Are there undecidable propositions?

Formalising a theorem by Martin-Löf in Coq

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Martin-Löf's Theorem

- "Is it conceivable that some propositions, or some problems, may be such that they just cannot be decided, or cannot be settled?"
- "Is it conceivable that a proposition may be such that it can neither be proved nor be disproved?"
- "Is it that conceivable that a proposition may be such that it can neither be known to be true nor be known to be false?"

P. Martin-Löf, "Verificationism then and now," in *Judgement and the Epistemic Foundation of Logic*, pp. 3-14, Springer, 2013

Martin-Löf's Theorem

According to Martin-Löf, on the *intuitionistic* interpretation of the key notions, the answer to those questions is **no**.

That is, "It is impossible to give a counterexample to the law of excluded middle in its positive formulation: every proposition can either be known to be true or known to be false"

P. Martin-Löf, "Verificationism then and now," in *Judgement and the Epistemic Foundation of Logic*, pp. 3-14, Springer, 2013

Modal Logic in Coq

Definition $\Box$ $(p\colon o)$:= fun $w \Rightarrow \forall$ $w1$, $(r\ w\ w1) \to (p\ w1)$.
Definition $\Diamond$ $(p\colon o)$:= fun $w \Rightarrow \exists$ $w1$, $(r\ w\ w1) \wedge (p\ w1)$.

Lemma $\mathit{example2}$: $[\mathit{mforall}\ p, \Diamond\ p\ m\!\to (m\neg\ (\Box\ (m\neg\ p)))]$.
Proof.
mv. intro. intro.
$\mathit{dia\_e}\ H$. //Diamond elimination rule
intro. assert($(m\neg\ x)\ w0$).
$\mathit{box\_e}\ H0\ H1$. //Box elimination rule
exact $H1$. $\mathit{contradiction}$.
Qed.

C. Benzmüller and B. W. Paleo, "Interacting with modal logics in the Coq proof assistant", in *International Computer Science Symposium in Russia*, pp. 398-411, Springer, 2015.

A potentially adequate formalisation

$\texttt{Axiom}\ \mathit{diaT} : \forall\ A,\ \mathit{True}\ (A \Rightarrow (\Diamond\ A)).$

$\texttt{Axiom}\ \mathit{A1} : \forall\ A\ B\ C,\ \mathit{True}\ (\ ((A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C)))).$

$\texttt{Axiom}\ \mathit{contrapositive} : \forall\ A\ B,\ \mathit{True}\ (\ ((A \Rightarrow B) \Rightarrow (\ \neg\ B \Rightarrow \neg\ A))\ ).$

$\texttt{Axiom}\ \mathit{mp} : \forall\ \{A\}\ \{B\},\ \mathit{True}\ (A \Rightarrow B) \rightarrow (\mathit{True}\ A) \rightarrow \mathit{True}\ B.$

Here, $\Diamond$ means “can be known”.

A potentially adequate formalisation

If a proposition cannot be known to be true then it can be known to be false.

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Lemma mlThirdLaw : ∀ p, True (((¬ (◇ p))) ⇒ ((◇ (¬ p)))).
Proof.
intros.
apply @mp with (A:=(¬ p ⇒ ◇ (¬ p))).
 - apply @mp with (A:=(¬ (◇ p)) ⇒ (¬ p)) .
     + apply A1.
     + apply @mp with (A:= (p ⇒ ◇ p)).
         × apply contrapositive.
         × apply diaT.
 - apply diaT.
Qed.
```

A potentially adequate formalisation

If a proposition cannot be known to be true then it can be known to be false.

