AMS 507

Chapter 4
Random Variables

Sections 1-5

Hongshik Ahn

4.1 Random Variables

- Def. Random variable (RV): a function from S to the real space $\mathbb{R}$.

$$X: S \to \mathbb{R}$$

$$P(X = x) = P(\{s \in S: X(s) = x\})$$

Example 4.1.1

Let X be the number of heads obtained in three tosses of a fair coin.

Outcome	X
HHH	
HHT	
HTH	
HTT	
THH	
THT	
TTH	
TTT	

Value of X	Event
X = 0	
X = 1	
X = 2	
X = 3	

$P(X = 0) =$

$P(X = 1) =$

$P(X = 2) =$

$P(X = 3) =$

Types of Random Variables

- Discrete Random Variables have a countable number of possible values.
- Continuous Random Variables can take on any value in an interval and cannot be enumerated.

eg) X: # heads obtained in three tosses of a coin: discrete
X: amount of precipitation produced by a storm: continuous

4.2 Discrete Random Variables

- Probability mass function (pmf)

1. $p(x) = P(X = x)$
2. $p(x) \geq 0 \quad \forall x$
3. $\sum_{\text{all } x} p(x) = 1$

Example 4.2.1

Which of the following is a pmf?

- $p(x) = \frac{x-1}{3}$ for $x = 0, 1, 2, 3$
- $p(x) = \frac{x^2}{12}$ for $x = 0, 1, 2, 3$

Cumulative Distribution Function

Def. Cumulative distribution function (cdf):

$$F(x) = P(X \leq x) \quad {}^{\forall}x$$

Theorem 4.2.1 A function $F(x)$ is a cdf $\Leftrightarrow$

1. $\lim\limits_{x \to -\infty} F(x) = 0$ and $\lim\limits_{x \to \infty} F(x) = 1$
2. $F(x)$ is nondecreasing
3. ${}^{\forall}x_0, \lim\limits_{x \downarrow x_0} F(x) = F(x_0)$: right continuous

Example 4.2.2

Foreign made cars: 30%. Four cars are selected at random.

X: number of foreign made cars. F: foreign made, D: domestic

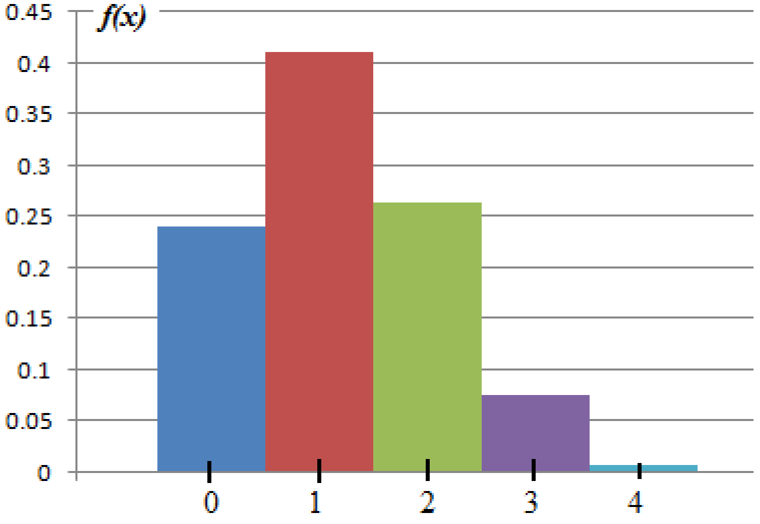
$P(X = 0) =$

$P(X = 1) =$

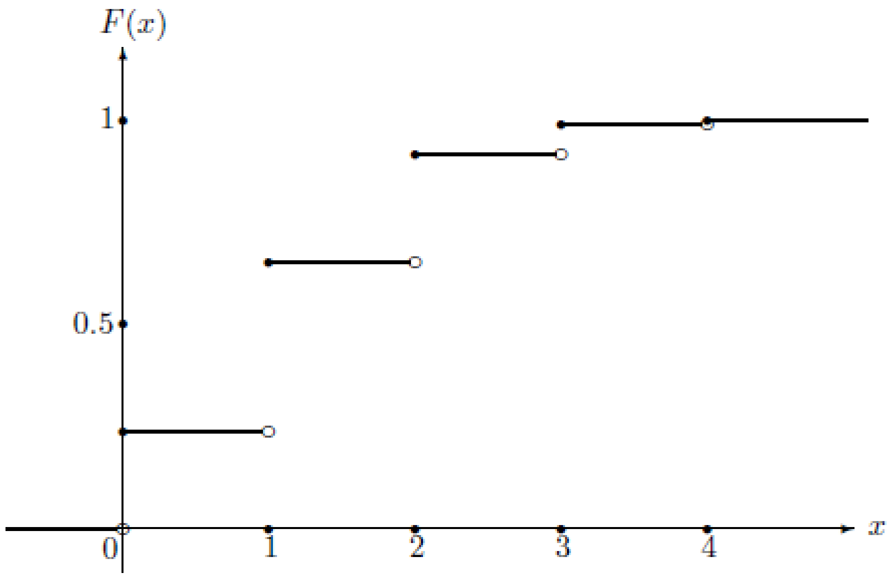
$P(X = 2) =$

$P(X = 3) =$

$P(X = 4) =$



Example 4.2.2 (continued)



x	$p(x)$	$F(x) = P(X \leq x)$

Example 3.3 (continued)

