\psi \\ -s\psi c\omega & s\psi s\omega & c\psi \end{bmatrix} \qquad ....(16)$$

Find the Rotation angles from a Rotational matrix

It is often the case that, given the numerical values of a rotational matrix which in the general form is:

$$ {}^A_B R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \qquad .... (17)$$

we need to know the angles φ, ψ, and ω.

To find these we need to know two things:

a) the methodology we use (i.e. Z-Y-X about fixed axes, Z-Y-X Euler, Z-Y-Z Euler), and
b) the values of the elements in the matrix

Even then there are nine equations and only three unknowns. So here is how it is done:

EXAMPLES

1. In the case of X-Y-Z rotation about fixed axes in equation (14), we can solve for angle φ by finding the arctangent of r_{21} over r_{11} (also using equation 17).

For the angle ω we can find the arctangent of r_{32} over r_{33}, **provided $\cos\psi \neq 0$**.

The angle ψ can be found by first taking the square root of the sum of the squares of r_{11} and r_{21} in order to find $\cos\psi$, and then finding the arc tangent of r_{31} over the computed cosine. Thus:

$$\varphi = \text{Atan2}(r_{21}, r_{11})^{*}$$
$$\omega = \text{Atan2}(r_{32}, r_{33})$$
$$\psi = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})$$

The last of the three equations always provide a solution $-90^0 \leq \psi \leq 90^0$. In case $\psi = \pm 90^0$ then the solution degenerates and we can only evaluate the sum or the difference of φ and ω. In such cases we normally set select $\varphi = 0$.

Hence:

$$\psi = \pm 90^0$$
$$\varphi = 0$$
$$\omega = \pm\, \text{Atan2}(r_{12}, r_{22}).$$

2. The case for Z-Y-X Euler angles is similar but note that the rotations are performed in different order and that they are also interpreted differently.

3. To conclude the case we will also examine the situation for the Z-Y-Z Euler angle using equation (16) in conjunction with the generic equation (17).

In this case it can be verified that if $\sin\psi \neq 0$ then,

$$\varphi = \text{Atan2}(r_{23}, r_{13})$$
$$\omega = \text{Atan2}(r_{32}, -r_{31})$$
$$\psi = \text{Atan2}(\sqrt{r_{31}^2 + r_{32}^2}, r_{33})$$

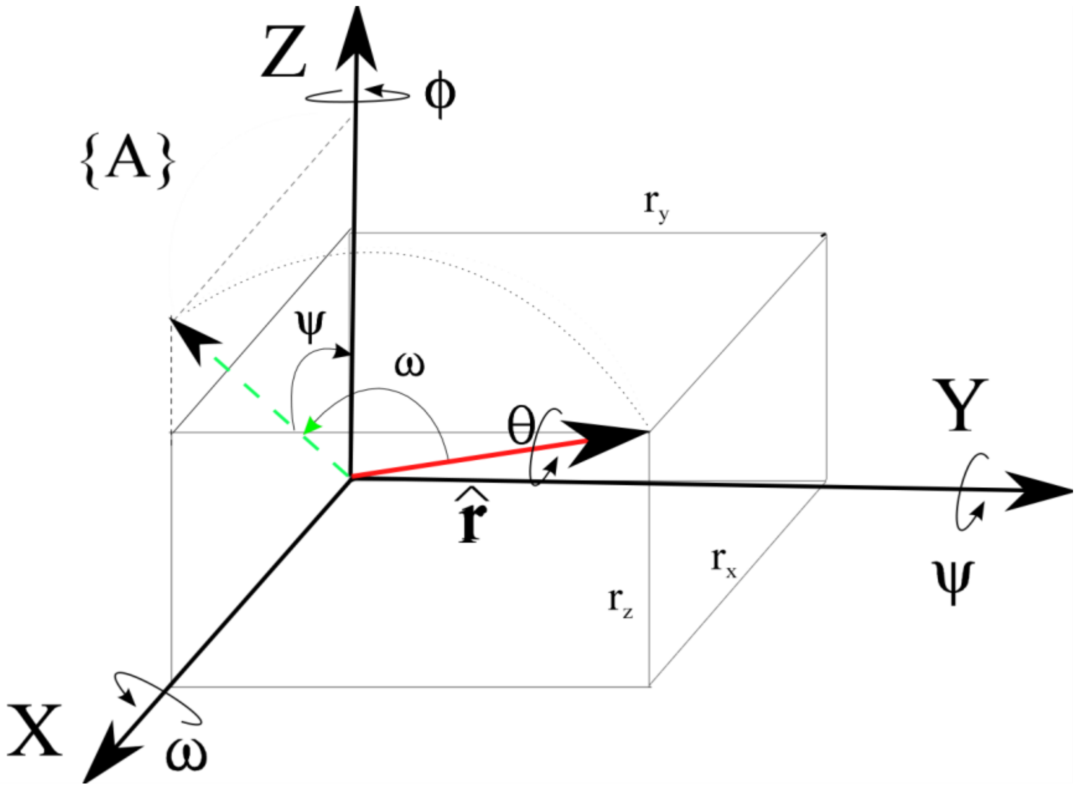
The last equation always provides a solution in the range of $0^0 \leq \psi \leq 180^0$. Once again, when $\psi = 0$ or 180^0 the solution degenerates and we can only evaluate the sum or the difference of φ and ω. In such cases we normally set select $\varphi = 0$. Hence:

