AMS 507

Chapter 5
Continuous Random Variables

Sections 1-3

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5.1 Introduction

- Definition: Random variable X is *continuous* if there exists a function, $f(x) \geq 0, f\colon \mathbb{R} \to \mathbb{R}^{+}$, such that, $\forall B \subset \mathbb{R}$, we have

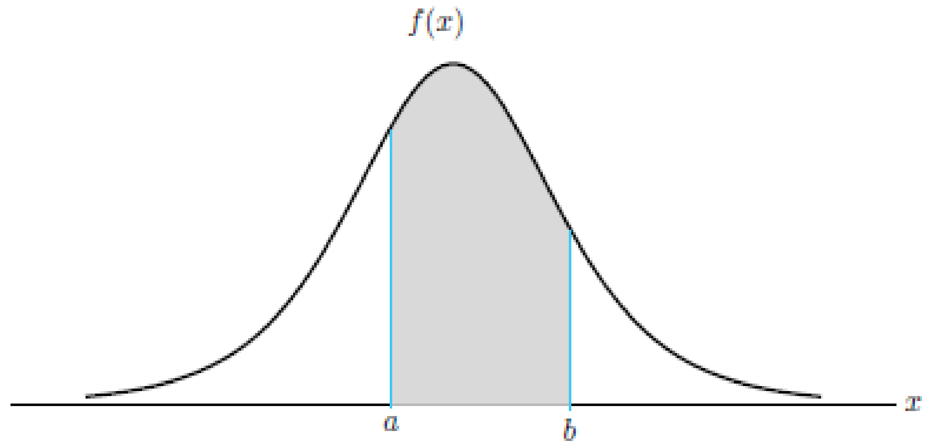
$$P(X \in B) = \int_{x \in B} f(x)dx$$

- The function $f(x)$ in the definition above is called *pdf* (probability density function).
- If $B = [a, b]$, then

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$$

- and

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$



The area under the graph of $f(x)$ representing $P(a \leq X \leq b)$

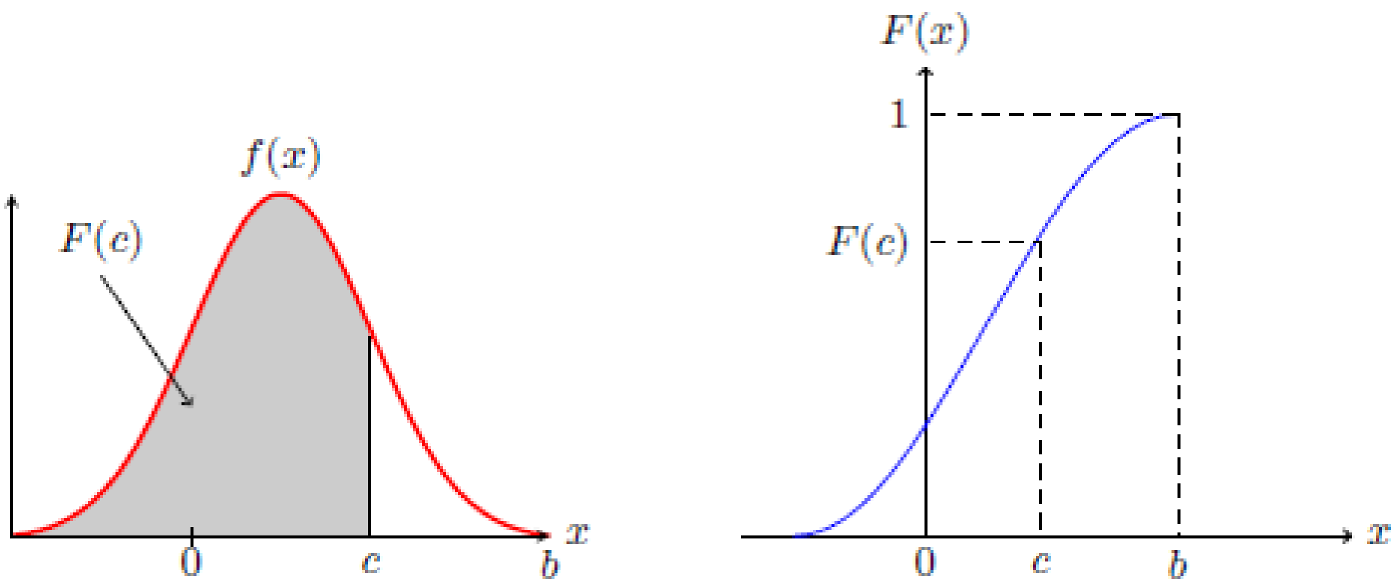
- If $a = b$, then

$$P(X = a) = \int_{a}^{a} f(x)dx = 0$$

Accordingly,
$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$

CDF of a Continuous R.V.