$p(0) =$

$p(1) =$

$p(2) =$

$p(3) =$

$p(4) =$

$F(0) =$

$F(1) =$

$F(2) =$

$F(3) =$

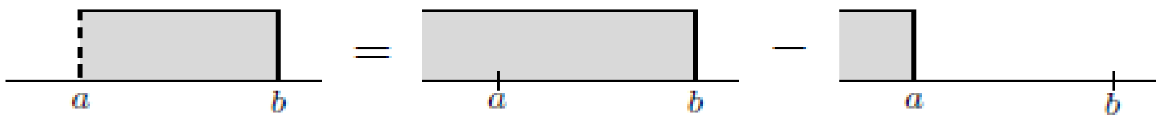
$F(4) =$

x	$p(x)$	$F(x)$
0	0.2401	0.2401
1	0.4116	0.6517
2	0.2646	0.9163
3	0.0756	0.9919
4	0.0081	1

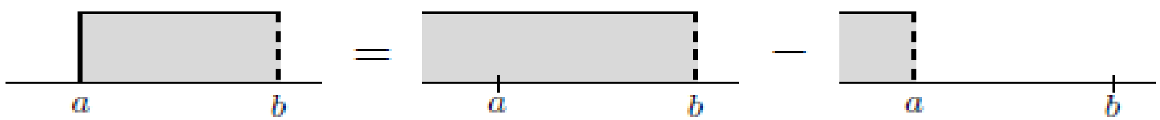
For $a < b$,

$$P(a \le X \le b) = P(X \le b) - P(X < a) = F(b) - F(a^-)$$

$$P(a < X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a)$$



$$P(a \le X < b) = P(X < b) - P(X < a) = F(b^-) - F(a^-)$$



$$P(a < X < b) = P(X < b) - P(X \le a) = F(b^-) - F(a)$$

Example 4.2.4

x	$p(x)$	$F(x)$
0	0.1	
1	0.2	
2	0.3	
3	0.2	
4	0.2	

$$P(1 \le X \le 3) =$$
$$P(1 < X \le 3) =$$
$$P(1 \le X < 3) =$$
$$P(1 < X < 3) =$$

Example 4.2.5

Tossing a coin until a head appears.

Let $p = P(H)$, and X: #tosses required to get a head.

$\Rightarrow P(X = x) =$

$P(X \leq x) =$

Is $F(x)$ a cdf?

4.3 Expected Value

Def Mean (Expected value) of a discrete r.v. X:

$$E(X) = \mu = \sum_{\text{all } x} x p(x)$$

Example 4.3.1

What is the expected number of heads in three tosses of a fair coin?

4.4 Expectation as a Function of RV's

$$E[h(X)] = \sum_{\text{all } x} h(x)p(x)$$

Example 4.4.1

In flipping 3 balanced coins find $E(X^3 - X)$.

Example 4.4.2

$h(x) = 10 + 2x + x^2$. Find $E[h(X)]$.

x	$p(x)$	$h(x)p(x)$
2	0.5	
3	0.3	
4	0.2	
Total	**1**	

Example 4.4.3

Let $Y = g(X) = aX + b$.

Then $E(Y) = aE(X) + b$.

<u>Proof</u>

Transformation of Discrete RV

X is a rv with cdf $F_X(x) \Rightarrow$ any function $Y = g(X)$ is a rv.

For all set A, $\mathrm{P(Y \in A)} = P[g(X) \in \mathrm{A}]$.

Let $\boldsymbol{\mathcal{X}}$ be the sample space of X and

$\boldsymbol{\mathcal{Y}}$ the sample space of Y. If g is 1-1,

$g^{-1}(A) = \{x \in \boldsymbol{\mathcal{X}}: g(x) \in A\}$

$g^{-1}(\{y\}) = \{x \in \boldsymbol{\mathcal{X}}: g(x) = y\}$

$P(Y \in A) = P[g(X) \in A] = P[X \in g^{-1}(A)]$

X is discrete $\Rightarrow \boldsymbol{\mathcal{X}}$ is countable $\Rightarrow$

$\boldsymbol{\mathcal{Y}} = \{y: y = g(x): x \in \boldsymbol{\mathcal{X}}\}$ is countable.

$\therefore Y$ is discrete.

$$p_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(\boldsymbol{\mathcal{Y}})} P(X = x)$$

$$= \sum_{x \in g^{-1}(\boldsymbol{\mathcal{Y}})} p_X(x) \text{ for } y \in \boldsymbol{\mathcal{Y}}$$

Example 4.4.4

Let X be a discrete rv with pmf

$$p_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

$x = 0, 1, \cdots, n.$

Let $Y = n - X$. Then

$p_Y(y) =$

4.5 Variance

- Variance of a discrete r.v. X:

$$\mathrm{Var}(X) = \sigma^2 = E[(X-\mu)^2] = \sum_{\text{all } x} (x-\mu)^2 p(x)$$

$$\mathrm{sd}(X) = \sigma = \sqrt{\mathrm{Var}(X)}$$

Alternatively $\mathrm{Var}(X) = E(X^2) - \mu^2$

$$= \sum_{\text{all } x} x^2 p(x) - \left[\sum_{\text{all } x} x p(x)\right]^2$$

Proof

4.5 Variance

$\mathrm{Var}(X) = E(X^2) - \mu^2$

<u>Proof</u>

Example 4.5.1

x	1	2	5	9
$p(x)$	0.3	0.4	0.2	0.1

$E(X) =$

$\mathrm{Var}(X) =$

Alternatively,
$\mathrm{Var}(X) =$

Let $Y = g(X) = aX + b$.

