Reinforcement Learning

Computer Engineering Department
Sharif University of Technology

Mohammad Hossein Rohban, Ph.D.

Spring 2025

Courtesy: Some slides are adopted from CS 285 Berkeley, and CS 234 Stanford, and Pieter Abbeel's compact series on RL.

Value Function

$$\mathbb{E}(g(X)) = \sum_{x_i \in D(X)} P(X=x_i)\, g(x_i)$$

- $V^*(s)$ = expected utility starting in s, and acting optimally in all subsequent actions.

$$V^*(s) = \max_{a_i's} \mathbb{E}\left(\sum_{t=0}^{\infty} \gamma^t R(s_t, a_t, s_{t+1}) \middle| s_0 = s\right)$$

H

$$\underbrace{R(s_0, a_0, s_1) + \gamma \underbrace{\sum_{t=1}^{\infty} \gamma^{t-1} R(s_t, a_t, s_{t+1})}_{V^*(s_1)}}$$

Bellman's Opt. Eq.

$$= \max_{a_0} \sum_{s_1 = s'} P(s' | s_0, a_0) \left[R(s_0, a_0, s') + \gamma V^*(s') \right]$$

Value Iteration

- $V_0^*(s)$ = optimal value for state s when H=0
  - $V_0^*(s) = 0 \quad \forall s$
- $V_1^*(s)$ = optimal value for state s when H=1
  - $V_1^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_0^*(s'))$
- $V_2^*(s)$ = optimal value for state s when H=2
  - $V_2^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_1^*(s'))$
- $V_k^*(s)$ = optimal value for state s when H = k
  - $V_k^*(s) = \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_{k-1}^*(s'))$

Value Iteration

Algorithm:

Start with $V_0^*(s) = 0$ for all s.

For k = 1, … , H:

For all states s in S:

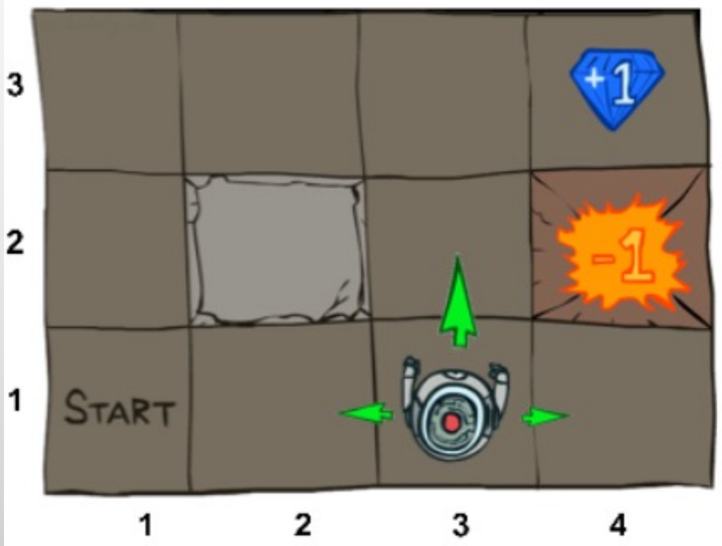
$$V_k^*(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V_{k-1}^*(s')\right)$$

$$\pi_k^*(s) \leftarrow \arg\max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V_{k-1}^*(s')\right)$$

