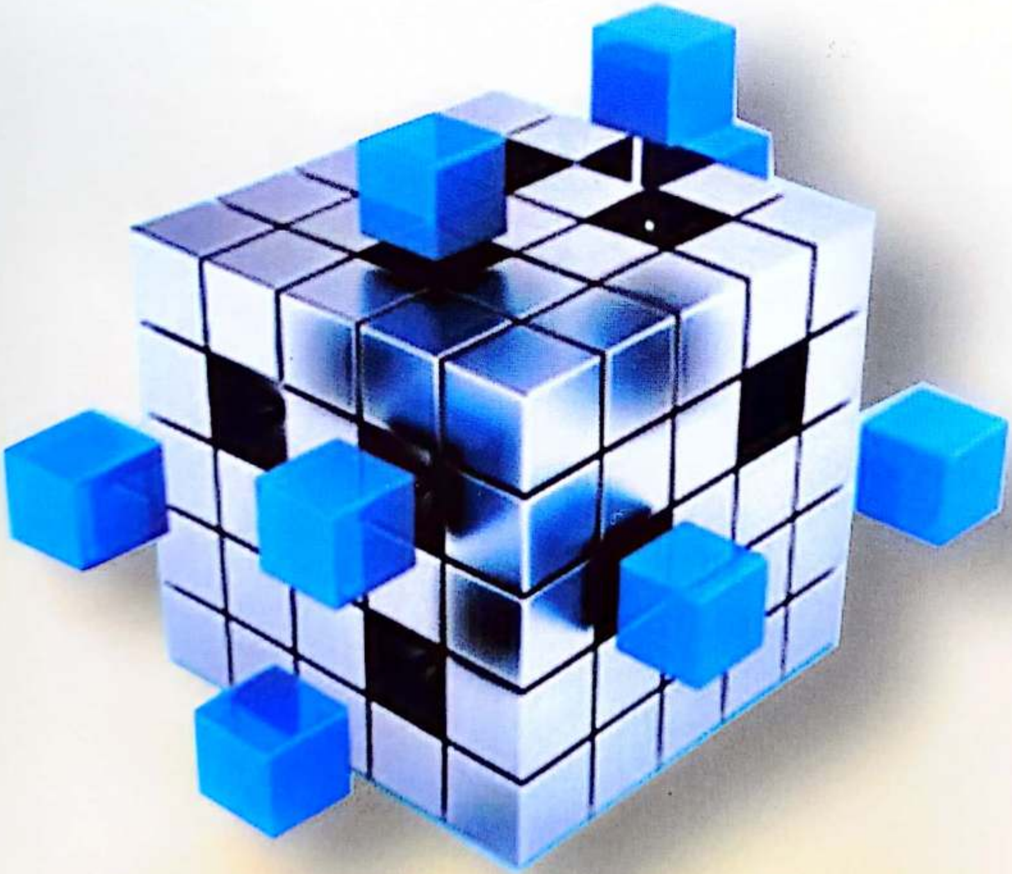
ASMITA

A Complete Solution Manual to

ALGEBRA AND GEOMETRY

(For all Engineering Programs of Pokhara University)



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Unit 1

Matrix and System of Linear Equations

Exercise 1.1

1. **Find the rank of following matrices.**

   a. $\begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} -1 & 2 & -2 \\ 4 & -3 & 4 \\ -2 & 4 & -4 \end{pmatrix}$

Applying $R_2 \to R_2 + 4R_1$, $R_3 \to R_3 - 2R_1$

$\sim \begin{pmatrix} -1 & 2 & -2 \\ 0 & 5 & -4 \\ 0 & 0 & 0 \end{pmatrix}$

which is the equivalent row echelon form of the given matrix and number of non - zero rows is 2.

   b. $\begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{pmatrix}$

Applying $R_2 \to R_2 + 2R_1$

Applying $R_3 \to R_3 + R_1$

$$\begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 10 & 10 \end{pmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} 1 & -2 & 3 & 4 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the equivalent row echelon form of the given matrix and number of non-zero rows is 2.

$\therefore$ $\rho(A) = 2$.

c. $\begin{pmatrix} 2 & 4 & -4 & 3 \\ 1 & -2 & -1 & 1 \\ 1 & 2 & -1 & 3 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & 4 & -4 & 3 \\ 1 & -2 & -1 & 1 \\ 1 & 2 & -1 & 3 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{1}{2} R_1$

$R_3 \to R_3 - \frac{1}{2} R_1$

$$\sim \begin{pmatrix} 2 & 4 & -4 & 3 \\ 0 & -4 & 1 & -0.5 \\ 0 & 0 & 1 & 1.5 \end{pmatrix}$$

which is the equivalent row echelon form of the given matrix and number of non-zero rows is 3.

$\therefore$ $\rho(A) = 3$.

d. $\begin{pmatrix} 1 & 3 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 3 & -2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_4 \to R_4 - 3R_1$

$$\sim \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & -6 & 1 & 0 \\ 0 & -6 & 3 & 0 \end{pmatrix}$$

Applying $R_3 \to R_3 - 3R_2$

$R_4 \to R_4 - 3R_2$

$$\sim \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -6 & 0 \end{pmatrix}$$

Applying $R_4 \to R_4 - \frac{3}{4} R_3$

$$\sim \begin{pmatrix} 1 & 3 & -2 & 1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the equivalent row echelon form of the given matrix and number of non-zero rows is 3.

$\therefore$ $\rho(A) = 3$.

e. $\begin{pmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{pmatrix}$

Solution:

$$\text{Let } A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 7 & 13 \\ 4 & -3 & -1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 5 & 7 \\ 0 & -5 & -7 \end{pmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\sim \begin{pmatrix} 2 & 1 & 3 \\ 0 & 5 & 7 \\ 0 & 0 & 0 \end{pmatrix}$$

which is the equivalent row echelon form of the given matrix and number of non-zero rows is 2.

$\therefore$ $\rho(A) = 2$.

f. $\begin{pmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$

Solution:

Let A = $\begin{pmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$

Applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - \frac{1}{2} R_1$

$R_4 \to R_4 - \frac{1}{2} R_1$

$$\sim \begin{pmatrix} 2 & -2 & 0 & 6 \\ 0 & 6 & 0 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{pmatrix}$$

Applying $R_3 \leftrightarrow R_4$

$$\sim \begin{pmatrix} 2 & -2 & 0 & 6 \\ 0 & 6 & 0 & -10 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the equivalent row echelon form of the given matrix and number of non-zero rows is 3.

$\therefore$ $\rho(A) = 3$.

2. **Solve by Gauss elimination method:**

a. $3x - 2y = 8, 5x + 3y = 7$

Solution:

The augmented matrix of given system of equations can be written as

$$\begin{pmatrix} 3 & -2 & : & 8 \\ 5 & 3 & : & 7 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{5}{3} R_1$

$$\sim \begin{pmatrix} 3 & -2 & : & 8 \\ 0 & \frac{19}{3} & : & \frac{-19}{3} \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$3x - 2y = 8$ .... (i)

$\frac{19}{3} y = -\frac{19}{3}$ .... (ii)

From (ii), we have

$y = -1$

Substituting the value of y in (ii), we get

$x = 2$

$\therefore$ $x = 2, y = -1$

b. $2x - y = 5, x - 2y = 1.$

Solution:

The augmented matrix of given system of equations can be written as

$$\begin{pmatrix} 2 & -1 & : & 5 \\ 1 & -2 & : & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - \frac{1}{2} R_1$

$$\sim \begin{pmatrix} 2 & -1 & : & 5 \\ 0 & \frac{-3}{2} & : & \frac{-3}{2} \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$2x - y = 5$ .... (i)

$-\frac{3}{2} y = -\frac{3}{2}$ .... (ii)

From (ii), we have

$y = 1$

Substituting the value of y in (ii), we get

$x = 2$

$\therefore$ $x = 2,\ y = 1$

c. $x - 2y - 3z = -1, 2x + y + z = 6, x + 3y - 2z = 13$

Solution:

The augmented matrix of given system of equations is

$$\begin{pmatrix} 1 & -2 & -3 & : & -1 \\ 2 & 1 & 1 & : & 6 \\ 1 & 3 & -2 & : & 13 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$\sim \begin{pmatrix} 1 & -2 & -3 & : & -1 \\ 0 & 5 & 7 & : & 8 \\ 0 & 5 & 1 & : & 14 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\sim \begin{pmatrix} 1 & -2 & -3 & : & -1 \\ 0 & 5 & 7 & : & 8 \\ 0 & 0 & -6 & : & 6 \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$x - 2y - 3z = -1$ .... (i)

$5y + 7z = 8$ .... (ii)

$-6z = 6$ .... (iii)

From (iii), we have

$z = -1$

Substituting the value of z in (ii), we get

$y = 3$

Substituting the value of y and z in (ii), we get

$x = 2$

$\therefore$ $x = 2, y = 3, z = -1$

d. $9y - 5x = 3, x + z = 1, z + 2y = 2$

Solution:

The augmented matrix of the given system of equations can be written as

$$\begin{pmatrix} -5 & 9 & 0 & : & 3 \\ 1 & 0 & 1 & : & 1 \\ 0 & 2 & 1 & : & 2 \end{pmatrix}$$

Applying $R_2 \to R_2 + \frac{1}{5} R_1$

$$\sim \begin{pmatrix} -5 & 9 & 0 & : & 3 \\ 0 & \frac{9}{5} & 1 & : & \frac{8}{5} \\ 0 & 2 & 1 & : & 2 \end{pmatrix}$$

Applying $R_3 \to R_3 - \frac{10}{9} R_2$

$$\sim \begin{pmatrix} -5 & 9 & 0 & : & 3 \\ 0 & \frac{9}{5} & 1 & : & \frac{8}{5} \\ 0 & 0 & \frac{-10}{9} & : & \frac{20}{9} \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$9y - 5x = 3$ .... (i)

$\frac{9}{5}y + z = \frac{8}{5}$ .... (ii)

$-\frac{10}{9}z = \frac{20}{9}$ .... (iii)

From (iii), we have

$z = -2$

Substituting the value of z in (ii), we get

$y = 2$

Substituting the value of y and z in (ii), we get

$x = 3$

$\therefore$ $x = 3, y = 2$ and $z = -2$

e. $x + y + z = 6, \quad x - y + z = 2, \quad 2x + y - z = 1$

Solution:

The augmented matrix of given system of equation is

$$\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 1 & : & 2 \\ 2 & 1 & -1 & : & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 0 & : & -4 \\ 0 & -1 & -3 & : & -11 \end{pmatrix}$$

Applying $R_3 \to R_3 - \frac{1}{2} R_2$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 0 & : & -4 \\ 0 & 0 & -3 & : & -9 \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$x + y + z = 6$ .... (i)

$-2y + 0z = -4$ .... (ii)

$-3z = -9$ .... (iii)

From (iii), we have

$z = 3$

Substituting the value of z in (ii), we get

$y = 2$

Substituting the value of y and z in (ii), we get

$x = 1$

$\therefore$ $x = 1, \ y = 2, z = 3.$

f. $x - 2y - z = -7, \quad 2x + y = z = 0, \quad 3x - 5y + 8z = 13$

Solution:

The augmented matrix of the given system of equation is

$$\sim \begin{pmatrix} 1 & -2 & -1 & : & -7 \\ 2 & 1 & 1 & : & 0 \\ 3 & -5 & 8 & : & 13 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & -2 & -1 & : & -7 \\ 0 & 5 & 3 & : & 14 \\ 0 & 1 & 11 & : & 34 \end{pmatrix}$$

Applying $R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 1 & -2 & -1 & : & -7 \\ 0 & 1 & 11 & : & 34 \\ 0 & 5 & 3 & : & 14 \end{pmatrix}$$

Applying $R_3 \to R_3 - 5R_2$

$$\sim\begin{pmatrix} 1 & -2 & -1 & : & -7 \\ 0 & 1 & 11 & : & 34 \\ 0 & 0 & -52 & : & -156 \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$x - 2y - z = -7$ ....(i)

$y + 11z = 34$ ....(ii)

$-52z = -156$ ....(iii)

From (iii), we have

$z = 3$

Substituting the value of z in (ii), we get

$y = 1$

Substituting the value of y and z in (ii), we get

$x = -2$

$\therefore$ $x = -2$, $y = 1$ and $z = 3$.

g. $\mathbf{2x + 6y = -11,\ 6x + 20y - 6z = -3,\ 6y - 18z = -1}$

Solution:

The augmented matrix of the given system of equation is

$$\sim\begin{pmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1$

$$\sim\begin{pmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{pmatrix}$$

Applying $R_3 \to R_3 - 3R_2$

$$\sim\begin{pmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$2x + 6y = -11$ ....(i)

$2y - 6z = 30$ ....(ii)

$0z = -91$ ....(iii)

Here (iii) is not true for any value of z, so the system of linear equations has no solution.

h. $\mathbf{2x + y + 4z = 1,\ x + y + z = 1,\ 4x + y + 10z = 1}$

Solution:

The augmented matrix of the given system of equation is

$$\sim\begin{pmatrix} 2 & 1 & 4 & : & 1 \\ 1 & 1 & 1 & : & 1 \\ 4 & 1 & 10 & : & 1 \end{pmatrix}$$

Applying $R_2 \leftrightarrow R_1$

$$\sim\begin{pmatrix} 1 & 1 & 1 & : & 1 \\ 2 & 1 & 4 & : & 1 \\ 4 & 1 & 10 & : & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 4R_1$

$$\sim\begin{pmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & -1 \\ 0 & -3 & 6 & : & -3 \end{pmatrix}$$

Applying $R_3 \to R_3 - 3R_1$

$$\sim\begin{pmatrix} 1 & 1 & 1 & : & 1 \\ 0 & -1 & 2 & : & -1 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

which is row echelon form (REF).

From above augmented matrix, we have

$x + y + z = 1$ ....(i)

$-y + 2z = -1$ ....(ii)

$0z = 0$ ....(iii)

Here (iii) is true for any real value of z, so let $z = k$.

Substituting the value of z in equation (ii), we get

$y = 2k + 1$

Substituting the value of y and z in equation (i), we get

$x = -3k$

$\therefore$ $x = -3k$, $y = 2k + 1$, $z = k$.

3. Test the consistency and solve.

a. $\mathbf{x + y + z = 6,\ x - y + z = 8,\ 5x - 9y = -3}$

Solution:

The given system of equation is

$x + y + z = 6,\ x - y + z = 8,\ 5x - 9y = -3.$

The matrix form of the above equation is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & -9 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \\ -3 \end{pmatrix}$$

$\Rightarrow$ $AX = b$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 5 & -9 & 0 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ b = \begin{pmatrix} 6 \\ 8 \\ -3 \end{pmatrix}$$

The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 1 & -1 & 1 & : & 8 \\ 5 & -9 & 0 & : & -3 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 5R_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 0 & : & 2 \\ 0 & -14 & -5 & : & -33 \end{pmatrix}$$

Applying $R_3 \to R_3 - 7R_2$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & -2 & 0 & : & 2 \\ 0 & -0 & -5 & : & -47 \end{pmatrix}$$

Here, $\rho(A) = 3$, $\rho[A : b] = 3$

i.e., $\rho(A) = \rho[A : b] = 3 =$ number of unknowns.

So, the system of linear equations is consistent and has a unique solution.

From third row, we have

$-5z = -47$

$z = \frac{47}{5}$

From second row, we have

$-2y = 2$

$y = -1$

From first row, we have

$x + y + z = 6$

Substituting the value of y and z, we have

$x = \frac{-12}{5}$

$\therefore \quad x = \frac{-12}{5}, y = -1$ and $z = \frac{47}{5}$

b. $\mathbf{x + y + z = -2, 2x + 8y + 5z = 5, x + 2y - z = 2}$

Solution:

The given system of equation is

$x + y + z = -2,\ 2x + 8y + 5z = 5,\ x + 2y - z = 2$

The matrix form of the above equation is

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 8 & 5 \\ 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}$$

$\Rightarrow AX = b$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 8 & 5 \\ 1 & 2 & -1 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ b = \begin{pmatrix} -2 \\ 5 \\ 2 \end{pmatrix}$$

The augmented matrix of the given system of equations is

$$\begin{pmatrix} 1 & 1 & 1 & : & -2 \\ 2 & 8 & 5 & : & 5 \\ 1 & 2 & -1 & : & 2 \end{pmatrix}$$

Applying $R_1 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & -2 \\ 0 & 6 & 3 & : & 9 \\ 0 & 1 & -2 & : & 4 \end{pmatrix}$$

Applying $R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & -2 \\ 0 & 1 & -2 & : & 4 \\ 0 & 6 & 3 & : & 9 \end{pmatrix}$$

Applying $R_3 \to R_3 - 6R_2$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & -2 \\ 0 & 1 & -2 & : & 4 \\ 0 & 0 & 15 & : & -15 \end{pmatrix}$$

Here, $\rho(A) = 3$, $\rho[A : b] = 3$

i.e., $\rho(A) = \rho[A : b] = 3 =$ number of unknowns.

So, the system of linear equations is consistent and has a unique solution.

From third row, we have

$15z = -15$

$z = -1$

From second row, we have

$y - 2z = 4$

$y = 2$

From first row, we have

$x + y + z = -2$

Substituting the value of y and z, we have

$x = -3$

$\therefore \quad x = -3, y = 2, z = -1.$

c. $\mathbf{x + 2y - 3z = 9, 2x - y + 2z = -8, 3x - y - 4z = 3}$

Solution:

The given system of equation is

$x + 2y - 3z = 9,\ 2x - y + 2z = -8,\ 3x - y - 4z = 3$

The matrix form of the above equation is

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -8 \\ 3 \end{pmatrix}$$

$\Rightarrow AX = b$

where

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & -1 & -4 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ b = \begin{pmatrix} 9 \\ -8 \\ 3 \end{pmatrix}$$

The augmented matrix of the given system of equations is

$$\begin{pmatrix} 1 & 2 & -3 & : & 9 \\ 2 & -1 & 2 & : & -8 \\ 3 & -1 & -4 & : & 3 \end{pmatrix}$$

Applying $R_1 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & 2 & -3 & : & 9 \\ 0 & -5 & 8 & : & -26 \\ 0 & -7 & 5 & : & -24 \end{pmatrix}$$

Applying $R_3 \to R_3 - \frac{7}{5} R_2$

$$\sim \begin{pmatrix} 1 & 2 & -3 & : & 9 \\ 0 & -5 & 8 & : & -26 \\ 0 & 0 & \frac{-31}{5} & : & \frac{62}{5} \end{pmatrix}$$

Here, $\rho(A) = 3$, $\rho[A : b] = 3$

i.e., $\rho(A) = \rho[A : b] = 3 =$ number of unknowns.

So, the system of linear equations is consistent and has a unique solution.

From third row, we have

$\frac{31}{5} z = \frac{62}{5}$

$z = -2$

From second row, we have

$-5y + 8z = -26$

$\Rightarrow$ $y = 2$

From first row, we have

$x + 2y - 3z = 9$

Substituting the value of y and z, we have

$x = -1$

$\therefore$ $x = -1, y = 2, z = -2$

d. $\mathbf{x + z = 1,\ 2y + z = 2;\ 5x - 9y + 3 = 0}$

Solution:

The given system of equation is

$x + y + z = -2,\ 2x + 8y + 5z = 5,\ x + 2y - z = 2$

The matrix form of the above equation is

$$\begin{pmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$\Rightarrow AX = b$

where

$$A = \begin{pmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ b = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

The augmented matrix of the given system of equations can be written as

$$\begin{pmatrix} -5 & 9 & 0 & : & 3 \\ 1 & 0 & 1 & : & 1 \\ 0 & 2 & 1 & : & 2 \end{pmatrix}$$

Applying $R_2 \to R_2 + \frac{1}{5} R_1$

$$\sim \begin{pmatrix} -5 & 9 & 0 & : & 3 \\ 0 & \frac{9}{5} & 1 & : & \frac{8}{5} \\ 0 & 2 & 1 & : & 2 \end{pmatrix}$$

Applying $R_3 \to R_3 - \frac{10}{9} R_2$

$$\sim \begin{pmatrix} -5 & 9 & 0 & : & 3 \\ 0 & \frac{9}{5} & 1 & : & \frac{8}{5} \\ 0 & 0 & \frac{-10}{9} & : & \frac{20}{9} \end{pmatrix}$$

which is row echelon form (REF).

Here, $\rho(A) = 3$, $\rho[A : b] = 3$

i.e., $\rho(A) = \rho[A : b] = 3 =$ number of unknowns.

So, the system of linear equations is consistent and has a unique solution.

From third row, we have

$-\frac{10}{9} z = \frac{20}{9}$

$z = -2$

From second row, we have

$\frac{9}{5} y + z = \frac{8}{5}$

Substituting the value of z, we get

$y = 2$

From first row, we have

$9y - 5x = 3$

Substituting the value of y, we get

$x = 3$

$x = 3, y = 2$ and $z = -2$

e. $\mathbf{x - y = 1,\ z + x = -6,\ x + y - 2z = 3}$

lution:

The given system of equation is

$x - y = 1,\ z + x = -6,\ x + y - 2z = 3$

The matrix form of the above equation is

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 3 \end{pmatrix}$$

$\Rightarrow AX = b$

where

$$A=\begin{pmatrix}1&-1&0\\1&0&1\\1&1&-2\end{pmatrix},\ X=\begin{pmatrix}x\\y\\z\end{pmatrix},\ b=\begin{pmatrix}1\\-6\\3\end{pmatrix}$$

The augmented matrix of the given system of equation can be written as

$$\begin{pmatrix}1&-1&0&:&1\\1&0&1&:&-6\\1&1&-2&:&3\end{pmatrix}$$

Applying $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$

$$\sim\begin{pmatrix}1&-1&0&:&1\\0&1&1&:&-7\\0&2&-2&:&2\end{pmatrix}$$

Applying $R_3\to R_3-2R_2$

$$\sim\begin{pmatrix}1&-1&0&:&1\\0&1&1&:&-7\\0&0&-4&:&16\end{pmatrix}$$

which is row echelon form (REF).

Here, $\rho(A)=3$, $\rho[A:b]=3$

i.e., $\rho(A)=\rho[A:b]=3=$ number of unknowns.

So, the system of linear equations is consistent and has a unique solution.

From third row, we have

$-4z=16$

$z=-4$

From second row, we have

$y+z=-7$

Substituting the value of z, we get

$y=-3$

From first row, we have

$x-y=1$

Substituting the value of y, we get

$x=-2$

$\therefore$ $x=-2$, $y=-3$ and $z=-4$.

4. Test the consistency and solve.

a. $x+y-3z=-1$, $2x-y+3z=4$, $15x-3y+9z=21$

Solution:

The given system of equation is

$x+y-3z=-1,\ 2x-y+3z=4,\ 15x-3y+9z=21$

The matrix form of the above equation is

$$\begin{pmatrix}1&1&-3\\2&-1&3\\15&-3&9\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1\\4\\21\end{pmatrix}$$

$\Rightarrow AX=b$

where

$$A=\begin{pmatrix}1&1&-3\\2&-1&3\\15&-3&9\end{pmatrix},\ X=\begin{pmatrix}x\\y\\z\end{pmatrix},\ b=\begin{pmatrix}-1\\4\\21\end{pmatrix}$$

The augmented matrix of the above system is

$$\begin{pmatrix}1&1&-3&:&-1\\2&-1&3&:&4\\15&-3&9&:&21\end{pmatrix}$$

Applying $R_2\to R_2-2R_1$ and $R_3\to R_3-15R_1$

$$\sim\begin{pmatrix}1&1&-3&:&-1\\0&-3&9&:&6\\0&-18&54&:&36\end{pmatrix}$$

Applying $R_3\to R_3-6R_2$

$$\sim\begin{pmatrix}1&1&-3&:&-1\\0&-3&9&:&6\\0&0&0&:&0\end{pmatrix}$$

Here, $\rho(A)=2$, $\rho[A:b]=2$

i.e., $\rho(A)=\rho[A:b]=2<3=$ number of unknowns.

So, the system of linear equations is consistent and has an infinitely many solutions.

From third row, we have

$0z=0$, which is true for all real values of z, so let $z=k$.

From second row, we have

$-3y+9z=6$

$y=3k-2$

From first row, we have

$x+y-3z=-1$

Substituting the value of y and z, we have

$x=1$

$\therefore$ $x=1$, $y=3k-2$ and $z=k$.

b. $x+y-2z=-2$, $x-2y+z=1$, $-2x+y+z=1$

Solution:

The given system of equation is

$x+y-2z=-2,\ x-2y+z=1,\ -2x+y+z=1$

The matrix form of the above equation is

$$\begin{pmatrix}1&1&-2\\1&-2&1\\-2&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

$\Rightarrow AX=b$

where

$$A=\begin{pmatrix}1&1&-2\\1&-2&1\\-2&1&1\end{pmatrix},\ X=\begin{pmatrix}x\\y\\z\end{pmatrix},\ b=\begin{pmatrix}-2\\1\\1\end{pmatrix}$$

The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 1 & -2 & : & -2 \\ 1 & -2 & 1 & : & 1 \\ -2 & 1 & 1 & : & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & -2 & : & -2 \\ 0 & -3 & 3 & : & 3 \\ 0 & 3 & -3 & : & -3 \end{pmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\sim \begin{pmatrix} 1 & 1 & -2 & : & -2 \\ 0 & -3 & 3 & : & 3 \\ 0 & 0 & 0 & : & 0 \end{pmatrix}$$

Here, $\rho(A) = 2$, $\rho[A : b] = 2$

i.e., $\rho(A) = \rho[A : b] = 2 < 3$ = number of unknowns.

So, the system of linear equations is consistent and has an infinitely many solutions

From third row, we have

$0z = 0$, which is true for all real values of z, so let $z = k$.

From second row, we have

$-3y + 3z = 3$

$y = k - 1$

From first row, we have

$x + y - 2z = -2$

Substituting the value of y and z, we have

$x = k - 1$

$\therefore$ $x = k - 1, y = k - 1$ and $z = k$.

5. Test the consistency and solve if has a solution.

a. $x + 2y - 3z = -1, 3x - y + 2z = 7, 5x + 3y - 4z = 2$

Solution:

The given system of equation is

$x + 2y - 3z = -1,\ 3x - y + 2z = 7,\ 5x + 3y - 4z = 2$

The matrix form of the above equation is

$$\begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

$\Rightarrow AX = b$

where

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -1 & 2 \\ 5 & 3 & -4 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ b = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 2 & -3 & : & -1 \\ 3 & -1 & 2 & : & 7 \\ 5 & 3 & -4 & : & 2 \end{pmatrix}$$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 5R_1$

$$\sim \begin{pmatrix} 1 & 2 & -3 & : & -1 \\ 0 & -7 & 11 & : & 10 \\ 0 & -7 & 11 & : & 7 \end{pmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\sim \begin{pmatrix} 1 & 2 & -3 & : & -1 \\ 0 & -7 & 11 & : & 10 \\ 0 & 0 & 0 & : & 17 \end{pmatrix}$$

Here, $\rho(A) = 2$, $\rho[A : b] = 3$

i.e., $\rho(A) \neq \rho[A : b]$.