Then $\mathrm{Var}(Y) = a^2\mathrm{Var}(X)$.

<u>Proof</u>

AMS 507

Chapter 4
Random Variables

Sections 6-8

Hongshik Ahn

4.6 Bernoulli and Binomial RV's

- Bernoulli Trial

a. Each trial yields one of two outcomes: Success (S) or Failure (F)

b. $P(S) = P(X = 1) = p;$
$P(F) = P(X = 0) = 1 - p$

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$= p^x(1-p)^{1-x}, \qquad x = 0, 1$$

c. Each trial is independent.

Bernoulli RV's

Let $X \sim \text{Bernoulli}(p)$. Then

$$p(x) = p^x(1-p)^{1-x}, \qquad x = 0, 1$$

$$E(X) = p$$

$$\text{Var}(X) = p(1-p)$$

<u>Proof</u>

Binomial Distribution

n: a fixed number of independent Bernoulli trials

p: the probability of success in each trial

X: # successes in n trials

X is a binomial random variable: $X \sim \mathrm{Bin}(n, p)$

If $Y_1, Y_2, \cdots, Y_n$ are independent Bernoulli(p), then

$\sum_{i=1}^{n} Y_i \sim \mathrm{Bin}(n, p)$

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \cdots, n$$

Show that $p(x)$ is a pmf.

$E(X) = np$, Var$(X) = np(1-p)$

$X \sim \mathrm{Bin}(n, p)$

$E(X) = np$, $\mathrm{Var}(X) = np(1-p)$

<u>Proof</u>

Example 4.6.1

Let X be the number of heads in 7 independent tosses of an unbiased coin. Then $X \sim \text{Bin}(7, 1/2)$.

$P(0 \leq X \leq 1) =$

$E(X) =$

$\text{Var}(X) =$

Example 4.6.2

If the probability is 0.1 that a certain device fails a comprehensive safety test, what are the probabilities that among 15 of such devices,

(a) at most two will fail?

(b) at least three will fail?

Example 4.6.3

30% of the automobiles in a certain city are foreign made. Four cars are selected at random.
X: #cars sampled that are foreign made

$P(X = 3) =$

$P(X \geq 3) =$

$P(X \leq 1) =$

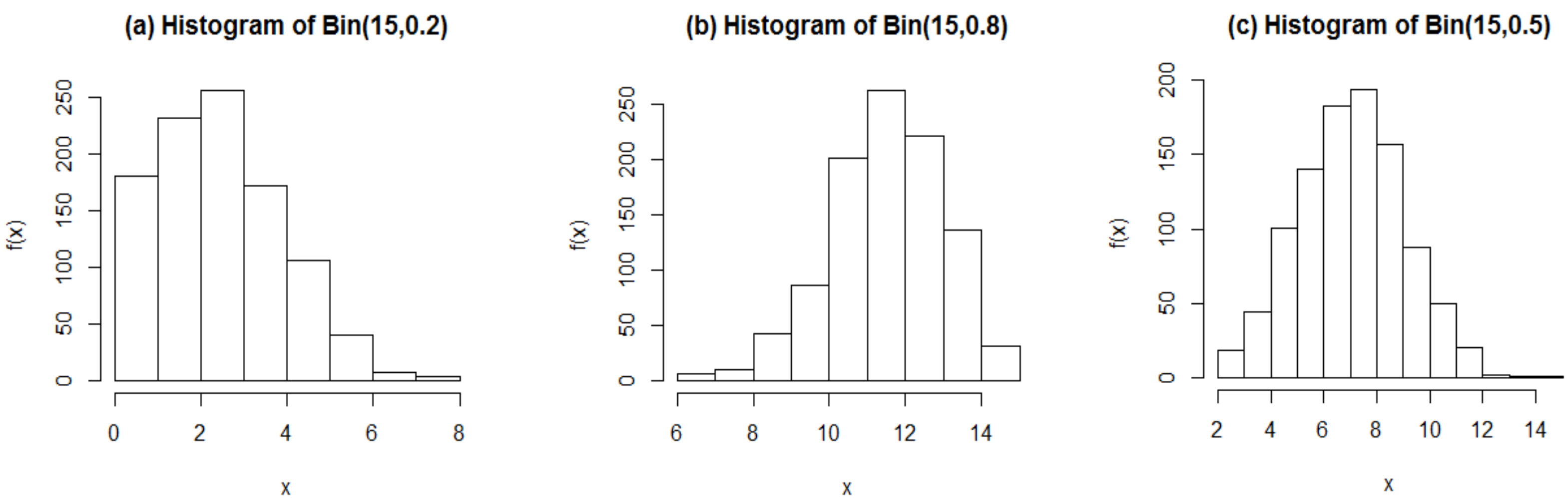
$P(X < 2) =$

Binomial distribution

$p < 0.5$: skewed to the right

$p > 0.5$: skewed to the left

$p = 0.5$: symmetric



4.7 Poisson RV's

- Properties:
  1. The probability that an event occurs in an interval is proportional to the length of the interval.
  2. Two events cannot occur at exactly the same instant.
  3. Events occur independently.
- The probability of x events occurring in a time period t for a Poisson random variable with parameter λ: $X \sim \text{Poisson}(\lambda)$

$$p(x) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 0, 1, \cdots$$

$$E(X) = \text{Var}(X) = \lambda$$

Poisson Distribution

Show that $p(x)$ is a pmf.