$\psi = 0$	$\psi = 180^0$
$\varphi = 0$	$\varphi = 0$
$\omega = \text{Atan2}(-r_{12}, r_{11})$	$\omega = \text{Atan2}(r_{12}, -r_{11})$

ROTATIONS ABOUT ARBITRARY AXES

* Atan2(y/x) evaluates $\tan^{-1}(y/x)$, but also uses the signs of x and y to evaluate the quadrant of the resulting angle. For example $\text{Atan2}(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = -135^{\circ}$, but $\text{Atan2}(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 45^{\circ}$. This difference would not be evaluated by the normal atan(y/x) function.

If a rotation is not performed about a principal axis (X, Y, or Z) but it is instead performed about a generalised arbitrary axis $\hat{r}$ (also called the **equivalent** axis) through an angle $\boldsymbol{\theta}$ we need to find the equivalent rotational matrix.



In engineering we often change the problem to one we already know how to solve.

To perform the rotation, we first align the arbitrary axis $\hat{r}$ with the principal **z**-axis, perform the rotation $\boldsymbol{\theta}$**,** and return the arbitrary axis $\hat{r}$ back to its former position by reversing the rotations. Hence looking at the diagram it is evident that in order to find the equivalent rotational matrix needed to:

- Rotate $\hat{r}$ about the X axis through an angle $\boldsymbol{\omega}$ so that $\hat{r}$ is on the XZ-plane.
- Rotate $\hat{r}$ about the Y axis through an angle $-\boldsymbol{\psi}$ so that $\hat{r}$ aligns with the Z axis.
- Perform the rotation through the angle $\boldsymbol{\theta}$.
- Rotate $\hat{r}$ about the Y axis through an angle $\boldsymbol{\psi}$ so that $\hat{r}$ is on the XZ plane.
- Rotate $\hat{r}$ about the X axis through an angle -$\boldsymbol{\omega}$ to put $\hat{r}$ back in its initial position.

Overall it can be shown that: $R({}^{A}\hat{r}, \theta) = R(x, -\omega)R(y, \psi)R(z, \theta)R(y, -\psi),R(x, \omega)$
or,

$$R({}^{A}\hat{r}, \theta) = \begin{bmatrix} r_x^2 v\theta + c\theta & r_x r_y v\theta - r_z s\theta & r_x r_z v\theta + r_y s\theta \\ r_x r_y v\theta + r_z s\theta & r_y^2 v\theta + c\theta & r_y r_z v\theta - r_x s\theta \\ r_x r_z v\theta - r_y s\theta & r_y r_z v\theta + r_x s\theta & r_z^2 v\theta + c\theta \end{bmatrix} \quad \text{........(18)}$$

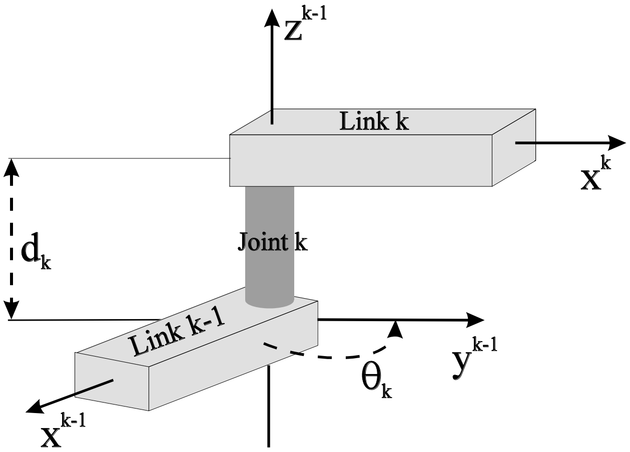
where $v\theta = 1 - \cos\theta$ and ${}^{A}\hat{r} = [r_x, r_y, r_z]^T$

ROBOTS LINKS AND JOINTS

Robot arms are essentially made up of chains of links. In controlling robots we need to know the positions of the links relative to one another. To do so we need to examine certain parameters associated with the physical design of robots.

Kinematic Parameters

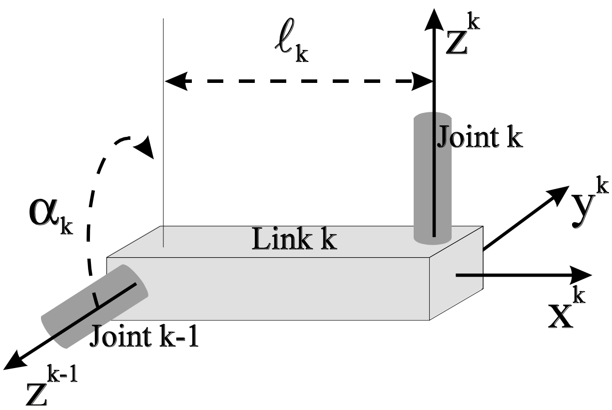
The position and orientation of two successive links is specified by two joint parameters.



