Model and Control of a differential drive mobile robot

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1 Kinematics of the Differential Drive Robot

$$v_0 = \frac{(\dot{\phi}_r + \dot{\phi}_l)r}{2} \tag{1}$$

$\mathbf{v_c}$ is the speed of the center of mass of the robot in the robot coordinates.

$$\mathbf{v}_c = \mathbf{v}_0 + \mathbf{d} \times \omega \tag{2}$$

$$\mathbf{v}_c = \begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ 0 \end{bmatrix} + \begin{bmatrix} -d \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{x}_r \\ \dot{y}_r + \omega d \\ 0 \end{bmatrix} \tag{3}$$

$$\omega_c = \omega_0 = \dot{\theta} = \frac{(\dot{\phi}_r - \dot{\phi}_l)r}{2b} \tag{4}$$

World coordinates

$$\dot{x} = \dot{x}_r \cos(\theta) - \dot{y}_r \sin(\theta) \tag{5}$$
$$\dot{y} = \dot{x}_r \sin(\theta) + \dot{y}_r \cos(\theta) \tag{6}$$

Solving

$$\dot{x}_r = \dot{x} \cos(\theta) + \dot{y} \sin(\theta) \tag{7}$$
$$\dot{y}_r = -\dot{x} \sin(\theta) + \dot{y} \cos(\theta) \tag{8}$$

Kinematic constraints of the differential drive in robot coordinates

$$\mathbf{v}_c = \begin{bmatrix} \frac{(\dot{\phi}_r + \dot{\phi}_l)r}{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x}\cos(\theta) + \dot{y}\sin(\theta) \\ -\dot{x}\sin(\theta) + \dot{y}\cos(\theta) + \omega d \\ 0 \end{bmatrix} \tag{9}$$

$$\omega_c = \begin{bmatrix} 0 \\ 0 \\ \frac{(\dot{\phi}_r - \dot{\phi}_l)r}{2b} \end{bmatrix} \tag{10}$$

That can be rewritten as:

$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) - d \cdot \dot{\theta} = 0 \tag{11}$$
$$\dot{x}\cos(\theta) + \dot{y}\sin(\theta) + b \cdot \dot{\theta} = r \cdot \dot{\phi}_r \tag{12}$$
$$\dot{x}\cos(\theta) + \dot{y}\sin(\theta) - b \cdot \dot{\theta} = r \cdot \dot{\phi}_r \tag{13}$$

In matrix form

$$A^T(q) \cdot \dot{q} = 0 \tag{14}$$

$$A^T(q) = \begin{bmatrix} \sin(\theta) & -\cos(\theta) & -d & 0 & 0 \\ \cos(\theta) & \sin(\theta) & b & -r & 0 \\ \cos(\theta) & \sin(\theta) & -b & 0 & -r \end{bmatrix} \tag{15}$$

2 Dynamics

The Langrage equations of nonholonomic system are given by [1]

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \mathbf{q}}\right)^T = \mathbf{S}(\mathbf{q})\tau + \mathbf{A}(\mathbf{q})\lambda \tag{16}$$

Where for the mobile differential drive robot we have

$$\mathbf{S}(\mathbf{q}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}; \tau = \begin{bmatrix} \tau_r \\ \tau_l \end{bmatrix}; \mathbf{q} = \begin{bmatrix} x \\ y \\ \theta \\ \phi_r \\ \phi_l \end{bmatrix} \tag{17}$$

Let us solve left side of the dynamic equation for each generalized coordinate of $\mathbf{q}$.