$$X \sim \mathrm{Poisson}(\lambda)$$

$$E(X) = \lambda$$

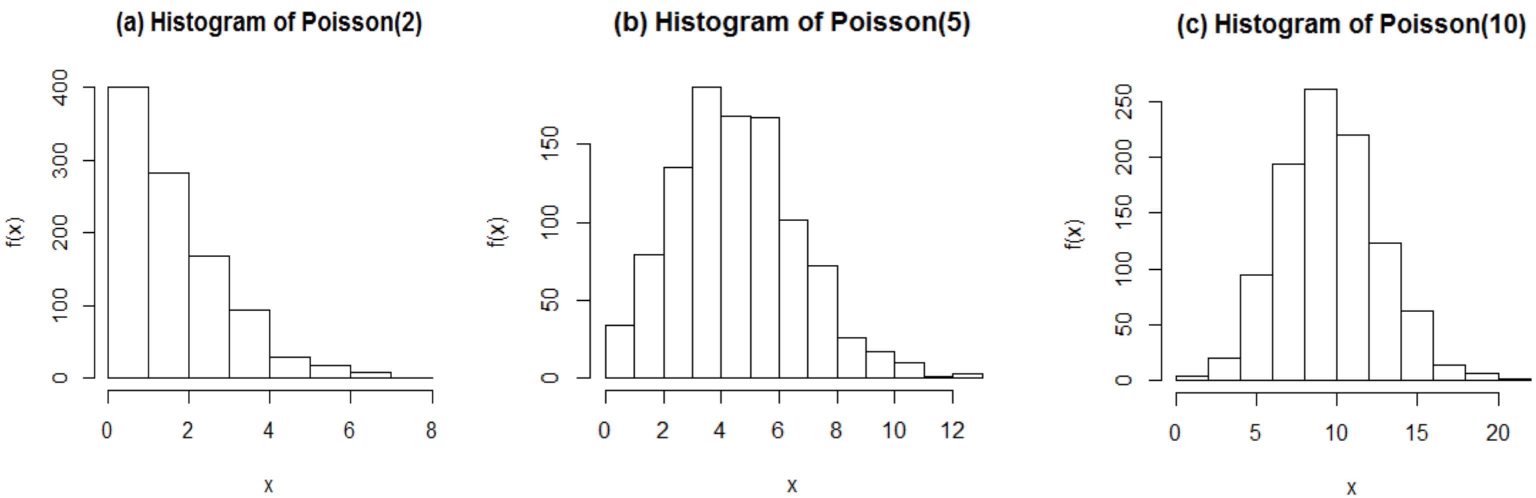
$$\mathrm{Var}(X) = \lambda$$

<u>Proof</u>

- Use of Poisson distributions, examples:
    - number of occurrences in a given time
    - distribution of bomb hits in an area
    - distribution of fish in a lake

Poisson Distribution

- The Poisson distribution is highly skewed for small values of λ.
- As λ increases, the distribution becomes more symmetric.



Example 4.7.1

A 911 operator handles 4 calls every 3 hours on average.

(a) What is the probability of no calls in the next hour?

(b) Find the probability of at most two calls in the next hour.

Example 4.7.2

On average, 12 cars pass a toll gate booth in a minute during rush hours.

(a) Probability that one car passes the booth in 3 seconds:

(b) Probability that at least two cars pass in 5 seconds:

(c) Probability that at most one car passes in 10 seconds:

Poisson Approximation to the Binomial

Let $X \sim \mathrm{Bin}(n, p)$.

Let $n \to \infty$ and $p \to 0$ such that np remains constant (np is a moderate number). Then

X is approximately $\mathrm{Poisson}(\lambda)$ such that $\lambda = np$.

$n \geq 20, p \leq 0.05$: acceptable

$n \geq 100, p \leq 0.01$: excellent approximation

$X \sim \mathrm{Bin}(n, p) \Longrightarrow E(X) = np = \lambda$

$\mathrm{Var}(X) = np(1 - p) = \lambda(1 - p) \to \lambda$ as $n \to \infty$ & $p \to 0$

Poisson Approximation to the Binomial

<u>Alternative Approach</u>

$X \sim \mathrm{Bin}(n, p), \lambda = np, n \to \infty$

$$\Longrightarrow P(X = x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{n-x} = \frac{n!}{(n-x)!\,x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{n(n-1)\cdots(n-x+1)}{n^x} \frac{\lambda^x}{x!} \frac{[(1-(\lambda/n)]^n}{[(1-(\lambda/n)]^x}, \qquad x = 0, 1, 2, \cdots$$

For $n \to \infty$,

$\left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda}$, $\frac{n(n-1)\cdots(n-x+1)}{n^x} \approx 1$, $\left(1 - \frac{\lambda}{n}\right)^x \approx 1$

Hence,

$$P(X = x) \approx \frac{\lambda^x e^{-\lambda}}{x!}, \ x = 0, 1, 2, \cdots$$

Poisson Approximation to the Binomial

$X \sim \text{Poisson}(\lambda)$

$$\Longrightarrow P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} = \frac{\lambda}{x} P(X = x - 1), \qquad x = 1, 2, \cdots$$

$Y \sim \text{Bin}(n, p)$

$$\Longrightarrow P(Y = y) = \binom{n}{y} p^y (1-p)^{n-y} = \frac{n - y + 1}{y} \cdot \frac{p}{1-p} P(Y = y - 1)$$

$$= \frac{np - p(y-1)}{y - yp} P(Y = y - 1) \approx \frac{\lambda}{y} P(Y = y - 1), \qquad y = 1, 2, \cdots, n$$

If $\lambda = np$ and $p \approx 0$,

$$P(Y = 0) = (1-p)^n = \left(1 - \frac{\lambda}{n}\right)^n \to e^{-\lambda} = P(X = 0)$$

as $n \to \infty$

Example 4.7.3

A publisher of mystery novels tries to keep its books free of typos. The probability of any given page containing at least one typo is 0.003 and errors are independent from page to page. What is the approximate probability that a 500 page book has

(a) exactly 1 page with typos?