θ_k is the <u>joint angle (or rotation)</u>. It is the rotation needed (about the Z^{k-1} axis) to make axis X^{k-1} parallel to X^k. (Variable for a revolute joint).

$\mathbf{d}_k$ is the <u>joint distance</u>. It is the translation along the Z^{k-1} axis needed to make axis X^{k-1} intersect with axis X^k. (Variable for a prismatic joint).

The position and orientation between two successive joints is specified by two link parameters.



ℓ_k (or $\mathbf{a}_k$) is the <u>link length</u>. It is the translation along X^k needed to make Z^{k-1} intersect Z^k.

α_k is the <u>link twist</u>. It is the rotation about X^k needed to make Z^{k-1} parallel to Z^k.
These parameters are always constant for non-flexible links and are specified in the design. The link twist is normally a multiple of $\pi/2$.

Assign Coordinate Frames to Links

To simplify the problem of assigning frames to links and to provide consistency of solution, a systematic algorithm (Denavit- Hartenberg) has been developed to do this.

The algorithm is as follows:

1. Number of joints from 1 to n starting with the base and ending with the robot tool (<u>yaw, pitch, and roll axes in **this** order</u>).

2. Assign a right-handed frame $\{F^0\}$ to the base of the robot, aligning Z^0 with the axis of joint 1. Set k = 1.

3. Align Z^k with the axis of joint k+1.

4. Locate origin of $\{F^k\}$ at the intersection of Z^k and Z^{k-1} axes. If they do not intersect, use the intersection of Z^k with a common normal between Z^k and Z^{k-1}. Select X^k to be orthogonal to both Z^k and Z^{k-1}. If Z^k is parallel with Z^{k-1}point X^k away from Z^{k-1}

5. Select Y^k to form a right-handed orthogonal co-ordinate frame $\{F^k\}$.

6. Set k = k+1. **If** k < n go to step 3 **Else**

7. Set the origin of $\{F^n\}$ at the tool point. Align Z^k with the vector pointing away from the tool and in the same direction as the fingers. Select Y^k and X^k to form a right-handed orthogonal frame, (infinite number of solutions).
8. Find parameters d_k, ℓ_k, α_k, θ_k for $1 \le k \le n$, and tabulate.
9. Evaluate the transformation matrices which link successive frames (i.e. how a frame is positioned and oriented relative to another).

To find the transformation matrices we use a generic procedure which consists of two rotations and two translations and involves the parameters of the links and joints that we have seen so far. The result is a representation known as the Denavit-Hartenberg (D-H) - first proposed in 1955.

$$
\begin{aligned}
{}^{k-1}_{k}T &= \mathrm{R}(z^{k-1},\theta_k)Tr(z^{k-1},d_k)Tr(x^k,\ell_k)\mathrm{R}(x^k,\alpha_k) \\
&= \left[\begin{array}{ccc:c} c\theta_k & -s\theta_k & 0 & 0 \\ s\theta_k & c\theta_k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]
\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_k \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]
\left[\begin{array}{ccc:c} 1 & 0 & 0 & \ell_k \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]
\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & c\alpha_k & -s\alpha_k & 0 \\ 0 & s\alpha_k & c\alpha_k & 0 \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \\
&= \left[\begin{array}{ccc:c} c\theta_k & -c\alpha_k s\theta_k & s\alpha_k s\theta_k & \ell_k c\theta_k \\ s\theta_k & c\alpha_k c\theta_k & -s\alpha_k c\theta_k & \ell_k s\theta_k \\ 0 & s\alpha_k & c\alpha_k & d_k \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] \qquad \cdots(19)
\end{aligned}
$$

where R( ) describes a pure rotation and Tr( ) a pure translation.

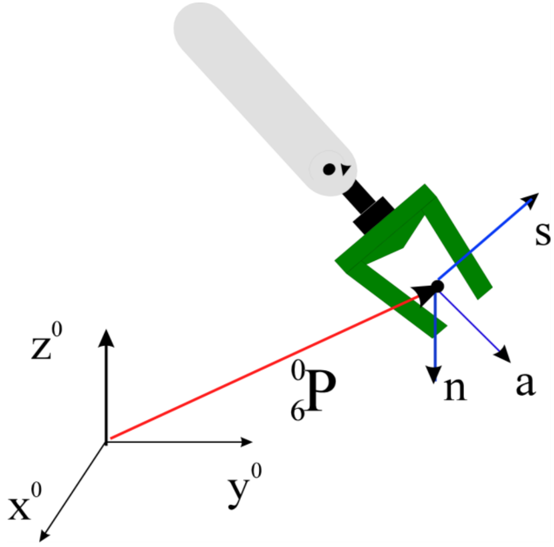
The order of multiplication of the matrices is indicative of how we progress from frame $\{F^{k-1}\}$ to frame $\{F^k\}$ (i.e the procedure describes frame $\{F^k\}$ relative to $\{F^{k-1}\}$). The procedure should be studied in conjunction with the preceding diagrams. The substitution of the corresponding link and joint parameters produces the final results for successive co-ordinate frames.