This is called a value update or Bellman update/back-up

Value Iteration

$$\underline{V_0}(s) \leftarrow 0$$



Noise = 0.2

Discount = 0.9

k = 0

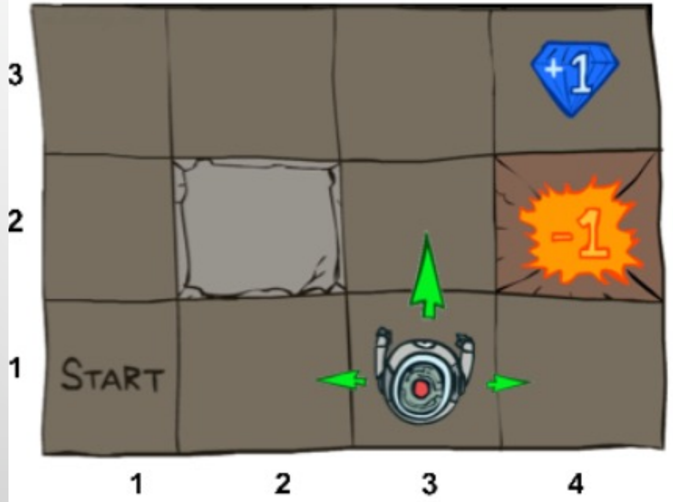
0.00	0.00	0.00	0.00
0.00		0.00	0.00
0.00	0.00	0.00	0.00

VALUES AFTER 0 ITERATIONS

Value Iteration

$$V_1(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_0(s'))$$

k = 0



0.00	0.00	0.00	0.00
0.00		0.00	0.00
0.00	0.00	0.00	0.00

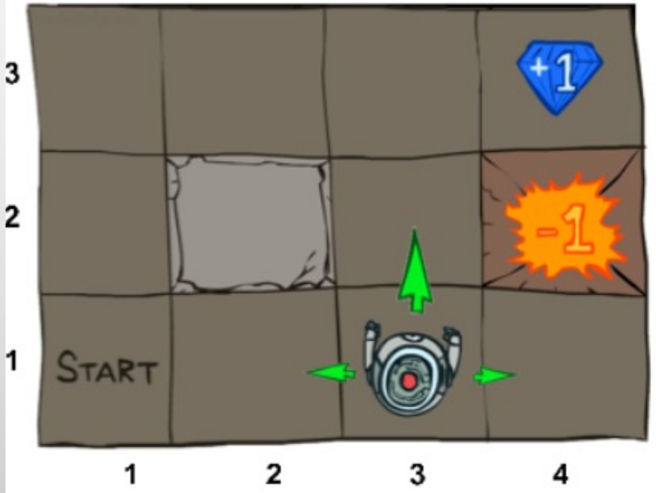
VALUES AFTER 0 ITERATIONS

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_2(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_1(s'))$$