(b) at most 2 pages with typos?

4.8 Other Discrete Probability Distribution

Hypergeometric RV

- Sampling w/o replacement

1) Finite population with N individuals (binomial: infinite population or with replacement)
2) S or F. There are m successes in the population.
3) Sample size: n. Each subset of size n is equally likely to be chosen.

$$P(X = x) = \frac{\binom{m}{x}\binom{N-m}{n-x}}{\binom{N}{n}}$$

$$\max\{0, n - N + m\} \leq x \leq \min\{n, m\}$$

$$(\because x \leq m \text{ and } n - x \leq N - m)$$

$$E(X) = \frac{nm}{N}, \qquad Var(X) = \frac{nm(N-m)(N-n)}{N^2(N-1)}$$

Example 4.8.4

A shipment of 25 CD's contains 5 that are defective. If 10 of them are randomly chosen without replacement, what is the probability that 2 of the 10 will be defective?

Hypergeometric RV

$$E(X) = \frac{nm}{N}, \qquad \mathrm{Var}(X) = \frac{nm(N-m)(N-n)}{N^2(N-1)}$$

Let $\frac{m}{N} = p$. Then

$E(X) = np, \mathrm{Var}(X) = np(1-p)\left(\frac{N-n}{N-1}\right)$

If $N \gg n$, then $\mathrm{Var}(X) \approx np(1-p)$

$\frac{N-n}{N-1}$: finite population correction factor

Binomial approximation to the hypergeometric, when

$\frac{n}{N} \leq 0.1$, and $p = \frac{m}{N}$ is not too close to either 0 or 1.

Example 4.8.5

From Example 4.8.4, for a lot of 100 CD's, 20 are defective. Find the probability that among a randomly selected sample of 10 CD's, 2 are defective, by using

(a) The formula for the hypergeometric distribution:

(b) The binomial approximation to the hypergeometric:

- Geometric RV: $X \sim \text{Geometric}(p)$

1. Each trial: Success (S) or Failure (F)
2. Independent trials
3. $P(S) = p$
4. Continues until the first success is observed

- Probability of getting the first success on the x-th trial:

$$p(x) = p(1-p)^{x-1}, x = 1, 2, \cdots$$

$$E(X) = \frac{1}{p}, \qquad \text{Var}(X) = \frac{1-p}{p^2}$$

Geometric RV

$$p(x) = p(1-p)^{x-1}, x = 1, 2, \cdots$$

Show $p(x)$ is a pmf.

Geometric RV

- CDF:

$$F(x) = \begin{cases} 1 - (1-p)^x, k \le x < k+1, k = 1, 2, \cdots \\ 0, \qquad x < 1 \end{cases}$$

Proof

Geometric RV

$$E(X) = \frac{1}{p}$$

Proof

Geometric RV

$$E(X) = \frac{1}{p}$$

<u>Alternative proof</u>

Geometric RV

$$Var(X) = \frac{1-p}{p^2}$$

Proof

Example 4.8.1

A fair die is tossed until a certain number appears. What is the probability that the first six appears at the fifth toss?

Negative Binomial RV: $X \sim \mathrm{NB}(r, p)$

- Independent Bernoulli Trials
- X: number of trials until r-th success
- $r = 1 \Longrightarrow X \sim \text{Geometric}(p)$

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, \cdots$$

$$E(X) = \frac{r}{p}$$

$$\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$$

Negative Binomial RV

$$E(X) = \frac{r}{p}$$

Proof

Negative Binomial RV

$$\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$$

<u>Proof</u>

Negative Binomial RV

- Let $X_1, X_2, \cdots, X_n$ be independent with
  $$X_i \sim \text{Geometric}(p), i = 1, 2, \cdots, n.$$

Then
$$Y = \sum_{i=1}^{n} X_i \sim \text{NB}(n, p).$$

- Let $X_1, X_2, \cdots, X_n$ be independent with
  $$X_i \sim \text{NB}(r, p), i = 1, 2, \cdots, n.$$

Then
$$Y = \sum_{i=1}^{n} X_i \sim \text{NB}(nr, p).$$

Example 4.8.2 (Ross Example)

A pipe-smoking mathematician carries, at all times, 2 matchboxes, 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match, he is equally likely to take it from either pocket. Consider the moment when the mathematician first discovers that one of his matchboxes is empty. If it is assumed that both matchboxes initially contained N matches, what is the probability that there are exactly k matches in the other box, $k = 0, 1, \cdots, N$?

Answer to Example 4.8.2

Example 4.8.3 (Ross Example)

Find the expected value and the variance of the number of times one must throw a die until the outcome 1 has occurred 4 times.