- CDF: $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$

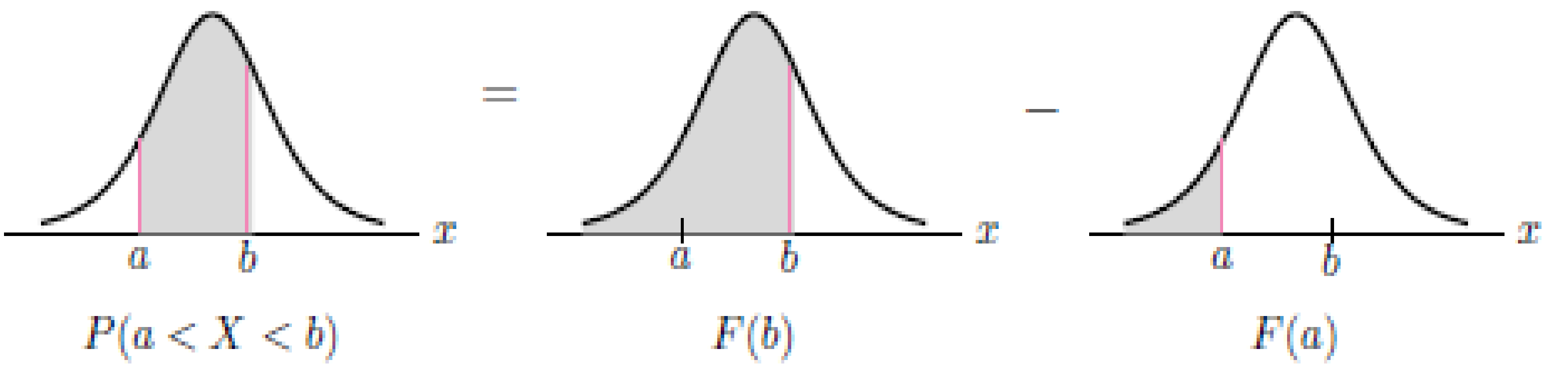


The pdf is the derivative of $F(x)$, i.e., $f(x) = F'(x)$

CDF of a Continuous R.V.

- For any interval [a, b],

$$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$$



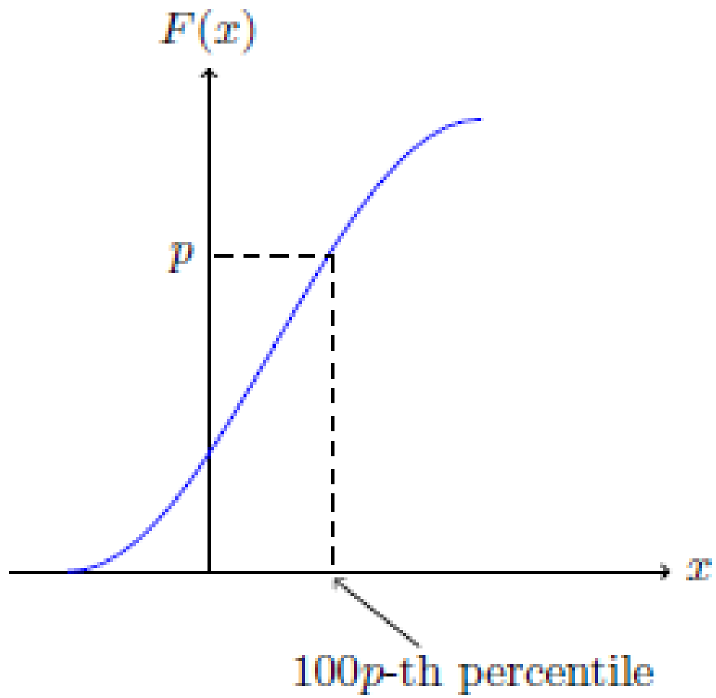
Relationship between cdf and pdf

- The pdf is the derivative of $F(x)$, i.e.,
$$f(x) = F'(x)$$
- The pdf is *not* a probability, so it can be bigger than 1.
- We get probabilities when the pdf is integrated.
- Also,

$$P\left(a - \frac{\varepsilon}{2} \leq X \leq a + \frac{\varepsilon}{2}\right) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)dx \cong \varepsilon f(a)$$