What does the D-H representation mean?

This representation contains all the information required for a robot arm to position and orient itself in order to manipulate objects within its environment. Equation (19) may be written in the following form:

$$
{}^{k-1}_{k}T = \left[\begin{array}{ccc:c} c\theta_k & -c\alpha_k s\theta_k & s\alpha_k s\theta_k & \ell_k c\theta_k \\ s\theta_k & c\alpha_k c\theta_k & -s\alpha_k c\theta_k & \ell_k s\theta_k \\ 0 & s\alpha_k & c\alpha_k & d_k \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc:c} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc:c} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ \hdashline 0 & 0 & 0 & 1 \end{array}\right]
$$

It is already known that p_x, p_y, and p_z are the three components of the position vector along the principal axes. Relating this to the diagram shown, p_x, p_y, and p_z are the components of the ${}^{0}_{6}P$ vector which represents the position of the end-effector relative to the base of the robot. The three vectors, **a**, **s**, and **n** are the three columns of the rotational matrix and are related in the following way:



a is the <u>approach vector</u> of the hand. It is parallel to the fingers and points away from the wrist. When assigning frames to robots, this vector always coincides with the z axis of the end-effector.

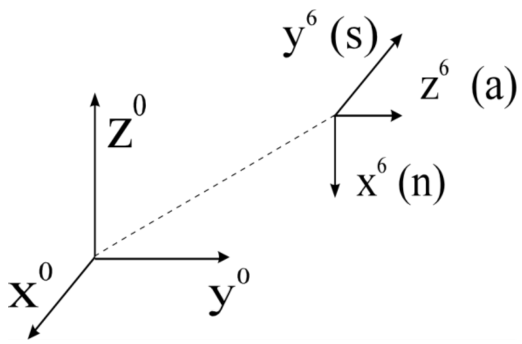
s is the <u>sliding vector</u>. It points in the direction of the finger motion as the fingers open and close. This vector always coincides with the y axis of the end-effector.

n is the <u>normal vector</u> of the hand (i.e. in a parallel finger hand is orthogonal to the fingers). From vectorial algebra $\mathbf{n} = \mathbf{s} \times \mathbf{a}$ (i.e. it is the cross-product of the other two vectors). This vector always coincides with the x axis of the end-effector.

In simple cases it is possible to evaluate the orientation of a robot-tool relative to its reference frame simply by examining the vectors **n**, **s**, and **a** of the rotational matrix. For example a rotational matrix

$$
{}^{0}_{6}R = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}
$$

corresponds to the opposite diagram.



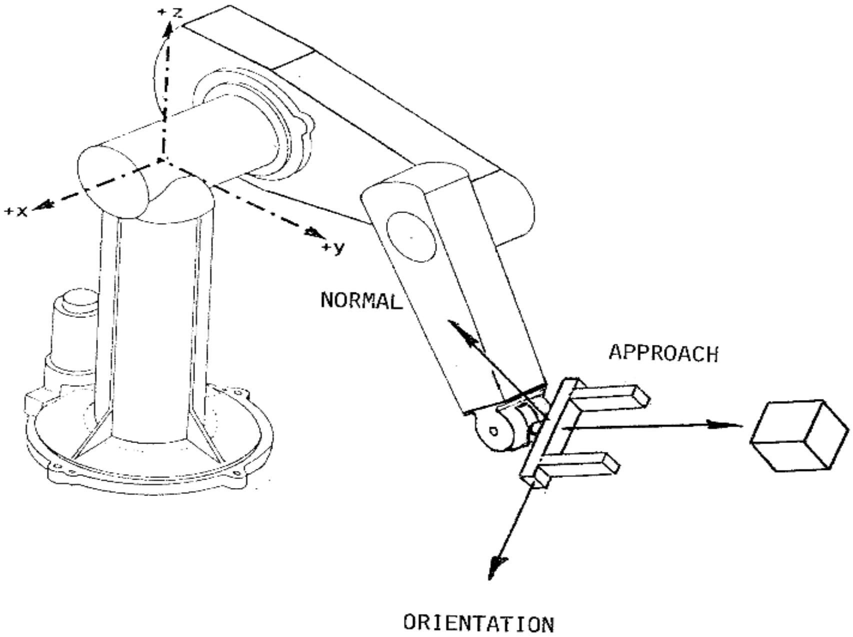
The rows of the matrix (from top to bottom)
refer to the axes of the reference frame, X, Y, Z in this order
The columns of the matrix from (left to right) refer to the axes of the referenced frame x, y, z in this order.

The matrix shows that:

1. The normal vector (first column) has only one component which is in the opposite direction of the reference z^0 axis (from the diagram the x^6 axis points in the opposite direction as the z^0 axis). The x^6 axis is the normal vector.

2. The sliding vector (middle column) has only one component which is in the opposite direction of the reference x^0 axis. The y^6 is the sliding vector.