AMS 507

Chapter 5
Continuous Random Variables

Sections 1-3

Hongshik Ahn

5.1 Introduction

- Definition: Random variable X is *continuous* if there exists a function, $f(x) \geq 0, f: \mathbb{R} \to \mathbb{R}^+$, such that, $\forall B \subset \mathbb{R}$, we have

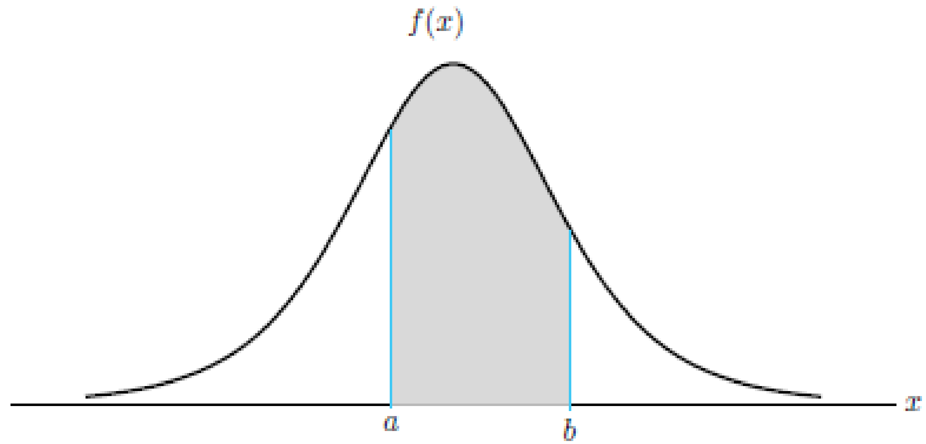
$$P(X \in B) = \int_{x \in B} f(x)dx$$

- The function $f(x)$ in the definition above is called *pdf* (probability density function).
- If $B = [a, b]$, then

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

- and

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)dx = 1$$



The area under the graph of $f(x)$ representing $P(a \leq X \leq b)$

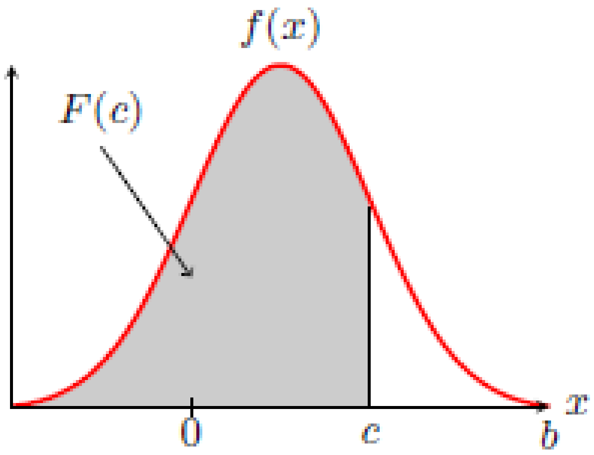
- If $a = b$, then

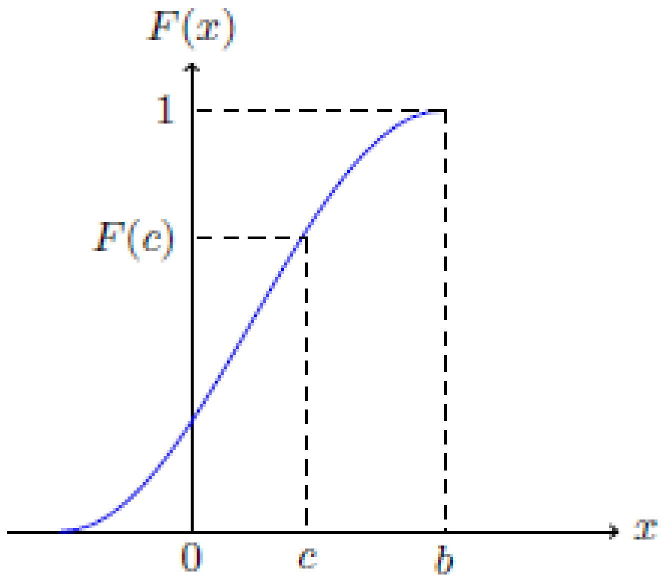
$$P(X = a) = \int_a^a f(x)dx = 0$$

Accordingly,
$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b)$

CDF of a Continuous R.V.

- CDF: $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$



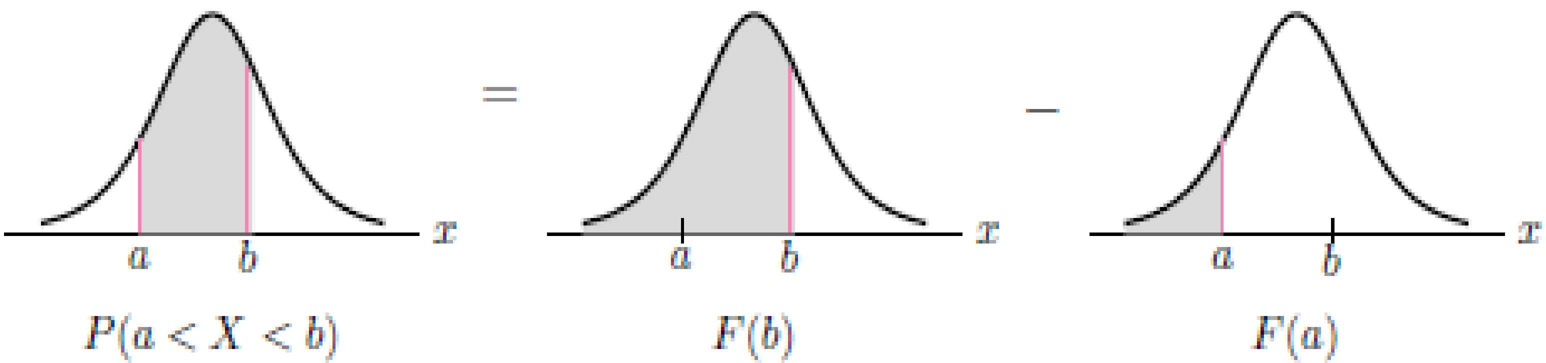


The pdf is the derivative of $F(x)$, i.e., $f(x) = F'(x)$

CDF of a Continuous R.V.