We know that $\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) - \mathcal{U}(q)$. However the mobile robot has a horizontal restricted dynamics that yields $\mathcal{U}(q) = 0$.

$$\mathcal{L}(q,\dot{q}) = \mathcal{T}(q,\dot{q}) = \frac{1}{2}m_r v_c^2 + \frac{1}{2}I_r\dot{\theta}^2 + \frac{1}{2}I_w\dot{\phi}_r^2 + \frac{1}{2}I_w\dot{\phi}_l^2 \tag{18}$$

$$v_c^2 = \langle \mathbf{v}_c, \mathbf{v}_c \rangle = \dot{x}^2 + \dot{y}^2 - 2d\sin(\theta)\dot{x}\dot{\theta} + 2d\cos(\theta)\dot{y}\dot{\theta} + \dot{\theta}^2 d^2$$
$$\dot{\theta}^2 = \frac{(\dot{\phi}_r - \dot{\phi}_l)^2 r^2}{4b^2}$$

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}m_r\left(\dot{x}^2 + \dot{y}^2 - 2d\sin(\theta)\dot{x}\dot{\theta} + 2d\cos(\theta)\dot{y}\dot{\theta} + \dot{\theta}^2 d^2\right) + \frac{1}{2}I_r\dot{\theta}^2 + \frac{1}{2}I_w\dot{\phi}_r^2 + \frac{1}{2}I_w\dot{\phi}_l^2 \tag{19}$$

- For x

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \left(\frac{\partial \mathcal{L}}{\partial x}\right) = m\ddot{x} - md\sin(\theta)\ddot{\theta} - md\cos(\theta)\dot{\theta}^2$$

- For y

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \left(\frac{\partial \mathcal{L}}{\partial y}\right) = m\ddot{y} - md\cos(\theta)\ddot{\theta} - md\sin(\theta)\dot{\theta}^2$$

- For θ

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \left(\frac{\partial \mathcal{L}}{\partial \theta}\right) = -md\sin(\theta)\ddot{x} + md\cos(\theta)\ddot{y} + I_r + md^2\ddot{\theta}$$

- For ϕ_r

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_r}\right) - \left(\frac{\partial \mathcal{L}}{\partial \phi_r}\right) = -\frac{mrd\sin(\theta)}{b}\ddot{x} + \frac{mrd\cos(\theta)}{b}\ddot{y} - \frac{mrd\cos(\theta)}{b}\dot{x}\dot{\theta} - \frac{mrd\sin(\theta)}{b}\dot{y}\dot{\theta} + \\ \left(I_w + \frac{md^2r^2}{4b^2} + \frac{I_r r^2}{4b^2}\right)\ddot{\phi}_r - \left(\frac{md^2r^2}{4b^2} + \frac{I_r r^2}{4b^2}\right)\ddot{\phi}_l\end{aligned} \tag{20}$$

- For ϕ_l

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_l}\right) - \left(\frac{\partial \mathcal{L}}{\partial \phi_l}\right) = \frac{mrd\sin(\theta)}{b}\ddot{x} - \frac{mrd\cos(\theta)}{b}\ddot{y} + \frac{mrd\cos(\theta)}{b}\dot{x}\dot{\theta} + \frac{mrd\sin(\theta)}{b}\dot{y}\dot{\theta} \\ + \left(I_w + \frac{md^2r^2}{4b^2} + \frac{I_r r^2}{4b^2}\right)\ddot{\phi}_l - \left(\frac{md^2r^2}{4b^2} + \frac{I_r r^2}{4b^2}\right)\ddot{\phi}_r\end{aligned} \tag{21}$$

The dynamical model can be written in matrix form where

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{S}(\mathbf{q})\tau + \mathbf{A}(\mathbf{q})\lambda \tag{22}$$

$$\mathbf{B}(\mathbf{q}) = \begin{bmatrix} m & 0 & -md\sin(\theta) & 0 & 0 \\ 0 & m & md\cos(\theta) & 0 & 0 \\ -md\sin(\theta) & md\cos(\theta) & I_r + md^2 & 0 & 0 \\ -\frac{r}{b}md\sin(\theta) & \frac{r}{b}md\cos(\theta) & 0 & \alpha_1 & \alpha_2 \\ \frac{r}{b}md\sin(\theta) & -\frac{r}{b}md\cos(\theta) & 0 & \alpha_2 & \alpha_1 \end{bmatrix} \tag{23}$$