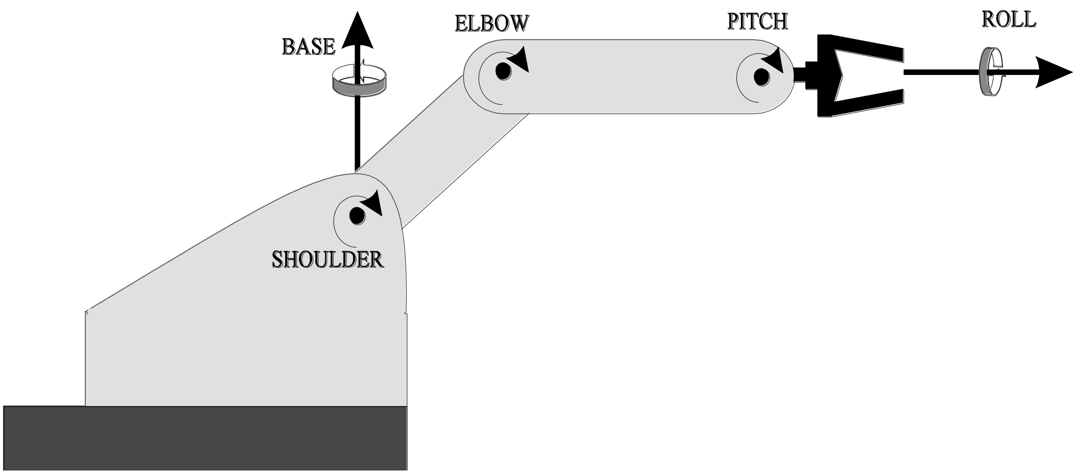
3. The approach vector **a** (third column) has only one component (two of the three elements are 0) which is in the same direction as the reference y^0 axis (from the diagram the z^6 axis points in the same direction as the y^0 axis).

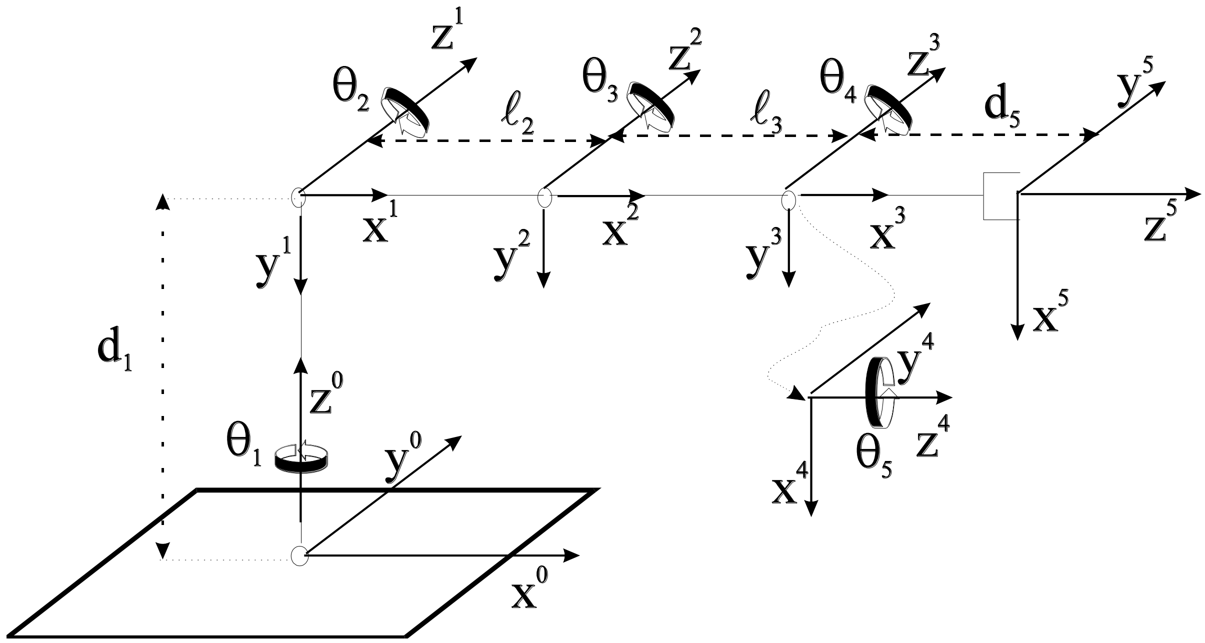


EXAMPLES

ALPHA II

Consider the ALPHA II robot shown in the diagram. The representation of its frames for all the joints are also displayed.





From the second diagram we can construct the following table:

KINEMATIC PARAMETERS FOR ALPHA II

Axis	θ_κ	d_k	ℓ_k	α_κ
1	θ_1	d_1 = 215mm	0	$-\pi/2$
2	θ_2	0	ℓ_2 = 177.8mm	0
3	θ_3	0	ℓ_3 = 177.8mm	0
4	θ_4	0	0	$\pi/2$
5	θ_5	d_5 = 96.5mm	0	0

We can now evaluate the transforms required to express the end-effector of the robot relative to its base. To do this, we substitute the above values into equation (19) for each of the axes of the robot. The result is as shown next:

$$
{}^0_1T = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad
{}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & \ell_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & \ell_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & \ell_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & \ell_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad
{}^3_4T = \begin{bmatrix} c\theta_4 & 0 & s\theta_4 & 0 \\ s\theta_4 & 0 & -c\theta_4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad
{}^4_5T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Clearly then ${}^0_5T = {}^0_1T.{}^1_2T.{}^2_3T.{}^3_4T.{}^4_5T$ is the matrix that describes the position and orientation of the end-effector (or tool) relative to the base. Such a matrix is also known as the arm matrix.