- For any interval [a, b],

$$P(a \le X \le b) = \int_a^b f(x)dx = F(b) - F(a)$$



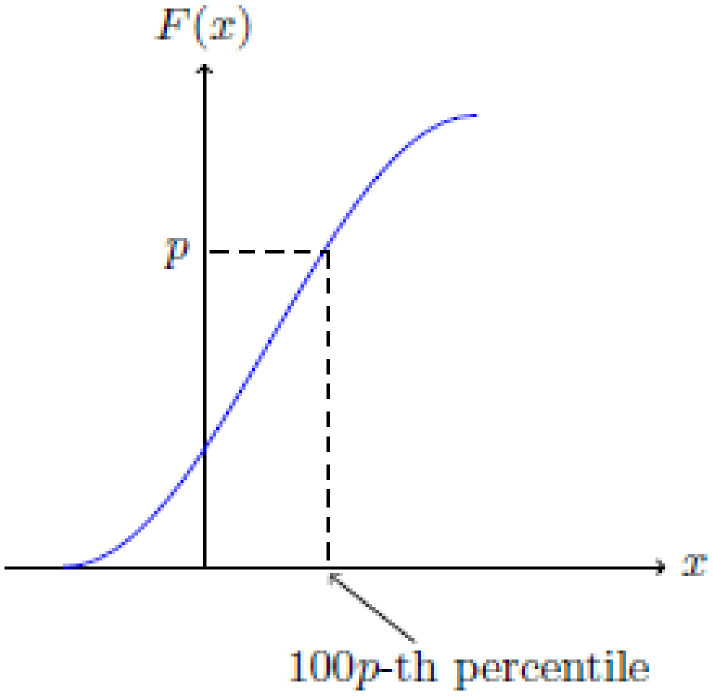
Relationship between cdf and pdf

- The pdf is the derivative of $F(x)$, i.e.,
$$f(x) = F'(x)$$
- The pdf is *not* a probability, so it can be bigger than 1.
- We get probabilities when the pdf is integrated.
- Also,

$$P\left(a - \frac{\varepsilon}{2} \leq X \leq a + \frac{\varepsilon}{2}\right) = \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f(x)dx \cong \varepsilon f(a)$$

Percentile

- The $100p$-th percentile: x such that $F(x) = p$



Example 5.1.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \le x < c \\ 0, & \text{otherwise} \end{cases}$$

(a) Find c.

Example 5.1.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \leq x < c \\ 0, & \text{otherwise} \end{cases}$$

(b) Find cdf $F(x)$.

Example 5.1.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} x^2, & 0 < x < 1 \\ x, & 1 \leq x < c \\ 0, & \text{otherwise} \end{cases}$$

(c) Find $P\left(\frac{1}{2} < X < \frac{5}{4}\right)$.

(d) Find the median.

Example 5.1.2

Let X be a continuous r.v. with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad -\infty < x < \infty$$

Find the pdf of $Y = X^2$.

5.2 Expectation and Variance

- Expectation (mean) of a continuous r.v. X:

$$\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$$

- For a real valued function g,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)\, f(x)dx$$

- If a and b are constants,

$$E(aX + b) = aE(X) + b$$

Example 5.2.1

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E(X)$.

(b) Find $E(4X + 5X^2)$.

Some Special Expectations

- n-th moment of X: $E(X^n)$
- n-th central moment of X: $E[(X-\mu)^n]$
- Variance (2nd central moment):
  $\sigma^2 = \mathrm{Var}(X) = E[(X-\mu)^2] = E(X^2) - \mu^2$

Some Special Expectations

- If a and b are constants,

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X)$$

Example 5.2.2

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} (x+1)/2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E(X)$.

(b) Find $\mathrm{Var}(X)$.