```
Lemma mlThirdLaw' : ∀ p, True ((( ¬ ( ◊ p))) ⇒ (( ◊ (¬ p)))).
Proof.
intro p.
pose proof (A1 (¬ ( ◊ p)) (¬ p) ( ◊ (¬ p)) ) as Line1.
pose proof (diaT p) as Line2.
pose proof (property11_p24 p ( ◊ p)) as Line3.
pose proof (mp Line3 Line2) as Line4.
pose proof (mp Line1 Line4) as Line5.
pose proof (diaT ( ¬ p)) as Line6.
pose proof (mp Line5 Line6) as Line7.
exact Line7.
Qed.
```

A potentially adequate formalisation

mlThirdLaw - Informal

1. $(\neg \Diamond p \Rightarrow \neg p) \Rightarrow ((\neg p \Rightarrow \Diamond (\neg p)) \Rightarrow (\neg \Diamond p \Rightarrow \Diamond (\neg p)))$ (A1)
2. $p \Rightarrow \Diamond p$ ($\Diamond$ T CS4 axiom)
3. $(p \Rightarrow \Diamond p) \Rightarrow (\neg \Diamond p \Rightarrow \neg p)$ (contrapositive)
4. $\neg \Diamond p \Rightarrow \neg p$ (Modus Ponens, 2, 3)
5. $(\neg p \Rightarrow \Diamond (\neg p)) \Rightarrow (\neg \Diamond p \Rightarrow \Diamond (\neg p))$ (Modus Ponens 4,1)
6. $\neg p \Rightarrow \Diamond (\neg p)$ ($\Diamond$ T CS4 axiom)
7. $\neg \Diamond p \Rightarrow \Diamond (\neg p)$ (Modus Ponens 6,7)

The proof above is by Monica Marcus, (personal communication, October 10, 2017).

A potentially adequate formalisation

Theorem $ml\_theorem$: $\forall\ A,\ \neg\ True\ ((\ \neg\ (\Diamond\ A) \wedge (\neg\ (\Diamond\ (\neg\ A)))))$.
Proof.
intro.
unfold not.
intro.
apply $fact\_2$ in H. destruct H.
apply $(fact\_4\ (\Diamond\ (\neg\ A)))$.
apply $fact\_2$.
exact $((conj\ (mp\ (mlThirdLaw\ A)\ H)\ H0))$.
Qed.

Important to consider: Law of excluded middle

We *know* $(p \vee \neg p)$ is not intuitionistically provable. (From external reasoning about intuitionistic logic).

We also know that $\neg (p \vee \neg p)$ is not intuitionistically provable. (because its negation is intuitionistically provable, below).

Lemma *not_not_excluded_middle* : $\forall\ p, \neg \neg (p \vee \neg p)$.
`Proof. firstorder. Qed.`

Questions?

Alternative formalisation

$\texttt{Inductive}\ Atoms :=$
$\quad |\ a :\ Atoms$
$\quad |\ S : Atoms \rightarrow Atoms.$

$\texttt{Inductive}\ proposition :=$
$\quad |\ Atom :\ Atoms \rightarrow proposition$
$\quad |\ Implies :\ proposition \rightarrow proposition \rightarrow proposition$
$\quad |\ Or :\ proposition \rightarrow proposition \rightarrow proposition$
$\quad |\ And :\ proposition \rightarrow proposition \rightarrow proposition$
$\quad |\ Falsum :\ proposition$
$\quad |\ Not :\ proposition \rightarrow proposition$
$\quad |\ K : proposition \rightarrow proposition.$ "K" for representing 'can be known' or 'knowable' is included here.

Alternative formalisation

Inductive Proof : *proposition* $\rightarrow$ Prop :=
| *atom_ev* : $\forall$ (A:*Atoms*), *AtomValuation A* = *true* $\rightarrow$ Proof (*Atom A*)
| *and_ev* : $\forall$ (p q : *proposition*), Proof p $\rightarrow$ Proof q $\rightarrow$ (Proof (*And* p q))
| *orl_ev* : $\forall$ (p q : *proposition*), Proof p $\rightarrow$ (Proof (*Or* p q))
| *orr_ev* : $\forall$ (p q : *proposition*), Proof q $\rightarrow$ (Proof (*Or* p q))
| *not_ev* : $\forall$ (p : *proposition*), Proof (*Implies* p *Falsum*) $\rightarrow$ (Proof (*Not* p))
| *K_ev* : $\forall$ (p : *proposition*), Proof p $\rightarrow$ (Proof (K p))
| *implies_ev* : $\forall$ (p q : *proposition*), *Implication_Is_True* p q $\rightarrow$ (Proof (*Implies* p q)).

Alternative formalisation

$\texttt{Axiom}$ $implies\_ev'$: $\forall$ p q, $Implication\_Is\_True$ p q $\leftrightarrow$ ($\exists$ (f: $\texttt{Proof}$ $p \to \texttt{Proof}$ q), $True$).

Alternative formalisation

Lemma $ml3rdLaw$: $\forall$ p, Proof (Not (K p)) $\rightarrow$ Proof (K ($Implies$ p $Falsum$)) .

Proof.

intro p.

intros.

apply K_ev. apply $implies\_ev$. apply $implies\_ev'$.

inversion H. inversion $H1$. apply $implies\_ev'$ in $H3$.

destruct $H3$.

assert(Proof p $\rightarrow$ Proof (K p)).

intros. apply K_ev in $H5$. exact $H5$.

assert(Proof p $\rightarrow$ Proof $Falsum$).

intros. apply x.

apply $H5$. exact $H6$. $\exists$ $H6$. apply I.

Qed.

Alternative formalisation

```
Theorem MLTheorem : ∀ A, ¬ Proof (( ¬ (K A) ∧ ( ¬ (K ( ¬
A))))).
Proof.
intro. unfold not. intro.
apply fact_2' in H. destruct H.
apply (non_contradiction (K ( ¬ A))).
split.
 - apply ml3rdLaw.
   apply not_ev.
   exact H.
 - apply not_ev.
   exact H0.
Qed.
```