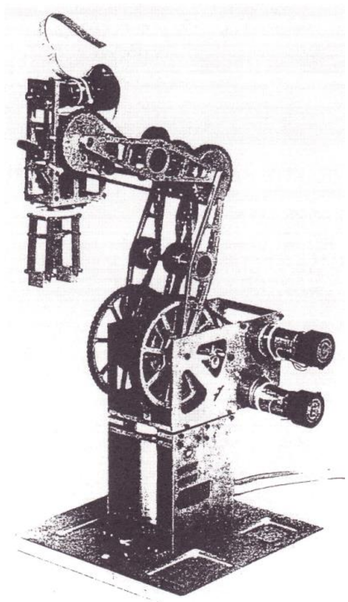
Note that:
${}^0_3T = {}^0_1T.{}^1_2T.{}^2_3T$ is the matrix which describes the position and orientation of the wrist relative to the base of the robot, and that

${}^3_5T = {}^3_4T.{}^4_5T$ is the matrix which describes the position and orientation of the end-effector relative to the wrist of the robot.

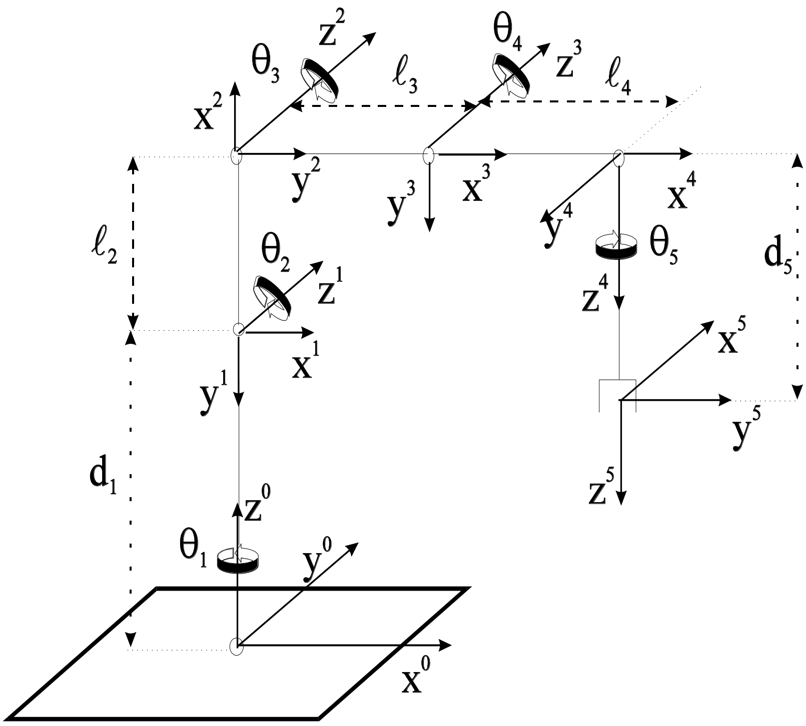
It is possible then, at any given time ***t***, to evaluate the relative position of the links of the robot if we know the variables **d** and **θ**.

In the general case distinction is not made between revolute and prismatic joints, and both variables **θ** and **d** are represented by the variable **q**. When the joint happens to be revolute, **q** represents **θ**, whilst in the case of a prismatic joint **q** will represent **d**.

RHINO XR3



Using the same procedure as for the previous example, the axes can be assigned as shown in the diagram below.
This assignment enables the tabulation of the kinematic parameters in the correct order. This, in turn, helps to find the matrices needed to establish the relative position of the links.



KINEMATIC PARAMETERS FOR RHINO XR3

Axis	**θ_k**	**d_k**	**ℓ_k**	**α_k**
1	θ_1	d_1 = 260.4mm	0	$-\pi/2$
2	θ_2	0	ℓ_2 = 228.6mm	0
3	θ_3	0	ℓ_3 = 228.6mm	0
4	θ_4	0	ℓ_4 = 95mm	$-\pi/2$
5	θ_5	d_5 = 171.5mm	0	0

giving the corresponding transformation matrices:

$$ {}_1^0T = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}_2^1T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & \ell_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & \ell_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}_3^2T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & \ell_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & \ell_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ {}_4^3T = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & \ell_4 c\theta_4 \\ s\theta_4 & 0 & c\theta_4 & \ell_4 s\theta_4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ {}_5^4T = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

WORKED EXAMPLES USING TRANSFORMS

1) At time, t_1, the robot angles θ_1, θ_2 and θ_3 are receptively 60^0, 30^0 and 45^0. Evaluate the transform that describes the position of the wrist relative to the base (use the ALPHA II parameters for your calculations).