Theorem 5.2.1 For a nonnegative continuous r.v. X,

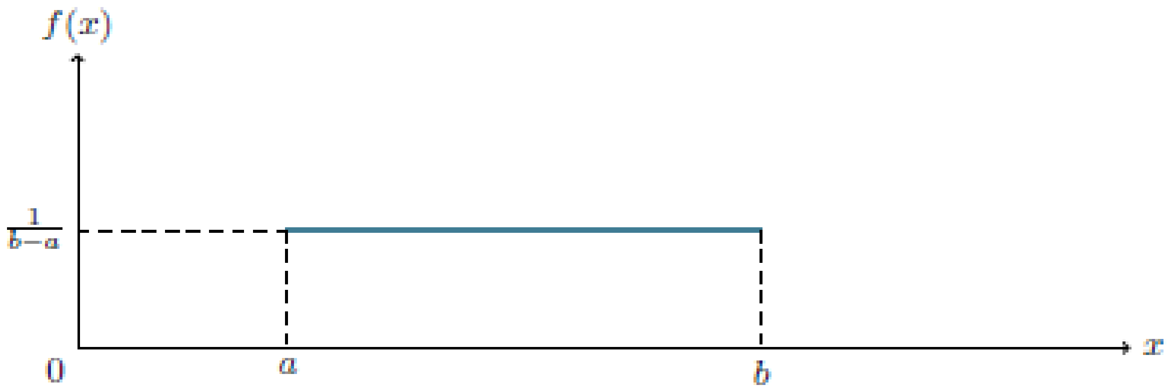
$$E(X) = \int_0^\infty P(X > x)dx$$

Proof

5.3 Uniform Distribution

- $X \sim$ Unif(a, b), where $a < b$ if it has pdf

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$



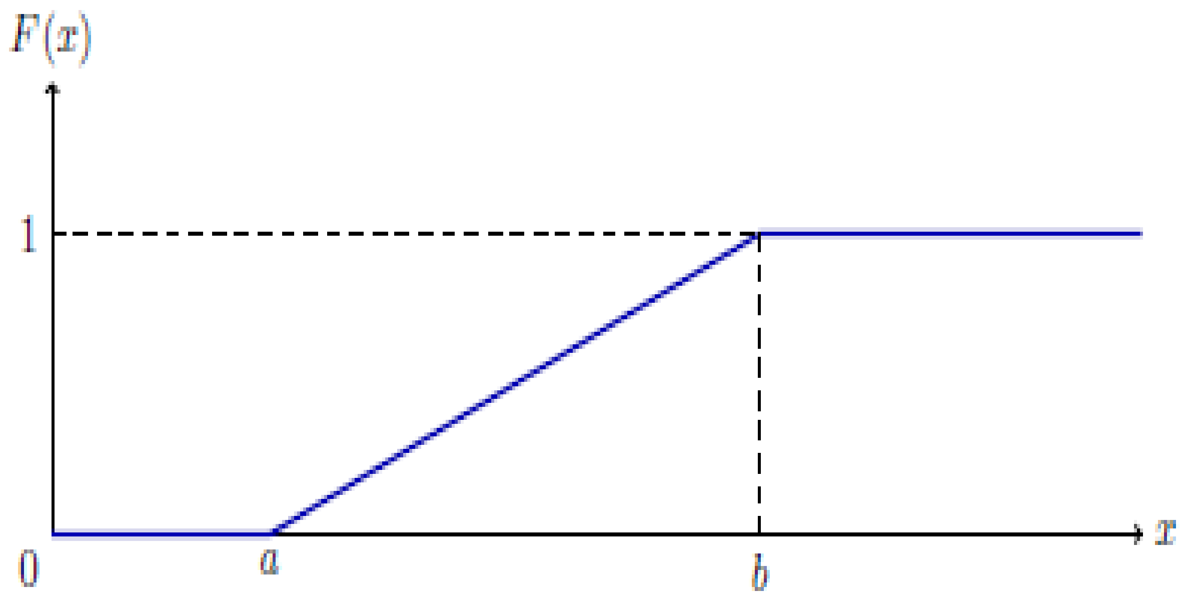
- When $a = 0$ and $b = 1$, i.e., $X \sim$ Unif(0, 1),

$$f(x) = 1, 0 < x < 1$$

- The cdf of $X \sim \text{Unif}(a, b)$ is

$$F(x) = P(X \leq x)$$

$$= \int_{-\infty}^{x} f(y)dy = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$



$X \sim \mathrm{Unif}(a, b)$, $a<b$

$$E(X) = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

<u>Proof</u>

$X \sim \text{Unif}(a, b)$, $a<b$

$$E(X) = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

If $X \sim \text{Uniform}(0, 1)$,

$$E(X) = \frac{1}{2}$$

$$\text{Var}(X) = \frac{1}{12}$$

Example 5.3.1

Let X ~Unif(0, 1), and $Y = e^X$.

(a) Find the cdf $F_Y(y)$ of Y.

(b) Find the pdf $f_Y(y)$ of Y.

(c) Find $E(e^X)$.

Random Number Generation

Theorem: Let a 1-1 function F be a cdf of a continuous r.v. X, then $F(X) \sim$ Unif$(0, 1)$.

- <u>Proof</u>

- Application: simulation

Example 5.3.2

Let X be a continuous r.v. with pdf

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\lambda > 0$. Then

$$F(x) = \int_{-\infty}^{x} f(y)\ dy = \int_{0}^{x} \lambda e^{-\lambda y}\ dy = -e^{-\lambda y}\Big|_{0}^{x} = 1 - e^{-\lambda x},$$
$$x > 0$$

Let $U = F(X)$. Then both U and $1 - U$ have Uniform(0, 1) distribution.

$$1 - u = F(x) \Rightarrow u = e^{-\lambda x}$$
$$x = F^{-1}(u) = -\frac{1}{\lambda}\log u$$

To generate a random number from the distribution of X, generate a random number u from Unif(0, 1) and find x using the above equation.

Example 5.3.3

Suppose that a subway train arrives at a certain stop at 10-minute intervals starting at 6:00 a.m. Every time Rachel takes this train, she arrives at the station at a different time, so her waiting time is random within the 10-minute interval. Her waiting time X is a continuous random variable.

(a) Find the probability that she waits from 1 to 4 minutes.

(b) Find the probability that she waits at least 6 minutes.

Answer to Example 5.3.3

$X \sim \text{Unif}(0, 10)$

(a) Find the probability that she waits from 1 to 4 minutes.

(b) Find the probability that she waits at least 6 minutes.