So, the system of linear equations is inconsistent and has no solution.

b. $x - 2y + z = 1, x + y - 2z = 1, -2x + y + z = 1$

Solution:

The given system of equation is

$x - 2y + z = 1,\ x + y - 2z = 1,\ -2x + y + z = 1$

The matrix form of the above equation is

$$\begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow AX = b$

where

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The augmented matrix of the above system is

$$\begin{pmatrix} 1 & -2 & 1 & : & 1 \\ 1 & 1 & -2 & : & 1 \\ -2 & 1 & 1 & : & 1 \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 + 2R_1$

$$\sim \begin{pmatrix} 1 & -2 & 1 & : & 1 \\ 0 & 3 & -3 & : & 0 \\ 0 & -3 & 3 & : & 3 \end{pmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\sim \begin{pmatrix} 1 & -2 & 1 & : & 1 \\ 0 & 3 & -3 & : & 0 \\ 0 & 0 & 0 & : & 3 \end{pmatrix}$$

Here, $\rho(A) = 2$, $\rho[A : b] = 3$

i.e., $\rho(A) \neq \rho[A : b]$.

So, the system of linear equations is inconsistent and has no solution.

6. Determine the value of p and q for which the system of linear equations $x+y+z=6$, $x+2y+5z=10$, $2x+3y+pz=q$ has

a. a unique solution

b. no solution

c. infinitely many solutions

Solution:

The given system of equation is

$$x+y+z=6,\ x+2y+5z=10,\ 2x+3y+pz=q$$

The matrix form of the above equation is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & p \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 6 \\ 10 \\ q \end{pmatrix}$$

$\Rightarrow AX = b$

where

$$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 2 & 3 & p \end{pmatrix},\ X=\begin{pmatrix} x \\ y \\ z \end{pmatrix},\ b=\begin{pmatrix} 6 \\ 10 \\ q \end{pmatrix}$$

The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 1 & 2 & 5 & : & 10 \\ 2 & 3 & p & : & q \end{pmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$\sim\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 4 & : & 4 \\ 0 & 1 & p-2 & : & q-12 \end{pmatrix}$$

Applying $R_3 \to R_3 - R_2$

$$\sim\begin{pmatrix} 1 & 1 & 1 & : & 6 \\ 0 & 1 & 4 & : & 4 \\ 0 & 0 & p-6 & : & q-16 \end{pmatrix}$$

Unique solution if rank of A = rank of $[A : b] = 3$

or, $p-6 \neq 0$ and $q-16 \neq 0$.

$\therefore$ $p \neq 6$ and $q \neq 16$.

No solution if rank of A $\neq$ rank of $[A : b]$

or, $p-6=0$ and $q-16 \neq 0$.

$\therefore$ $p=6$ and $q \neq 16$.

Infinitely many solutions if rank of A = rank of $[A : b] < 3$

or, $p-6=0$ and $q-16=0$.

$\therefore$ $p=6$ and $q=16$.

Unit 2

Vector Space

Exercise 2.1

1. Let $V = \mathbb{R}^2$ be a vector space. Show that $W = \{(x, y) : 2x + y = 0\}$ is a subspace of $V = \mathbb{R}^2$.

Solution

We have $V = \mathbb{R}^2$ and $W = \{(x, y) : 2x + y = 0\}$

Clearly, W is a subset of V.

Let $w_1 = (x_1, y_1)$, $w_2 = (x_2, y_2)$, $w_1, w_2 \in W$ and $c_1, c_2 \in \mathbb{R}$

Then, $2x_1 + y_1 = 0$ and $2x_2 + y_2 = 0$

Also, $c_1w_1 + c_2w_2 = c_1(x_1, y_1) + c_2(x_2, y_2)$
$= (c_1x_1, c_1y_1) + (c_2x_2, c_2y_2)$
$= (c_1x_1 + c_1x_2,\ c_2y_1 + c_2y_2)$

Now, $2(c_1x_1 + c_2x_2) + (c_1y_1 + c_2y_2) = c_1(2x_1 + y_1) + c_2(2x_2 + y_2)$
$= c_1 \cdot 0 + c_2 \cdot 0$
$= 0$

So, $c_1w_1 + c_2w_2 \in W$ for all $w_1, w_2 \in W$ and $c_1, c_2 \in \mathbb{R}$.

Hence, W is subspace of V.

2. Let $V = \mathbb{R}^3$ be a vector space. Show that $W = \{(x, y, z) : x - 2y + z = 0\}$ is a subspace of $V = \mathbb{R}^3$.

Solution

We have $V = \mathbb{R}^3$ and $W = \{(x, y, z) : x - 2y + z = 0\}$

Clearly, W is a subset of V.

Let $w_1 = (x_1, y_1, z_1)$, $w_2 = (x_2, y_2, z_2)$ such that $w_1, w_2 \in W$

Also, let $c_1, c_2 \in \mathbb{R}$

Then, $x_1 - 2y_1 + z_1 = 0$ and $x_2 - 2y_2 + z_2 = 0$

Then, $c_1w_1 + c_2w_2 = c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2)$
$= (c_1x_1, c_1y_1, c_1z_1) + (c_2x_2, c_2y_2, c_2z_2)$
$= (c_1x_1 + c_2x_2,\ c_1y_1 + c_2y_2,\ c_1z_1 + c_2z_2)$

Now, $c_1x_1 + c_2x_2 - 2(c_1y_1 + c_2y_2) + c_1z_1 + c_2z_2$
$= c_1(x_1 - 2y_1 + z_1) + c_2(x_2 - 2y_2 + z_2)$
$= c_1 \cdot 0 + c_2 \cdot 0$
$= 0$

So, $c_1w_1 + c_2w_2 \in W$ for all $w_1, w_2 \in W$ and $c_1, c_2 \in \mathbb{R}$

Hence, W is a subspace of V.

3. **Let $V = \mathbb{R}^3$ be a vector space. Show that $W = \{(x, y, z) : x - y$ subspace of $V = \mathbb{R}^3$.**

Solution

We have, $V = \mathbb{R}^3$ and $W = \{(x, y, z) : x - y = 0\}$

Clearly W is subset of V.

Let $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$ such that $w_1, w_2 \in W$, then

$x_1 - y_1 = 0$, $x_2 - y_2 = 0$

Also, let $c_1, c_2 \in \mathbb{R}$, then

$$\begin{aligned} c_1w_1 + c_2w_2 &= c_1(x_1, y_1, z_1) + c_2(x_2, y_2, z_2) \\ &= (c_1x_1, c_1y_1, c_1z_1) + (c_2x_2, c_2y_2, c_2z_2) \\ &= (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, c_1z_1 + c_2z_2) \end{aligned}$$

$$\begin{aligned} \text{Now, } c_1x_2 + c_2x_2 - (c_1y_1 + c_2y_2) &= c_1(x_1 - y_1) + c_2(x_2 - y_2) \\ &= c_1(x_1 - y_1) + c_2(x_2 - y_2) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

So, $c_1w_1 + c_2w_2 \in W$ for all $w_1, w_2 \in W$ and $c_1, c_2 \in \mathbb{R}$.

Hence, W is a subspace of V.

4. **Let $V = \mathbb{R}^3$ be a vector space. Let $W_1 = \{(x, 0, z) : x, z \in \mathbb{R}\}$ $W_2 = \{(x, y, 0) : x, y \in \mathbb{R}$. Show that:**

   a. **W_1 is a vector subspace of $\mathbb{R}^3$.**

Solution

We have, $V = \mathbb{R}^3$ and $W_1 = \{(x, 0, z) : x, z \in \mathbb{R}\}$

Clearly, W_1 is subset of V.

Let $w_1 = (x_1, 0, z_1)$ and $w_2 = (x_2, 0, z_2)$

Such that $w_1, w_2 \in W_1$, then

$x_1, z_1 \in \mathbb{R}$, $x_2, z_2 \in \mathbb{R}$

Also, let $c_1, c_2 \in \mathbb{R}$, then

$$\begin{aligned} c_1w_1 + c_2w_2 &= c_1(x_1, 0, z_1) + c_2(x_2, 0, z_2) \\ &= (c_1x_1, 0, c_1z_1) + (c_2x_2, 0, c_2z_2) \\ &= (c_1x_1 + c_2x_2, 0, c_1z_1 + c_2z_2) \end{aligned}$$

Here, $c_1x_1 + c_2x_2$ and $c_1z_1 + c_2z_2 \in \mathbb{R}$.

So, $c_1w_1 + c_2w_2 \in W_1$, for all $w_1, w_2 \in W_1$ and $c_1, c_2 \in \mathbb{R}$

Hence, w is a subspace of V over a field $\mathbb{R}$.

   b. **W_2 is a vector subspace of $\mathbb{R}^3$.**

Solution

We have, $V = \mathbb{R}^3$ and $W_2 = \{(x, y, 0) : x, y \in \mathbb{R}\}$

Clearly, W_2 is subset of V

Let $w_1 = (x_1, y_1, 0)$, $w_2 = (x_2, y_2, 0)$ such that $w_1, w_2 \in W_2$ then

$x_1, y_1 \in \mathbb{R}$ and $x_2, y_2 \in \mathbb{R}$

Also, let $c_1, c_2 \in \mathbb{R}$, then

$$\begin{aligned} c_1w_1 + c_2w_2 &= c_1(x_1, y_1, 0) + c_2(x_2, y_2, 0) \\ &= (c_1x_1, c_1y_1, 0) + (c_2x_2, c_2y_2, 0) \\ &= (c_1x_1 + c_2x_2, c_1y_1 + c_2y_2, 0) \end{aligned}$$

Here, $c_1x_2 + c_2x_2 \in \mathbb{R}$ and $c_1y_1 + c_2y_2 \in \mathbb{R}$.

So, $c_1w_1 + c_2w_2 \in W_2$ for all $w_1, w_2 \in W_2$ and $c_1, c_2 \in \mathbb{R}$.

Hence, W_2 is subspace of V over a field $\mathbb{R}$.

   c. **$W_1 \cap W_2$ is a vector subspace of $\mathbb{R}^3$.**

Solution

We have, $V = \mathbb{R}^3$ and $W_1 \cap W_2 = \{(x, 0, 0) : x \in \mathbb{R})$

Clearly, $W_1 \cap W_2$ is subset of V.

Let $u_1 = (x_1, 0, 0)$, $u_2 = (x_2, 0, 0)$ such that

$u_1, u_2 \in W_1 \cap W_2$, then $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$.

Also, let $c_1, c_2 \in \mathbb{R}$, then

$$\begin{aligned} c_1u_1 + c_2u_2 &= c_1(x_1, 0, 0) + c_2(x_2, 0, 0) \\ &= (c_1x_1, 0, 0) + (c_2x_2, 0, 0) \\ &= (c_1x_1 + c_2x_2, 0, 0) \end{aligned}$$

Here, $c_1x_1 + c_2x_2 \in \mathbb{R}$

So, $c_1u_1 + c_2u_2 \in W_1 \cap W_2$ for all $u_1, u_2 \in W_1 \cap W_2$ and $c_1, c_2 \in \mathbb{R}$.

Hence, $W_1 \cap W_2$ is subspace of V over a field $\mathbb{R}$.

5. **Determine whether the following vectors are linearly dependent or independent.**

   a. **(2, 3), (−1, 2)**

Solution

Let $v_1 = (2, 3)$, $v_2 = (-1, 2)$

The matrix A with given vectors as column vectors is

$A = [v_1 \quad v_2]$

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 3 & 2 \end{bmatrix} R_1 \to \frac{R_1}{2}$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{7}{2} \end{bmatrix} R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} R_2 \to \frac{2}{7} R_2$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 \to R_1 + \frac{1}{2} R_2$$

Thus, the row reduced echelon form has pivot in each column. So, ve... linearly independent.

b. (1, 2), (2, 0)

Solution

Let $v_1 = (1, 2)$, $v_2 = (2, 0)$

The matrix A with given vectors as column vectors is

$A \quad = [v_1 \ \ v_2]$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} R_2 \to -\frac{1}{4} R_2$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 \to R_1 - 2R_2$$

Thus, the row reduced echelon form has pivot in each column, so vectors are independent.

c. (1, 2, 1), (2, 1, 0), (1, –1, 2)

Solution

Let $v_1 = (1, 2, 1)$, $v_2 = (2, 1, 0)$, $v_3 = (1, -1, 2)$

The matrix A with given vectors as column vector is

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -3 \\ 0 & -2 & 1 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix} R_2 \to -\frac{1}{3} R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \frac{1}{3} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

Which shows that row reduced echelon form has pivot in every column, so vectors are linearly independent.

d. (1, 1, 0), (1, 1, 1), (1, 0, 1)

Solution

Let $v_1 = (1, 1, 0)$, $v_2 = (1, 1, 1)$, $v_3 = (1, 0, 1)$, then

$A \quad = [v_1 \ \ v_2 \ \ v_3]$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} R_1 \to R_1 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to -R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 \to R_2 - R_3$$

Which shows that row reduced echelon has pivot in every column, so vectors are linearly independent.

e. (1, 2, 3), (2, 4, 6), (3, 6, 9)

Solution

Let $v_1 = (1, 2, 3)$, $v_2 = (2, 4, 6)$, $v_3 = (3, 6, 9)$

$$A = [v_1 \ \ v_2 \ \ v_3] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

Thus, the row reduced echelon form of A has no pivot in each column, so vectors are linearly dependent.

f. (1, 1, 1), (1, 2, 3), (–1, 1, 3)

Solution

Let $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, $v_3 = (-1, 1, 3)$

$A \quad = [v_1 \ \ v_2 \ \ v_3]$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

Thus, the row reduced echelon form of A has no pivot in each column, so vectors are linearly dependent.

6. **Show that the following vectors span $\mathbb{R}^2$.**

a. **(1, 2) , (3, 1)**

Solution

Let $v_1 = (1, 2)$, $v_2 = (3, 1)$

$A = [v_1 \quad v_2]$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} R_2 \to -\frac{1}{5}R_2$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 \to R_1 - 3R_2$$

Thus, the equivalent row reduced echelon form of matrix A has

i. pivot element in each column, so given vectors are linearly independent.

ii. pivot element in each row, so $\{v_1, v_2\}$ is span of $\mathbb{R}^2$.

b. **(1, 2) , (2, 1)**

Solution

Let $v_1 = (1, 2)$, $v_2 = (2, 1)$

$A = [v_1 \quad v_2]$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} R_2 \to \frac{R_2}{-3}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 \to R_1 - 2R_2$$

Which is equivalent matrix of A in row reduced echelon form has

i. each column has pivot in row reduced echelon form, so the given vectors are linearly independent.

ii. each row has a pivot, so $\{v_1, v_2\}$ is span of $\mathbb{R}^2$.

c. **(1, 0) , (0, 1)**

Solution

Let $v_1 = (1, 0)$, $v_2 = (0, 1)$

$A = [v_1 \quad v_2]$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Which is in the form of row reduced echelon form. Also

i. Each column has pivot, so vectors are linearly independent.

ii. each row has a pivot, so $\{v_1, v_2\}$ is span of $\mathbb{R}^2$

7. **Show that the following vectors span $\mathbb{R}^3$.**

a. **(1, 0, 0) , (0, 1, 0) , (0, 0, 1)**

Solution

Let $A = [v_1, v_2, v_3]$

Where, $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is in row reduced echelon form has

i. each column has pivot, so vectors are linearly independent.

ii. each row has a pivot, so given vector is span of $\mathbb{R}^3$.

b. **(1, 0, 1) , (0, 1, – 1) , (0, 0, 1)**

Solution

Let $A = [v_1, v_2, v_3]$

Where, $v_1 = (1, 0, 1)$, $v_2 = (0, 1, -1)$, $v_3 = (0, 0, 1)$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to R_3 + R_2$$

Thus, the equivalent row reduced echelon form of matrix A has:

i. pivot element in each column, so vectors are linearly independent.

ii. pivot element in each row, so vector generate $\mathbb{R}^3$.

Check whether the following vectors form a basis of $\mathbb{R}^2$ or not.

a. **(1, 0) , (0, 1)**

Solution

Consider the matrix A with given vectors as column vectors

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Which is in row reduced echelon form has

i. pivot element in each column, so vectors are linearly independent.

ii. pivot element in each row, so vector generate $\mathbb{R}^2$.

Thus, given set of vectors form basis of $\mathbb{R}^2$.

b. **(1, 1) , (–1, 1)**

Solution

Consider the matrix A with given vectors as column vectors

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} R_2 \to \frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_1 \to R_1 + R_2$$

Thus, the equivalent row reduced echelon form of matrix A has

i. pivot element in each column, so vectors are linearly independent

ii. pivot element in each row, so vector generate $\mathbb{R}^2$.

Thus, given set of vectors form basis of $\mathbb{R}^2$.

c. (1, 1) , (2, 2)

Solution

Consider the matrix A with given vectors as column vectors

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Thus, the equivalent row reduced echelon form of matrix A has no pivot ele each column, so vectors are linearly dependent.

So, given set of vectors does not form basis of $\mathbb{R}^2$.

d. (1, –2) , (0 , 0)

Solution

Consider the matrix A with given vectors as column vectors

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 + 2R_1$$

Thus, the equivalent row reduced echelon form of matrix A has no pivot e each column, so vectors are linearly dependent.

So, given set of vectors does not form basis of $\mathbb{R}^2$.

9. Check whether the following vectors form a basis of $\mathbb{R}^3$.

a. (1, 1, 0) , (1, 0, 1) , (3, 1, 2)

Solution

Consider the matrix A with given vectors as column vectors

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & 2 \\ 0 & 1 & 2 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

Thus, the row reduced echelon form equivalent to matrix A has no pivot in element in each column, so vectors are linearly dependent. Thus, given vectors does not form basis of $\mathbb{R}^3$.

b. (1, 1, 0) , (1, 0, 1), (3, 1, 1)

Solution

The matrix A with given vectors as column vectors is

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} R_2 \to (-) R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to (-) R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \end{matrix}$$

Which is equivalent matrix of A in row reduced echelon form has

i. pivot in each column, so vectors are linearly independent.

ii. pivot in each row, so given vectors generate $\mathbb{R}^3$.

Hence, given set of vectors form basis of $\mathbb{R}^3$.

c. (1, 0, 0) , (0, 1, 0) , (0, 0, 1)

Solution

Consider the matrix A with given vectors as column vectors

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is in the form of row reduced echelon form has

i. each column has a pivot, so the given vectors are linearly independent.

ii. Each row has a pivot, so given vectors generate $\mathbb{R}^3$.

Thus, the given vectors are set of linearly independent vectors generating $\mathbb{R}^3$. So, it forms basis for $\mathbb{R}^3$.

d. **(1, 2, 0), (0, 1, 1), (1, 0, 3)**

Solution

The matrix A with given vectors as column vector is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{bmatrix} R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} R_3 \to \left(-\frac{1}{2}\right) R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 3R_3 \end{matrix}$$

Thus, the row reduced echelon form equivalent to matrix A has

i. pivot element in each column, so the given vectors are linearly independent

ii. pivot in each row, so the vectors generate $\mathbb{R}^3$.

Thus, the given vectors form basis of $\mathbb{R}^3$.

e. **(1, 2, 0), (0, 1, 1)**

Solution

The matrix A with given vectors as column vector is

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

Which is equivalent matrix of A row reduced echelon form of given matrix has

i. pivot in each column, so vectors are linearly independent.

ii. no pivot in third row, so given vectors can not generate $\mathbb{R}^3$.

Thus, the given vectors does not form basis of $\mathbb{R}^3$.

f. **(1, 2, 2), (2, 5, 4), (2, 7, 4)**

Solution

The matrix A with given vectors as column vector is

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 7 \\ 2 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

Thus, the row reduced echelon form of A has no pivot in each column, so vectors linearly dependent.

So, the given vectors does not form basis for $\mathbb{R}^3$.

0. **Determine whether the following transformation is linear or not?**

a. **$T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + 2y - z, 0)$**

olution

We have the transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x + 2y - z, 0)$

Let $v_1 = (x_1, y_1, z_1)$, $v_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ then

$T(v_1) = T(x_1, y_1, z_1) = (x_1 + 2y_1 - z_1, 0)$

$T(v_2) = T(x_2, y_2, z_2) = (x_2 + 2y_2 - z_2, 0)$

$v_1 + v_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$$\begin{aligned} T(v_1 + v_2) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2 + 2(y_1 + y_2) - (z_1 + z_2), 0) \\ &= (x_1 + 2y_1 - z_1 + x_2 + 2y_2 - z_2, 0) \\ &= (x_1 + 2y_1 - z_1, 0) + (x_2 + 2y_2 - z_2, 0) \\ &= T(v_1) + T(v_2) \end{aligned}$$

For any scalar k

$kv = k(x, y, z) = (kx, ky, kz)$, then

$$\begin{aligned} T(kv) &= T(kx, ky, kz) \\ &= (kx + 2ky - kz, 0) \\ &= k(x + 2y - z, 0) \\ &= k\,T(v) \end{aligned}$$

Here, $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in \mathbb{R}^3$

$T(kv) = kT(v)$ for all $v \in \mathbb{R}^3$ and for any scalar k.

Hence, T is linear.

b. **$T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)$**

olution

We have the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x + y, x - y)$

Let $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$, then

$T(v_1) = T(x_1, y_1) = (x_1 + y_1, x_1 - y_1)$

$T(v_2) = T(x_2, y_2) = (x_2 + y_2, x_2 - y_2)$

$v_1 + v_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$\begin{aligned} T(v_1 + v_2) &= T(x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2 + y_1 + y_2, x_1 + x_2 - y_1 - y_2) \\ &= (x_1 + y_1 + x_2 + y_2, x_1 - y_1 + x_2 - y_2) \\ &= (x_1 + y_1, x_1 - y_1) + (x_2 + y_2, x_2 - y_2) \\ &= T(v_1) + T(v_2) \end{aligned}$$

For any scalar k

$kv = k(x, y) = (kx, ky)$

$$\begin{aligned} T(kv) &= T(kx, ky) \\ &= (kx + ky, kx - ky) \\ &= k(x + y, x - y) \\ &= k\,T(v) \end{aligned}$$

$T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in \mathbb{R}^2$ and $T(kv) = k\,T(v)$ for all $v \in \mathbb{R}^2$ and for any scalar k.

Hence, T is linear transformation.

c. $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x - 2$

Solution

We have, transformation $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x - 2$

Let $v_1 = x_1$, $v_2 = x_2 \in \mathbb{R}$ then

$T(v_1) = T(x_1) = x_1 - 2$

$T(v_2) = T(x_2) = x_2 - 2$

$v_1 + v_2 = x_1 + x_2$

$T(v_1 + v_2) = T(x_1 + x_2) = x_1 + x_2 - 2 \neq T(v_1) + T(v_2)$

Hence, T is not linear.

d. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x + 3y, x - 2y)$

Solution

Given transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (2x + 3y, x - 2y)$

Let $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$, then

$T(v_1) = T(x_1, y_1) = (2x_1 + 3y_1, x_1 - 2y_1)$

$T(v_2) = T(x_2, y_2) = (2x_2 + 3y_2, x_2 - 2y_2)$

$v_1 + v_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$
\begin{aligned}
T(v_1 + v_2) &= T(x_1 + x_2, y_1 + y_2) \\
&= (2x_1 + 2x_2 + 3y_1 + 3y_2, x_1 + x_2 - 2y_1 - 2y_2) \\
&= (2x_1 + 3y_1 + 2x_2 + 3y_2, x_1 - 2y_1 + x_2 - 2y_2) \\
&= (2x_1 + 3y_1, x_1 - 2y_1) + (2x_2 + 3y_2, x_2 - 2y_2) \\
&= T(v_1) + T(v_2)
\end{aligned}
$$

For any scalar k

$kv = k(x, y) = (kx, ky)$

$$
\begin{aligned}
T(kv) &= T(kx, ky) \\
&= (2kx + 3ky, kx - 2ky) \\
&= k\,(2x + 3y, x - 2y) \\
&= k\,T(v)
\end{aligned}
$$

$\therefore$ $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in \mathbb{R}^2$ and $T(kv) = k\,T(v)$ for all $v \in$ for any scalar k.

Hence, T is linear transformation.

e. $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = |x + y|$

Solution

Given that $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = |x + y|$

Let $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$, then

$T(v_1) = T(x_1, y_1) = |x_1 + y_1|$

$T(v_2) = T(x_2, y_2) = |x_2 + y_2|$

$v_1 + v_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$
\begin{aligned}
T(v_1 + v_2) &= T(x_1 + x_2, y_1 + y_2) \\
&= |x_1 + x_2 + y_1 + y_2| = |x_1 + y_1 + x_2 + y_2| \\
&\neq |x_1 + y_1| + |x_2 + y_2| = T(v_1) + T(v_2)
\end{aligned}
$$

$\therefore$ $T(v_1 + v_2) \neq T(v_1) + T(v_2)$ for all $v_1, v_2 \in \mathbb{R}^2$.

Hence, T is not linear transformation.

f. $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, xy)$

Solution

Given transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, xy)$

Let $v_1, v_2 \in \mathbb{R}^2$, where $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$, then

$T(v_1) = T(x_1, y_1) = (x_1, x_1y_1)$

$T(v_2) = T(x_2, y_2) = (x_2, x_2y_2)$

$v_1 + v_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

$$
\begin{aligned}
T(v_1 + v_2) &= (x_1 + x_2, (x_1 + x_2)(y_1 + y_2)) \\
&= (x_1 + x_2, x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2) \\
&\neq T(v_1) + T(v_2)
\end{aligned}
$$

Hence, T is not linear.

g. $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x - 2y, y + 2z, x + 3z)$

Solution

Given transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x - 2y, y + 2z, x + 3z)$

Let $v_1, v_2 \in \mathbb{R}^3$, where $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$, then

$T(v_1) = T(x_1, y_1, z_1) = (x_1 - 2y_1, y_1 + 2z_1, x_1 + 3z_1)$

$T(v_2) = T(x_2, y_2, z_2) = (x_2 - 2y_2, y_2 + 2z_2, x_2 + 3z_2)$

$v_1 + v_2 = (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$$
\begin{aligned}
T(v_1 + v_2) &= (x_1 + x_2 - 2(y_1 + y_2), y_1 + y_2 + 2(z_1 + z_2), x_1 + x_2 + 3\,(z_1 + z_2)) \\
&= (x_1 - 2y_1 + x_2 - 2y_2, y_1 + 2z_1 + y_2 + 2z_2, x_1 + 3z_1 + x_2 + 3z_2) \\
&= (x_1 - 2y_1, y_1 + 2z_1, x_1 + 3z_1) + (x_2 - 2y_2, y_2 + 2z_2, x_2 + 3z_2) \\
&= T(v_1) + T(v_2)
\end{aligned}
$$

For any scalar k

$$
\begin{aligned}
kv &= (kx - 2ky, ky + 2kz, kz + 3kz) \\
&= k(x - 2y, y + 2z, x + 3z) \\
&= k\,T(v)
\end{aligned}
$$

$\therefore$ $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in \mathbb{R}^3$.

and $T(kv) = k\,T(v)$ for all $v \in \mathbb{R}^3$ and for any scalar k.