Solution

The task requires to know ${}_3^0T = {}_1^0T.{}_2^1T.{}_3^2T$. This is obtained as shown:

$$ {}_3^0T = \begin{bmatrix} 0.5 & 0 & -0.866 & 0 \\ 0.866 & 0 & 0.5 & 0 \\ 0 & -1 & 0 & 215 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.866 & -0.5 & 0 & 153.98 \\ 0.5 & 0.866 & 0 & 88.9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \times \begin{bmatrix} 0.707 & -0.707 & 0 & 125.72 \\ 0.707 & 0.707 & 0 & 125.72 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1294 & -0.4829 & -0.866 & 100 \\ 0.2240 & -0.8364 & 0.5 & 173.19 \\ -0.9660 & -0.259 & 0 & -45.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2) If the above relationship is given, find the value of θ_2 if $\theta_1 = 60^0$ and $\theta_3 = 45^0$.

Solution

Known	Required
${}^{0}_{3}T,\ {}^{0}_{1}T,\ {}^{2}_{3}T$	${}^{1}_{2}T,\ \theta_2$

It is known that: $${}^{0}_{3}T = {}^{0}_{1}T.{}^{1}_{2}T.{}^{2}_{3}T \quad .....(1)$$

Rearranging the equation we have: $${}^{0}_{1}T^{-1}.{}^{0}_{3}T.{}^{2}_{3}T^{-1} = {}^{1}_{2}T \quad .....(2)$$
but since ${}^{0}_{1}T^{-1} = {}^{1}_{0}T$ and ${}^{2}_{3}T^{-1} = {}^{3}_{2}T$, (2) becomes:

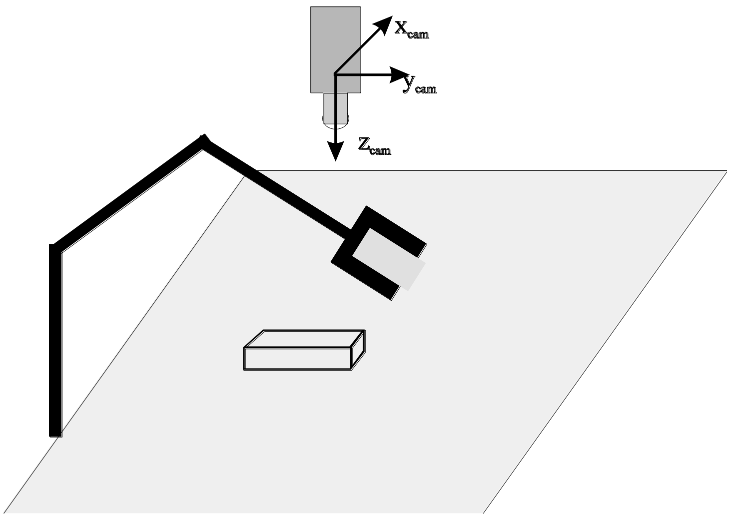
$${}^{1}_{2}T = {}^{1}_{0}T.{}^{0}_{3}T.{}^{3}_{2}T \quad .....(3)$$

Final substitution of the known values produces the result:

$${}^{1}_{2}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 153.98 \\ 0.5 & 0.866 & 0 & 88.9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & \ell_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & \ell_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad .....(4)$$

Finally from elements r_{11} and r_{21} of the matrix above it can be verified that $\boldsymbol{\theta_2 = 45^0}$

3) The camera -robot arrangement shown in the diagram is set-up so that the camera can see both the base of the robot and the workspace (end-effector and cuboid). A frame of orthonormal axes have been attached to the camera as shown. The matrix T_1 describes the robot's base as seen by the camera. If the matrix T_2 describes the cuboid's co-ordinate system (attached to the centre of the cuboid) as seen by the camera, evaluate:



a) The distance of the cuboid from the base of the robot.
b) The orientation of the gripper if the y-axis of the tool is to be aligned with the y-axis of the cuboid whilst the gripper grasps the object from the top.
c) The transform matrix which describes the position and orientation of the gripper relative to the base of the robot when the robot has just acquired the part.

The two matrices are: $T_1 = \begin{bmatrix} -1 & 0 & 0 & -10 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & 1 & 0 & -8 \\ 1 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

<u>Solution</u>

a) To find the matrix that relates the robot base to the cube, the method for compound transformations can be used: ${}^{r}_{cub}T = {}^{r}_{cam}T \cdot {}^{cam}_{cub}T$