$$\alpha_1 = I_w + \frac{mr^2d^2}{4b^2} + \frac{I_r r^2}{4b^2}$$

$$\alpha_2 = -\left(\frac{md^2r^2}{4b^2} + \frac{I_r r^2}{4b^2}\right)$$

$$\mathbf{n}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} -\dot{\theta}^2 md\cos(\theta) \\ -\dot{\theta}^2 md\sin(\theta) \\ 0 \\ -\frac{mrd\cos(\theta)}{b}\dot{x}\dot{\theta} - \frac{mrd\sin(\theta)}{b}\dot{y}\dot{\theta} \\ \frac{mrd\cos(\theta)}{b}\dot{x}\dot{\theta} + \frac{mrd\sin(\theta)}{b}\dot{y}\dot{\theta} \end{bmatrix} \tag{24}$$

3 Reduced Dynamic Model

From equations (11) to (13) one can derive a relation between the pseudo-velocity vector $\mathbf{v} = [\mathbf{v}_c, \dot{\theta}]^T$ and the generalized velocities $\dot{\mathbf{q}}$ where

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q})\mathbf{v} \tag{25}$$

Solving for the differential drive robot we have

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} \frac{br\cos(\theta)+dr\sin(\theta)}{2b} & \frac{br\cos(\theta)-dr\sin(\theta)}{2b} \\ \frac{br\sin(\theta)-dr\cos(\theta)}{2b} & \frac{br\sin(\theta)+dr\cos(\theta)}{2b} \\ \frac{r}{2b} & -\frac{r}{2b} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

From equation (14) one can see that $\mathbf{G}(\mathbf{q})$ columns are the null spaces of $\mathbf{A}(\mathbf{q})^{\mathbf{T}}$ so that $\mathbf{A}(\mathbf{q})^{\mathbf{T}} \cdot \mathbf{G}(\mathbf{q}) = \mathbf{G}(\mathbf{q})^T \cdot \mathbf{A}(\mathbf{q}) = 0$.

The Lagrange multipliers in equation (35) can be eliminated premultiplying both sides of the equation by $\mathbf{G}(\mathbf{q})^{\mathbf{T}}$. This leads to the reduced dynamic model

$$\mathbf{G}(\mathbf{q})^{\mathbf{T}}(\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q}, \dot{\mathbf{q}})) = \mathbf{G}(\mathbf{q})^{\mathbf{T}}\mathbf{S}(\mathbf{q})\tau \tag{27}$$

Where the differentiation of equation (25) in respect to time yields

$$\ddot{\mathbf{q}} = \dot{\mathbf{G}}(\mathbf{q})\mathbf{v} + \mathbf{G}(\mathbf{q})\dot{\mathbf{v}}$$

The reduced dynamic model can be rewritten as

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{v}} + \mathbf{m}(\mathbf{q}, \mathbf{v}) = \mathbf{G}^{\mathbf{T}}(\mathbf{q})\mathbf{S}(\mathbf{q})\tau \tag{28}$$

if we premultiply equation (27) by $\mathbf{G}^{\mathbf{T}}(\mathbf{q})\mathbf{B}(\mathbf{q})$ and define

$$\mathbf{M}(\mathbf{q}) = \mathbf{G}^{\mathbf{T}}(\mathbf{q})\mathbf{B}(\mathbf{q})\mathbf{G}(\mathbf{q}) \tag{29}$$

$$\mathbf{m}(\mathbf{q}, \mathbf{v}) = \mathbf{G}^{\mathbf{T}}(\mathbf{q})\mathbf{B}(\mathbf{q})\dot{\mathbf{G}}(\mathbf{q})\mathbf{v} + \mathbf{G}^{\mathbf{T}}(\mathbf{q})\ \mathbf{n}(\mathbf{q}, \mathbf{G}(\mathbf{q})\mathbf{v}). \tag{30}$$