Hence, T is linear transformation.

Exercise 2.2

1. Find the eigen value and eigen vector of the following matrices.

a. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Solution

Let the given matrix is $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

The characteristic matrix $= A - \lambda I$

$$
= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix}
$$

Then, characteristics equation is

$|A - \lambda I| = 0$

or, $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$

or, $(2-\lambda)^2 - 1 = 0$

or, $4 - 4\lambda + \lambda^2 - 1 = 0$

or, $\lambda^2 - 4\lambda + 3 = 0$

or, $\lambda^2 - 3\lambda - \lambda + 3 = 0$

or, $\lambda(\lambda - 3) - 1(\lambda - 3) = 0$

or, $(\lambda - 3)(\lambda - 1)$

or, $\lambda = 1, 3$

Thus, eigen value of matrix A are 1, 3.

Let, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector of A

Then, from $(A - \lambda I)X = 0$

or, $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ ...(1)

Which is a system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 1$, from (1)

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

Which is a system of homogeneous with coefficient matrix.

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} R_2 \to R_2 - R_1$

Which is row reduced form of coefficient matrix of homogeneous system.

So, by Gauss elimination method,

From R_2, y is free, say $y = k_1$

From R_1, $x + y = 0 \Rightarrow x = -y \Rightarrow x = -k_1$

Thus the eigen vector is

$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda = 3$, from (1)

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

Which is system of homogeneous equation with coefficient matrix.

$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 + R_1$

Which is row reduced echelon form of coefficient matrix of homogeneous system

So by Gauss elimination method,

From R_2: y is free, say $y = k_2$

From R_1: $x - y = 0$ i.e., $x = y$ i.e., $x = k_2$

Thus, eigen vector is

$X = \begin{bmatrix} k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b. $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Solution

Let the given matrix is

$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

The characteristic matrix is $= A - \lambda I$

$= \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{pmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{pmatrix}$

Then, characteristic equation is

$|A - \lambda I| = 0$

$= \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)(2-\lambda) - 4 = 0$

or, $10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

or, $\lambda^2 - 6\lambda - \lambda + 6 = 0$

or, $\lambda(\lambda - 6) - 1(\lambda - 6) = 0$

or, $(\lambda - 1)(\lambda - 6) = 0$

or, $\lambda = 1, 6$

Thus, eigen values of matrix A are $\lambda = 1, 6$.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector of A

Then from $AX = \lambda X$

or, $(A - \lambda I)X = 0$

or, $\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$ ...(1)

Which is a system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 1$, from (1)

$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

Which is a system of homogeneous equation with coefficient matrix

$A = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} R_1 \to \frac{1}{4}R_1$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - R_1$

From R_2: y is free, say $y = k_1$

From R_1, $x + y = 0 \Rightarrow x = -y \Rightarrow x = -k_1$

Thus, eigen vector is

$$X = \begin{bmatrix} -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

When $\lambda = 6$, from (1)

$$\begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -1 & 4 \\ 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 \\ 1 & 4 \end{bmatrix} R_1 \to -R_1$$

$$\sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - R_1$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method,

From R_2: y is free, say $y = k_2$

From R_1: $x - 4y = 0$ or $x = 4k_2$

Thus, eigen vector is $X = k_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

c. $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$$

The characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)^2 - 1^2 = 0$

or, $(3-\lambda-1)(3-\lambda+1) = 0$

or, $(2-\lambda)(4-\lambda) = 0$

or, $\lambda = 2, 4$

Thus, eigen values of matrix A are $\lambda = 2, 4$.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector of A

Then from

$(A - \lambda I)X = 0$

$$\therefore \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \quad \ldots(1)$$

Which is system of homogeneous equation. To solve this system of Gauss elimination method reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 2$, from (1)

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - R_1$$

From R_2: y is free, say $y = k_1$

From R_1: $x + y = 0 \Rightarrow x = -y$, $x = -k_1$

Thus, eigen vector is

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

When $\lambda = 4$, From (1)

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} R_1 \to (-) R_1$$

$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - R_1$$

From R_2: y is free say $y = k_2$

From R_1: $x - y = 0 \Rightarrow x = y \Rightarrow x = k_2$

Thus, the eigen vector is

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

d. $\begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix}$$

Then characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix} = 0$

or, $(-\lambda)(-1-\lambda) - 6 = 0$

or, $\lambda^2 + \lambda - 6 = 0$

or, $\lambda^2 + 3\lambda - 2\lambda - 6 = 0$

or, $\lambda(\lambda + 3) - 2(\lambda + 3) = 0$

or, $(\lambda - 2)(\lambda + 3) = 0$

or, $\lambda = -3, 2$

Thus, eigen values of matrix A are $\lambda = -3, 2$

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector of A.

Then from $AX = \lambda X$

or, $(A - \lambda I)X = 0$

or, $\begin{bmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation. To solve this system b elimination method reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = -3$, then from (1)

$\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation with coefficient matrix

$A = \begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} R_1 \to \frac{1}{3}R_1$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - 2R_1$

Which is row reduced echelon form of coefficient matrix of homogeneous

So, by Gauss elimination method,

From R_2: y is free, say $y = k_1$

From R_1: $x + y = 0 \Rightarrow x = -k_1$

Thus, eigen vector $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda = 2$, then from (1)

$\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation with coefficient matrix.

$A = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -3/2 \\ 2 & -3 \end{bmatrix} R_1 \to \frac{R_1}{-2}$

$\sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} R_2 + R_2 - 2R_1$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method

From R_2: y is free, say $y = k_2$

From R_1: $x - \frac{3}{2}y \Rightarrow x = \frac{3}{2}k_2$

Thus, eigen vector $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/2\, k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = k_2\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

e. $\begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$

Solution

Let the given matrix is

$A = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$

Then characteristic matrix is

$A - \lambda I = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 2-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

or, $-(2-\lambda)(2+\lambda) - 4 = 0$

or, $-4 + \lambda^2 - 4 = 0$

or, $\lambda^2 = 8$

or, $\lambda = \pm 2\sqrt{2}$

Thus eigen values of matrix A are $\lambda = \pm 2\sqrt{2}$.

f. $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$

Solution

Let the given matrix is

$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

The characteristic matrix $= A - \lambda I$

$= \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix}$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)(2-\lambda) - 6 = 0$

or, $2 - \lambda - 2\lambda + \lambda$ [illegible]

or, $\lambda^2 - 4\lambda + \lambda - 4 = 0$

or, $\lambda(\lambda - 4) + 1(\lambda - 4) = 0$

or, $(\lambda + 1)(\lambda - 4) = 0$

or, $\lambda = -1, 4$

Thus, eigen values of matrix A are $\lambda = -1, 4$

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector of A.

Then from $AX = \lambda X$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \ldots (1)$$

Which is a system of homogeneous equation. To solve this system by Ga elimination method, reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = -1$, from 1

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} R_1 \to \frac{1}{2}R_1$$

$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - 3R_1$$

From R_2: y is free, say $y = k_1$

From R_1: $x + y = 0 \Rightarrow x = -k_1$

Thus, eigen vector is $X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda = 4$, from (1)

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2/3 \\ 3 & -2 \end{bmatrix} \sim R_1 \to \frac{R_1}{-3}$$

$$\sim \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - 3R_1$$

From R_2: y is free, say $y = k_2$

From R_1: $x - \frac{2}{3}y = 0 \Rightarrow x = \frac{2}{3}k_2$

Eigen vector is

$$X = k_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = k_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

2. **Find the eigen value and eigen vector of the following matrices.**

a. $\begin{pmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix}$$

The characteristic equation of matrix A is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$

or, $(4 - \lambda)\{(3-\lambda)(1-\lambda) - 8\} - 2\{-5(1-\lambda) + 4\} - 2\{-20 + 2(3-\lambda)\} = 0$

or, $(4-\lambda)(3 - 3\lambda - \lambda + \lambda^2 - 8) - 2(-5 + 5\lambda + 4) - 2(-20 + 6 - 2\lambda) = 0$

or, $(4-\lambda)(\lambda^2 - 4\lambda - 5)\ 2(5\lambda - 1) - 2(-2\lambda - 14) = 0$

or, $-\lambda^3 + 8\lambda^2 - 17\lambda + 10 = 0$

Solving we get

$\lambda = 1, 2, 5$

Thus eigen value of matrix A are 1, 2, 5

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vectors of matrix A.

Then from $(A - \lambda I)X = 0$

$$\begin{bmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 1$, from (1)

$$\begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 4 & 0 \\ -5 & 2 & 2 \\ 3 & 2 & -2 \end{bmatrix} R_1 \leftrightarrow R_3$$

$$\sim\begin{bmatrix}1 & -2 & 0\\ -5 & 2 & 2\\ 3 & 2 & -2\end{bmatrix} R_1 \to \left(-\frac{1}{2}\right) R_1$$

$$\sim\begin{bmatrix}1 & -2 & 0\\ 0 & -8 & 2\\ 0 & 8 & -2\end{bmatrix} \begin{matrix}R_2 \to R_2 + 5R_1\\ R_3 \to R_3 - 3R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & -2 & 0\\ 0 & 1 & -1/4\\ 0 & 8 & -2\end{bmatrix} R_2 \to \left(-\frac{1}{8}\right) R_2$$

$$\sim\begin{bmatrix}1 & -2 & 0\\ 0 & 1 & -1/4\\ 0 & 0 & 0\end{bmatrix} R_3 \to R_3 - 8R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous sys.

So by Gauss elimination method

From R_3: z is free, say $z = k_1$

From R_2: $y - \frac{z}{4} = 0 \Rightarrow y = \frac{k_1}{4}$

From R_1: $x - 2y = 0 \Rightarrow x = 2y \Rightarrow x = \frac{1}{2}k_1$

Thus, the eigen vector, $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$$= \begin{bmatrix}1/2\,k_1\\ 1/4\,k_1\\ k_1\end{bmatrix}$$

$$= k_1\begin{bmatrix}1/2\\ 1/4\\ 1\end{bmatrix} = k_1\begin{bmatrix}2\\ 1\\ 4\end{bmatrix}$$

When $\lambda = 2$, from (1)

$$\begin{bmatrix}2 & 2 & -2\\ -5 & 1 & 2\\ -2 & 4 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix}2 & 2 & -2\\ -5 & 1 & 2\\ -2 & 4 & -1\end{bmatrix}$$

$$\sim\begin{bmatrix}1 & 1 & -1\\ -5 & 1 & 2\\ -2 & 4 & -1\end{bmatrix} R_1 \to \frac{1}{2}R_1$$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & 6 & -3\\ 0 & 6 & -3\end{bmatrix} \begin{matrix}R_2 \to R_2 + 5R_1\\ R_3 \to R_3 + 2R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -1/2\\ 0 & 6 & -3\end{bmatrix} R_2 \to \frac{1}{6}R_2$$

$$\sim\begin{bmatrix}1 & 1 & -1\\ 0 & 1 & -1/2\\ 0 & 0 & 0\end{bmatrix} R_3 \to R_3 - 6R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method

From R_3: z is free, say $z = k_2$

From R_2: $y - \frac{1}{2}z = 0 \Rightarrow y = \frac{z}{2} \Rightarrow y = \frac{k_2}{2}$

From R1: $x + y - z = 0 \Rightarrow x = z - y \Rightarrow x = k_2 - \frac{1}{2}k_2 \Rightarrow x = \frac{1}{2}k_2$

Thus, eigen vectors

$$X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1/2\,k_2\\ 1/2\,k_2\\ k_2\end{bmatrix} = k_2\begin{bmatrix}1/2\\ 1/2\\ 1\end{bmatrix}$$

When $\lambda = 5$, from (1)

$$\begin{bmatrix}-1 & 2 & -2\\ -5 & -2 & 2\\ -2 & 4 & -4\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix}-1 & 2 & -2\\ -5 & -2 & 2\\ -2 & 4 & -4\end{bmatrix}$$

$$\sim\begin{bmatrix}-1 & -2 & 2\\ -5 & -2 & 2\\ -2 & 4 & -4\end{bmatrix} R_1 \to (-1) R_1$$

$$\sim\begin{bmatrix}-1 & -2 & 2\\ 0 & -12 & 12\\ 0 & 0 & 0\end{bmatrix} \begin{matrix}R_2 \to R_2 + 5R_1\\ R_3 \to R_3 + 2R_1\end{matrix}$$

$$\sim\begin{bmatrix}-1 & -2 & 2\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix} R_2 \to \left(\frac{-1}{12}\right) R_2$$

From R_3: z is free, say $z = k_3$

From R_2: $y - z = 0 \Rightarrow y = z \Rightarrow y = k_3$

From R_3: $x - 2y + 2z = 0 \Rightarrow x - 2k_3 + 2k_3 = 0 \Rightarrow x = 0$

Thus, eigen vector $X = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ k_3\\ k_3\end{bmatrix} = k_3\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$

b. $\begin{pmatrix}-9 & 2 & 6\\ 5 & 0 & -3\\ -16 & 4 & 11\end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix}-9 & 2 & 6\\ 5 & 0 & -3\\ -16 & 4 & 11\end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix}-9-\lambda & 2 & 6\\ 5 & \lambda & -3\\ -16 & 4 & 11-\lambda\end{bmatrix}$$

Then characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} -9-\lambda & 2 & 6 \\ 5 & \lambda & -3 \\ -16 & 4 & 11-\lambda \end{vmatrix} = 0$

or, $(-9-\lambda)\{-\lambda(11-\lambda)+12\} - 2\{5(11-\lambda)-48\} + 6(20-16\lambda\} = 0$

or, $-(\lambda+9)(-11\lambda+\lambda^2+12) - 2(55-5\lambda-48) + 120 - 96\lambda = 0$

or, $11\lambda^2 + 99\lambda - \lambda^3 - 9\lambda^2 - 12\lambda - 108 - 14 + 10\lambda + 120 - 96\lambda = 0$

or, $-\lambda^3 + 2\lambda^2 + \lambda^2 - 2 = 0$

Solving, we get

$\lambda = -1, 1, 2$

Thus, eigen values of matrix A are $\lambda = -1, 1, 2$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of A

Then

$(A - \lambda I)X = 0$

or, $\begin{bmatrix} -9-\lambda & 2 & 6 \\ 5 & \lambda & -3 \\ -16 & 4 & 11-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$ ...(1)

Which is a system of homogeneous equation. To solve this system by G elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = -1$, then from (1)

$\begin{bmatrix} -8 & 2 & 6 \\ 5 & 1 & -3 \\ -16 & 4 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Which is system of homogeneous equation with coefficient matrix

$A = \begin{bmatrix} -8 & 2 & 6 \\ 5 & 1 & -3 \\ -16 & 4 & 12 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1/4 & -3/4 \\ 5 & 1 & -3 \\ -16 & 4 & 12 \end{bmatrix} R_1 \to \frac{R_1}{-8}$

$\sim \begin{bmatrix} 1 & 1/4 & -3/4 \\ 0 & 9/4 & 3/4 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 + 16R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & 1/4 & -3/4 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to \frac{4}{9} R_2$

Which is row reduced echelon form of coefficient matrix of homogeneous sys

So, by Gauss elimination method,

From R_3: z is free, say $z = k_1$

From R_2: $y + \frac{1}{3}z = 0 \Rightarrow y = -\frac{k_1}{3}$

From R_1: $x - \frac{1}{4}y - \frac{3}{4}z = 0$

or, $x = \frac{1}{4}y + \frac{3}{4}z$

or, $x = \frac{1}{4}\left(-\frac{k_1}{3}\right) + \frac{3}{4}k_1$

or, $x = \frac{2}{3}k_1$

Thus, the eigen vector,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/3\, k_1 \\ -1/3\, k_1 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} 2/3 \\ -1/3 \\ 1 \end{bmatrix}$

$\therefore$ $x = k_1 \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

When $\lambda = 1$, then from (1)

$\begin{bmatrix} -10 & 2 & 6 \\ 5 & -1 & -3 \\ -16 & 4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation with coefficient matrix.

$A = \begin{bmatrix} -10 & 2 & 6 \\ 5 & -1 & -3 \\ -16 & 4 & 10 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1/5 & -3/5 \\ 5 & -1 & -3 \\ -16 & 4 & 10 \end{bmatrix} R_1 \to \frac{R_1}{-10}$

$\sim \begin{bmatrix} 1 & -1/5 & -3/5 \\ 0 & 0 & 0 \\ 0 & 4/5 & 2/5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 + 16R_1 \end{matrix}$

$\sim \begin{bmatrix} 1 & -1/5 & -3/5 \\ 0 & 4/5 & 2/5 \\ 0 & 0 & 0 \end{bmatrix} R_2 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & -1/5 & -3/5 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method

From R_1 : z is free, say $z = k_2$

From R_2: $y + \frac{1}{2}z = 0 \Rightarrow y = -\frac{k_2}{2}$

From R_3: $x - \frac{1}{5}y - \frac{3}{5}z = 0$

or, $x = \frac{1}{5}y + \frac{3}{5}z$

or, $x = \frac{1}{5}\left(-\frac{k_2}{2}\right) + \frac{3}{5}k_2$

or, $x = \frac{1}{2}k_2$

Thus, the eigen vector

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2\, k_2 \\ -1/2\, k_2 \\ 1 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

When $\lambda = 2$, then from (1)

$$\begin{bmatrix} -11 & 2 & 6 \\ 5 & -2 & -3 \\ -16 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix

$$A = \begin{bmatrix} -11 & 2 & 6 \\ 5 & -2 & -3 \\ -16 & 4 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2/11 & -6/11 \\ 5 & -2 & -3 \\ -16 & 4 & 9 \end{bmatrix} R_1 \to \frac{R_1}{-11}$$

$$\sim \begin{bmatrix} 1 & -2/11 & -6/11 \\ 0 & -12/11 & -3/11 \\ 0 & 12/11 & 3/11 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 + R_3 + 16R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2/11 & -6/11 \\ 0 & 1 & 1/4 \\ 0 & 12/11 & 3/11 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2/11 & -6/11 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - \frac{12}{11}R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system

So by Gauss elimination method

From R_3: z is free, say $z = k_3$

From R_2: $y + \frac{1}{4}z = 0 \Rightarrow y = -\frac{k_3}{4}$

From R_1: $x - \frac{2}{11}y - \frac{6}{11}z = 0$

or, $x = \frac{2}{11}y + \frac{6}{11}z$

or, $x = \frac{2}{11}\left(-\frac{k_3}{4}\right) + \frac{6}{11}k_3$

or, $x = \frac{1}{2}k_3$

Thus, eigen vector is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2\, k_3 \\ -1/4\, k_3 \\ k_3 \end{bmatrix} = k_3\begin{bmatrix} 1/2 \\ -1/4 \\ 1 \end{bmatrix}$$

$$\therefore \quad X = k_3\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

c. $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)\{(2-\lambda)(1-\lambda) - 0\} - 1\{(1-\lambda) - 0\} + 1 \times 0 = 0$

or, $(2-\lambda)(2 - 3\lambda + \lambda^2) - 1 + \lambda = 0$

or, $4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$

or, $-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

Solving, we get

$\lambda = 1, 1, 3$

Thus, eigen values of matrix A are $\lambda = 1, 3$

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of A

Then, from $(A - \lambda I)X = 0$

or, $\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad \ldots(1)$

Which is system of homogeneous equation. To solve this system by Gauss elimination method reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 1$, from (1)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to R_2 - R_1$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

From R_1: $x + y + z = 0$, from R_3: z is free, from R_2: y is free

If we put $z = 0$, we get $x = -y$

If we put $y = 0$, we get $x = -z$

Since R_3, z is free, say $z = k_1$

Then eigen vector

$$X = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Since R_2: y is free, say $y = k_2$

Then eigen vector

$$X = \begin{bmatrix} -k_2 \\ k_2 \\ 0 \end{bmatrix} = k_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ Eigen vector at z = 1 are 1

$$X = k_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \text{ and } k_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

When $\lambda = 3$, then from (1)

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} R_2 \to R_2 + R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} R_2 \to \frac{1}{2}R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 2R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system

So by Gauss elimination method,

From R_2: $z = 0$

From R_1: $x - y = 0 \Rightarrow x = y$ and y is free

If $y = k_3$, $x = -k_3$

Thus, eigen vector $X = \begin{bmatrix} -k_3 \\ k_3 \\ 0 \end{bmatrix} = k_3\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

d. $\begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$

Solution

Let given matrix is

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix}$$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)\{(1-\lambda)(-1-\lambda) - 3\} - 1\{-2(-1-\lambda) - 6\} + 1\{-2 - 2(1-\lambda)\} = 0$

or, $(2-\lambda)(\lambda^2 - 4) - 1(2\lambda - 4) + 1(2\lambda - 4) = 0$

or, $2\lambda^2 - 8 - \lambda^3 + 4\lambda - 2\lambda + 4 + 2\lambda - 4 = 0$

or, $-\lambda^3 + 2\lambda^2 + 4\lambda - 8 = 0$

Solving, we get $\lambda = -2, 2, 2$

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector of A

Then from $AX = \lambda X$

$(A - \lambda I)X = 0$

or, $\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ...(1)

Which is a system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = -2$; from (1)

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 4 & -2 & 2 \\ 1 & 3 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -14 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 1/7 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to \frac{R_2}{-14}$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So, by Gauss elimination method

From R_3: z is free, say $z = k_1$

From R_2: $y + \frac{1}{7}z = 0 \Rightarrow y = -\frac{z}{7} \Rightarrow y = -\frac{k_1}{7}$

From R_1: $x + 3y + z = 0$

$\Rightarrow$ $x = -3y - z$

$\Rightarrow$ $x = -3\left(-\frac{k_1}{7}\right) - k_1$

$\Rightarrow$ $x = -\frac{4}{7}k_1$

Thus, eigen vector,

$$X = \begin{bmatrix} -4/7\,k_1 \\ -1/7\,k_1 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} -4/7 \\ -1/7 \\ 1 \end{bmatrix} = k_1\begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$$

When $\lambda = 2$, from (1)

$$\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix

$$A = \begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 2 \\ 1 & 3 & -3 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & -4 \end{bmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & -4 \end{bmatrix} R_2 \to \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 4R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous s...

So, by Gauss elimination method

From R_3: z is free, say $z = k_2$

From R_2: $y - z = 0 \Rightarrow y = z \Rightarrow y = k_2$

From R_1: $x - y + z = 0 \Rightarrow x - k_2 + k_2 = 0 \Rightarrow x = 0$

Thus, eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

e. $\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix}$$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$

or, $(-9-\lambda)\{(3-\lambda)(7-\lambda) - 32\} - 4\{-8(7-\lambda) + 64\} + 4\{-64 + 16(3-\lambda)\} = 0$

or, $-9(9+\lambda)(\lambda^2 - 10\lambda - 11) - 4(8\lambda + 8) + 4(-16 - 16\lambda) = 0$

or, $-\lambda^3 + \lambda^2 + 101\lambda + 99 - 32\lambda - 32 - 64 - 64\lambda = 0$

or, $-\lambda^3 + \lambda^2 + 5\lambda + 3 = 0$

or, $\lambda = -1, -1, 3$

Thus, eigen values of matrix A are $\lambda = -1, -1, 3$.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of the matrix A.

Then, from $(A - \lambda I)X = 0$

or, $\begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ...(1)

Which is system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = -1$, then from (1)

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & -1/2 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} R_1 \to \frac{R_1}{-8}$$

$$\sim \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 8R_1 \\ R_3 \to R_3 + 16R_1 \end{matrix}$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method

From R_3: z is free

From R_2: y is free

From R_1: $x - \frac{1}{2}y - \frac{1}{2}z = 0$

Case I: let $z = 0$, $y = k_1$, then $x = \frac{k_1}{2}$

Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1/2 \\ k_1 \\ 0 \end{bmatrix} = k_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = k_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Case II: let $y = 0$, $z = k_2$ then $x = \frac{k_2}{2}$

Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_2/2 \\ 0 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

When $\lambda = 3$, then from (1)

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/3 & -1/3 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} R_1 \to \frac{R_1}{-12}$$

$$\sim \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & -8/3 & 4/3 \\ 0 & 8/3 & -4/3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 8R_1 \\ R_3 + R_3 + 16R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & -1/2 \\ 0 & 8/3 & -4/3 \end{bmatrix} R_2 \to \frac{3}{-8} R_2$$

$$\sim \begin{bmatrix} 1 & -1/3 & -1/3 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - \frac{8}{3} R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous sy

So, by Gauss elimination method

From R_3: z is free, say $z = k_3$

From R_2: $y - \frac{1}{2}z = 0 \Rightarrow y = \frac{1}{2}z \Rightarrow y = \frac{1}{2}k_3$

From R_1: $x - \frac{1}{3}y - \frac{1}{3}z = 0$

or, $x = \frac{1}{3}y + \frac{1}{3}z$

or, $x = \frac{1}{3}\frac{1}{2}k_3 + \frac{1}{3}k_3$

or, $x = \frac{1}{2}k_3$

Thus, eigen vector is

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2\,k_3 \\ 1/2\,k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1/2 \\ 1/2 \\ 1 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

f. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)(1-\lambda)(1-\lambda) = 0$

or, $\lambda = 1, 1, 1$

Thus, eigen value of matrix A is $\lambda = 1$.

Let, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of A

Then, from $(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots(1)$$

Which is system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 1$, from (1)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$\lambda = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Where, x and z are free. If we consider $x = k$, $z = k$, then

From R_1: $y = 0$

Thus, eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 0 \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

g. $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}$

Then, the characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

or, $(3-\lambda)\{(5-\lambda)(3-\lambda)-1\} + 1\{-(3-\lambda)+1\} + 1\{1-(5-\lambda)\} = 0$

or, $(3-\lambda)(\lambda^2 - 8\lambda + 14) + (\lambda - 2) + (\lambda - 4) = 0$

or, $3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda + \lambda - 2 + \lambda - 4 = 0$

or, $-\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$

Solving we get, $\lambda = 2, 3, 6$

Thus, eigen values of matrix A are $\lambda = 2, 3, 6$.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of matrix A.

Then, $(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \ldots(1)$$

Which is system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 2$, from equation (1), we have

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_1 + R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method.

