

## 1 Hit-and-Run

The Hit-and-Run algorithm used for sampling in a convex body  $K$ , was introduced by Smith in 1984 [5]. The mixing time is known to be  $\mathcal{O}^*(n^2 R^2 / r^2)$ , where  $R$  and  $r$  are the radii of the inscribed and circumscribed ball of  $K$  respectively [1, 3]. In the case of the muscles of a limb, we are interested in the polygon  $P$  that is given by the set of all possible activations  $\mathbf{a} \in \mathbb{R}^n$  that satisfy

$$\mathbf{f} = A\mathbf{a}, \mathbf{a} \in [0, 1]^n,$$

where  $\mathbf{f} \in \mathbb{R}^m$  is a fixed force vector and  $A = J^{-T} R F_m \in \mathbb{R}^{m \times n}$ .  $P$  is bounded by the unit  $n$ -cube since all variables  $a_i$ ,  $i \in [n]$  are bounded by 0 and 1 from below, above respectively.

Consider the following  $1 \times 3$  example.

$$1 = \frac{10}{3}a_1 - \frac{53}{15}a_2 + 2a_3$$

$$a_1, a_2, a_3 \in [0, 1],$$

the set of feasible activations is given by the shaded set in figure ??.

### FIGURE 1

The Hit-and-Run walk on  $P$  is defined as follows (it works analogously for any convex body).

- (a) Find a given starting point  $\mathbf{p}$  of  $P$ .
- (b) Generate a random direction through  $\mathbf{p}$  (uniformly at random over all directions).
- (c) Choose the next point of the sampling algorithm uniformly at random from the segment of the line in  $P$ .
- (d) Repeat from (b) the above steps with the new point as the starting point.

**FIGURE 2:** *Figures in 2D or 3D? Better in 3D to have the overview.*

- (a) *figure inner point*
- (b) *figure choice of direction*
- (c) *figure line segment*
- (d) *figure choice of new point*
- (e) *figure of distribution after some sampling*

The implementation of this algorithm is straight forward except for the choice of the random direction. How do we sample uniformly at random (u.a.r.) from all directions in  $P$ ? Suppose that  $\mathbf{q}$  is a direction in  $P$  and  $\mathbf{p} \in P$ . Then by definition of  $P$ ,  $\mathbf{q}$  must satisfy  $\mathbf{f} = A(\mathbf{p} + \mathbf{q})$ . Since  $\mathbf{p} \in P$ , we know that  $\mathbf{f} = A\mathbf{p}$  and therefore

$$\mathbf{f} = A(\mathbf{p} + \mathbf{q}) = \mathbf{f} + A\mathbf{q}$$

and hence

$$A\mathbf{q} = 0.$$

**FIGURE 3:**  $q$  must be parallel to the plane given by  $1 = \frac{10}{3}a_1 - \frac{53}{15}a_2 + 2a_3$ .

We therefore need to choose directions uniformly at random from all directions in the vectorspace

$$V = \{\mathbf{q} \in \mathbb{R}^n | A\mathbf{q} = 0\}.$$

As shown by Marsaglia this can be done as follows [4].

- (a) Find an orthonormal basis  $b_1, \dots, b_r \in \mathbb{R}^n$  of  $A\mathbf{q} = 0$ .
- (b) Choose  $(\lambda_1, \dots, \lambda_r) \in \mathcal{N}(0, 1)^r$  (from the Gaussian distribution).
- (c)  $\sum_{i=1}^r \lambda_i b_i$  is a u.a.r. direction.

A basis of a vectorspace  $V$  is a minimal set of vectors that generate  $V$ , and it is orthonormal if the vectors are pairwise orthogonal (perpendicular) and have unit length. Using basic linear algebra one can find a basis for  $V = \{A\mathbf{q} = 0\}$  and orthogonalize it with the well known Gram-Schmidt method (for details see e.g. [2]). Note that in order to get the desired u.a.r. distribution the basis needs to be orthonormal. For the limb case we can safely assume that the rows of  $A$  are linearly independent and hence the number of basis vectors is  $n - m$ .

**FIGURE 4**

- (a) *Some basis*
- (b) *orthonormal basis*

## References

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