AMS 507

Chapter 4
Random Variables

Sections 1-5

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4.1 Random Variables

- <u>Def.</u> Random variable (RV): a function from S to the real space $\mathbb{R}$.

$$X\colon S \to \mathbb{R}$$

$$P(X = x) = P(\{s \in S : X(s) = x\})$$

Example 4.1.1

Let X be the number of heads obtained in three tosses of a fair coin.

Outcome	X
HHH	
HHT	
HTH	
HTT	
THH	
THT	
TTH	
TTT	

Value of X	Event
X = 0	
X = 1	
X = 2	
X = 3	

$P(X = 0) =$

$P(X = 1) =$

$P(X = 2) =$

$P(X = 3) =$

Types of Random Variables

- Discrete Random Variables have a countable number of possible values.
- Continuous Random Variables can take on any value in an interval and cannot be enumerated.

eg) X: # heads obtained in three tosses of a coin: discrete
X: amount of precipitation produced by a storm: continuous

4.2 Discrete Random Variables

- Probability mass function (pmf)

1. $p(x) = P(X = x)$
2. $p(x) \geq 0 \quad {}^{\forall}x$
3. $\sum_{\text{all } x} p(x) = 1$

Example 4.2.1

Which of the following is a pmf?

- $p(x) = \frac{x-1}{3}$ for $x = 0, 1, 2, 3$
- $p(x) = \frac{x^2}{12}$ for $x = 0, 1, 2, 3$

Cumulative Distribution Function

<u>Def.</u> Cumulative distribution function (cdf):

$$F(x) = P(X \leq x) \quad {}^{\forall}x$$

Theorem 4.2.1 A function $F(x)$ is a cdf $\Leftrightarrow$

1. $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$
2. $F(x)$ is nondecreasing
3. ${}^{\forall}x_0, \lim_{x \downarrow x_0} F(x) = F(x_0)$: right continuous

Example 4.2.2

Foreign made cars: 30%. Four cars are selected at random.

X: number of foreign made cars. F: foreign made, D: domestic

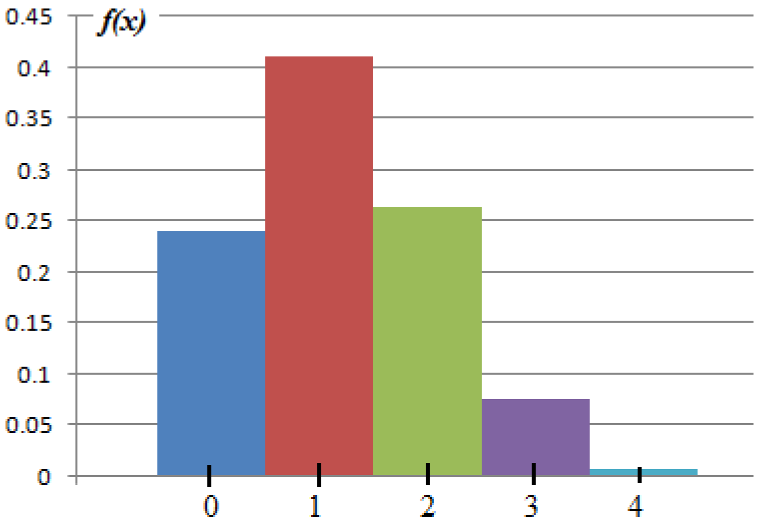
$P(X = 0) =$

$P(X = 1) =$

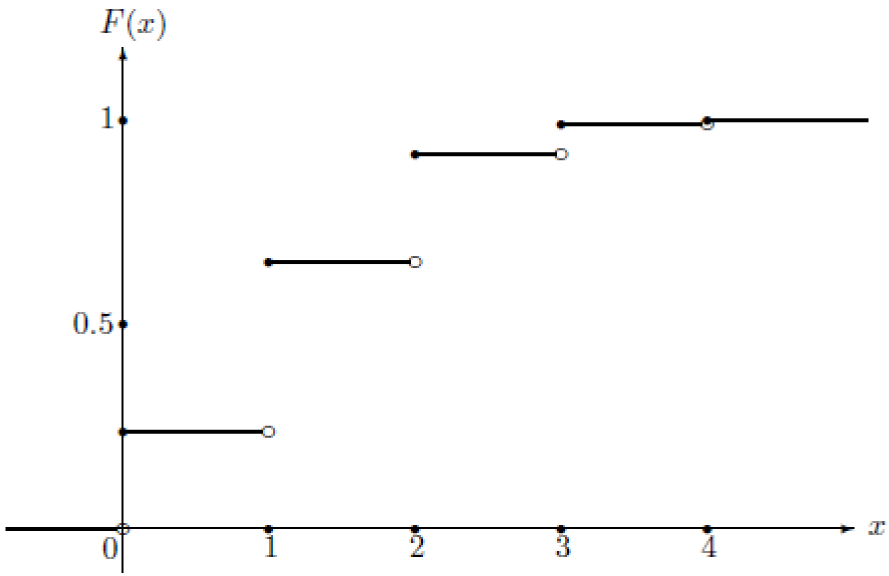
$P(X = 2) =$

$P(X = 3) =$

$P(X = 4) =$



Example 4.2.2 (continued)



x	$p(x)$	$F(x) = P(X \leq x)$

Example 3.3 (continued)