From R_3: z is free, say $z = k_1$

From R_2: $y = 0$

From R_1: $x - y + z = 0 \Rightarrow x = -2 \Rightarrow x = -k_1$

$\therefore$ Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda = 3$, then from (1)

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is system of homogeneous with coefficient matrix.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} R_1 \to -R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} R_2 \to -R_2$$

$$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method

From R_3: z is free, say $z = k_2$

From R_2: $y - z = 0 \Rightarrow y = z \Rightarrow y = k_2$

From R_1: $-x + 2y - z = 0 \Rightarrow x = 2y - z \Rightarrow x = 2k_2 - k_2 = k_2$

Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_2 \\ k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

When $\lambda = 6$, then from (1)

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & -1 & -1 \\ -3 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ -3 & -1 & 1 \\ 1 & -1 & -3 \end{bmatrix} R_1 \to -R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & -2 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} R_2 \to \frac{R_2}{2}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 2R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method

From R_3: z is free, say $z = k_3$

From R_2: $y + 2z = 0 \Rightarrow y = -2z \Rightarrow y = -2k_3$

From R_1: $x = -y - z \Rightarrow x = 2k_3 - k_3 = k_3$

$\therefore$ Eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_3 \\ -2k_3 \\ k_3 \end{bmatrix} = k_3\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

h. $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Solution

Let the given matrix is $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)\{(5-\lambda)(1-\lambda) - 1\} - 1\{(1-\lambda) - 3\} + 3\{1 - 3(5-\lambda)\} = 0$

or, $(1-\lambda)(\lambda^2 - 6\lambda + 4) + \lambda + 2 + 9\lambda - 42 = 0$

or, $\lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 + 9\lambda - 42 = 0$

or, $-\lambda^3 + 7\lambda^2 - 36 = 0$

Solving we get, $\lambda = -2, 3, 6$

Thus, eigen values of A are $\lambda = -2, 3, 6$.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of matrix A.

Then from $[A - \lambda I]\, X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots(1)$$

Which is a system of homogeneous equation. To solve this system by Gaus elimination method we reduce $(A - \lambda I)$ into row reduced echelon form

When $\lambda = -2$, from (1)

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1/3 & 1 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} R_1 \to \left(\frac{1}{3}\right) R_1$$

$$\sim \begin{bmatrix} 1 & 1/3 & 1 \\ 0 & 20/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to \left(\frac{3}{20}\right) R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system. so by Gauss elimination method

From R_3: z is free, say $z = k_1$

From R_2: $y = 0$

From R_1: $x + \frac{1}{3}y + z = 0$

or, $x + z = 0$

or, $x = -z$

or, $x = -k_1$

Thus, eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -k_1 \\ 0 \\ k_1 \end{bmatrix} = k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda = 3$, from (1)

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -5 & -5 \end{bmatrix} R_2 \to \frac{R_2}{5}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 5R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system. So by Gauss elimination method

From R_3: z is free, say $z = k_2$

From R_2: $y + z = 0 \Rightarrow y = -z \Rightarrow y = -k_2$

From R_1: $x + 2y + z = 0$

or, $x - 2y - z$

or, $x = -2(-k_2) - k_2$

or, $x = k_2$

Thus, eigen vector $X = \begin{bmatrix} k_2 \\ -k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

When $\lambda = 6$, then from (1)

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ -5 & 1 & 3 \\ 3 & 1 & -5 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 5R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{bmatrix} R_2 \to \left(-\frac{1}{4}\right) R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 4R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So by Gauss elimination method

From R_3: z is free, say $z = k_3$

From R_2: $y - 2z = 0 \Rightarrow y = 2k_3$

From R_1: $x - y + z = 0 \Rightarrow x = y - z \Rightarrow x = k_3$

Thus, eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_3 \\ 2k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

i. $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}$

Then, the characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

or, $(6-\lambda)\{(3-\lambda)^2 - 1\} + 2\{2 - (3-\lambda) + 2\}\{2 - 2(3-\lambda)\} = 0$

or, $(6-\lambda)(9 - 6\lambda + \lambda^2 - 1) + 2(-6 + 2\lambda + 2) + 2(2 - 6 + 2\lambda) = 0$

or, $(6-\lambda)(\lambda^2 - 6\lambda + 8) + 2(2\lambda - 4) + 2(2\lambda - 4) = 0$

or, $6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$

or, $-\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

Solving, we get $\lambda = 2, 2, 8$

Thus, eigen values of A are $\lambda = 2, 2, 8$.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be eigen vector of matrix A

Then from $(A - \lambda I)X = 0$

or, $\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ...(1)

Which is a system of homogeneous equation. To solve this system by Gauss elimination method, we reduce $(A - \lambda I)$ into row reduced echelon form.

When $\lambda = 2$, then from (1)

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 1/2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} R_1 \to \left(\frac{1}{4}\right) R_1$$

$$\sim \begin{bmatrix} 1 & -1/2 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.

So

From R_3: z is free

From R_2: y is free

From R_1: $x - \frac{1}{2}y + \frac{1}{z} = 0$

Case I: If we consider, $z = 0$ and $y = k_1$ then

$x = \frac{1}{2}y$

i.e., $x = \frac{k_1}{2}$

Then, eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_1/2 \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Case II: If we consider $y = 0$ and $z = k_2$, then

$x = -\frac{1}{2}z$

i.e., $x = -\frac{1}{2}k_2$

Then, eigen vector $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1/2\, k_2 \\ 0 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

When $\lambda = 8$, then from (1)

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which is system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} R_1 \to \left(-\frac{1}{2}\right) R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{matrix} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} R_2 \to \left(-\frac{1}{3}\right) R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 3R_2$$

Which is row reduced echelon form of coefficient matrix of homogeneous system.
So by Gauss elimination method,

From R_3: z is free, say $z = k_3$

From R_2: $y + z = 0 \Rightarrow y = -z \Rightarrow y = -k_3$

From R_1: $x + y - z = 0 \Rightarrow x = -y + z \Rightarrow x = k_3 + k_3 \Rightarrow x = 2k_3$

Thus, eigen vector is $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2k_3 \\ -k_3 \\ k_3 \end{bmatrix} = k_3\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

3. Verify Cayley-Hamilton theorem for the following matrices.

a. $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

Characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}$$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 2-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)(3-\lambda) + 1 = 0$

or, $6 - 2\lambda - 3\lambda + \lambda^2 + 1 = 0$

or, $\lambda^2 - 5\lambda + 7 = 0$ ...(1)

We have to show $\lambda = A$ satisfies the equation (1), i.e., $A^2 - 5A + 7I = 0$

L.H.S. $= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4-1 & -2-3 \\ 2+3 & -1+9 \end{bmatrix} - \begin{bmatrix} 10 & -5 \\ 5 & 15 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 3-10+7 & -5+5+0 \\ 5-5+0 & 8-15+7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $A^2 - 5A + 7I = 0$

Hence, Cayley-Hamilton theorem is verified.

b. $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

Characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{bmatrix}$$

The characteristic equation is

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)^2 - 6 = 0$

or, $1 - 2\lambda + \lambda^2 - 6 = 0$

or, $\lambda^2 - 2\lambda - 5 = 0$ ...(1)

We have to show $\lambda = A$ satisfies the equation (1), i.e., $A^2 - 2A - 5I = 0$

L.H.S. $= \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $A^2 - 2A - 5I = 0$

Hence, Cayley-Hamilton theorem is verified.

c. $\begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}$

Solution

Let given matrix be

$$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 5 \\ 2 & 3-\lambda \end{bmatrix}$$

Then, characteristic equation is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 5 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

or, $(1-\lambda)(3-\lambda) - 10 = 0$

or, $3 - 4\lambda + \lambda^2 - 10 = 0$

or, $\lambda^2 - 4\lambda - 7 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies the equation (1), i.e., $A^2 - 4A - 7I = 0$

L.H.S. $= \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} - 4\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 11 & 20 \\ 8 & 19 \end{bmatrix} - \begin{bmatrix} 4 & 20 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $A^2 - 4A - 7I = 0$

Hence, Caley-Hamilton theorem is verified.

d. $\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$

Then the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)(1-\lambda) + 1 = 0$

or, $3 - 3\lambda - \lambda + \lambda^2 + 1 = 0$

or, $\lambda^2 - 4\lambda + 4 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies the equation (i), i.e., $A^2 - 4A + 4I = 0$

L.H.S. $= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} - 4\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & -4 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 12 & -4 \\ 4 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $A^2 - 4A + 4I = 0$

Hence, Cayley-Hamilton theorem is verified.

e. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution

Let the given matrix be $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$

Then the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)(1-\lambda)(1-\lambda) = 0$

or, $(1-\lambda)^3 = 0$

or, $1 - 3\lambda + 3\lambda^2 - \lambda^3 = 0$

or, $-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies the equation (1), i.e., $-A^3 + 3A^2 - 3A + 1 = 0$

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

L.H.S. $= -A^3 + 3A^2 - 3A + I$

$= -\begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $-A^3 + 3A^2 - 3A + I = 0$

Hence, Cayley-Hamilton theorem is verified.

f. $\begin{pmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 4 & -1 \\ 1 & -1-\lambda & 1 \\ -1 & 2 & 3-\lambda \end{bmatrix}$$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 3-\lambda & 4 & -1 \\ 1 & -1-\lambda & 1 \\ -1 & 2 & 3-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)\{(1-\lambda)(3-\lambda)-2\}^2 - 4\{3-\lambda+1\} - 1\{2+(1-\lambda)\} = 0$

or, $(3-\lambda)(-3+\lambda-3\lambda+\lambda^2-2) - 4(4-\lambda) - (1-\lambda) = 0$

or, $(3-\lambda)(\lambda^2-2\lambda-5) - 16 + 4\lambda - 1 + \lambda = 0$

or, $3\lambda^2 - 6\lambda - 15 - \lambda^3 + 2\lambda^2 + 5\lambda - 17 + 5\lambda = 0$

or, $\lambda^3 + 5\lambda^2 + 4\lambda - 32 = 0$

or, $\lambda^3 - 5\lambda^2 - 4\lambda + 32 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies equation (1), i.e., $A^3 - 5A^2 - 4A + 32I = 0$

Now,

$$A^2 = A \cdot A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 6 & -2 \\ 1 & 7 & 1 \\ -4 & 0 & 12 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 14 & 6 & -2 \\ 1 & 7 & 1 \\ -4 & 0 & 12 \end{bmatrix}\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 50 & 46 & -14 \\ 9 & -1 & 9 \\ -24 & 8 & 40 \end{bmatrix}$$

Now, L.H.S. $= A^3 - 5A^2 - 4A + 32I$

$$= \begin{bmatrix} 50 & 46 & -14 \\ 9 & -1 & 9 \\ -24 & 8 & 40 \end{bmatrix} - 5\begin{bmatrix} 14 & 6 & -2 \\ 1 & 7 & 1 \\ -4 & 0 & 12 \end{bmatrix} - 4\begin{bmatrix} 3 & 4 & -1 \\ 1 & -1 & 1 \\ -1 & 2 & 3 \end{bmatrix} + 32\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 50-70-12+32 & 46-30-16+0 & -14+10+4 \\ 9-5-4+0 & -1-35+4+32 & 9-5-4+0 \\ -24+20+4+0 & 8+0-8+0 & 40-60-12+32 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $A^3 - 5A^2 - 4A + 32I = 0$

Hence, Cayley-Hamilton theorem is verified.

4. Verify Cayley Hamilton theorem for the following matrices. Also, find the inverse of the matrix using this theorem.

a. $\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}$

Solution

Let given matrix be

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-\lambda & 3 \\ 1 & 1-\lambda \end{bmatrix}$$

Then the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 4-\lambda & 3 \\ 1 & 1-\lambda \end{vmatrix} = 0$

or, $(4-\lambda)(1-\lambda) - 3 = 0$

or, $4 - 4\lambda - \lambda + \lambda^2 - 3 = 0$

or, $\lambda^2 - 5\lambda + 1 = 0$ ...(1)

We have to show $\lambda = A$ satisfies the equation (i),

i.e., $A^2 - 5A + I = 0$ ...(2)

L.H.S. $= \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} - 5\begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 19 & 15 \\ 5 & 4 \end{bmatrix} - \begin{bmatrix} 20 & 15 \\ 5 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2 - 5A + I = 0$

Hence, Cayley-Hamilton theorem is verified.

Again, from equation (2)

$A^2 - 5A + I = 0$

or, $A(A - 5I) = -I$

or, $A^{-1}A(A - 5I) = -A^{-1}I$

or, $-A^{-1} = A - 5I$

or, $A^{-1} = 5I - A$

$$= 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

$\therefore$ $A^{-1} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

b. $\begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 5 \\ 3 & 2-\lambda \end{bmatrix}$$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & 5 \\ 3 & 2-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)(2-\lambda) - 15 = 0$

or, $2 - \lambda - 2\lambda + \lambda^2 - 15 = 0$

or, $\lambda^2 - 3\lambda - 13 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies equation (1)

i.e., L.H.S. $= A^2 - 3A - 13I$

$$= \begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} - 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 15 \\ 9 & 19 \end{bmatrix} - \begin{bmatrix} 3 & 15 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2 - 3A - 13I = 0$

Hence, Cayley- Hamilton theorem is verified.

Now, for inverse, from equation (2), we have

$A(A - 3I) = 13I$

or, $A^{-1}A(A - 3I) = A^{-1}(13I)$

or, $13A^{-1} = A - 3I$

or, $A^{-1} = \frac{1}{13}(A - 3I)$

$$= \frac{1}{13}\left\{\begin{bmatrix} 1 & 5 \\ 3 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$\therefore$ $A^{-1} = \frac{1}{13}\begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix}$

c. $\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 3-\lambda & 4 \\ 2 & 3-\lambda \end{bmatrix}$$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 3-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)^2 - 8 = 0$

or, $9 - 6\lambda + \lambda^2 - 8 = 0$

or, $\lambda^2 - 6\lambda + 1 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies the equation (1)

i.e., $A^2 - 6A + I = 0$ ...(2)

L.H.S. $= A^2 - 6A + I$

$$= \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} - 6\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 24 \\ 12 & 17 \end{bmatrix} - \begin{bmatrix} 18 & 24 \\ 12 & 18 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2 - 6A + I = 0$

Hence, Cayley-Hamilton theorem is verified.

For A^{-1}, from equation (2), we have

$A(A - 6I) = -I$

or, $A^{-1}A(A - 6I) = -A^{-1}I$

or, $-A^{-1} = A - 6I$

or, $A^{-1} = 6I - A$

or, $A^{-1} = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$

d. $\begin{pmatrix} 3 & 2 \\ 1 & 5 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 2 \\ 1 & 5-\lambda \end{bmatrix}$$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 3-\lambda & 2 \\ 1 & 5-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)(5-\lambda)-2=0$

or, $15-3\lambda-5\lambda+\lambda^2=0$

or, $\lambda^2-8\lambda+13=0$ ...(1)

We have to show that $\lambda = A$ satisfies the equation (1)

i.e., $A^2-8A+13I=0$ ...(2)

L.H.S. $= A^2-8A+13I$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} - 8\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} + 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 16 \\ 8 & 27 \end{bmatrix} - \begin{bmatrix} 24 & 16 \\ 8 & 40 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2-8A+13I=0$

Hence, Cayley-Hamilton theorem is verified.

From equation (2), we can write

$A(A-8I) = -13I$

or, $A^{-1}A(A-8I) = A^{-1}(-13I)$

or, $A-8I = -13A^{-1}$

or, $A^{-1} = -\frac{1}{13}(A-8I)$

$$= -\frac{1}{13}\left\{\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} - 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= -\frac{1}{13}\begin{bmatrix} -5 & 2 \\ 1 & -3 \end{bmatrix}$$

$\therefore$ $A^{-1} = \frac{1}{13}\begin{bmatrix} 5 & -2 \\ -1 & 3 \end{bmatrix}$

e. $\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$$

The characteristic matrix $= A-\lambda I$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 3 & -5-\lambda \end{bmatrix}$$

Then, characteristic equation is

$|A-\lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & 2 \\ 3 & -5-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)(-5-\lambda)-6=0$

or, $-5-\lambda+5\lambda+\lambda^2-6=0$

or, $\lambda^2-4\lambda-11=0$ ...(1)

We have to show that $\lambda = A$ satisfies equation (1)

i.e., $A^2+4A-11I=0$ ...(2)

L.H.S. $= A^2+4A-11I$

$$= \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} - 11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -8 \\ -12 & 31 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 12 & -20 \end{bmatrix} - \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ $A^2+4A-11I=0$

Hence, Cayley=Hamilton theorem is verified.

From equation (2), we have

$A(A+4I) = 11I$

or, $A^{-1}A(A+4I) = A^{-1}(11I)$

or, $A^{-1} = \frac{1}{11}(A+4I)$

$$= \frac{1}{11}\left\{\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{11}\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$$

5. Verify Cayley Hamilton theorem for the following matrices. Also, find the inverse of the matrix using this theorem.

a. $\begin{pmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}$$

The characteristic matrix $= A-\lambda I$

$$= \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7-\lambda & 2 & 1 \\ 0 & 3-\lambda & -1 \\ -3 & 4 & -2-\lambda \end{bmatrix}$$

Then, the characteristic equation is

$|A-\lambda I| = 0$

or, $\begin{vmatrix} 7-\lambda & 2 & 1 \\ 0 & 3-\lambda & -1 \\ -3 & 4 & -2-\lambda \end{vmatrix} = 0$

or, $(7-\lambda)\{(3-\lambda)(-2-\lambda)+4\} - 3\{-2-(3-\lambda)\} = 0$

or, $(7-\lambda)(-6-3\lambda+2\lambda+\lambda^2+4) + 3(2+3-\lambda) = 0$

or, $(7-\lambda)(\lambda^2-\lambda-2) + 3(5-\lambda) = 0$

or, $7\lambda^2-7\lambda-14-\lambda^3+\lambda^2+2\lambda+15-3\lambda = 0$

or, $\lambda^3 - 8\lambda^2 + 8\lambda - 1 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies equation (1)

i.e., $A^3 - 8A^2 + 8A - I = 0$ ...(2)

Now,

$$A^2 = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix}\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 46 & 24 & 3 \\ 3 & 5 & -1 \\ -15 & -2 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 46 & 24 & 3 \\ 3 & 5 & -1 \\ -15 & -2 & -3 \end{bmatrix}\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} = \begin{bmatrix} 313 & 176 & 16 \\ 24 & 17 & 0 \\ -96 & -48 & -7 \end{bmatrix}$$

Now, L.H.S. of equation (2)

L.H.S. $= A^3 - 8A^2 + 8A - I$

$$= \begin{bmatrix} 313 & 176 & 16 \\ 24 & 17 & 0 \\ -96 & -48 & -7 \end{bmatrix} - 8\begin{bmatrix} 46 & 24 & 3 \\ 3 & 5 & -1 \\ -15 & -2 & -3 \end{bmatrix} + 8\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $A^3 - 8A^2 + 8A - I = 0$

Hence, Cayley-Hamilton theorem is verified.

Now, from (2), we have

$A(A^2 - 8A + 8I) = I$

or, $A^{-1}A\,(A^2 - 8A - 8I) = A^{-1}I$

or, $A^{-1} = A^2 - 8A - 8I$

$$= \begin{bmatrix} 46 & 24 & 3 \\ 3 & 5 & -1 \\ -15 & -2 & -3 \end{bmatrix} - 8\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 46 & 24 & 3 \\ 3 & 5 & -1 \\ -15 & -2 & -3 \end{bmatrix} - \begin{bmatrix} 56 & 16 & 8 \\ 0 & 24 & -8 \\ -24 & 32 & -16 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$\therefore$ $A^{-1} = \begin{bmatrix} -2 & 8 & -5 \\ 3 & -11 & 7 \\ 9 & -34 & 21 \end{bmatrix}$

b. $\begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I$

$$= \begin{bmatrix} 3-\lambda & -3 & 4 \\ 2 & -3-\lambda & 4 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 3-\lambda & -3 & 4 \\ 2 & -3-\lambda & 4 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)\,\{(-3-\lambda)(1-\lambda) + 4\} - 2\{-3(1-\lambda) + 4\} = 0$

or, $(3-\lambda)\,\{-3 + 3\lambda - \lambda + \lambda^2 + 4) - 2(-3 + 3\lambda + 4) = 0$

or, $(3-\lambda)\,(\lambda^2 + 2\lambda + 1) - 2(3\lambda + 1) = 0$

or, $3\lambda^2 + 6\lambda + 3 - \lambda^3 - 2\lambda^2 - \lambda - 6\lambda - 2 = 0$

or, $\lambda^3 - \lambda^2 + \lambda - 1 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies equation (1)

i.e., $A^3 - A^2 + A - I = 0$ ...(2)

Now,

$$A^2 = A\cdot A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^3 = A^2\cdot A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 4 \\ -2 & 3 & -3 \end{bmatrix}$$

Now, from (2)

L.H.S. $= A^3 - A^2 + A - I$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 4 \\ -2 & 3 & -3 \end{bmatrix} - \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, Cayley-Hamilton theorem is verified.

From (2), we can write

$A(A^2 - A - I) = I$

or, $A^{-1}A\,(A^2 - A - I) = A^{-1}I$

or, $A^{-1} = A^2 - A - I$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

c. $\begin{pmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 1 \\ 0 & -2-\lambda & 1 \\ -2 & -3 & 0-\lambda \end{bmatrix}$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & -2-\lambda & 1 \\ -2 & -3 & 0-\lambda \end{vmatrix} = 0$

or, $(1-\lambda)\{\lambda(2+\lambda)+3\} - 2(-1+2+\lambda) = 0$

or, $(1-\lambda)(\lambda^2+2\lambda+3) - 2(\lambda+1) = 0$

or, $\lambda^2 + 2\lambda + 3 - \lambda^3 - 2\lambda^2 - 3\lambda - 2 - 2 = 0$

or, $\lambda^3 + \lambda^2 + 3\lambda - 1 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies equation (1)

i.e., $A^3 + A^2 + 3A - I = 0$ ...(2)

Now,

$$A^2 = A \cdot A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -2 \\ -2 & 8 & -5 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -2 \\ -2 & 8 & -5 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -3 \\ 2 & 6 & -1 \\ 8 & 1 & 6 \end{bmatrix}$$

From (2) L.H.S. $= A^3 - 2A^2 - A + 2I$

$$\text{L.H.S.} = \begin{bmatrix} -1 & 5 & -3 \\ 2 & 6 & -1 \\ 8 & 1 & 6 \end{bmatrix} - 2\begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -2 \\ -2 & 8 & -5 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 & -3 \\ 2 & 6 & -1 \\ 8 & 1 & 6 \end{bmatrix} + \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -2 \\ -2 & 8 & -5 \end{bmatrix} + \begin{bmatrix} 3 & -3 & 3 \\ 0 & -6 & 3 \\ -6 & -9 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $A^3 + A^2 + 3A - I = 0$

Hence, Cayley Hamilton theorem is verified.

From equation (2), we have

$I = A(A^2 + A + 3I)$

or, $A^{-1}I = A^{-1}A(A^2 + A + 3I)$

or, $A^{-1} = A^2 + A + 3I$

$$= \begin{bmatrix} -1 & -2 & 0 \\ -2 & 1 & -2 \\ -2 & 8 & -5 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -2 & -3 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -2 & 2 & -1 \\ -4 & 5 & -2 \end{bmatrix}$$

d. $\begin{pmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 6-\lambda & 2 & 3 \\ 3 & 1-\lambda & 1 \\ 10 & 3 & 4-\lambda \end{bmatrix}$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 6-\lambda & 2 & 3 \\ 3 & 1-\lambda & 1 \\ 10 & 3 & 4-\lambda \end{vmatrix} = 0$

or, $(6-\lambda)\{(1-\lambda)(4-\lambda) - 3\} - 2\{3(4-\lambda) - 10\} + 3\{9 - 10(1-\lambda)\} = 0$

or, $(6-\lambda)(\lambda^2 - 5\lambda + 1) - 2(2 - 3\lambda) + 3(10\lambda + 1) = 0$

or, $6\lambda^2 - 30\lambda + 6 - \lambda^3 + 5\lambda^2 - \lambda - 4 + 6\lambda + 30\lambda - 3 = 0$

or, $-\lambda^3 + 11\lambda^2 + 5\lambda - 1 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies equation (1)

i.e., $-A^3 + 11A^2 + 5A - I = 0$ ...(2)

Now,

$$A^2 = A \cdot A = \begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix}\begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 72 & 23 & 32 \\ 31 & 10 & 14 \\ 109 & 35 & 49 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 72 & 23 & 32 \\ 31 & 10 & 14 \\ 109 & 35 & 49 \end{bmatrix}\begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 821 & 263 & 367 \\ 356 & 14 & 159 \\ 1249 & 400 & 558 \end{bmatrix}$$

Now, from equation (2)

L.H.S. $= -A^3 + 11A^2 + 5A - I$

$$= -\begin{bmatrix} 821 & 263 & 367 \\ 356 & 14 & 159 \\ 1249 & 400 & 558 \end{bmatrix} + 11\begin{bmatrix} 72 & 23 & 32 \\ 31 & 10 & 14 \\ 109 & 35 & 49 \end{bmatrix} + 5\begin{bmatrix} 6 & 2 & 3 \\ 3 & 1 & 1 \\ 10 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 821 & 263 & 367 \\ 356 & 14 & 159 \\ 1249 & 400 & 558 \end{bmatrix}\begin{bmatrix} 792 & 253 & 352 \\ 341 & 110 & 154 \\ 1199 & 385 & 539 \end{bmatrix} + \begin{bmatrix} 30 & 10 & 15 \\ 15 & 5 & 5 \\ 50 & 15 & 20 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $-A^3 + 11A^2 + 5A - I = 0$

Hence, Cayley-Hamilton theorem is verified.

From (2), we can write

$I = A(-A^2 + 11A + 5I)$

$A^{-1}I = A^{-1}A(-A^2 + 11A + 5I)$

or, $A^{-1} = -A^2 + 11A + 5I$

or, $A^{-1} = -\begin{bmatrix} 72 & 23 & 32 \\ 31 & 10 & 14 \\ 109 & 35 & 49 \end{bmatrix} + \begin{bmatrix} 66 & 22 & 33 \\ 33 & 11 & 11 \\ 110 & 33 & 44 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$\therefore$ $A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 6 & -3 \\ 1 & -2 & 0 \end{bmatrix}$

e. $\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$

Solution

Let the given matrix is

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 2 \\ 2 & 0-\lambda & -2 \\ 0 & 1 & 1-\lambda \end{bmatrix}$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 3-\lambda & 0 & 2 \\ 2 & 0-\lambda & -2 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$

or, $(3-\lambda)\{-\lambda(1-\lambda)+2\} - 2(0-2) = 0$

or, $(3-\lambda)(-\lambda+\lambda^2+2) + 4 = 0$

or, $3\lambda^2 - 3\lambda + 6 - \lambda^2 - \lambda^3 - 2\lambda + 4 = 0$

or, $\lambda^3 - 4\lambda^2 + 5\lambda - 10 = 0$ ...(1)

We have to show that $\lambda = A$ satisfies the equation (1)

i.e., $A^3 - 4A^2 + 5A - 10I = 0$ ...(2)

Now,

$A^2 = A \cdot A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 8 \\ 6 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 9 & 2 & 8 \\ 6 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 31 & 8 & 22 \\ 14 & 2 & 18 \\ 8 & -1 & 1 \end{bmatrix}$

Now, from equation (2)

L.H.S. $= A^3 - 4A^2 + 5A - 10I$

$= \begin{bmatrix} 31 & 8 & 22 \\ 14 & 2 & 18 \\ 8 & -1 & 1 \end{bmatrix} - 4\begin{bmatrix} 9 & 2 & 8 \\ 6 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix} + 5\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} - 10\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 31 & 8 & 22 \\ 14 & 2 & 18 \\ 8 & -1 & 1 \end{bmatrix} - \begin{bmatrix} 36 & 8 & 32 \\ 24 & -8 & 8 \\ 8 & 4 & -4 \end{bmatrix} + \begin{bmatrix} 15 & 0 & 10 \\ 10 & 0 & -10 \\ 0 & 5 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ $A^3 - 4A^2 + 5A - 10I = 0$

Hence, Cayley- Hamilton theorem is verified.