k = 1



Noise = 0.2
Discount = 0.9

Value Iteration

$$V_2(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_1(s'))$$

k = 2

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 3

Noise = 0.2

Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

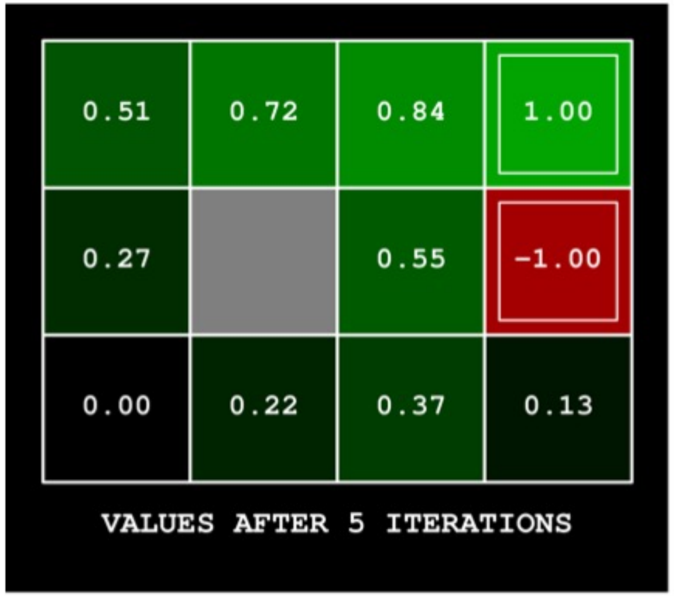
k = 4

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 5

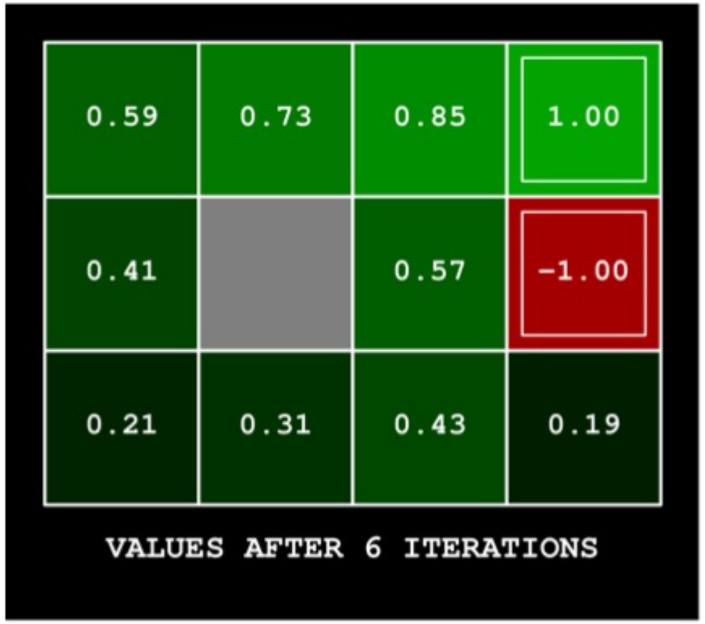


Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 6



Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 7

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 8

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 9

Noise = 0.2

Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 10

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)\left(R(s,a,s') + \gamma V_k(s')\right)$$

k = 11

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 12

Noise = 0.2
Discount = 0.9

Value Iteration

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma V_k(s'))$$

k = 100

Noise = 0.2

Discount = 0.9

Q-Values

- $Q^*(s, a) =$ expected utility starting in s, taking action a, and (thereafter) acting optimally

$$V^*(s) = \max_{a'} Q^*(s, a')$$

$$\max_{a'} Q^*(s', a')$$

$$Q^*(s,a) = \sum_{s'} P(s'|s,a)\left[R(s,a,s') + \gamma \underbrace{V^*(s')}_{\max_{a'} Q^*(s',a')}\right]$$

- Bellman Equation: $$Q^*(s,a) = \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma \max_{a'} Q^*(s',a'))$$

- Q-value Iteration: $$Q^*_{k+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma \max_{a'} Q^*_k(s',a'))$$

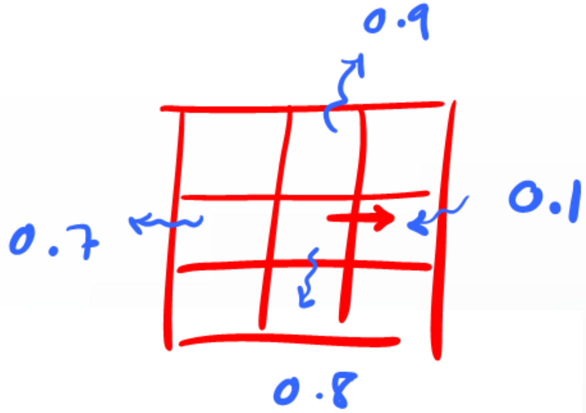
Q-Value Iteration

$$Q^*_{k+1}(s,a) \leftarrow \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma \max_{a'} Q^*_k(s',a'))$$

k = 100

Noise = 0.2
Discount = 0.9

Policy Evaluation



- Recall value iteration:

$$V_k^*(s) \leftarrow \max_a \sum_{s'} P(s'|s,a) \left(R(s,a,s') + \gamma V_{k-1}^*(s')\right)$$