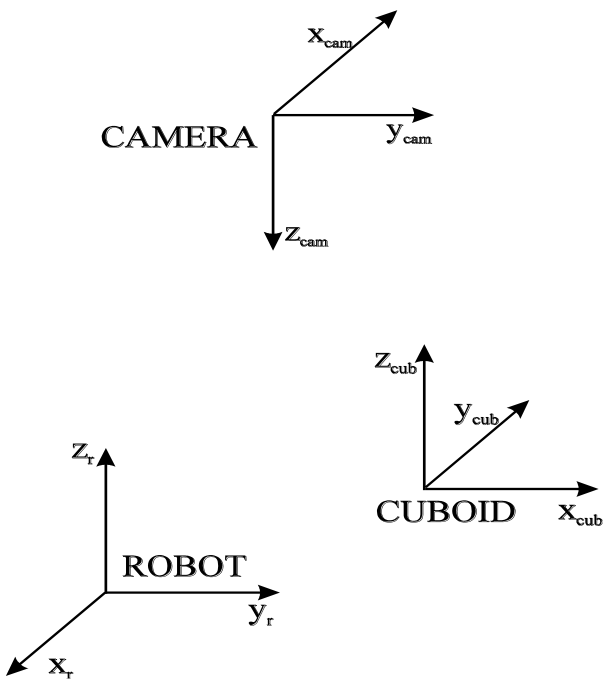
It is known that ${}^{cam}_{cub}T = T_2$ and that ${}^{cam}_{r}T = T_1$. Hence ${}^{r}_{cub}T = {}^{r}_{cam}T \cdot {}^{cam}_{cub}T = T_1^{-1} \cdot T_2$

$$ {}^{r}_{cub}T = T_1^{-1} \cdot T_2 = \begin{bmatrix} -1 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -8 \\ 1 & 0 & 0 & 0.5 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

The distance between the robot base and the cuboid co-ordinate frame is given as:

$$ d = \sqrt{p_x^2 + p_y^2 + p_z^2} = \sqrt{(-2)^2 + (1.5)^2 + (1)^2} = 2.693 $$

b) From the information known, the simple representation of the co-ordinate frames, shown opposite, is deduced:



It is clear now that we need to find the rotation of the robot gripper relative to the cuboid. To acquire the cuboid the robot must align its end-effector so that the z-axis of the cuboid and the end-effector's approach axis are coincident and antiparallel. Also the sliding vector of the end-effector is coincident with the y-axis of the cuboid. Hence for the robot's tool we have:

$\mathbf{a} = [0, 0, -1]^T$ and $\mathbf{s} = [0, 1$[*] $, 0]^T$.

The third vector **n** can be found using the cross product of **s** and **a**. Thus:

$$ n = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ s_x & s_y & s_z \\ a_x & a_y & a_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} $$

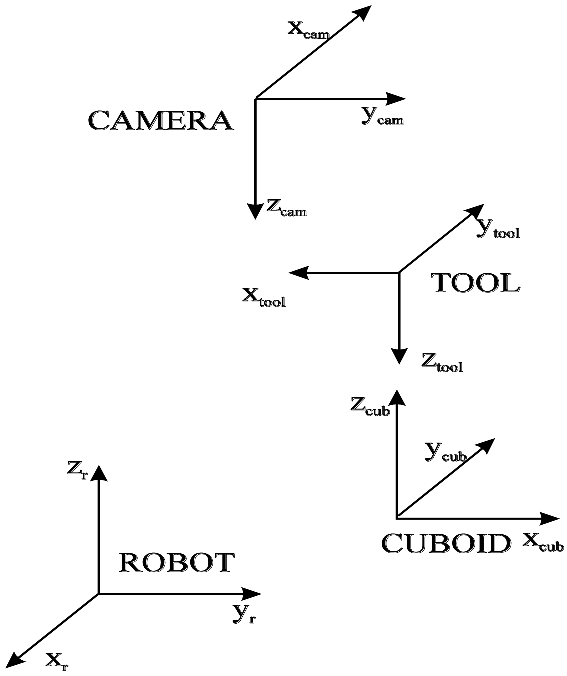
[*] For a two finger symmetrical gripper a sliding vector which is coincident and antiparallel to the y-axis of the cuboid may also be chosen. In this case the sliding vector **s** would be $\mathbf{s} = [0,-1,0]^T$.

Thus the following rotational matrix

$$ {}^{cub}_{tool}R = \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} $$

is obtained.

This corresponds to an overall diagrammatic representation of orientation as shown opposite.



c) The transform describing the position of the gripper relative to the cuboid at the time of grasping is:

$$ {}^{cub}_{tool}T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

hence,

$$ {}^{r}_{tool}T = {}^{r}_{cub}T . {}^{cub}_{tool}T = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -2 \\ -1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

a result that agrees fully with the diagram shown here.

Another way of looking at the problem is the following:

As stated earlier, when the robot grasps the object, the robot must align its end-effector so that the z-axis of the cuboid and the end-effector’s approach axis are coincident and antiparallel. Also the sliding vector of the end-effector must be coincident with the y-axis of the cuboid.

During grasping, the position of the tool frame must be the same as that of the cuboid frame.

Looking at the diagram, it should be obvious that all the above can be accomplished if:

- The frame of the end-effector (originally assumed to have the same orientation as the cuboid) is rotated by 180^0 about the y-axis of the cuboid.

- The end-effector is moved to the position of the cuboid.

Which, in terms of the transformation matrix at the time of grasping, results in:

$$ {}^{cub}_{tool}T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

which, of course, results in the same overall answer:

$$ {}^{r}_{tool}\mathrm{T} = {}^{r}_{cub}\mathrm{T}.{}^{cub}_{tool}\mathrm{T} = \begin{bmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -2 \\ -1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$