6. **Diagonalize the following matrices.**

a. $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

First we need to determine eigen values and eigen vectors of A.

Let λ and X are eigen value and eigen vector if A respectively.

Then, the characteristic equation of A is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)(2-\lambda) - 4 = 0$

or, $10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$

or, $\lambda^2 - 7\lambda + 6 = 0$

or, $\lambda^2 - 6\lambda - \lambda + 6 = 0$

or, $\lambda(\lambda - 6) - 1(\lambda - 6) = 0$

or, $(\lambda - 1)(\lambda - 6) = 0$

$\therefore$ $\lambda = 1, 6$

Since, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is eigen vector of A, then

$(A - \lambda I)X = 0$

or, $\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ...(1)

When $\lambda = 1$, then from (1)

$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation with coefficient matrix.

$A = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $R_2 \to 4R_1$

From R_2: y is free, say $y = k_1$

From R_1: $x + y = 0 \Rightarrow x = -y \Rightarrow x = -k_1$

Eigen vector $X = k_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda = 6$, then from (1)

$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -4 \\ -1 & 4 \end{bmatrix} R_1 \leftrightarrow R_2$$
$$\sim \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 + R_1$$

From R_2: y is free, say $y = k_2$

From R_1: $x - 4y = 0 \Rightarrow x = 4y \Rightarrow x = 4k_2$

Eigen vector $X = k_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

Let $P = \begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$ so that $P^{-1} = -\frac{1}{5}\begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}$, where P is model matrix

We have, $A = PDP^{-1} \Rightarrow D = P^{-1}AP$

So, diagonal matrix $D = P^{-1}AP$

$$= -\frac{1}{5}\begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & 1 \end{bmatrix}$$
$$= -\frac{1}{5}\begin{bmatrix} 1 & -4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 24 \\ 1 & 6 \end{bmatrix}$$
$$= -\frac{1}{5}\begin{bmatrix} -5 & 0 \\ 0 & -30 \end{bmatrix}$$
$$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

b. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

Then, characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)^2 - 1 = 0$

or, $4 - 4\lambda + \lambda^2 - 1 = 0$

or, $\lambda^2 - 4\lambda + 3 = 0$

or, $\lambda^2 - 3\lambda - \lambda + 3 = 0$

or, $\lambda(\lambda - 3) - 1(\lambda - 3) = 0$

or, $(\lambda - 3)(\lambda - 1) = 0$

or, $\lambda = 1, 3$

$\therefore$ Eigen values of matrix A are $\lambda = 1, 3$.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be eigen vector of A then

$(A - \lambda I)X = 0$

or, $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ...(1)

When $\lambda = 1$, then from (1)

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - R_1$$

From R_2: y is free, say $y = k_1$

From R_1 $x + y = 0 \Rightarrow x = -y \Rightarrow x = -k_1$

$\therefore$ $X = k_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda = 3$, then from (1)

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which is a system of homogeneous equation with coefficient matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} R_1 \leftrightarrow R_2$$
$$\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 + R_1$$

From R_2: y is free, say $y = k_2$

From R_1: $x - y = 0 \Rightarrow x = y \Rightarrow x = k_2$

$\therefore$ $X = k_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Let $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ so that $P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

Where P is model matrix

We have $A = PDP^{-1} \Rightarrow D = P^{-1}AP$

So, the diagonal matrix is

$D = P^{-1}AP$

$$= -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$
$$= -\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & 3 \end{bmatrix}$$
$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 \\ 0 & -6 \end{bmatrix}$$
$$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

c. $\begin{pmatrix} 2 & 1 \\ 5 & 6 \end{pmatrix}$

Solution

Let the given matrix be

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix}$$

The characteristic matrix $= A - \lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 5 & 6-\lambda \end{bmatrix}$

Then, the characteristic equation is

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 2-\lambda & 1 \\ 5 & 6-\lambda \end{vmatrix} = 0$

or, $(2-\lambda)(5-\lambda) - 5 = 0$

or, $12 - 12\lambda - 6\lambda + \lambda^2 - 5 = 0$

or, $\lambda^2 - 8\lambda + 7 = 0$

or, $\lambda^2 - 7\lambda - \lambda + 7 = 0$

or, $\lambda(\lambda - 7) - 1(\lambda - 7) = 0$

or, $(\lambda - 1)(\lambda - 7) = 0$

or, $\lambda = 1, 7$

$\therefore$ The eigen values of matrix A are $\lambda = 1, 7$.

Let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ be the eigen vector of A then

$(A - \lambda I)X = 0$

or, $\begin{bmatrix} 2-\lambda & 1 \\ 5 & 6-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ... (1)

When $\lambda = 1$, then from (1)

$\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation with coefficient matrix

$A = \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - 5R_1$

From R_2: y is free, say $y = k_1$

From R_1: $x + y = 0 \Rightarrow x = -y \Rightarrow x = -k_1$

$\therefore$ Eigen vector $X = k_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

When $\lambda = 7$, then from (1)

$\begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Which is a system of homogeneous equation with coefficient matrix.

$A = \begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1/5 \\ 5 & -1 \end{bmatrix} R_1 \to -\frac{1}{5}R_1$

$\sim \begin{bmatrix} 1 & -1/5 \\ 0 & 0 \end{bmatrix} R_2 \to R_2 - 5R_1$

From R_2: y is free, say $y = k_2$

From R_1: $x - \frac{1}{5}y = 0 \Rightarrow x = \frac{k_2}{5}$

$\therefore$ Eigen vector $X = k_2 \begin{bmatrix} 1/5 \\ 1 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\therefore$ The model matrix $P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}$

Then, $P^{-1} = -\frac{1}{6} \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix}$

Now, the diagonal matrix D corresponding to A

$D = P^{-1}AP$

$= -\frac{1}{6} \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}$

$= -\frac{1}{6} \begin{bmatrix} 5 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 7 \\ 1 & 35 \end{bmatrix}$

$= -\frac{1}{6} \begin{bmatrix} -6 & 0 \\ 0 & -42 \end{bmatrix}$

$\therefore D = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$

Unit 3

Linear Programming Problems

Exercise 3.1

1. **By using the simplex method, solve the following LPP.**

   a. **Maximize $Z = 25x_1 + 30x_2$ subject to the constraints**

   $20x_1 + 30x_2 \le 6905$ $\quad x_1 + 4x_2 \le 120$ $\quad x_1, x_2 \ge 0$

Solution

Introducing S_1 and S_2 as slack variables, we can write above LPP in standard form system of linear equation as

$$Z - 25x_1 + 30x_2 + 0S_1 + 0S_2 = 0$$
$$20x_1 + 30x_2 + S_1 + 0S_2 = 6905$$
$$x_1 + 4x_2 + 0S_1 + S_2 = 120$$

The first simplex table of above system is

B	Z	x_1	$x_2 \downarrow$	S_1	S_2	b	Ratio
	1	–25	–30	0	0	0	
S_1	0	20	30	1	0	6905	$\frac{6905}{30} = 230.16$
$\leftarrow S_2$	0	1	4	0	1	120	$\frac{120}{4} = 30$

Basic variables : $S_1 = 6905$, $S_2 = 120$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_2, pivot row: R_3, Pivot element: 4

Applying $R_1 \to R_1 + \frac{15}{2}R_3$, $R_2 \to R_2 - \frac{15}{2}R_3$

B	Z	$x_1 \downarrow$	x_2	S_1	S_2	b	Ratio
	1	–35/2	0	0	15/2	900	
S_1	0	25/2	0	1	–15/2	6005	$\frac{6005}{25/2} = 480.4$
$\leftarrow x_2$	0	1	4	0	1	120	$\frac{120}{4} = 30$

Basic variables : $S_1 = 6005$, $4x_2 = 1200$, i.e., $x_2 = 30$

Non-basic variables: $x_1 = S_2 = 0$

For these values, $Z = 900$

Pivot column: Column of x_1, Pivot row: R_3, Pivot element: 1

Applying $R_1 \to R_1 + \frac{35}{2}R_3$, $R_2 \to R_2 - \frac{25}{2}R_3$

B	Z	x_1	x_2	S_1	S_2	b
	1	0	35/2	0	25	3000
S_1	0	0	–25/2	1	–20	4505
x_1	0	1	4	0	1	120

Since there are no negative numbers in first row, so simplex process is completed.

From the last table,

$x_1 = 120$, $S_1 = 4505$, $x_2 = S_2 = 0$

$\therefore$ Max $Z = 3000$ at (120, 0).

   b. **Maximize $Z = 3x_1 + x_2$ subject to the constraints**

   $2x_1 + x_2 \le 6$ $\quad x_1 + 3x_2 \le 9$ $\quad x_1, x_2 \ge 0$

Solution

Introducing S_1 and S_2 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 3x_1 - x_2 + 0S_1 + 0S_2 = 0$$
$$2x_1 + x_2 + S_1 + 0S_2 = 6$$
$$x_1 + 3x_2 + 0S_1 + S_2 = 9$$

The first simplex table of above system is

B	Z	$x_1 \downarrow$	x_2	S_1	S_2	b	Ratio
	1	–3	–1	0	0	0	
$\leftarrow S_1$	0	2	1	1	0	6	6/2 = 3
S_2	0	1	3	0	1	9	9/1 = 9

Basic variables : $S_1 = 6$, $S_2 = 9$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_2, pivot element: 2

Applying $R_1 \to R_1 + \frac{3}{2}R_2$, $R_3 \to R_3 - \frac{1}{2}R_2$

B	Z	x_1	$x_2 \downarrow$	S_1	S_2	b
	1	0	1/2	3/2	0	9
x_1	0	2	1	1	0	6
S_2	0	0	5/2	–1/2	1	6

Since there are no negative numbers in front row so, simplex process is completed.

From the last table,

$2x_1 = 5$, $S_2 = 6$, $x_2 = 0$, $S_1 = 0$

i.e., $x_1 = 3$, $x_2 = 0$

$\therefore$ Max $Z = 9$ at (3, 0)

Note: Exercise 3.1 Question No. 1(c) has been moved to Exercise 3.2 Q.No. 3(f)

c. **Maximize $Z = 2x_1 + 2x_2$ subject to the constraints**

$x_1 + 2x_2 \le 20 \qquad 2x_1 + x_2 \le 20 \qquad x_1 \ge 0 \qquad x_2 \ge 0$

Solution

Introducing S_1 and S_2 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 2x_1 - 2x_2 + 0S_1 + 0S_2 = 0$$
$$x_1 + 2x_2 + S_1 + 0S_2 = 20$$
$$2x_1 + x_2 + 0S_1 + S_2 = 20$$

The first simplex table of above system is

B	Z	$x_1 \downarrow$	x_2	S_1	S_2	b	Ratio
	1	−2	−2	0	0	0	
S_1	0	1	2	1	0	20	20/1 = 20
$\leftarrow S_2$	0	2	1	0	1	20	20/2 = 10 min

Basic variables : $S_1 = 20$, $S_2 = 20$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 2

Applying $R_1 \to R_1 + R_3$, $R_2 \to R_2 - \frac{1}{2}R_3$

B	Z	x_1	$x_2 \downarrow$	S_1	S_2	b	Ratio
	1	0	−1	0	1	20	
$\leftarrow S_1$	0	0	3/2	1	−1/2	10	$\frac{10}{3/2} = 6.66$
x_1	0	2	1	0	1	20	20/1 = 20

Basic variables : $S_1 = 10$, $2x_1 = 20$, i.e., $x_1 = 10$

Non-basic variables: $x_2 = S_2 = 0$

For these values, $Z = 20$

Pivot column: column of x_2, pivot row: R_2, Pivot element: 3/2

Applying $R_1 \to R_1 + \frac{2}{3}R_2$, $R_3 \to R_3 - \frac{2}{3}R_2$

B	Z	x_1	x_2	S_1	S_2	b
	1	0	0	2/3	2/3	80/3
x_2	0	0	3/2	1	−1/2	10
x_1	0	2	0	−2/3	4/3	40/3

Since there are no negative number in first row, so simplex process is completed.

From the last table,

$$2x_1 = \frac{40}{3} \text{ and } \frac{3}{2}x_2 = 10$$

i.e. $x_1 = \frac{20}{3}$ and $x_2 = \frac{20}{3}$

$\therefore$ Maximize $Z = \frac{80}{3}$ at $\left(\frac{20}{3}, \frac{20}{3}\right)$

d. **Maximize $Z = 30x_1 + 20x_2$ subject to the constraints**

$-x_1 + x_2 \le 5 \qquad 2x_1 + x_2 \le 10 \qquad x_1, x_2 \ge 0$

Solution

Introducing S_1 and S_2 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 30x_1 - 20x_2 + 0S_1 + 0S_2 = 0$$
$$-x_1 + x_2 + S_1 + 0S_2 = 5$$
$$2x_1 + x_2 + 0S_1 + S_2 = 10$$

The first simplex table of above system is

B	Z	$x_1 \downarrow$	x_2	S_1	S_2	b	Ratio
	1	−30	−20	0	0	0	
S_1	0	−1	1	1	0	5	–
$\leftarrow S_2$	0	2	1	0	1	10	10/2 = 5 min

Basic variables : $S_1 = 5$, $S_2 = 10$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 2

Applying $R_1 \to R_1 + 15R_3$, $R_2 \to R_2 \to \frac{1}{2}R_3$

B	Z	x_1	$x_2 \downarrow$	S_1	S_2	b	Ratio
	1	0	−5	0	15	150	
$\leftarrow S_1$	0	0	3/2	1	1/2	10	$\frac{10}{3/2} = 6.66$ min
x_1	0	2	1	0	1	10	10/1 = 10

Basic variables : $S_1 = 10$, $2x_1 = 10$ or $x_1 = 5$

Non-basic variables: $x_2 = S_2 = 0$

For these values, $Z = 150$

Pivot column: column of x_2, pivot row: R_2, Pivot element: $\frac{3}{2}$

Applying $R_1 \to R_1 + \frac{10}{3}R_2$, $R_3 \to R_3 - \frac{2}{3}R_2$

B	Z	x_1	x_2	S_1	S_2	b
	1	0	0	10/3	50/3	550/3
x_2	0	0	3/2	1	1/2	10
x_1	0	2	0	−2/3	2/3	10/3

Since there are no negative numbers in first row, so simplex process is completed.

From the first table,

$$2x_1 = \frac{10}{3} \text{ and } \frac{3}{2}x_2 = 10$$

i.e. $x_1 = \frac{5}{3}$ and $x_3 = \frac{20}{3}$

$\therefore$ Max $Z = \frac{550}{3}$ at $\left(\frac{5}{3}, \frac{20}{3}\right)$.

e. **Minimize $Z = 5x_1 - 20x_2$ subject to the constraints**

$-2x_1 + 10x_2 \le 5 \quad 2x_1 + 5x_2 \le 10 \quad x_1, x_2 \ge 0$

Solution

Let $Z = \bar{Z} = -Z$, then objective function becomes Max $\bar{Z} = -Z = = -5x_1 + 20x_2$ under thee given constraints.

Introducing S_1 and S_2 as slack variables, we can write above LPP in standard form as system of linear equation as

$$\bar{Z} + 5x_1 - 20x_2 + 0S_1 + 0S_2 = 0$$
$$-2x_1 + 10x_2 + S_1 + 0S_2 = 5$$
$$2x_2 + 5x_2 + 0S_1 + S_2 = 10$$

The first simplex table of above system is

B	$\bar{Z}$	x_1	$x_2\downarrow$	S_1	S_2	b	Ratio
	1	5	–20	0	0	0	
$\leftarrow S_1$	0	–2	10	1	0	5	5/10 = 0.5
S_2	0	2	5	0	1	10	10/5 = 2

Basic variables : $S_1 = 5$, $S_2 = 10$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $\bar{Z} = 0$

Pivot column: column of x_2, pivot row: R_2, Pivot element: 10

Applying $R_1 \to R_1 + 2R_2$, $R_3 \to R_3 - \frac{1}{2}R_2$

Now, second simplex table is

B	$\bar{Z}$	x_1	x_2	S_1	S_2	b
	1	1	0	2	0	10
x_2	0	–2	10	1	0	5
S_2	0	3	0	–1/2	1	15/2

Since there are no negative numbers in first row, so simplex process is completed.

From the last table

$x_1 = 0$ and $10x_2 = 5$, i.e., $x_2 = \frac{1}{2}$

By minimum-maximum theorem

$\therefore$ Min $Z = -$ Max $\bar{Z} = -10$ at $\left(0, \frac{1}{2}\right)$

f. **Maximize $Z = 300x_1 - 500x_2$ subject to the constraints**

$2x_1 + 8x_2 \le 60 \quad 2x_1 + x_2 \le 30 \quad 4x_1 + 4x_2 \le 60 \quad x_1, x_2 \ge 0$

Solution

Introducing S_1, S_2 and S_3 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 300x_1 + 500x_2 + 0S_1 + 0S_2 + 0S_3 = 0$$
$$2x_1 + 8x_2 + S_1 + 0S_2 + 0S_3 = 60$$
$$2x_1 + x_2 + 0S_1 + S_2 + 0S_3 = 30$$
$$4x_1 + 4x_2 + 0S_1 + 0S_2 + S_3 = 60$$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	S_1	S_2	S_3	b	Ratio
	1	–300	500	0	0	0	0	
S_1	0	2	8	1	0	0	60	60/2 = 30
S_2	0	2	1	0	1	0	30	30/2 = 15
$\leftarrow S_3$	0	4	4	0	0	1	60	60/4 = 15

Basic variables : $S_1 = 60$, $S_2 = 30$, $S_3 = 60$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_4, Pivot element: 4

Applying $R_1 \to R_1 + \frac{150}{2}R_4$, $R_2 \to R_2 - \frac{1}{2}R_4$, $R_3 \to R_3 - \frac{1}{2}R_4$

B	Z	x_1	x_2	S_1	S_2	S_3	b
	1	0	800	0	0	150/2	4500
S_1	0	0	6	1	0	–1/2	898/15
S_2	0	0	–1	0	1	–1/2	445/15
x_1	0	4	4	0	0	1	60

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table, $4x_1 = 60$ i.e., $x_1 = 15$ and $x_2 = 0$.

$\therefore$ Max. $Z = 4500$ at $(15,0)$.

g. **Maximize $Z = 90x_1 + 50x_2$ subject to the constraints**

$x_1 + 3x_2 \le 18 \quad x_1 + x_2 \le 10 \quad 3x_1 + x_2 \le 24 \quad x_1, x_2, x_3 \ge 0$

Solution

Introducing S_1, S_2 and S_3 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 90x_1 - 50x_2 + 0S_1 + 0S_2 + 0S_3 = 0$$
$$x_1 + 3x_2 + S_1 + 0S_2 + 0S_3 = 18$$
$$x_1 + x_2 + 0S_1 + S_2 + 0S_3 = 10$$
$$3x_1 + x_2 + 0S_1 + 0S_2 + S_3 = 24$$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	S_1	S_2	S_3	b	Ratio
	1	–90	–50	0	0	0	0	
S_1	0	1	3	1	0	0	18	18/1 = 18
S_2	0	1	1	0	1	0	10	10/1 = 10
$\leftarrow S_3$	0	3	1	0	0	1	24	24/3 = 8

Basic variables : $S_1 = 18$, $S_2 = 10$, $S_3 = 24$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_4, Pivot element: 3

Applying $R_1 \to R_1 + 30R_4$, $R_2 \to R_2 - \frac{1}{3}R_4$, $R_3 \to R_3 - \frac{1}{3}R_4$

The second simplex table of above system is

B	Z	x_1	$x_2\downarrow$	S_1	S_2	S_3	b	Ratio
	1	0	−20	0	0	30	720	
S_1	0	0	8/3	1	0	−1/3	10	$\frac{10}{8/3} = 3.75$
$\leftarrow S_2$	0	0	2/3	0	0	−1/3	2	$\frac{2}{2/3} = 3$
x_1	0	3	1	0	0	1	24	$\frac{24}{3} = 8$

Basic variables : $S_1 = 10$, $S_2 = 2$, $3x_1 = 24$, i.e., $x_1 = 8$

Non-basic variables: $x_2 = S_3 = 0$

For these values, $Z = 720$

Pivot column: column of x_2, pivot row: R_3, Pivot element: $\frac{2}{3}$

Applying $R_1 \to R_1 + 30R_3$, $R_2 \to R_2 - 4R_3$, $R_4 \to R_4 - \frac{3}{2}R_3$

The third simplex table of above system is

B	Z	x_1	x_2	S_1	S_2	S_3	b
	1	0	0	0	30	20	780
S_1	0	0	0	1	−4	1	2
x_2	0	0	2/3	0	1	−1/3	2
x_1	0	3	0	1	−3/2	3/2	21

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table,

$3x_1 = 21$ and $\frac{2}{3}x_2 = 2$

i.e., $x_1 = 7$ and $x_2 = 3$

$\therefore$ Max $Z = 780$ at $(7, 3)$.

h. **Maximize $Z = 5x_1 + 3x_2$ subject to the constraints**

$4x_1 + 2x_2 \le 10$ $\quad 2x_1 + 2x_2 \le 8$ $\quad x_1, x_2 \ge 0$

Solution

Introducing S_1 and S_2 as slack variables, we can write above LPP in standard form system of linear equation as

$$Z - 5x_1 - 3x_2 + 0S_1 + 0S_2 = 0$$
$$4x_1 + 2x_2 + S_1 + 0S_2 = 10$$
$$2x_1 + 2x_2 + 0S_1 + S_2 = 8$$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	S_1	S_2	b	Ratio
	1	−5	−3	0	0	0	
$\leftarrow S_1$	0	4	2	1	0	10	10/4 = 2.5
S_2	0	2	2	0	1	8	8/2 = 4

Basic variables : $S_1 = 10$, $S_2 = 8$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_2, Pivot element: 4

Applying $R_1 \to R_1 + \frac{5}{4}R_2$, $R_3 \to R_3 - \frac{1}{2}R_2$

The second simplex table of above system is

B	Z	x_1	$x_2\downarrow$	S_1	S_2	b	Ratio
	1	0	−1/2	5/4	0	25/2	
x_1	0	4	2	1	0	10	10/2 = 5
$\leftarrow S_2$	0	0	1	−1/2	1	3	3/1 = 3

Basic variables : $4x_1 = 10$, i.e., $x_1 = \frac{5}{2}$, $S_2 = 3$

Non-basic variables: $x_2 = S_2 = 0$

For these values, $Z = \frac{25}{2}$

Pivot column: column of x_2, pivot row: R_3, Pivot element: 1

Applying $R_1 \to R_1 + \frac{1}{2}R_3$, $R_2 \to R_2 - 2R_3$

The second simplex table of above system is

B	Z	x_1	x_2	S_1	S_2	b
	1	0	0	1/2	1/2	14
x_1	0	4	0	−2	−2	4
x_2	0	0	1	−1/2	1	3

Since there are no negative numbers in first row, so simplex process is completed.

From the last table,

$4x_1 = 4$ and $x_2 = 3$

i.e., $x_1 = 1$ and $x_2 = 3$

$\therefore$ Max $Z = 14$ at $(1, 3)$.

2. **Solve the following LPP by using simplex for three decision variables.**

a. **Maximize $Z = 4x_1 + x_2 + 2x_3$ subject to the constraints**

$x_1 + x_2 + x_3 \le 1$ $\quad x_1 + x_2 - x_3 \le 0$ $\quad x_1, x_2, x_3 \ge 0$

Solution

Introducing S_1 and S_2 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 4x_1 - x_2 - 2x_3 + 0S_1 + 0S_2 = 0$$
$$x_1 + x_2 + x_3 + S_1 + 0S_2 = 1$$
$$x_1 + x_2 - x_3 + 0S_1 + S_2 = 0$$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	x_3	S_1	S_2	b	Ratio
	1	−4	−1	−2	0	0	0	
S_1	0	1	1	1	1	0	1	1/1 = 1
$\leftarrow S_2$	0	1	1	−1	0	1	0	−1/0 = 0

Basic variables : $S_1 = 1$, $S_2 = 0$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 1

Applying $R_1 \to R_1 + 4R_3$, $R_2 \to R_2 - R_3$

The first simplex table of above system is

B	Z	x_1	x_2	$x_3\downarrow$	S_1	S_2	b	Ratio
	1	0	3	–6	0	4	0	
$\leftarrow S_1$	0	0	0	2	1	–1	1	1/2 = 0.5
x_1	0	1	1	–1	0	1	0	–

Basic variables : $S_1 = 1, x_1 = 0$

Non-basic variables: $x_1 = S_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_3, pivot row: R_2, Pivot element: 2

Applying $R_1 \to R_1 + 3R_2,\ R_3 \to R_3 + \frac{1}{2}R_2$

The second simplex table of above system is

B	Z	x_1	x_2	x_3	S_1	S_2	b
	1	0	3	0	3	1	3
x_3	0	0	0	2	1	–1	1
x_1	0	1	1	0	1/2	1/2	1/2

Since there are no negative numbers in first row, so simplex process is completed.

From the last table, $x_1 = \frac{1}{2},\ x_2 = 0,\ 2x_3 = 1$

i.e., $x_1 = \frac{1}{2},\ x_2 = 0,\ x_3 = \frac{1}{2}$

$\therefore$ Max $Z = 3$ at $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$.

b. Maximize $Z = 4x_1 - 10x_2 - 20x_3$ subject to the constraints

$3x_1 + 4x_2 + 5x_3 \le 60$ $\quad$ $2x_1 + x_2 \le 20$

$2x_1 + 3x_3 \le 30$ $\quad$ $x_1, x_2 \ge 0$

Solution

Introducing S_1, S_2 and S_3 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 4x_1 + 10x_2 + 20x_3 + 0S_1 + 0S_2 + 0S_3 = 0$$
$$3x_1 + 4x_2 + 5S_3 + S_1 + 0S_2 + 0S_3 = 60$$
$$2x_1 + x_2 + 0x_3 + 0S_1 + S_2 + 0S_3 = 20$$
$$2x_1 + 0x_2 + 3x_3 + 0S_1 + 0S_2 + S_3 = 30$$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	x_3	S_1	S_2	S_3	b	Ratio
	1	–4	10	20	0	0	0	0	
S_1	0	3	4	5	1	0	0	60	60/3 = 20
$\leftarrow S_2$	0	2	1	0	0	1	0	20	20/2 = 10
S_3	0	2	0	3	0	0	1	30	30/2 = 15

Basic variables : $S_1 = 60, S_2 = 20, S_3 = 30$

Non-basic variables: $x_1 = x_2 = x_3 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 2

Applying $R_1 \to R_1 + 2R_3,\ R_2 \to R_2 - \frac{3}{2}R_3,\ R_4 \to R_4 - R_3$

The second simplex table of above system is

B	Z	x_1	x_2	x_3	S_1	S_2	S_3	b
	1	0	12	20	0	2	0	40
S_1	0	0	5/2	5	1	–3/2	0	30
x_1	0	2	1	0	0	1	0	20
S_3	0	0	–1	3	0	–1	1	10

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table.