The state space reduced model can be found by solving for $\dot{\mathbf{q}}$ and $\dot{\mathbf{v}}$

$$\dot{\mathbf{q}} = \mathbf{G}(\mathbf{q})\mathbf{v} \tag{31}$$

$$\dot{\mathbf{v}} = -\mathbf{M^{-1}}(\mathbf{q})\mathbf{m}(\mathbf{q},\mathbf{v}) + \mathbf{M^{-1}}(\mathbf{q})\mathbf{G^T}(\mathbf{q})\mathbf{S}(\mathbf{q})\tau \tag{32}$$

4 DC Motor Model

DC motor dynamics are given by

$$\begin{cases} L\dfrac{di}{dt} + Ri + K_w\dot{\phi} = E \\ I_s\ddot{\phi} - K_t i + \mathcal{V}\dot{\phi} - \tau = 0 \end{cases} \tag{33}$$

as in [1, p.198] where E and i denote the armature voltage and current, R and L are respectively the armature resistance and inductance while I_s is the motor shaft inertia, $\mathcal{V}$ the viscous friction coefficient and τ the dynamic load applied to the motor. K_t is the motor torque constant and K_w the voltage constant.

A reduced order model can be achieved for the dynamic behaviour since the electric time constant $\frac{L}{R}$ can be neglected if compared to the mechanical time constant $\frac{I}{\mathcal{V}}$. Hence, we consider $L = 0$ and the first equation yields

$$i = \frac{E - K_w\dot{\phi}}{R}$$

that can be replaced on the second equation

$$I_s\ddot{\phi} - K_t\left(\frac{E - K_w\dot{\phi}}{R}\right) + \mathcal{V}\dot{\phi} - \tau = 0$$

That gives us

$$I_s\ddot{\phi} + \left(\frac{K_tK_w}{R} + \mathcal{V}\right)\dot{\phi} - \frac{K_t}{R}E = \tau \tag{34}$$

Therefore the new dynamic model for the mobile robot including the DC motor dynamics is

$$\mathbf{B}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{S}(\mathbf{q})\mathbf{E} + \mathbf{A}(\mathbf{q})\lambda \tag{35}$$

$$\mathbf{B}(\mathbf{q}) = \begin{bmatrix} m & 0 & -md\sin(\theta) & 0 & 0 \\ 0 & m & md\cos(\theta) & 0 & 0 \\ -md\sin(\theta) & md\cos(\theta) & I_r + md^2 & 0 & 0 \\ -\frac{r}{b}md\sin(\theta) & \frac{r}{b}md\cos(\theta) & 0 & \alpha_1 & \alpha_2 \\ \frac{r}{b}md\sin(\theta) & -\frac{r}{b}md\cos(\theta) & 0 & \alpha_2 & \alpha_1 \end{bmatrix} \tag{36}$$

$$\alpha_1 = I_w + I_s + \frac{mr^2d^2}{4b^2} + \frac{I_rr^2}{4b^2}$$

$$\alpha_2 = -\left(\frac{md^2r^2}{4b^2} + \frac{I_rr^2}{4b^2}\right)$$

$$\mathbf{n}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} -\dot{\theta}^2md\cos(\theta) \\ -\dot{\theta}^2md\sin(\theta) \\ 0 \\ -\frac{mrd\cos(\theta)}{b}\dot{x}\dot{\theta} - \frac{mrd\sin(\theta)}{b}\dot{y}\dot{\theta} + \left(\frac{K_tK_w}{R}\right)\dot{\phi}_r \\ \frac{mrd\cos(\theta)}{b}\dot{x}\dot{\theta} + \frac{mrd\sin(\theta)}{b}\dot{y}\dot{\theta} + \left(\frac{K_tK_w}{R}\right)\dot{\phi}_l \end{bmatrix} \tag{37}$$

$$\mathbf{S}(\mathbf{q}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{K_t}{R} & 0 \\ 0 & \frac{K_t}{R} \end{bmatrix}; \mathbf{E} = \begin{bmatrix} E_r \\ E_l \end{bmatrix} \tag{38}$$

References

[1] B. Siciliano, P. L. Sciavicco, L. Vilani and G. Oriolo, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.