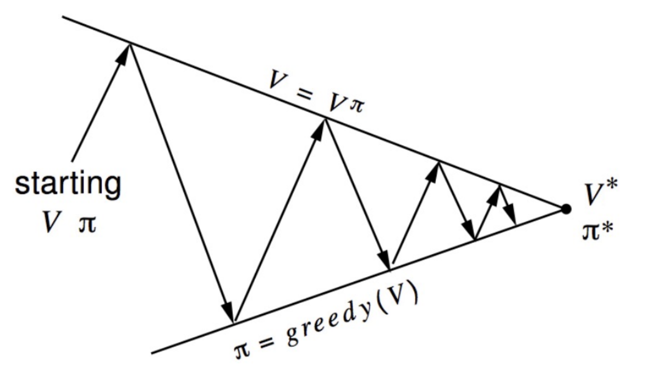
- Policy evaluation for a given $\pi(s)$:

$$V_k^\pi(s) \leftarrow \sum_{s'} P(s'|s,\pi(s))(R(s,\pi(s),s') + \gamma V_{k-1}^\pi(s'))$$

At convergence:

$$\forall s \quad V^\pi(s) \leftarrow \sum_{s'} P(s'|s,\pi(s))(R(s,\pi(s),s') + \gamma V^\pi(s))$$

Policy Iteration



- One iteration of policy iteration
  - Policy evaluation for current policy π_k :
    - Iterate until convergence

Eval

$$V_{i+1}^{\pi_k}(s) \leftarrow \sum_{s'} P(s'|s, \pi_k(s)) \left[R(s, \pi(s), s') + \gamma V_i^{\pi_k}(s') \right]$$

  - Policy improvement: find the best action according to one-step look-ahead

Improvement

$$\pi_{k+1}(s) \leftarrow \arg\max_a \sum_{s'} P(s'|s,a) \left[R(s,a,s') + \gamma V^{\pi_k}(s') \right]$$

- Repeat until policy converges
  - At convergence: optimal policy; and converges faster than value iteration under some conditions

One-step look ahead improves the policy

- Consider an alternative policy $\pi^{(1)}_{(k+1)}(t,s)$ that takes the prescribed actions in $\pi_{k+1}(s)$ **only at time t = 0**; and stays the same as $\pi_k(s)$ in later times.
- The value function V(s) for this new time-dependent policy is larger than or equal to V(s) for the original policy $\pi_k(s)$ for all s. Why?
- Now let $\pi^{(2)}_{(k+1)}(t,s)$, which takes the prescribed action in $\pi_{k+1}(s)$ **only at times t = 0 and t = 1**, and stays the same as $\pi_k(s)$ in later times.
- Similarly, V(s) gets improved for $\pi^{(2)}_{(k+1)}(t,s)$ compared to $\pi^{(1)}_{(k+1)}(t,s)$ for all s.
- Repeating this argument $\pi^{(\infty)}_{(k+1)}(t,s)$ becomes the same as $\pi_{k+1}(s)$.

An Example

Let this be the initial policy π_0, show how policy improvement, makes this policy better.

+1	→	+1
-1	↑	-1
-1	→	-1
-1	↑	-1
-1	←	-1
-1	↑	-1
-1	→	-1

Planning vs. Learning

$E(g(X)) \approx \frac{1}{n}\sum_{i=1}^{n} g(X_i)$ $\quad X_1 \cdots X_n \overset{iid}{\sim} X$

- Assumed to have access to the dynamics P(s' | s, a).
- We don't have access to this in the real world.
- We need to estimate (or learn) the value functions.

$$Q^*(s,a) = \sum_{s'} P(s'|s,a)(R(s,a,s') + \gamma \max_{a'} Q^*(s',a'))$$

Monte-Carlo Prediction

Monte Carlo Methods - Introduction

- Experience samples to learn without a model
- MC methods require only experience— sample sequences of states, actions, and rewards from actual or simulated interaction with an environment.
- We can learn with samples: **episodes**!

We don't have access to

$$P(s'|s,a)$$

→ Model Free Learning!