$2x_1 = 20,\ x_2 = 0,\ x_3 = 0$ i.e., $x_1 = 10,\ x_2 = 0,\ x_3 = 0$

$\therefore$ Max $Z = 40$ at $(10, 0, 0)$.

c. Maximize $Z = 3x_1 + 2x_2 + x_3$ subject to the constraints

$4x_1 + x_2 + x_3 \le 30$ $\quad$ $2x_1 + 3x_2 + x_3 \le 60$

$x_1 + 2x_2 + 3x_3 \le 40$ $\quad$ $x_1, x_2, x_3 \ge 0$

Solution

Introducing S_1, S_2 and S_3 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 3x_1 - 2x_2 - x_3 + 0S_1 + 0S_2 + 0S_3 = 0$$
$$4x_1 + x_2 + x_3 + S_1 = 30$$
$$2x_1 + 3x_2 + x_3 + S_2 = 60$$
$$x_1 + 2x_2 + 3x_3 + S_3 = 40$$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	x_3	S_1	S_2	S_3	b	Ratio
	1	–3	–2	–1	0	0	0	0	
$\leftarrow S_1$	0	4	1	1	1	0	0	30	30/4 = 7.5
S_2	0	2	3	1	0	1	0	60	60/2 = 30
S_3	0	1	2	3	0	0	1	40	40/1 = 40

Basic variables : $S_1 = 30, S_2 = 60, S_3 = 40$

Non-basic variables: $x_1 = x_2 = x_3 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_2, Pivot element: 4

Applying $R_1 \to R_1 + \frac{3}{4}R_2,\ R_3 \to R_3 - \frac{1}{2}R_2,\ R_4 \to R_4 - \frac{1}{4}R_2$

The second simplex table of above system is

B	Z	x_1	$x_2\downarrow$	x_3	S_1	S_2	S_3	b	Ratio
	1	0	–5/4	–1/4	3/4	0	0	45/2	
x_1	0	4	1	1	1	0	0	30	30/1 = 30
$\leftarrow S_2$	0	0	5/2	1/2	–1/2	1	0	45	$\frac{45}{5/2} = 18$
S_3	0	0	7/4	11/4	–1/4	0	1	65/2	$\frac{65/2}{7/4} = 18.57$

Basic variables : $4x_1 = 30$, i.e., $x_1 = 15/2$, $S_2 = 45$ and $S_3 = \frac{65}{2}$

Non-basic variables: $x_1 = x_2 = x_3 = 0$

For these values, $Z = \frac{45}{2}$

Pivot column: column of x_2, pivot row: R_3, Pivot element: $\frac{5}{2}$

Applying $R_1 \to R_1 + \frac{1}{2}R_3$, $R_2 \to R_2 - \frac{2}{5}R_3$, $R_4 \to R_4 - \frac{7}{10}R_3$

The third simplex table of above system is

B	Z	x_1	x_2	x_3	S_1	S_2	S_3	[illegible]
	1	0	0	0	1/2	1/2	0	[illegible]
x_1	0	4	0	4/5	12/10	−2/5	0	[illegible]
x_2	0	0	5/2	1/2	−1/2	1	0	[illegible]
S_3	0	0	0	14/20	1/10	−7/10	1	[illegible]

Since, there are no negative numbers in first row, so simplex process is complete.
From the last table,

$4x_1 = 12, \frac{5}{2}x_2 = 45, x_3 = 0$

i.e. $x_1 = 31, x_2 = 18, x_3 = 0$

$\therefore$ Max $Z = 45$ at $(3, 18, 0)$.

d. **Maximize $Z = 3x_1 + 4x_2 + x_3$ subject to the constraints**

$x_1 + 2x_2 + 3x_3 \le 90$ $\quad$ $3x_1 + x_2 + 2x_3 \le 80$

$2x_1 + x_2 + x_3 \le 60$ $\quad$ $x_1, x_2, x_3 \ge 0$

Solution

Introducing S_1, S_2 and S_3 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 3x_1 - 4x_2 - x_3 + 0S_1 + 0S_2 + 0S_3 = 0$$
$$x_1 + 2x_2 + 3x_3 + S_1 + 0S_2 + 0s_3 = 90$$
$$3x_1 + x_2 + 2x_3 + 0S_1 + S_2 + 0S_3 = 80$$
$$2x_1 + x_2 + x_3 + 0S_1 + 0S_2 + S_3 = 60$$

The first simplex table of above system is

B	Z	x_1	$x_2 \downarrow$	x_3	S_1	S_2	S_3	b	Ratio
	1	−3	−4	−1	0	0	0	0	
$\leftarrow S_1$	0	1	2	3	1	0	0	90	90/2 = 4
S_2	0	3	1	2	0	1	0	80	80/1 = 8
S_3	0	2	1	1	0	0	1	60	60/1 = 6

Basic variables : $S_1 = 90, S_2 = 80, S_3 = 60$

Non-basic variables: $x_1 = x_2 = x_3 = 0$

For these values, $Z = 0$

Pivot column: column of x_2, pivot row: R_2, Pivot element: 2

Applying $R_1 \to R_1 + 2R_2$, $R_3 \to R_3 - \frac{1}{2}R_2$, $R_4 \to R_4 - \frac{1}{2}R_2$

The second simplex table of above system is

B	Z	$x_1 \downarrow$	x_2	x_3	S_1	S_2	S_3	b	Ratio
	1	−1	0	7	2	0	0	180	
x_2	0	1	2	3	1	0	0	90	90/1 = 9
S_2	0	5/2	0	1/2	−1/2	1	0	35	$\frac{35}{5/2}$ = 1
$\leftarrow S_3$	0	3/2	0	−1/2	−1/2	0	1	15	$\frac{15}{3/2}$ = 1

Basic variables : $x_2 = 45, S_2 = 35, S_3 = 15$

Non-basic variables: $x_1 = x_2 = x_3 = 0$

For these values, $Z = 180$

Pivot column: column of x_1, pivot row: R_4, Pivot element: $\frac{3}{2}$

Applying $R_1 \to R_1 + \frac{2}{3}R_4$, $R_2 \to R_2 - \frac{2}{3}R_4$, $R_3 \to R_3 - \frac{5}{3}R_4$

The third simplex table of above system is

B	Z	x_1	x_2	x_3	S_1	S_2	S_3	b
	1	0	0	20/3	5/3	0	2/3	190
x_2	0	0	2	10/3	4/3	0	−2/3	80
S_2	0	0	0	4/3	1/3	1	−5/3	10
x_1	0	3/2	0	−1/2	−1/2	0	1	15

Since, there are no negative numbers in first row, so simplex process is completed.
From the last table,

$\frac{3}{2}x_1 = 15, 2x_2 = 80, x_3 = 0$

i.e. $x_1 = 10, x_2 = 40, x_3 = 0$

$\therefore$ Max $Z = 190$ at $(10, 40, 0)$.

e. **Max $Z = 3x_1' + 5x_2 + 4x_3$ subject to the constraint**

$2x_1 + 3x_3 \le 8$ $\quad$ $5x_1 + 2x_2 + 2x_3 \le 10$

$5x_2 + 4x_3 \le 15$ $\quad$ $x_1, x_2, x_3 \ge 0$

Solution

Introducing S_1, S_2 and S_3 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z - 3x_1 - 5x_2 - 4x_3 + 0S_1 + 0S_2 + 0S_3 = 0$$
$$2x_1 + 0x_2 + 3x_3 + S_1 + 0S_2 + 0S_3 = 8$$
$$5x_1 + 2x_2 + 2x_3 + 0S_1 + S_2 + 0S_3 = 10$$
$$0x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + S_3 = 15$$

The first simplex table of above system is

B	Z	x_1	$x_2 \downarrow$	x_3	S_1	S_2	S_3	b	Ratio
	1	−3	−5	−4	0	0	0	0	
S_1	0	2	0	3	1	0	0	8	-
S_2	0	5	2	2	0	1	0	10	10/2 = 5
$\leftarrow S_3$	0	0	5	4	0	0	1	15	15/5 = 3

Basic variables : $S_1 = 8, S_2 = 10, S_3 = 15$

Non-basic variables: $x_1 = x_2 = x_3 = 0$

For these values, $Z = 0$

Pivot column: column of x_2, pivot row: R_4, Pivot element: 5

Applying $R_1 \to R_1 + R_4$, $R_3 \to R_3 - \frac{2}{5}R_4$

The second simplex table of above system is

B	Z	$x_1\downarrow$	x_2	x_3	S_1	S_2	S_3	b	Ratio
	1	–3	0	0	0	0	1	15	
S_1	0	2	0	3	1	0	0	8	8/2 = 4
$\leftarrow S_2$	0	5	0	2/5	0	1	–2/5	4	4/5 = 0.8
x_2	0	0	5	4	0	0	1	15	–

Basic variables : $S_1 = 8$, $S_2 = 4$, $5x_2 = 15$, i.e., $x_2 = 3$

Non-basic variables: $x_1 = x_3 = S_3 = 0$

For these values, $Z = 15$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 5

Applying $R_1 \rightarrow R_1 + \frac{3}{5}R_3$, $R_2 \rightarrow R_2 - \frac{2}{5}R_3$

The third simplex table of above system is

B	Z	x_1	x_2	x_3	S_1	S_2	S_3	b
	1	0	0	6/25	0	3/5	19/25	87/
S_1	0	0	0	71/25	1	–2/5	4/25	32/
x_1	0	5	0	2/5	0	1	–2/5	4
x_2	0	0	5	4	0	0	1	15

Since, there are no negative numbers in first row, so simplex process is complete.

From the last table,

$5x_1 = 4$, $5x_2 = 15$, $x_3 = 0$

i.e. $x_1 = \frac{4}{5}$, $x_2 = 3$, $x_3 = 0$

$\therefore$ Max $Z = \frac{87}{5}$ at $\left(\frac{4}{5}, 3, 0\right)$.

Note: Exercise 3.1 Question No. 3(a) has been moved to Exercise 3.2 Q.No. 4

3. A company manufactures two products A and B. Both products processed on two machines m_1 and m_2.

	M_1	M_2
A	**6 hrs/units**	**2 hrs/units**
B	**4 hrs/units**	**4 hrs/units**
Availability	**7200 hrs/month**	**4000 hrs/month**

The profit per unit for A is Rs 100 and for is Rs 80. Find out the mo production of A and B to maximize the profit by simplex method.

Solution

Let monthly production of product A and B be x_1 and x_2 respectively.

So, from the given data

Maximize $Z = 100x_1 + 80x_2$

Subject to the constraints $6x_1 + 4x_2 \le 7200$, $2x_1 + 4x_2 \le 4000$, with $x_1, x_2 \ge 0$

Introducing S_1 and S_2 as slack variables, the given LPP in standard form c written as system of linear equations as

$Z - 100x_1 - 80x_2 + 0S_1 + 0S_2 = 0$

$6x_1 + 4x_2 + S_1 + 0S_2 = 7200$

$2x_1 + 4x_2 + 0S_1 + S_2 = 4000$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	S_1	S_2	b	Ratio
	1	–100	–80	0	0	0	
$\leftarrow S_1$	0	6	4	1	0	7200	$\frac{7200}{6} = 1200$
S_2	0	2	4	0	1	4000	$\frac{4000}{2} = 2000$

Basic variables : $S_1 = 7200$, $S_2 = 4000$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_2, Pivot element: 6

Applying $R_1 \rightarrow R_1 + \frac{50}{3}R_2$, $R_3 \rightarrow R_3 - \frac{1}{3}R_2$

The next simplex table is

B	Z	x_1	$x_2\downarrow$	S_1	S_2	b	Ratio
	1	0	–40/3	50/3	0	120000	
x_1	0	6	4	1	0	7200	
$\leftarrow S_2$	0	0	8/3	–1/3	1	1600	

Basic variables : $6x_1 = 7200$, i.e., $x_1 = 1200$, $S_2 = 1600$

Non-basic variables: $x_2 = S_1 = 0$

For these values, $Z = 120000$

Pivot column: column of x_2, pivot row: R_3, Pivot element: $\frac{8}{3}$

Applying $R_1 \rightarrow R_1 + 5R_3$, $R_2 \rightarrow R_2 - \frac{3}{2}R_3$

The next simplex table is

B	Z	x_1	x_2	S_1	S_2	b
	1	0	0	15	5	128000
x_1	0	6	0	3/2	–3/2	4800
x_2	0	0	8/3	–1/3	1	1600

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table,

$6x_1 = 4800$, $\frac{8}{3}x_2 - 1600$

i.e., $x_1 = 800$, $3x_2 = 600$

$\therefore$ Maximum $Z = 128000$ at $(800, 600)$

Hence, the monthly production of A and B must be 800 units and 600 units to maximize the profit.

The second simplex table of above system is

B	Z	$x_1\downarrow$	x_2	x_3	S_1	S_2	S_3	b	Ratio
	1	–3	0	0	0	0	1	15	
S_1	0	2	0	3	1	0	0	8	8/2 = 4
$\leftarrow S_2$	0	5	0	2/5	0	1	–2/5	4	4/5 = 0.8
x_2	0	0	5	4	0	0	1	15	–

Basic variables : $S_1 = 8$, $S_2 = 4$, $5x_2 = 15$, i.e., $x_2 = 3$

Non-basic variables: $x_1 = x_3 = S_3 = 0$

For these values, $Z = 15$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 5

Applying $R_1 \to R_1 + \frac{3}{5}R_3$, $R_2 \to R_2 - \frac{2}{5}R_3$

The third simplex table of above system is

B	Z	x_1	x_2	x_3	S_1	S_2	S_3	b
	1	0	0	6/25	0	3/5	19/25	87/
S_1	0	0	0	71/25	1	–2/5	4/25	32/
x_1	0	5	0	2/5	0	1	–2/5	4
x_2	0	0	5	4	0	0	1	15

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table,

$5x_1 = 4$, $5x_2 = 15$, $x_3 = 0$

i.e. $x_1 = \frac{4}{5}$, $x_2 = 3$, $x_3 = 0$

$\therefore$ Max $Z = \frac{87}{5}$ at $\left(\frac{4}{5}, 3, 0\right)$.

Note: Exercise 3.1 Question No. 3(a) has been moved to Exercise 3.2 Q.No. 4

3. A company manufactures two products A and B. Both products are processed on two machines m_1 and m_2.

	M_1	M_2
A	**6 hrs/units**	**2 hrs/units**
B	**4 hrs/units**	**4 hrs/units**
Availability	**7200 hrs/month**	**4000 hrs/month**

The profit per unit for A is Rs 100 and for is Rs 80. Find out the most production of A and B to maximize the profit by simplex method.

Solution

Let monthly production of product A and B be x_1 and x_2 respectively.

So, from the given data

$$\text{Maximize } Z = 100x_1 + 80x_2$$

Subject to the constraints $6x_1 + 4x_2 \le 7200$, $2x_1 + 4x_2 \le 4000$, with $x_1, x_2 \ge 0$

Introducing S_1 and S_2 as slack variables, the given LPP in standard form can be written as system of linear equations as

$$Z - 100x_1 - 80x_2 + 0S_1 + 0S_2 = 0$$
$$6x_1 + 4x_2 + S_1 + 0S_2 = 7200$$
$$2x_1 + 4x_2 + 0S_1 + S_2 = 4000$$

The first simplex table of above system is

B	Z	$x_1\downarrow$	x_2	S_1	S_2	b	Ratio
	1	–100	–80	0	0	0	
$\leftarrow S_1$	0	6	4	1	0	7200	$\frac{7200}{6} = 1200$
S_2	0	2	4	0	1	4000	$\frac{4000}{2} = 2000$

Basic variables : $S_1 = 7200$, $S_2 = 4000$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = 0$

Pivot column: column of x_1, pivot row: R_2, Pivot element: 6

Applying $R_1 \to R_1 + \frac{50}{3}R_2$, $R_3 \to R_3 - \frac{1}{3}R_2$

The next simplex table is

B	Z	x_1	$x_2\downarrow$	S_1	S_2	b	Ratio
	1	0	–40/3	50/3	0	120000	
x_1	0	6	4	1	0	7200	
$\leftarrow S_2$	0	0	8/3	–1/3	1	1600	

Basic variables : $6x_1 = 7200$, i.e., $x_1 = 1200$, $S_2 = 1600$

Non-basic variables: $x_2 = S_1 = 0$

For these values, $Z = 120000$

Pivot column: column of x_2, pivot row: R_3, Pivot element: $\frac{8}{3}$

Applying $R_1 \to R_1 + 5R_3$, $R_2 \to R_2 - \frac{3}{2}R_3$

The next simplex table is

B	Z	x_1	x_2	S_1	S_2	b
	1	0	0	15	5	128000
x_1	0	6	0	3/2	–3/2	4800
x_2	0	0	8/3	–1/3	1	1600

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table,

$6x_1 = 4800$, $\frac{8}{3}x_2 - 1600$

i.e., $x_1 = 800$, $3x_2 = 600$

$\therefore$ Maximum $Z = 128000$ at $(800, 600)$

Hence, the monthly production of A and B must be 800 units and 600 units to maximize the profit.

Exercise 3.2

1. **Write the dual to the following problems.**

a. **Maximize $Z = x_1 - x_2 + 3x_3$ subject to the constraints**

$x_1 + x_2 + x_3 \le 10$ $\quad$ $2x_1 - x_2 - x_3 \le 2$

$2x_1 - 2x_2 - 3x_3 \le 6$ $\quad$ $x_1, x_2, x_3 \ge 0$

Solution

The objective function is

$$\text{Max } Z = x_1 - x_2 + 3x_3 = [1\ \ -1\ \ 3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = CX$$

Where, $C = [1\ \ -1\ \ 3]$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

and the given constraints are

$x_1 + x_2 + x_3 \le 10$

$2x_1 - x_2 - x_3 \le 2$

$2x_1 - 2x_2 - 3x_3 \le 6$

i.e. $\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 10 \\ 2 \\ 6 \end{bmatrix}$

or, $AX \le B$.

Where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & -2 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 2 \\ 6 \end{bmatrix}$

Then its dual is minimize $Z^* = B^T y$ such that $A^T y \ge C^T$

For $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\text{Minimize } Z^* = B^T y = [10\ \ 2\ \ 6]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 10y_1 + 2y_2 + 6y_3$$

Subject to constraints $A^T y \ge C^T$

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & -1 & -2 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ge \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

or, $y_1 + 2y_2 + 2y_3 \ge 1$

or, $y_1 - y_2 - 2y_3 \ge -1$

$y_1 - y_2 - 3y_3 \ge 3$

$\therefore$ The dual is minimize $Z^* = 10y_1 + 2y_2 + 6y_3$.

Subject to constraints $y_1 + 2y_2 + 2y_3$ 1, $y_1 - y_2 - 2y_3 \ge -1$, $y_1 - y_2 - 3y_3 \ge 3$,

$y_1, y_2, y_3 \ge 0$.

b. **Maximize $Z = 3x_1 + 4x_2 + x_3$ subject to the constraints**

$x_1 + 2x_2 + 3x_3 \le 90$ $\quad$ $2x_1 + x_2 + x_3 \le 60$

$3x_1 + x_2 + 2x_3 \le 80$ $\quad$ $x_1, x_2, x_3 \ge 0$

Solution

The objective function is

$$\text{Max } Z = 3x_1 + 4x_2 + x_3 = [3\ \ 4\ \ 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = CX$$

Where, $C = [3\ \ 4\ \ 1]$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

and the given constraints are

$x_1 + 2x_2 + 3x_3 \le 90$

$2x_1 + x_2 + x_3 \le 60$

$3x_1 + x_2 + 2x_3 \le 80$

or, $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \le \begin{bmatrix} 90 \\ 60 \\ 80 \end{bmatrix}$

or, $AX \le B$.

Where, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $B = \begin{bmatrix} 90 \\ 60 \\ 80 \end{bmatrix}$

Then its dual is minimize $Z^* = B^T y = [90\ \ 60\ \ 80]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 90y_1 + 60y_2 + 80y_3$

Subject to constraints $A^T y \ge C^T$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ge \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

or, $y_1 + 2y_2 + 3y_3 \ge 3$

$2y_1 + y_2 + y_3 \ge 4$

$3y_1 + y_2 + 2y_3 \ge 1$

$y_1, y_2, x_3 \ge 0$

$\therefore$ The dual is Min $Z^* = 90y_1 + 60y_2 + 80y_3$, subject to the constraints $y_1 + 2y_2 + 3y_3 \ge 3$, $2y_1 + y_2 + y_3 \ge 4$, $3y_1 + y_2 + 2y_3 \ge 1$ with $y_1, y_2, y_3 \ge 0$.

2. **The primal LPP are given below. Construct the dual problem and solve by using simplex method.**

a. **Minimize $Z = 8x_1 + 9x_2$ subject to**

$x_1 + x_2 \ge 5$ $\quad$ $3x_1 + x_2 \ge 21$ $\quad$ $x_1 \ge 0$ $\quad$ $x_2 \ge 0$

Solution

The objective function is

$$\text{Minimize } Z = 8x_1 + 9x_2 = [8\ \ 9]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = CX$$

Where, $C = [8\ \ 9]$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Subject to

$x_1 + x_2 \ge 5$

$3x_1 + x_2 \ge 21$

or, $\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 5 \\ 21 \end{bmatrix}$

or, $AX \geq B$,

Where, $A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 21 \end{bmatrix}$

Then its dual is maximize $Z^* = B^T y = [5 \ \ 21] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 5y_1 + 21y_2$

Subject to constraints $A^T y \leq C^T$

$$\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq \begin{bmatrix} 8 \\ 9 \end{bmatrix}$$

or, $y_1 + 3y_2 \leq 8$

$y_1 + y_2 \leq 9$ with $y_1, y_2 \geq 0$

∴ Required dual of given primal LPP is Max $Z^* = 5y_1 + 21y_2$

Subject to $y_1 + 3y_2 \leq 8$, $y_1 + y_2 \leq 9$, with $y_1, y_2 \geq 0$

Introducing x_1 and x_2 as slack variables, we can write above LPP in standard form system of linear equation as

$$Z^* - 5y_1 - 21y_2 + 0x_1 + 0x_2 = 0$$
$$y_1 + 3y_2 + x_1 + 0x_2 = 8$$
$$y_1 + y_2 + 0x_1 + x_2 = 9$$

The first simplex table of above system is

B	Z^*	y_1	$y_2\downarrow$	x_1	x_2	b	Ratio
	1	–5	–21	0	0	0	
$\leftarrow x_1$	0	1	3	1	0	8	8/3 = 2.66
x_2	0	1	1	0	1	9	9/1 = 9

Basic variables : $x_1 = 8$, $x_2 = 9$

Non-basic variables: $y_1 = y_2 = 0$

For these values, $Z^* = 0$

Pivot column: column of y_2, pivot row: R_2, Pivot element: 3

Applying $R_1 \to R_1 + 7R_2$, $R_3 \to R_3 - \frac{1}{3}R_2$

The second simplex table of above system is

B	Z^*	y_1	y_2	x_1	x_2	b
	1	2	0	7	0	56
y_2	0	1	3	1	0	8
x_2	0	2/3	0	–1/3	1	19/3

Since, there are no negative numbers in first row, so simplex process is completed

From the last table,

$y_1 = 0$ and $3y_2 = 8$, i.e., $y_2 = 8/3$

∴ Max $Z^* = 56$ at $\left(0, \frac{8}{3}\right)$

and $x_1 = 7$, $x_2 = 0$

∴ Minimum $Z = 56$ at (7, 0).

b. Minimize $Z = 3x_1 + 2x_2$ subject to constraints

$7x_1 + 2x_2 \geq 30 \quad 5x_1 + 4x_2 \geq 20 \quad 2x_1 + 8x_2 \geq 16 \quad x_1, x_2 \geq 0$

Solution

Here, the objective function is Min $Z = 3x_1 + 2x_2$

Subject to $7x_1 + 2x_2 \geq 30$, $5x_1 + 4x_2 \geq 20$, $2x_1 + 8x_2 \geq 16$, $x_1, x_2 \geq 0$

By the theorem of duality of the LPP, its dual problem is defined by

Max $Z^* = 30y_1 + 20y_2 + 16y_3$

Subject to $7y_1 + 5y_2 + 2y_3 \leq 3$, $2y_1 + 4y_2 + 8y_3 \leq 2$ with $y_1, y_2 \geq 0$

Introducing x_1 and x_2 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z^* - 30y_1 - 20y_2 - 16y_3 + 0x_1 + 0x_2 = 0$$
$$7y_1 + 5y_2 + 2y_3 + x_1 + 0x_2 = 3$$
$$2y_1 + 4y_2 + 8y_3 + 0x_1 + x_2 = 2$$

and $y_1, y_2, y_3, x_1, x_2 \ 0$

The first simplex table of above system is

B	Z	$y_1\downarrow$	y_2	y_3	x_1	x_2	b	Ratio
	1	–30	–20	–16	0	0	0	
$\leftarrow x_1$	0	7	5	2	1	0	3	3/7 = 0.42
x_2	0	2	4	8	0	1	2	2/2 = 1

Basic variables : $x_1 = 3$, $x_2 = 2$

Non-basic variables: $y_1 = y_2 = y_3 = 0$

For these values, $Z^* = 0$

Pivot column: column of y_1, pivot row: R_2, Pivot element: 7

Applying $R_1 \to R_1 + \frac{30}{7}R_2$, $R_3 \to R_3 - \frac{2}{7}R_2$

B	Z	y_1	y_2	$y_3\downarrow$	x_1	x_2	b	Ratio
	1	0	10/7	–52/7	30/7	0	90/7	
y_1	0	7	5	2	1	0	3	$\frac{3/7}{2/7} = 1.5$
$\leftarrow x_2$	0	0	18/7	52/7	–2/7	1	8/7	$\frac{8/7}{52/7} = 0.15$

Basic variables : $7y_1 = 3$, i.e., $y_1 = \frac{3}{7}$ and $x_2 = \frac{8}{7}$

Non-basic variables: $y_2 = y_3 = x_1 = 0$

For these values, $Z = \frac{90}{7}$

Pivot column: column of y_3, pivot row: R_3, Pivot element: $\frac{52}{7}$

Applying $R_1 \to R_1 + R_3$, $R_2 \to R_2 - \frac{7}{26}R_3$

B	Z^*	y_1	y_2	$y_3\downarrow$	x_1	x_2	b
	1	0	4	0	4	1	14
y_1	0	7	56/13	0	14/13	–7/26	35/13
y_3	0	0	18/7	52/7	–2/7	1	8/7

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table,

$7y_1 = \frac{35}{13}, y_2 = 0, \frac{52}{7} y_3 = \frac{8}{7}$

i.e., $y_1 = \frac{5}{13}, y_2 = 0, y_3 = \frac{2}{13}$

$\therefore$ Max $Z^* = 14$ at $\left(\frac{5}{13}, 0, \frac{2}{13}\right)$.

c. **Minimize $Z = 21x_1 + 50x_2$ subject to the constraints**

$2x_1 + 5x_2 \geq 12 \quad 3x_1 + 7x_2 \geq 17 \quad x_1, x_2 \geq 0$

Solution

Here, the given primal LPP is minimize $Z = 21x_1 + 50x_2$

Subject to constraints $2x_1 + 5x_2 \geq 12, \quad 3x_1 + 7x_2 \geq 17, \quad x_1, x_2 \geq 0$

By the theorem of duality of the LPP, its dual problem is defined by

Max $Z^* = 12y_1 + 17y_2$

Subject to constraints: $2y_1 + 3y_2 \leq 21, \quad 5y_1 + 7y_2 \leq 50, \quad y_1, y_2 \geq 0$

Introducing x_1 and x_2 as slack variables, we can write above LPP in standard form system of linear equation as

$$Z^* - 12y_1 - 17y_2 + 0x_1 + 0x_2 = 0$$
$$2y_1 + 3y_2 + x_1 + 0x_2 = 21$$
$$5y_1 + 7y_2 + 0x_1 + x_2 = 50$$

The first simplex table of above system is

B	Z^*	y_1	$y_2\downarrow$	x_1	x_2	b	Ratio
	1	–12	–17	0	0	0	
$\leftarrow x_1$	0	2	3	1	0	21	21/3 = 7
x_2	0	5	7	0	1	50	50/7 = 7.14

Basic variables : $x_1 = 21, x_2 = 50$

Non-basic variables: $y_1 = y_2 = 0$

For these values, $Z^* = 0$

Pivot column: column of y_2, pivot row: R_2, Pivot element: 3

Applying $R_1 \to R_1 + \frac{17}{3} R_2, R_3 \to R_3 - \frac{7}{3} R_2$

The second simplex table of above system is

B	Z^*	$y_1\downarrow$	y_2	x_1	x_2	b	Ratio
	1	–2/3	0	17/3	0	119	
y_2	0	2	3	1	0	21	21/2 = 10.5
$\leftarrow x_2$	0	1/3	0	–7/3	1	1	$\frac{1}{1/3} = 3$

Basic variables : $x_2 = 1, 3y_2 = 21$, i.e., $y_2 = 7$

Non-basic variables: $y_1 = x_1 = 0$

For these values, $Z^* = 119$

Pivot column: column of y_1, pivot row: R_3, Pivot element: $\frac{1}{3}$

Applying $R_1 \to R_1 + 2R_3, R_2 \to R_2 - 6R_3$

The third simplex table of above system is

B	Z^*	y_1	y_2	x_1	x_2	b
	1	0	0	1	2	121
y_2	0	0	3	15	–6	15
y_1	0	1/3	0	–7/3	1	1

Since there are no negative numbers in first row, so simplex process is completed.