Percentile

- The $100p$-th percentile: x such that $F(x) = p$



Example 5.1.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \le x < c \\ 0, & \text{otherwise} \end{cases}$$

(a) Find c.

Example 5.1.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \le x < c \\ 0, & \text{otherwise} \end{cases}$$

(b) Find cdf $F(x)$.

Example 5.1.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \le x < c \\ 0, & \text{otherwise} \end{cases}$$

(c) Find $P\left(\frac{1}{2} < X < \frac{5}{4}\right)$.

(d) Find the median.

Example 5.1.2

Let X be a continuous r.v. with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

Find the pdf of $Y = X^2$.

5.2 Expectation and Variance

- Expectation (mean) of a continuous r.v. X:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$$

- For a real valued function g,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f(x) dx$$

- If a and b are constants,

$$E(aX + b) = aE(X) + b$$

Example 5.2.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E(X)$.

(b) Find $E(4X + 5X^2)$.

Some Special Expectations

- n-th moment of X: $E(X^n)$
- n-th central moment of X: $E[(X-\mu)^n]$
- Variance (2nd central moment):
  $\sigma^2 = \mathrm{Var}(X) = E[(X-\mu)^2] = E(X^2) - \mu^2$

Some Special Expectations

- If a and b are constants,

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

Example 5.2.2

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} (x+1)/2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E(X)$.

(b) Find $\mathrm{Var}(X)$.