Monte-Carlo prediction

- Suppose we wish to estimate $V_\pi(s)$, the value of a state s under policy π.

- The **first-visit mc** method estimates $V_\pi(s)$ as *the average of the returns* following first visits to s.

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

  $V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

  $Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

  Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

  $G \leftarrow 0$

  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

    $G \leftarrow \gamma G + R_{t+1}$

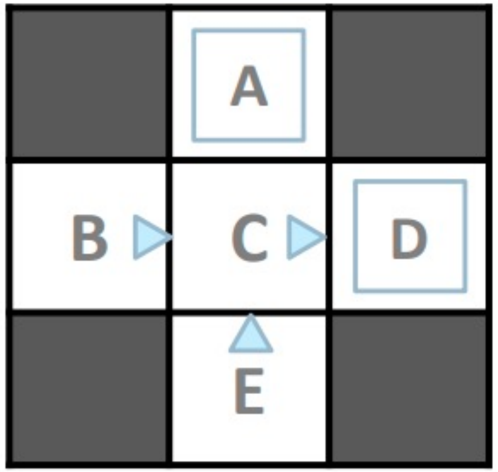
    Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

      Append G to $Returns(S_t)$

      $V(S_t) \leftarrow \text{average}(Returns(S_t))$

Episodes: another example

Input Policy π



Assume: $\gamma = 1$

Observed Episodes (Training)

Episode 1

B, east, C, -1
C, east, D, -1
D, exit, x, +10

Episode 2

B, east, C, -1
C, east, D, -1
D, exit, x, +10

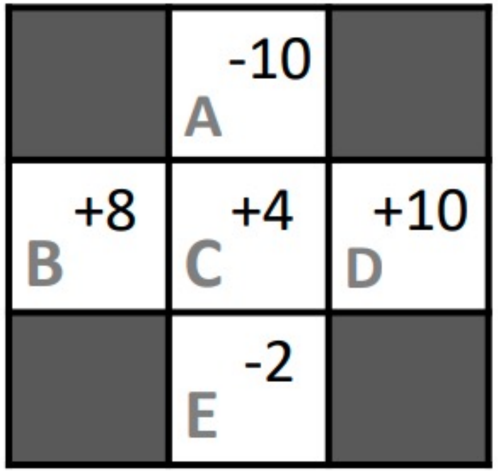
Episode 3

E, north, C, -1
C, east, D, -1
D, exit, x, +10

Episode 4

E, north, C, -1
C, east, A, -1
A, exit, x, -10

Output Values



Every Visit Monte-Carlo Policy

Initialize $N(s) = 0$, $G(s) = 0\ \forall s \in S$
Loop

- Sample episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots, s_{i,T_i}$
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots \gamma^{T_i - 1} r_{i,T_i}$ as return from time step t onwards in ith episode
- For each time step t until T_i (the end of the episode i)
    - state s is the state visited at time step t in episodes i
    - Increment counter of total visits: $N(s) = N(s) + 1$
    - Increment total return $G(s) = G(s) + G_{i,t}$
    - Update estimate $V^\pi(s) = G(s)/N(s)$