From the last table

$\frac{1}{3} y_1 = 1, 3y_2 = 15$ and $x_1 = 1, x_2 = 2$

i.e. $y_1 = 3, y_2 = 5$

$\therefore$ Max $Z^* = 121$ at $y_1 = 3, y_2 = 5$ and min $Z = 121$ at $x_1 = 1, x_2 = 2$.

d. **Minimize $Z = 20x_1 + 30x_2$ subject to**

$x_1 + 4x_2 \geq 8 \quad x_1 + x_2 \geq 5 \quad 2x_1 + x_2 \geq 7 \quad x_1 \geq 0 \quad x_2 \geq 0$

Solution

By the theorem of duality of the LPP, its dual problem is defined by

Max $Z^* = 8y_1 + 5y_2 + 7y_3$

Subject to $y_1 + y_2 + 2y_3 \leq 20, \quad 4y_1 + y_2 + y_3 \leq 30$ with $y_1, y_2, y_3 \leq 0$

Introducing x_1 and x_2 as slack variables, we can write above LPP in standard form as system of linear equation as

$$Z^* - 8y_1 - 5y_2 - 7y_3 + 0x_1 + 0x_2 = 0$$
$$y_1 + y_2 + 2y_3 + x_1 + 0x_2 = 20$$
$$4y_1 + y_2 + y_3 + 0x_1 + x_2 = 30$$

The first simplex table of above system is

B	Z^*	$y_1\downarrow$	y_2	y_3	x_1	x_2	b	Ratio
	1	–8	–5	–7	0	0	0	
x_1	0	1	1	2	1	0	20	20/1 = 20
$\leftarrow x_2$	0	4	1	1	0	1	30	30/4 = 7.5

Basic variables : $x_1 = 20, x_2 = 30$

Non-basic variables: $y_1 = y_2 = y_3 = 0$

For these values, $Z^* = 0$

Pivot column: column of y_1, pivot row: R_3, Pivot element: 4

Applying $R_1 \to R_1 + 2R_3, R_2 \to R_2 - \frac{1}{4} R_3$

The second simplex table of above system is

B	Z^*	y_1	y_2	$y_3\downarrow$	x_1	x_2	b	Ratio
	1	0	–3	–5	0	2	60	
$\leftarrow x_1$	0	0	3/4	7/4	1	–1/4	25/2	$\frac{25/2}{7/4} = 7.14$
y_1	0	4	1	1	0	1	30	30/1 = 30

Basic variables : $x_1 = \frac{25}{2}, 4y_1 = 30$, i.e., $y_1 = \frac{15}{2}$

Non-basic variables: $x_2 = y_2 = y_3 = 0$

For these values, $Z^* = 0$

Pivot column: column of y_3, pivot row: R_2, Pivot element: $\frac{7}{4}$

Applying $R_1 \to R_1 + \frac{2}{7}R_2$, $R_3 \to R_3 - \frac{4}{7}R_2$

The third simplex table of above system is

B	Z^*	y_1	$y_2 \downarrow$	y_3	x_1	x_2	b	Ratio
	1	0	–6/7	0	20/7	9/7	670/7	
$\leftarrow y_3$	0	0	3/4	7/4	1	–1/4	25/2	$\frac{25/2}{3/4} = 16.66$
y_1	0	4	4/7	0	–4/7	8/7	160/7	$\frac{160/7}{4/7} = 40$

Basic variables : $\frac{7}{4}y_3 = \frac{25}{2}$, $4y_1 = \frac{160}{7}$, i.e., $y_3 = \frac{50}{7}$, $y_1 = \frac{40}{7}$

Non-basic variables: $y_2 = x_2 = x_3 = 0$

For these values, $Z^* = \frac{670}{7}$

Pivot column: column of y_2, pivot row: R_2. Pivot element: $\frac{3}{4}$

Applying $R_1 \to R_1 + \frac{8}{7}R_2$, $R_3 \to R_3 - \frac{16}{21}R_2$

The fourth simplex table of above system is

B	Z^*	y_1	$y_2 \downarrow$	y_3	x_1	x_2	b
	1	0	0	8/7	4	1	110
y_2	0	0	3/4	7/4	1	–1/4	25/2
y_1	0	4	0	–16/21	–4/3	31/21	40/3

Since, there are no negative numbers in first row, so simplex process is completed

From the last table

$4y_1 = \frac{40}{3}$, $\frac{3}{4}y_2 = \frac{25}{2}$, $y_3 = 0$ and $x_1 = 4$, $x_2 = 1$

i.e., $y_1 = \frac{10}{3}$, $y_2 = \frac{50}{3}$, $y_3 = 0$

$\therefore$ Max $Z^* = 110$ at $\left(\frac{10}{3}, \frac{50}{3}, 0\right)$ and mini $Z = 110$ at $x_1 = 4$ and $x_2 = 1$.

e. Minimize $Z = 8x_1 + 9x_2$ subject to constraints

$x_1 + 3x_2 \geq 4$ $\quad 2x_1 + x_2 \geq 5$ $\quad x_1, x_2 \geq 0$

Solution

Here, min $Z = 8x_1 + 9x_2$

Subject to $\quad x_1 + 3x_2 \geq 4$

$\quad 2x_1 + x_2 \geq 5$ with $x_1, x_2 \geq 0$

By the theorem of duality of the LPP, its dual problem is defined by

Maximum $Z = 4y_1 + 5y_2$

Subject to $\quad y_1 + 2y_2 \leq 8$

$\quad 3y_1 + y_2 \leq 9$ with $y_1, y_2 \geq 0$, y_1, y_2 are dual variable.

Using slack variables as x_1, x_2 the given LPP in standard form can be written system of linear equation as

Max $\quad Z' - 4y_1 - 5y_2 + 0x_1 + 0x_2 = 0$

$\quad y_1 + 2y_2 + x_1 + 0x_2 = 8$

$\quad 3y_1 + y_2 + 0x_1 + x_2 = 9$

and $\quad y_1, y_2, x_1, x_2 \geq 0$

The first simplex table is

B	Z'	y_1	$y_2 \downarrow$	x_1	x_2	b	Ratio
	1	–4	–5	0	0	0	
$\leftarrow x_1$	0	1	2	1	0	8	$\frac{8}{2} = 4$
x_2	0	3	1	0	1	9	$\frac{9}{1} = 9$

Basic variables: $x_1 = 8$, $x_2 = 9$

Non basic variables: $y_1 = y_2 = 0$

For these values, $Z' = 0$

Pivot column: column of y_2; pivot row: R_2, pivot element: 2

Applying $R_1 \to R_1 + \frac{5}{2}R_2$, $R_3 \to R_3 - 2R_2$

The next table is

B	Z'	$y_1 \downarrow$	y_2	x_1	x_2	b	Ratio
	1	$\frac{-3}{2}$	0	$\frac{5}{2}$	0	20	
y_2	0	1	2	1	0	8	$\frac{8}{1} = 8$
$\leftarrow x_2$	0	$\frac{5}{2}$	0	$\frac{-1}{2}$	1	5	$\frac{5}{5/2} = 2$

Basic variables: $y_2 = 8$, $x_2 = 5$

Non basic variables: $x_1 = y_1 = 0$

For these values, $Z' = 40$

Pivot column: column of y_1, pivot row: R_3, pivot element: $\frac{5}{2}$

Applying $R_1 \to R_1 + \frac{3}{5}R_3$, $R_2 \to R_2 - \frac{2}{5}R_3$

The next table is

B	Z'	y_1	y_2	x_1	x_2	b
	1	0	0	$\frac{11}{5}$	$\frac{3}{5}$	23
y_2	0	0	2	$\frac{6}{5}$	$\frac{-2}{5}$	6
y_1	0	$\frac{5}{2}$	0	$\frac{-1}{2}$	1	5

Since there is no negative number in the first row, so the simplex process is completed.

From last table

$\frac{5}{2}y_1 = 5$, $2y_2 = 6$ and $x_1 = \frac{11}{5}$, $x_2 = \frac{3}{5}$

i.e. $y_1 = 2$, $y_2 = 3$

Max $Z' = 23$ at $y_1 = 2$ and $y_2 = 3$, by duality relation,

Min $Z = 23$ at $x_1 = \frac{11}{5}$ and $x_2 = \frac{3}{5}$

f. **Minimize $Z = 2x_1 + 9x_2 + x_3$ subject to constraints**

$x_1 + 4x_2 + 2x_3 \geq 5 \quad 3x_1 + x_2 + 2x_3 \geq 4 \quad x_1, x_2, x_3 \geq 0$

Solution

Here minimize $Z = 2x_1 + 9x_2 + x_3$

Subject to $x_1 + 4x_2 + 2x_3 \geq 5, \quad 3x_1 + x_2 + 2x_3 \geq 4$ with $x_1, x_2, x_3 \geq 0$

By the theorem of duality of the LPP, its dual problem is defined by

Maximum $Z^* = 5y_1 + 4y_2$

Subject to $y_1 + 3y_2 \leq 2, 4y_1 + y_2 \leq 9, 2y_1 + 2y_2 \leq 1$

With $y_1, y_2 \geq 0$, y_1, y_2 are dual variables.

Using slack variables on x_1, x_2 and x_3 the given LPP in standard form can be written as system of linear equation as

$$Z^* - 5y_1 - 4y_2 + 0x_1 + 0x_2 + 0x_3 = 0$$
$$y_1 + 3y_2 + x_1 + 0x_2 + 0x_3 = 2$$
$$4y_1 + y_2 + 0x_1 + x_2 = 0x_3 = 9$$
$$2y_1 + 2y_2 + 0x_1 + 0x_2 + x_3 = 1$$

The first simplex table of above system is

B	Z^*	$y_1\downarrow$	y_2	x_1	x_2	x_3	b	Ratio
	1	−5	−4	0	0	0	0	
x_1	0	1	3	1	0	0	2	2/1 = 2
x_2	0	4	2	0	1	0	9	9/4 = 2.25
$\leftarrow x_3$	0	2	2	0	0	1	1	1/2 = 0.5

Basic variables : $x_1 = 2, x_2 = 9, x_3 = 1$

Non-basic variables: $y_1 = y_2 = 0$

For these values, $Z^* = 0$

Pivot column: column of y_1, pivot row: R_4, Pivot element: 2

Applying $R_1 \to R_1 + \frac{5}{2}R_4, R_2 \to R_2 - \frac{1}{2}R_4, R_3 \to R_3 - 2R_4$

The second simplex table of above system is

B	Z^*	y_1	y_2	x_1	x_2	x_3	b
	1	0	1	0	0	5/2	5/[illegible]
x_1	0	0	2	1	0	−1/2	3/[illegible]
x_2	0	0	−2	0	1	−2	7
y_1	0	2	2	0	0	1	1

Since there are no negative terms in first row, so simplex process is completed.

From last table,

$2y_1 = 1$, i.e., $y_1 = \frac{1}{2}, y_2 = 0$

$\therefore$ Max $Z^* = \frac{5}{2}$ at $\left(\frac{1}{2}, 0\right)$.

$x_1 = 0, x_2 = 0$ and $x_3 = \frac{5}{2}$

$\therefore$ Minimum $Z = \frac{5}{2}$ at $\left(0, 0, \frac{5}{2}\right)$

3. **Solve the following LPP by Big M method.**

a. **Maximize $Z = 5x_1 - 2x_2$ subject to the constraints**

$3x_1 - 4x_2 \leq 2, \quad x_1 + 2x_2 \geq 4, \quad x_2 \leq 4, \qquad x_1, x_2 \geq 0.$

Solution

Let S_1, S_3 be slack variables, S_2 be surplus variable, and a_1 be artificial variables. Then, the given objective function with penalty M due to artificial variable a_1

$$Z = 5x_1 - 2x_2 - Ma_1$$

With the above variables the LPP can be written as system of linear equations

$$Z - 5x_1 + 2x_2 + 0S_1 + Ma_1 + 0S_2 + 0S_3 = 0$$
$$3x_1 - 4x_2 + S_1 + 0a_1 + 0S_2 + 0S_3 = 2$$
$$x_1 + 2x_2 + 0S_1 + a_1 - S_2 + 0S_3 = 4$$
$$0x_1 + x_2 + 0S_1 + 0a_1 + 0S_2 + S_3 = 4$$

Now, the preliminary simplex table is

Z	x_1	x_2	S_1	a_1	S_2	S_3	b
1	−5	2	0	M	0	0	0
0	3	−4	1	0	0	0	2
0	1	2	0	1	−1	0	4
0	0	1	0	0	0	1	4

Applying $R_1 \to R_3 - MR_3$

B	Z	x_1	$x_2\downarrow$	S_1	a_1	S_2	S_3	b	Ratio
	1	−5−M	2−2M	0	0	M	0	−4M	
S_1	0	3	−4	1	0	0	0	2	–
$\leftarrow a_1$	0	1	2	0	1	−1	0	4	4/2 = 4
S_3	0	0	10	0	0	0	1	4	4/1 = 4

Basic variables : $S_1 = 2, a_1 = 4, S_3 = 4$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = -4m$

Pivot column: column of x_2, pivot row: R_4, Pivot element: 2

Applying $R_1 \to R_1 + (M-1)R_3, R_2 \to R_2 + 2R_3, R_4 \to R_4 - \frac{1}{2}R_3$

The simplex table is

B	Z	$x_1\downarrow$	x_2	S_1	a_1	S_2	S_3	b	Ratio
	1	−6	0	0	M−1	1	0	−4	
$\leftarrow S_1$	0	5	0	1	2	−2	0	10	10/5 = 2
x_2	0	1	2	0	1	−1	0	4	4/1 = 4
S_3	0	−1/2	0	0	−1/2	1/2	1	2	–

Basic variables : $S_1 = 10, x_2 = 2, S_3 = 2$

Non-basic variables: $x_1 = S_2 = a_1 = 0$

a_1 is the non basic variable, so we remove the column of a_1

For these values, $Z = -4$

Pivot column: column of x_1, pivot row: R_2, Pivot element: 5

Applying $R_1 \to R_1 + \frac{6}{5}R_2, R_3 \to R_3 - \frac{1}{5}R_2, R_4 \to R_4 \to R_4 + \frac{1}{10}R_2$

The next simplex table is

B	Z	x_1	x_2	S_1	$S_2\downarrow$	S_3	b	Ratio
	1	0	0	6/5	7/5	0	8	
x_1	0	5	0	1	−2	0	10	–
x_2	0	0	2	−1/5	−3/5	0	2	–
$\leftarrow S_3$	0	0	0	1/10	3/10	1	3	$\frac{3}{3/10} = 10$

Basic variables : $5x_1 = 10$, $2x_2 = 2$, $S_3 = 3$; i.e., $x_1 = 2$ $x_2 = 1$, $S_3 = 3$

Non-basic variables: $S_1 = S_2 = 0$

For these values, $Z = 8$

Pivot column: column of S_2, pivot row: R_4, Pivot element: $\frac{3}{10}$

Applying $R_1 \to R_1 + \frac{14}{3} R_4$, $R_2 \to R_2 + \frac{20}{3} R_4$ and $R_3 \to R_3 + 2R_4$

The next simplex table is

B	Z	x_1	x_2	S_1	S_2	S_3	b
	1	0	0	5/3	0	14/3	22
x_1	0	5	0	5/3	0	20/3	30
x_2	0	0	2	0	0	2	8
S_2	0	0	0	1/10	3/10	1	3

Since there are no negative numbers in first row, so simplex process is completed.

From last table.

$5x_1 = 30$, $2x_2 = 8$, $\frac{3}{10} S_2 = 3$

i.e., $x_1 = 6$, $x_2 = 4$, $S_2 = 10$

$\therefore$ Max. $Z = 22$ at (6, 4).

b. Maximize $Z = -3x_1 + 7x_2$ subject to the constraints

$2x_1 + 3x_2 \le 5$, $5x_1 + 2x_2 \ge 3$, $x_2 \le 1$, $x_1, x_2 \ge 0$.

Solution

Let S_1, S_3 be slack variables, S_2 be surplus variable, and a_1 be artificial variable. Then, the given objective function with penalty M due to artificial variable a_1

$$Z = -3x_1 + 7x_2 - Ma_1$$

With the above variables the LPP can be written as system of linear equations

$$Z + 3x_1 - 7x_2 + 0S_1 + Ma_1 + 0S_2 + 0S_3 = 0$$
$$2x_1 + 3x_2 + S_1 + 0a_1 + 0S_2 + 0S_3 = 5$$
$$5x_1 + 2x_2 + 0S_1 + a_1 - S_2 + 0S_3 = 3$$
$$0x_1 + x_2 + 0S_1 + 0a_1 + 0S_2 + S_3 = 1$$

Now, the preliminary simplex table is

Z	x_1	x_2	S_1	a_1	S_2	S_3	b
1	3	−7	0	M	0	0	0
0	2	3	1	0	0	0	5
0	5	2	0	1	−1	0	3
0	0	1	0	0	0	1	1

Applying $R_1 \to R_1 - MR_3$

The simplex table is

B	Z	$x_1\downarrow$	x_2	S_1	a_1	S_2	S_3	b	Ratio
	1	3−5M	−7−2M	0	0	M	0	−3M	
S_1	0	2	3	1	0	0	0	5	5/2 = 2.5
$\leftarrow a_1$	0	5	2	0	1	−1	0	3	3/5 = 0.6
S_3	0	0	1	0	0	0	1	1	–

Basic variables : $S_1 = 5$, $a_1 = 3$, $S_3 = 1$

Non-basic variables: $x_1 = x_2 = S_2 = 0$

For these values, $Z = -3M$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 5

Applying $R_1 \to R_1 + \left(\frac{5M-3}{5}\right) R_3$, $R_2 \to R_2 - \frac{2}{5} R_3$

The next simplex table is

B	Z	x_1	$x_2\downarrow$	S_1	a_1	S_2	S_3	b	Ratio
	1	0	$-\frac{41}{5}$	0	$M-\frac{3}{5}$	$\frac{3}{5}$	0	$-\frac{9}{5}$	
S_1	0	0	$\frac{11}{5}$	1	$-\frac{2}{5}$	$\frac{2}{5}$	0	$\frac{19}{5}$	$\frac{19/5}{11/5} = 1.7$
x_1	0	5	2	0	1	−1	0	3	$\frac{3}{2} = 1.5$
$\leftarrow S_3$	0	0	1	0	0	0	1	1	$\frac{1}{1} = 1$

Basic variables : $S_1 = \frac{19}{5}$, $5x_1 = 3$, i.e., $x = \frac{3}{5}$, $S_3 = 1$

Non-basic variables: $x_2 = S_2 = a_1 = 0$

For these values, $Z = -\frac{9}{5}$

a_1 is the non basic variables, so we remove the column of a_1.

Pivot column: column of x_2, pivot row: R_4, Pivot element: 1

Applying $R_1 \to R_1 + \frac{41}{5} R_4$, $R_2 \to R_2 - \frac{11}{5} R_4$, $R_3 \to R_3 - 2R_4$

The next simplex table is

B	Z	x_1	x_2	S_1	S_2	S_3	b
	1	0	0	0	3/5	41/5	32/5
S_1	0	0	0	1	2/5	−11/5	8/5
x_1	0	5	0	0	−1	−2	1
x_2	0	0	1	0	0	1	1

Since there are no negative numbers in first row, so simplex process is completed.

From last table.

$5x_1 = 1$, $x_2 = 1$ and $S_1 = \frac{8}{5}$

i.e., $x_1 = \frac{1}{5}$, $x_2 = 1$

$\therefore$ Max. $Z = \frac{32}{5}$ at $\left(\frac{1}{5}, 1\right)$.

c. **Maximize $Z = 4x_1 + 2x_2$ subject to the constraints**

$3x_1 + x_2 \le 27,\ x_1 + x_2 \ge 21,\qquad x_1, x_2 \ge 0.$

Solution

Let S_1 be slack variable, S_2 be surplus variable, and a_1 be artificial variables. In the given objective function with penalty M due to artificial variable a_1

$Z = 4x_1 + 2x_2 - Ma_1$

With the above variables the LPP can be written as system of linear equations

$Z - 4x_1 - 2x_2 + 0S_1 + Ma_1 + 0S_2 = 0$

$3x_1 + x_2 + S_1 + 0a_1 + 0S_2 = 27$

$x_1 + x_2 + 0S_1 + a_1 - S_2 = 21$

Now, the preliminary simplex table is

Z	x_1	x_2	S_1	a_1	S_2	b
1	-4	-2	0	M	0	0
0	3	1	1	0	0	27
0	1	1	0	1	-1	21

Applying $R_1 \to R_1 - MR_3$

B	Z	$x_1\downarrow$	x_2	S_1	a_1	S_2	b	Ratio
	1	-4-M	-2-M	0	0	M	21M	
$\leftarrow S_1$	0	3	1	1	0	0	27	27/3 = 9
a_1	0	1	1	0	1	-1	21	21/1 = 21

Basic variables : $S_1 = 27$, $a_1 = 21$

Non-basic variables: $x_1 = x_2 = S_2 = 0$

For these values, $Z = -21M$

Pivot column: column of x_1, pivot row: R_2, Pivot element: 3

Applying $R_1 \to R_1 + \frac{(4+M)}{3}R_2$, $R_3 \to R_3 - \frac{1}{3}R_2$

The simplex table is

B	Z	x_1	$x_2\downarrow$	S_1	a_1	S_2	b	Ratio
	1	0	$\frac{-2-2M}{3}$	$\frac{4+M}{3}$	0	M	36–12M	
S_1	0	3	1	1	0	0	27	27/1 = 27
$\leftarrow a_1$	0	0	2/3	-1/3	1	-1	12	$\frac{12}{2/3} = 18$

Basic variables : $3x_1 = 27$, i.e., $x_1 = 9$, $a_1 = 12$

Non-basic variables: $x_2 = S_1 = S_2 = 0$

For these values, $Z = 36 - 12M$

Pivot column: column of x_2, pivot row: R_3, Pivot element: $\frac{2}{3}$

Applying $R_1 \to R_1 + (1 + M)R_3$, $R_2 \to R_2 - \frac{3}{2}R_3$

The next simplex table is

B	Z	x_1	x_2	S_1	a_1	$S_2\downarrow$	b	Ratio
	1	0	0	1	1+M	-1	48	
$\leftarrow x_1$	0	3	0	3/2	-3/2	3/2	9	$\frac{12}{3/2} = 8$
x_2	0	0	2/3	-1/3	1	-1	12	–

Basic variables : $3x_1 = 9$, $\frac{2}{3}x_2 = 12$ i.e., $x_1 = 3$, $x_2 = 18$

Non-basic variables: $= S_1 = S_2 = a_1 = 0$

For these values, $Z = 48$

a_1 is the non-basic variable, so we remove the column of a_1

Pivot column: column of S_2, pivot row: R_2, Pivot element: $\frac{3}{2}$

Applying $R_1 \to R_1 + \frac{2}{3}R_2$, $R_3 \to R_3 + \frac{2}{3}R_2$

The next simplex table is

B	Z	x_1	x_2	S_1	S_2	b
	1	2	0	2	0	54
S_2	0	3	0	3/2	3/2	9
x_2	0	2	2/3	2/3	0	18

Since, there is no negative numbers in first row, so simplex process is completed.

From the last table,

$x_1 = 0$ and $\frac{2}{3}x_2 = 18$, i.e., $x_2 = 27$

$\therefore$ Max. $Z = 54$ at $x_1 = 0$ and $x_2 = 27$

d. **Minimize $Z = 12x_1 + 20x_2$ subject to the constraints**

$6x_1 + 8x_2 \ge 100,\ 7x_1 + 12x_2 \ge 120,\qquad x_1, x_2 \ge 0.$

Solution

Let $\bar{Z} = -Z$, then Min $Z = -$ (Max $\bar{Z}$) and LPP becomes maximize $\bar{Z} = -12x_1 - 20x_2$ under the given constraints.