Theorem 5.2.1 For a nonnegative continuous r.v. X,

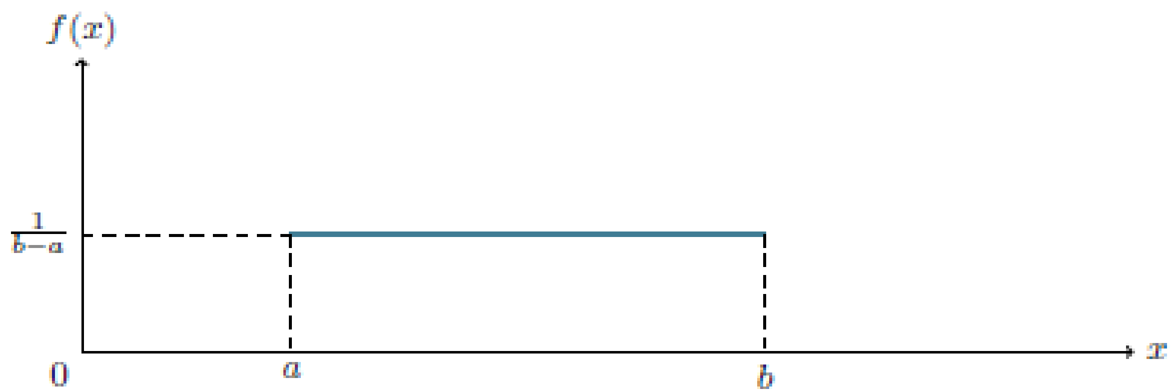
$$E(X) = \int_0^\infty P(X > x)dx$$

Proof

5.3 Uniform Distribution

- $X \sim$ Unif(a, b), where $a < b$ if it has pdf

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$



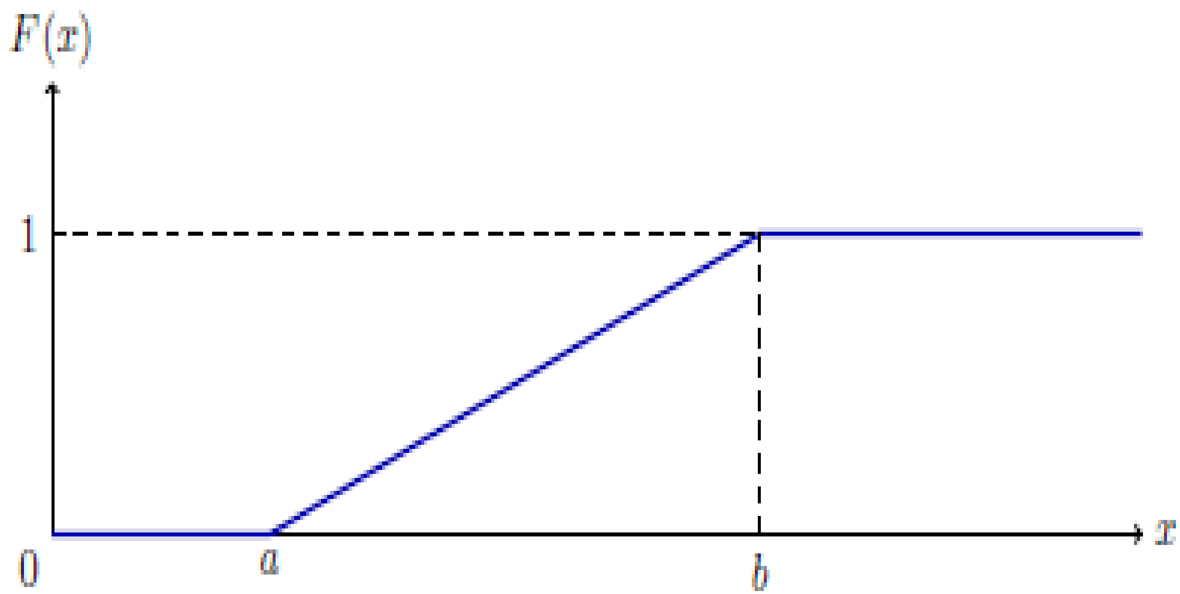
- When $a = 0$ and $b = 1$, i.e., $X \sim$ Unif(0, 1),

$$f(x) = 1, 0 < x < 1$$

- The cdf of $X \sim \text{Unif}(a, b)$ is

$$F(x) = P(X \le x)$$

$$= \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \ge b \end{cases}$$



$X \sim \text{Unif}(a, b), a<b$

$$E(X) = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

<u>Proof</u>

$X \sim \mathrm{Unif}(a, b)$, $a<b$

$$E(X) = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

If $X \sim \mathrm{Uniform}(0, 1)$,

$$E(X) = \frac{1}{2}$$

$$\mathrm{Var}(X) = \frac{1}{12}$$

Example 5.3.1

Let $X \sim$Unif(0, 1), and $Y = e^X$.

(a) Find the cdf $F_Y(y)$ of Y.

(b) Find the pdf $f_Y(y)$ of Y.

(c) Find $E(e^X)$.

Random Number Generation

Theorem: Let a 1-1 function F be a cdf of a continuous r.v. X, then $F(X) \sim$ Unif$(0, 1)$.

- Proof

- Application: simulation

Example 5.3.2

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$. Then

$$F(x) = \int_{-\infty}^{x} f(y)\ dy = \int_0^x \lambda e^{-\lambda y}\ dy = -e^{-\lambda y}\Big|_0^x = 1 - e^{-\lambda x},$$
$$x > 0$$

Let $U = F(X)$. Then both U and $1 - U$ have Uniform(0, 1) distribution.

$$1 - u = F(x) \Rightarrow u = e^{-\lambda x}$$
$$x = F^{-1}(u) = -\frac{1}{\lambda}\log u$$

To generate a random number from the distribution of X, generate a random number u from Unif(0, 1) and find x using the above equation.

Example 5.3.3

Suppose that a subway train arrives at a certain stop at 10-minute intervals starting at 6:00 a.m. Every time Rachel takes this train, she arrives at the station at a different time, so her waiting time is random within the 10-minute interval. Her waiting time X is a continuous random variable.

(a) Find the probability that she waits from 1 to 4 minutes.

(b) Find the probability that she waits at least 6 minutes.

Answer to Example 5.3.3

$X \sim \text{Unif}(0, 10)$

(a) Find the probability that she waits from 1 to 4 minutes.

(b) Find the probability that she waits at least 6 minutes.