$p(0) =$

$p(1) =$

$p(2) =$

$p(3) =$

$p(4) =$

$F(0) =$

$F(1) =$

$F(2) =$

$F(3) =$

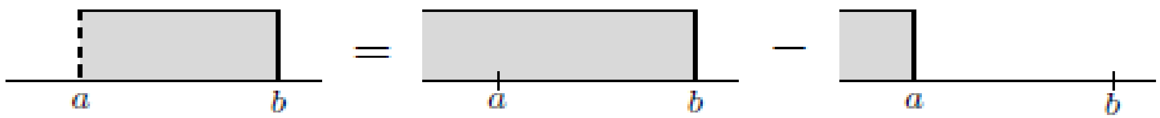
$F(4) =$

x	$p(x)$	$F(x)$
0	0.2401	0.2401
1	0.4116	0.6517
2	0.2646	0.9163
3	0.0756	0.9919
4	0.0081	1

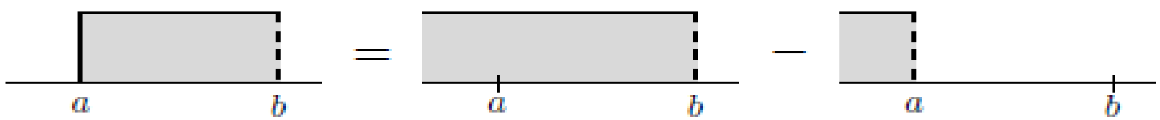
For $a < b$,

$$P(a \le X \le b) = P(X \le b) - P(X < a) = F(b) - F(a^-)$$

$$P(a < X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a)$$



$$P(a \le X < b) = P(X < b) - P(X < a) = F(b^-) - F(a^-)$$



$$P(a < X < b) = P(X < b) - P(X \le a) = F(b^-) - F(a)$$

Example 4.2.4

x	$p(x)$	$F(x)$
0	0.1	
1	0.2	
2	0.3	
3	0.2	
4	0.2	

$P(1 \leq X \leq 3) =$

$P(1 < X \leq 3) =$

$P(1 \leq X < 3) =$

$P(1 < X < 3) =$

Example 4.2.5

Tossing a coin until a head appears.

Let $p = P(H)$, and X: #tosses required to get a head.

$\Rightarrow P(X = x) =$

$P(X \leq x) =$

Is $F(x)$ a cdf?

4.3 Expected Value

Def Mean (Expected value) of a discrete r.v. X:

$$E(X) = \mu = \sum_{\text{all } x} x p(x)$$

Example 4.3.1

What is the expected number of heads in three tosses of a fair coin?

4.4 Expectation as a Function of RV's

$$E[h(X)] = \sum_{\text{all } x} h(x)p(x)$$

Example 4.4.1

In flipping 3 balanced coins find $E(X^3 - X)$.

Example 4.4.2

$h(x) = 10 + 2x + x^2$. Find $E[h(X)]$.

x	$p(x)$	$h(x)p(x)$
2	0.5	
3	0.3	
4	0.2	
Total	**1**	

Example 4.4.3

Let $Y = g(X) = aX + b$.

Then $E(Y) = aE(X) + b$.

<u>Proof</u>

Transformation of Discrete RV

X is a rv with cdf $F_X(x) \Rightarrow$ any function $Y = g(X)$ is a rv.
For all set A, $\mathrm{P(Y \in A)} = P[g(X) \in \mathrm{A}]$.
Let $\boldsymbol{\mathcal{X}}$ be the sample space of X and
$\boldsymbol{\mathcal{Y}}$ the sample space of Y. If g is 1-1,
$g^{-1}(A) = \{x \in \boldsymbol{\mathcal{X}}: g(x) \in A\}$
$g^{-1}(\{y\}) = \{x \in \boldsymbol{\mathcal{X}}: g(x) = y\}$
$P(Y \in A) = P[g(X) \in A] = P[X \in g^{-1}(A)]$
X is discrete $\Rightarrow \boldsymbol{\mathcal{X}}$ is countable $\Rightarrow$
$\boldsymbol{\mathcal{Y}} = \{y: y = g(x): x \in \boldsymbol{\mathcal{X}}\}$ is countable.
$\therefore Y$ is discrete.

$$p_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(\boldsymbol{\mathcal{Y}})} P(X = x)$$

$$= \sum_{x \in g^{-1}(\boldsymbol{\mathcal{Y}})} p_X(x) \text{ for } y \in \boldsymbol{\mathcal{Y}}$$

Example 4.4.4

Let X be a discrete rv with pmf

$$p_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$x = 0, 1, \cdots, n.$

Let $Y = n - X$. Then

$p_Y(y) =$

4.5 Variance

- <u>Variance</u> of a discrete r.v. X:

$$\mathrm{Var}(X) = \sigma^2 = E[(X-\mu)^2] = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

$$\mathrm{sd}(X) = \sigma = \sqrt{\mathrm{Var}(X)}$$

Alternatively $\mathrm{Var}(X) = E(X^2) - \mu^2$

$$= \sum_{\text{all } x} x^2 p(x) - \left[\sum_{\text{all } x} x p(x)\right]^2$$

<u>Proof</u>

4.5 Variance

$\mathrm{Var}(X) = E(X^2) - \mu^2$

<u>Proof</u>

Example 4.5.1

x	1	2	5	9
$p(x)$	0.3	0.4	0.2	0.1

$E(X) =$

$\mathrm{Var}(X) =$

Alternatively,
$\mathrm{Var}(X) =$

Let $Y = g(X) = aX + b$.

Then $\text{Var}(Y) = a^2\text{Var}(X)$.

Proof