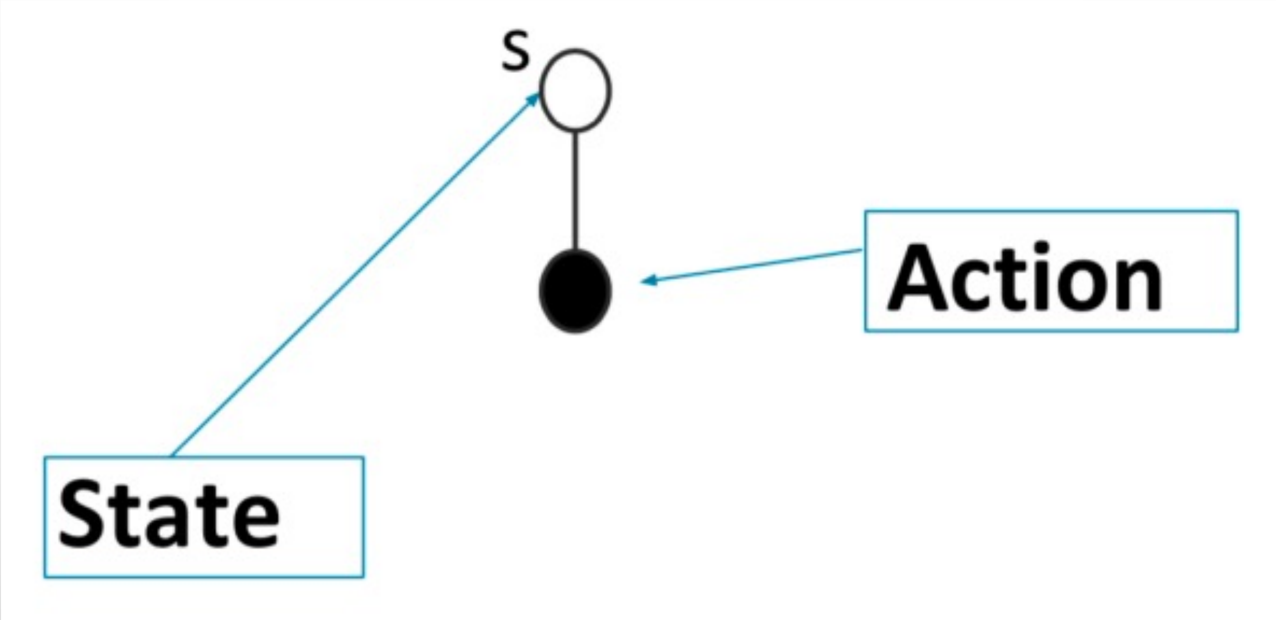
Incremental Monte-Carlo Policy

After each episode $i = s_{i,1}, a_{i,1}, r_{i,1}, s_{i,2}, a_{i,2}, r_{i,2}, \ldots$

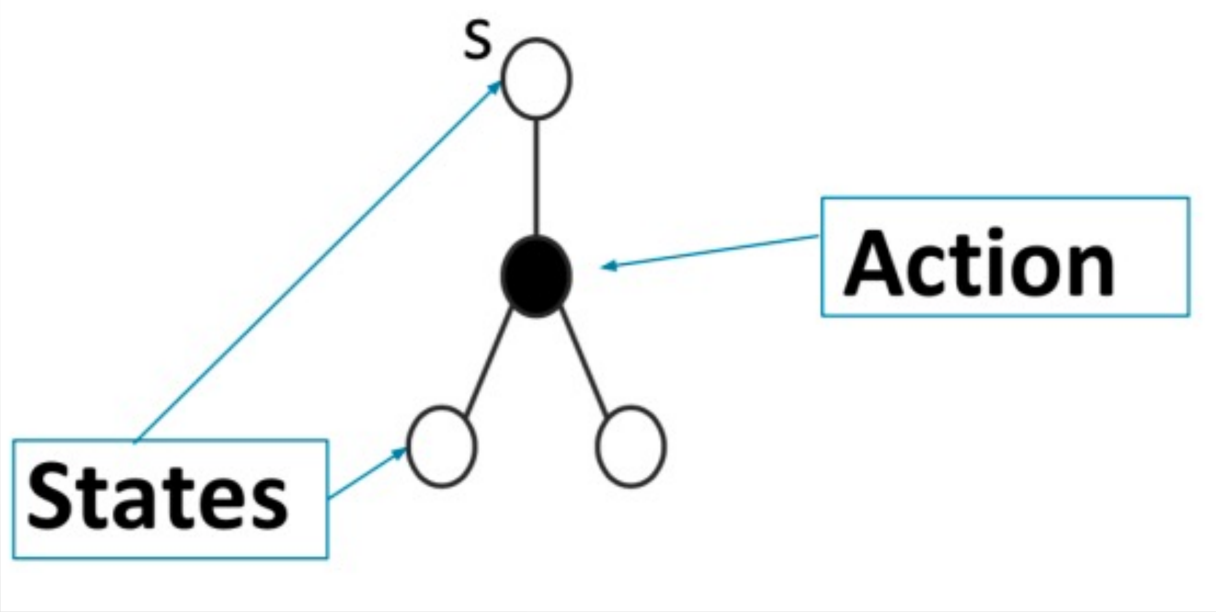
- Define $G_{i,t} = r_{i,t} + \gamma r_{i,t+1} + \gamma^2 r_{i,t+2} + \cdots$ as return from time step t onwards in ith episode
- For state s visited at time step t in episode i
  - Increment counter of total visits: $N(s) = N(s) + 1$
  - Update estimate

$$V^\pi(s) = V^\pi(s)\frac{N(s)-1}{N(s)} + \frac{G_{i,t}}{N(s)} = V^\pi(s) + \frac{1}{N(s)}(G_{i,t} - V^\pi(s))$$

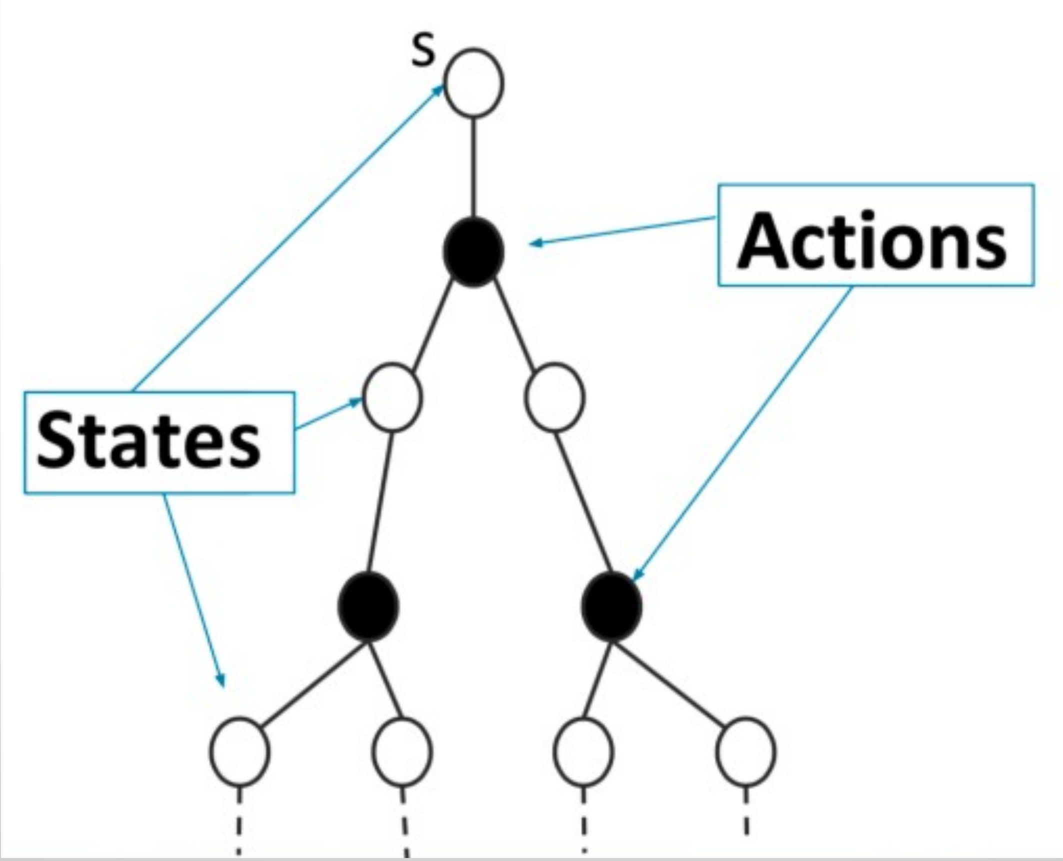
Policy Evaluation Diagram



Policy Evaluation Diagram

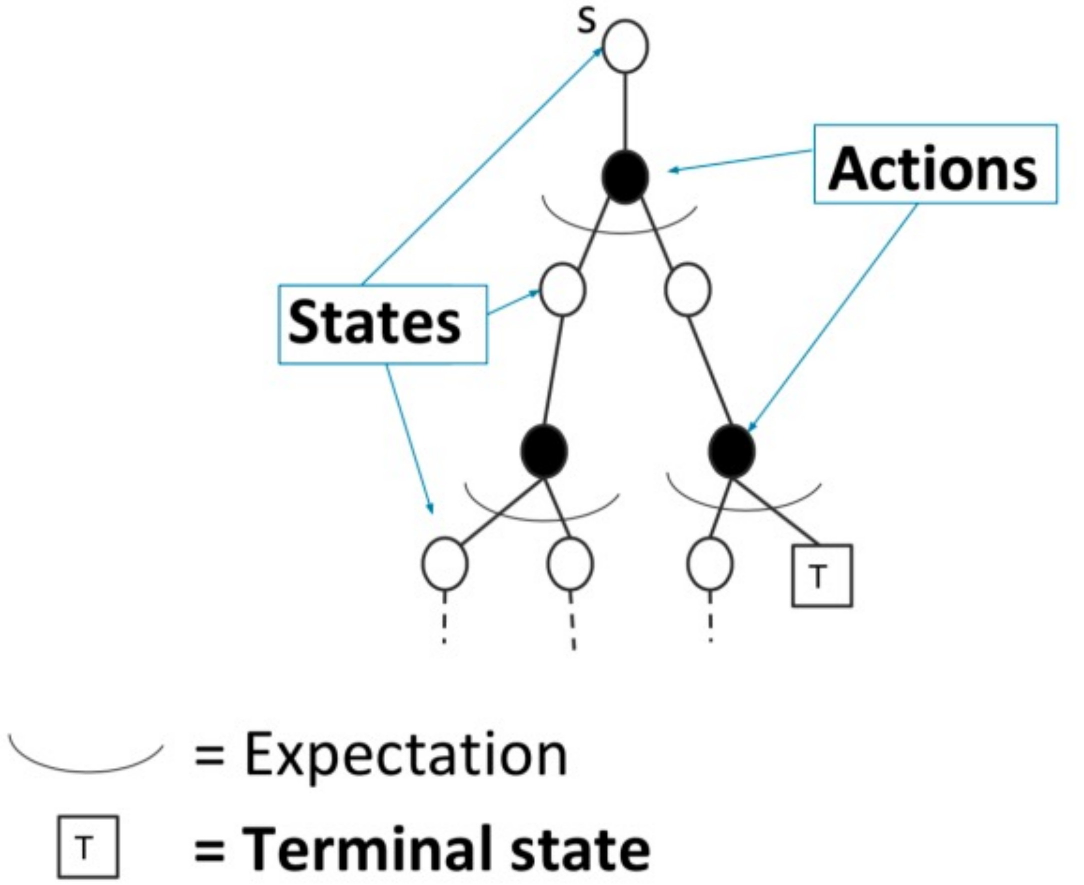


Policy Evaluation Diagram



Policy Evaluation Diagram

$$V^{\pi}(s) = V^{\pi}(s) + \alpha(G_{i,t} - V^{\pi}(s))$$



Policy Evaluation Diagram

$$V^{\pi}(s) = V^{\pi}(s) + \alpha(G_{i,t} - V^{\pi}(s))$$

MC updates the value estimate using a **sample** of the return to approximate an expectation