Introducing surplus variables S_1, S_2 and artificial variables a_1 and a_2. Thus the given objective function can be written as $\bar{Z} = -12x_1 - 20x_2 - Ma_1 - Ma_2$

With the above variables the LPP can be written as system of linear equations

$\bar{Z} + 12x_1 + 20x_2 + Ma_1 + Ma_2 + 0S_1 + 0S_2 = 0$

$6x_1 + 8x_2 + a_1 + 0a_2 - S_1 + 0S_2 = 100$

$7x_1 + 12x_2 + 0a_1 + a_2 + 0S_1 - S_2 = 120$

Now, the preliminary simplex table is

$\bar{Z}$	x_1	x_2	a_1	a_2	S_1	S_2	b
1	12	20	M	M	0	0	0
0	6	8	1	0	-1	0	100
0	7	12	0	1	0	-1	120

Applying $R_1 \to R_1 - MR_2 - MR_3$

B	Z	x_1	$x_2\downarrow$	a_1	a_2	S_1	S_2	b	Ratio
	1	12-13M	20-20M	0	0	M	M	–220M	
a_1	0	6	8	1	0	–1	0	100	100/8 = 12
$\leftarrow a_2$	0	7	12	0	1	0	–1	120	120/12 =

Basic variables : $a_1 = 100$, $a_2 = 120$

Non-basic variables: $S_1 = S_2 = x_1 = x_2 = 0$

For these values, $\bar{Z} = -220$ M

Pivot column: column of x_2, pivot row: R_3, Pivot element: 12

Applying $R_1 \to R_1 + \frac{(5M-5)}{3} R_3$, $R_2 \to R_2 - \frac{2}{3} R_3$

The simplex table is

B	$\bar{Z}$	$x_1\downarrow$	x_2	a_1	a_2	S_1	S_2	b	Ratio
	1	$\frac{1-4M}{3}$	0	0	$\frac{5M-5}{3}$	M	$\frac{5-2M}{3}$	–20M –200	
$\leftarrow a_1$	0	4/3	0	1	–2/3	–1	2/3	20	$\frac{20}{4/3} = 15$
x_2	0	7	12	0	1	0	–1	120	$\frac{120}{7} = 17.1$

Non-basic variables: $a_1 = 20$, $12x_2 = 120$, i.e., $x_2 = 10$

For these values, $x_1 = S_1 = S_2 = a_2 = 0$

Since a_2 is non-basic variable, so we remove the column of a_2.

Pivot column: column of x_1, pivot row: R_2, Pivot element: $\frac{4}{3}$

Applying $R_1 \to R_1 + \left(\frac{4M-1}{4}\right) R_2$, $R_3 \to R_3 - \frac{21}{4} R_2$

The simplex table is

B	$\bar{Z}$	$x_1\downarrow$	x_2	a_1	S_1	S_2	b
	1	0	0	$\frac{4M-1}{4}$	1	$\frac{3}{2}$	–205
x_1	0	4/3	0	1	–1	2/3	20
x_2	0	0	12	–21/4	21/4	–9/2	15

Since, there are no negative numbers in first row, so simplex process is complete.

From last table.

$\frac{4}{3} x_1 = 20$, $12x_2 = 15$

i.e., $x_1 = 15$, $x_2 = \frac{5}{4}$

$\therefore$ Min $Z = -(\text{Max } \bar{Z}) = 205$ at $\left(15, \frac{5}{4}\right)$.

e. **Maximize $Z = -2x_1 - x_2$ subject to the constraints**

$3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$, $x_1, x_2 \geq 0$.

Solution

Let S_1 be surplus variable, S_2 be slack variable and a_1, a_2 be artificial variables. Then the given objective function with penalty M due to artificial variables a_1 and a_2.

$$Z = -2x_1 - x_2 - Ma_1 - Ma_2$$

With the above variables the LPP can be written as system of linear equations.

$$Z + 2x_1 + x_2 + Ma_1 + Ma_2 + 0S_1 + 0S_2 = 0$$
$$3x_1 + x_2 + a_1 + 0a_2 + 0S_1 + 0S_2 = 3$$
$$4x_1 + 3x_2 + 0a_1 + a_2 - S_1 + 0S_2 = 6$$
$$x_1 + 2x_2 + 0a_1 + 0a_2 + 0S_1 + S_2 = 4$$

Now, the preliminary table is

Z	x_1	x_2	a_1	a_2	S_1	S_2	b
1	2	1	M	M	0	0	0
0	3	1	1	0	0	0	3
0	4	3	0	1	–1	0	6
0	1	2	0	0	0	1	4

Applying $R_1 \to R_1 - MR_2 - MR_3$

B	Z	$x_1\downarrow$	x_2	a_1	a_2	S_1	S_2	b	Ratio
	1	2–7M	1–4M	0	0	M	0	–9M	
$\leftarrow a_1$	0	3	1	1	0	0	0	3	3/3 = 1
a_2	0	4	3	0	1	–1	0	6	6/4 = 1.5
S_2	0	1	2	0	0	0	1	4	4/1 = 4

Basic variables : $a_1 = 3$, $a_2 = 6$, $S_2 = 4$

Non-basic variables: $x_1 = x_2 = S_1 = 0$

For these values, $Z = -9m$

Pivot column: column of x_1, pivot row: R_2, Pivot element: 3

Applying $R_1 \to R_1 + \frac{(7M-2)}{3} R_2$, $R_3 \to R_3 - \frac{4}{3} R_2$, $R_4 \to R_4 - \frac{1}{3} R_2$

The simplex table is

B	Z	x_1	$x_2\downarrow$	a_1	a_2	S_1	S_2	b	Ratio
	1	0	$\frac{1-5M}{3}$	$\frac{7M-2}{3}$	0	M	0	–2M –2	
x_1	0	3	1	1	0	0	0	3	3/1 = 3
$\leftarrow a_2$	0	0	5/3	– 4/3	1	–1	0	2	$\frac{2}{5/3} = 1.2$
S_2	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	0	0	1	3	$\frac{3}{5/3} = 1.8$

Basic variables : $3x_1 = 3$, i.e., $x_1 = 1$, $a_2 = 2$, $S_2 = 3$

Non-basic variables: $x_2 = a_2 = S_1 = 0$

Since, a_1 is the non-basic variables, so we remove the column of a_1.

For these values, $Z = -2M - 2$

Pivot column: column of x_1, pivot row: R_3, Pivot element: $\frac{5}{3}$

Applying $R_1 \to R_1 + \frac{(5M-1)}{5} R_3$, $R_2 \to R_2 - \frac{3}{5} R_3$, $R_4 \to R_4 - R_3$

The next simplex table is

B	Z	x_1	x_2	a_2	S_1	S_2	b
	1	0	0	$\frac{5M-1}{5}$	$\frac{1}{5}$	0	$\frac{12}{5}$
x_1	0	3	0	–3/5	3/5	0	9/5
x_2	0	0	5/3	1	–1	0	2
S_2	0	0	0	–1	1	1	1

Since, there are no negative numbers in first row, so simplex process is completed

From last table,

$3x_1 = \frac{9}{5}, \frac{5}{3}x_2 = 2$ and $S_2 = 1$

i.e., $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$

$\therefore$ Max. $Z = -\frac{12}{5}$ at $\left(\frac{3}{5}, \frac{6}{5}\right)$.

f. Minimize $Z = 5x_1 + 7x_2$ subject to the constraints

$2x_1 + 3x_2 \geq 6$ $\quad$ $2x_1 + 5x_2 \leq 27$

$3x_1 - x_2 \leq 15$ $\quad$ $x_1 \geq 0$

$-x_1 + x_2 \leq 4$ $\quad$ $x_2 \geq 0$

Solution

Given LPP is $Z = 5x_1 + 7x_2$

Subject to constraints $2x_1 + 3x_2 \geq 6, 2x_1 + 5x_2 \leq 27, 3x_1 - x_2 \leq 15, -x_1 + x_2 \leq 4, x_1, x_2 \geq 0$

Let $Z = -Z$, then min $Z = -$ (Max Z) and LPP becomes maximize $Z = -5x_1 - 7x_2$.

Introducing S_1 as surplus variable, a_1 artificial variable and S_2, S_3, S_4 as slack variables. Thus the given objective function can be written as $Z = -5x_1 - 7x_2 - Ma_1$

The above variables the LPP can be written as system of linear equations.

$$Z + 5x_1 + 7x_2 + Ma_1 + 0S_1 + 0S_2 + 0S_3 + 0S_4 = 0$$
$$2x_1 + 3x_2 - S_1 + a_1 + 0S_2 + 0S_3 + 0S_4 = 6$$
$$2x_1 + 5x_2 + 0S_1 + 0a_1 + S_2 + 0S_3 + 0S_4 = 27$$
$$3x_1 - x_2 + 0S_1 + 0a_1 + 0S_2 + S_3 + 0S_4 = 15$$
$$x_1 + x_2 + 0S_1 + 0a_1 + 0S_2 + 0S_3 + S_4 = 4$$

Now, the preliminary simplex table is

Z	x_1	x_2	S_1	a_1	S_2	S_3	S_4	b
1	5	7	0	M	0	0	0	0
0	2	3	–1	1	0	0	0	6
0	2	5	0	0	1	0	0	27
0	3	–1	0	0	0	1	0	15
0	–1	1	0	0	0	0	1	4

Applying $R_1 \rightarrow R_1 - MR_2$

B	Z	x_1	$x_2 \downarrow$	S_1	a_1	S_2	S_3	S_4	b	Ratio
	1	5–2M	7–3M	M	0	0	0	0	–6M	
$\leftarrow a_1$	0	2	3	–1	1	0	0	0	6	6/3 = 2
S_2	0	2	5	0	0	1	0	0	27	27/5 = 5.4
S_3	0	3	–1	0	0	0	1	0	15	
S_4	0	–1	1	0	0	0	0	1	4	4/1 = 4

Basic variables : $a_1 = 6, S_2 = 27, S_3 = 15, S_4 = 4$

Non-basic variables: $x_1 = x_2 = 0$

For these values, $Z = -6m$

Pivot column: column of x_2, pivot row: R_2, Pivot element: 3

Applying $R_1 \rightarrow R_1 + \left(\frac{3M-7}{3}\right)R_2$, $R_3 \rightarrow R_3 - \frac{5}{3}R_2$, $R_4 \rightarrow R_4 + \frac{1}{3}R_2$, $R_5 \rightarrow R_5 - \frac{1}{3}R_2$

B	Z	x_1	x_2	S_1	a_1	S_2	S_3	S_4	b
	1	1/3	0	7/3	$\frac{3M-7}{3}$	0	0	0	6
x_2	0	2	3	–1	1	0	0	0	6
S_2	0	–4/3	0	5/3	–5/3	1	0	0	17
S_3	0	11/3	0	–1/3	1/3	0	1	0	17
S_4	0	–5/3	0	1/3	–1/3	0	0	1	2

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table

$x_1 = 0, 3x_2 = 6 \Rightarrow x_2 = 2$

$\therefore$ Min $Z = -$(Max Z) = 14 at (0, 2).

4. A firm produces two products P and Q. Daily production upper limit is 600 units for total production. The least production is 300 units per day. The machine hours consumption for P is 6 per unit and for Q 2 per Q. The 1200 machines are used daily. Manufacturing cost per P and Q are respectively Rs 50 and Rs 20. Find the optimal solution by using simplex method.

Solution

Let x_1 be number of units of P per day and x_2 be number of units of Q per day.

Minimum $Z = 50x_1 + 20x_2$

Subject to constraints $x_1 + x_2 \leq 600$, $x_1 + x_2 \geq 300$, $6x_1 + 2x_2 \geq 1200$, $x_1, x_2 \geq 0$

Let $\overline{Z} = -Z$, then min $Z = -$ (max $\overline{Z}$) and LPP becomes maximize $\overline{Z} = -50x_1 - 20x_2$

Introducing S_1 as slack variables, S_1, S_3 as surplus variables and a_1, a_2 as artificial variables, then the given objective function can be written as

$$\text{Max } \overline{Z} = -51x_1 - 20x_2 - Ma_1 - Ma_2$$

With the above variables, the LPP can be written as the system of linear equation as

$$\text{Max } Z + 50x_1 + 20x_2 + 0S_1 + 0S_2 + 0S_3 + Ma_1 + Ma_2 = 0$$
$$x_1 + x_2 + S_1 + 0S_2 + 0S_3 + 0a_1 + 0a_2 = 600$$
$$x_1 + x_2 + 0S_1 + S_2 + 0S_3 + a_1 + 0a_2 = 300$$
$$6x_1 + 2x_2 + 0S_1 + 0S_2 - S_3 + 0a_1 - a_2 = 1200$$

Now, the preliminary simplex table is

Z	x_1	x_2	S_1	S_2	S_3	a_1	a_2	b
1	50	20	0	0	0	M	M	0
0	1	1	1	0	0	0	0	600
0	1	1	0	–1	0	1	0	300
0	6	2	0	0	–1	0	1	1200

Applying $R_1 \to R_1 - M(R_3 + R_4)$

B	Z	$x_1\downarrow$	x_2	S_1	S_2	S_3	a_1	a_2	b	Ratio
	1	50–7M	20–3M	0	M	M	0	0	–1500M	
S_1	0	1	1	1	0	0	0	0	600	$\frac{600}{1} =$ [illegible]
a_1	0	1	1	0	–1	0	1	0	300	$\frac{300}{1} =$ [illegible]
$\leftarrow a_2$	0	6	2	0	0	–1	0	1	1200	$\frac{1200}{6} =$ [illegible]

Basic variables : $S_1 = 600$, $a_1 = 300$, $a_2 = 1200$

Non-basic variables: $x_1 = x_2 = 0 = S_2 = S_3$

For these values, $Z = -1500$ M

Pivot column: column of x_1, pivot row: R_4. Pivot element: 6

Applying $R_1 \to R_1 + \left(\frac{7M - 50}{6}\right) R_4$, $R_2 \to R_2 - \frac{1}{6} R_4$, $R_3 \to R_3 - \frac{1}{6} R_4$

B	Z	x_1	$x_2\downarrow$	S_1	S_2	S_3	a_1	a_2	b	Ratio
	1	0	$\frac{10-2M}{3}$	0	–M	$\frac{50-M}{6}$	0	7M – 50	–100M – 10000	
S_1	0	0	2/3	1	0	1/6	0	$0 - \frac{1}{6}$	400	$\frac{400}{2/3} = 600$
$\leftarrow a_1$	0	0	2/3	0	–1	1/6	1	$0 - \frac{1}{6}$	100	$\frac{100}{2/3} = 150$
x_1	0	6	2	0	0	–1	0	1	1200	$\frac{1200}{2} = 600$

Basic variables : $S_1 = 400$, $a_1 = 100$, $6x_1 = 1200$, i.e., $x_1 = 600$

Non-basic variables: $x_2 = S_2 = S_3 = a_2 = 0$

a_2 is the non-basic variables, so we remove the column of a_2.

Pivot column: column of x_2, pivot row: R_3, Pivot element: $\frac{2}{3}$

Applying $R_1 \to R_1 + (M - 5)R_3$, $R_2 \to R_2 - R_3$, $R_4 \to R_4 - 3R_3$

B	Z	x_1	x_2	S_1	S_2	S_3	b
	1	0	0	0	5	15/2	10500
S_1	0	0	0	1	1	0	300
x_2	0	0	2/3	0	–1	1/6	1000
x_1	0	6	0	0	3	–3/2	900

Since, there are no negative numbers in first row, so simplex process is completed.

From the last table,

$6x_1 = 900$, $\frac{2}{3}x_2 = 1000$, $S_1 = 300$

i.e., $x_1 = 150$, $x_2 = 150$

$\therefore$ Minimum $Z = 10500$ at $(150, 150)$.

5. One unit of product A contributes Rs 7 and requires 3 unit of raw materials and 2 hrs of labour. One unit of product B contributes Rs 5 and requires one unit of raw material and one hour labour. Availability of raw materials at present is 48 units and there are 40 hours of labour.

a. Formulate this problem as a LPP.

b. Write its dual and solve by simplex method.

Solution

a. Let unit of product A be x_1 and unit of product B be x_2 then by question, the information is listed as follows:

	Product A	Product B	No. of units
Availability of raw materials	3	1	48
Availability of time (in hrs.)	2	1	40
Availability of amount (in Rs.)	7	5	

Maximum $Z = 7x_1 + 5x_2$

Subject to the constraints $3x_1 + x_2 \le 48$, $2x_1 + x_2 \le 40$, $x_1, x_2 \ge 0$

b. Now objective function can be written as

$$\text{Max } Z = 7x_1 + 5x_2 = [7\ \ 5]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = CX$$

where $C = [7\ \ 5]$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Under the constraints

$3x_1 + x_2 \le 48$

$2x_1 + x_2 \le 40$ with $x_1, x_2 \ge 0$

or, $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 48 \\ 40 \end{bmatrix}$

or, $AX \le B$

Where, $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $B = \begin{bmatrix} 48 \\ 40 \end{bmatrix}$

Then its dual is minimize $Z^* = B^TY$, where $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

or, Minimize $Z^* = [48\ \ 40]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\therefore$ Minimize $Z^* = 48y_1 + 40y_2$

Under the constraints

$A^TY \ge C^T$

or, $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \ge \begin{bmatrix} 7 \\ 5 \end{bmatrix}$

or, $3y_1 + 2y_2 \ge 7$

$y_1 + y_2 \ge 5$

$\therefore$ Required dual is minimize $Z^* = 48y_1 + 40y_2$

Under the constraints $3y_1 + 2y_2 \ge 7$, $y_1 + y_2 \ge 5$, $y_1, y_2 \ge 0$.

Let $Z = -Z$, then min $Z^* = -(\max Z)$ and objective function becomes

Maximize $Z = -48y_1 - 40y_2$

Introducing surplus variables x_1, x_2 and artificial variables a_1 and a_2. Then the given objective function can be written as $Z = -48y_1 - 40y_2 - Ma_1 - Ma_2$ with the above variables the LPP can be written as system of linear equations.

$$Z + 48y_1 + 40y_2 + Ma_1 + Ma_2 + 0x_1 + 0x_2 = 0$$
$$3y_1 + 2y_2 + a_1 + 0a_2 - x_1 + 0x_2 = 7$$
$$y_1 + y_2 + 0a_1 + a_2 + 0x_1 - x_2 = 5$$

Now, the preliminary simplex table is

Z	y_1	y_2	a_1	a_1	x_1	x_2	b
1	48	40	M	M	0	0	0
0	3	2	1	0	–1	0	7
0	1	1	0	1	0	–1	5

Applying $R_1 \to R_1 - M(R_2 + R_3)$

B	Z	$y_1\downarrow$	y_2	a_1	a_1	x_2	x_3	b	Ratio
	1	48–4M	40–3M	0	0	M	M	–12M	
$\leftarrow a_1$	0	3	2	1	0	–1	0	7	7/3 = 2.33
a_2	0	1	1	0	1	0	–1	5	5/1 = 5

Basic variables : $a_1 = 7$, $a_2 = 5$

Non-basic variables: $x_1 = x_2 = y_1 = y_2 = 0$

For these values, $Z = -12M$

Pivot column: column of y_1, pivot row: R_2, Pivot element: 3

Applying $R_1 \to R_1 + \frac{(4M-48)}{3} R_2$, $R_3 \to R_3 - \frac{1}{3} R_2$

The simplex table is

B	Z	y_1	$y_2\downarrow$	a_1	a_1	x_2	x_3	b	Ratio
	1	0	$\frac{24-M}{3}$	$\frac{4M-48}{3}$	0	$\frac{48-M}{3}$	M	$\frac{-8M-336}{3}$	
$\leftarrow y_1$	0	3	2	1	0	–1	0	7	7/2 = 3.5
a_2	0	0	1/3	–1/3	1	1/3	–1	8/3	$\frac{8/3}{1/3} = 8$

Basic variables : $3y_1 = 7$, i.e., $y_1 = \frac{7}{3}$ and $a_2 = \frac{8}{3}$

Non-basic variables: $y_2 = x_1 = x_2 = a_1 = 0$

For these values, $\bar{Z} = \frac{-8M - 336}{3}$

Since a_1 is the non-basic variables, so we remove the column of a_1.

Pivot column: column of y_2, pivot row: R_2, Pivot element: 2

Applying $R_1 \to R_1 + \left(\frac{M-24}{6}\right) R_2$, $R_3 \to R_3 - \frac{1}{6} R_2$

The next simplex table is

B	$\bar{Z}$	y_1	y_2	a_1	$x_2\downarrow$	x_3	b	Ratio
	1	$\frac{M-24}{2}$	0	0	$\frac{40-M}{2}$	0	$\frac{-3M-280}{3}$	
y_2	0	3	2	0	–1	0	7	–
$\leftarrow a_2$	0	–1/2	0	1	1/2	–1	3/2	$\frac{3/2}{1/2} = 3$

Basic variables : $y_2 = 7/2$, $a_2 = 3/2$

Non-basic variables: $x_1 = x_2 = y_1 = 0$

For these values, $\bar{Z} = \frac{-3M - 280}{2}$

Pivot column: column of x_1, pivot row: R_3, Pivot element: $\frac{1}{2}$

Applying $R_1 \to R_1 + (M - 40)R_3$, $R_2 \to R_2 + 2R_3$

The next simplex table is

B	$\bar{Z}$	y_1	y_2	a_1	x_2	x_3	b
	1	8	0	M–40	0	40	200
y_2	0	2	2	2	0	–2	10
x_1	0	–1/2	0	1	1/2	–1	3/2

Since there are no negative numbers in first row, so simplex process is completed.

From last table,

$2y_2 = 10$, i.e., $y_2 = 5$ and $y_1 = 0$

$\therefore$ Maximize $\bar{Z} = 200$ at ($y_1 = 0$ and $y_2 = 5$) and from the same table

$x_1 = 0$ and $x_2 = 40$

So minimize $Z = -200$ at $x_1 = 0$ and $x_2 = 40$.

6. A company produces three products P, Q and R from three raw materials A, B and C. One unit of product P requires 2 units of A and 3 units of B. A units of product Q requires 2 units of B and 4 units of C. The company has 8 units of A. 10 units of material B and 15 units of material C available to it. Profits per units of product P, Q and R are Rs 3, Rs 5 and Rs 4 respectively.

a. Formulate this problem as an LPP.

b. How many units of each product should be produced to maximize profit.

c. Write the dual of this problem.

Solution

a. Let x_1, x_2 and x_3 be units of each product to be produce to maximize profit.

	Product P	Product Q	Product R	Units
Raw materials A	2	0	3	8
Raw materials B	5	2	2	10
Raw materials C	0	5	4	15
Profit (in Rs)	3	5	4	

LPP: Maximize $Z = 3x_1 + 5x_2 + 4x_3$

Subject to constraints

$2x_1 + 0x_2 + 3x_3 \le 8$,

$5x_1 + 2x_2 + 2x_3 \le 10$

$0x_1 + 5x_2 + 4x_3 \le 15$

with $x_1, x_2, x_3 \ge 0$

b. Introducing S_1, S_2, S_3 as slack variables, the given LPP in standard form can be written as system of linear equation as

$$Z - 3x_1 - 5x_2 - 4x_3 + 0S_1 + 0S_2 + 0S_3 = 0$$
$$2x_1 + 0x_2 + 3x_3 + S_1 + 0S_2 + 0S_3 = 8$$
$$5x_1 + 2x_2 + 2x_3 + 0S_1 + S_2 + 0S_3 = 10$$
$$0x_1 + 5x_2 + 4x_3 + 0S_1 + 0S_2 + S_3 = 15$$

The simplex table is

B	Z	x_1	$x_2 \downarrow$	x_3	S_1	S_2	S_3	b	Ratio
	1	-3	-5	-4	0	0	0	0	
S_1	0	2	0	3	1	0	0	8	-
S_2	0	5	2	2	0	1	0	10	10/2 = 5
$\leftarrow S_3$	0	0	5	4	0	0	1	15	15/5 = 3

Basic variables : $S_1 = 8, S_2 = 10, S_3 = 15$

Non-basic variables: $x_1 = x_2 = x_3 = 0$

For these values, $Z = 0$

Pivot column: column of x_2, pivot row: R_4, Pivot element: 5

Applying $R_1 \to R_1 + R_4$, $R_3 \to R_3 - \frac{2}{5} R_4$

The next simplex table is

B	Z	$x_1 \downarrow$	x_2	x_3	S_1	S_2	S_3	b	Ratio
	1	-3	0	0	0	0	1	15	
S_1	0	2	0	3	1	0	0	8	8/2 = 4
$\leftarrow S_2$	0	5	0	2/5	0	0	-2/5	4	4/5 = 0.8
x_2	0	0	5	4	0	0	1	15	-

Basic variables : $S_1 = 8, S_2 = 4, 5x_2 = 15$, i.e., $x_2 = 3$

Non-basic variables: $x_1 = x_3 = S_3 = 0$

For these values, $Z = 15$

Pivot column: column of x_1, pivot row: R_3, Pivot element: 5

Applying $R_1 \to R_1 + \frac{3}{5} R_3$, $R_2 \to R_2 - \frac{2}{5} R_3$

The next simplex table is

B	Z	x_1	x_2	x_3	S_1	S_2	S_3	b
	1	0	0	6/25	0	3/5	19/25	87/5
S_1	0	0	0	71/25	1	-2/5	4/5	32/5
x_1	0	5	0	2/5	0	1	-2/5	4
x_2	0	0	5	4	0	0	1	15

Since there are no negative numbers in first row, so simplex process is completed.

From the last table,

$5x_1 = 4, 5x_2 = 15, x_3 = 0$

i.e., $x_1 = \frac{4}{5}, x_2 = 3, x_3 = 0$

$\therefore$ Max $Z = \frac{87}{5}$ at $\left(\frac{4}{5}, 3, 0\right)$

c. By the theorem of duality of the LPP its dual problem is defined by

minimize $Z^* = 8y_1 + 10y_2 + 15y_3$

Subject to constraints

$2y_1 + 5y_2 \geq 3$

$2y_2 + 5Y_3 \geq 5$

$3y_1 + 2y_2 + 4y_3 \geq 4$

With $y_1, y_2, y_3 \geq 0$

7. **The XYZ Plastic Company has received a government contract to produce different plastic values. The values must be highly pressure and heat resistant and the company has developed a three stage production process that will provide the values with the necessary properties involving work in three different chambers.**

Chamber 1 provides the necessary pressure resistant and can process values for 1200 minutes each week. Chamber 2 provides heat resistance and can process values for 900 minutes per week. Chamber 3 tests the values and can work 1300 minutes per week. The three value types and the time in minutes required in each chamber are

Value type	Time required in		
	Chamber 1	Chamber 2	Chamber 3
A	5	7	4
B	3	2	10
C	2	4	5

The government will buy all values that can be produced and the company will receive the following profit margins on each value.

Value A : Rs 15

Value B : Rs 13.50

Value C : Rs 10

How many values of each type should the company produce each week in order to maximum profits? Write the dual of the given LPP and give its economic interpretation.

Solution

Let x_1, x_2 and x_3 be each type of value the company should produce each week in order to maximum profits.

Time required in	Values Type			Time required in each week
	A	B	C	
Chamber 1	5	3	2	1200
Chamber 2	7	2	4	900
Chamber 3	4	10	5	1300
Profit in (Rs)	15	13.5	10	

Objective function in maximize $Z = 15x_1 + 13.5x_2 + 10x_3$

The constraints are $5x_1 + 3x_2 + 2x_3 \leq 1200$, $7x_1 + 2x_2 + 4x_3 \leq 900$, $4x_1 + 10x_2 + 5x_3 \leq 1300$, with $x_1, x_2, x_3 \geq 0$.

By the theorem of duality of LPP, its dual problem is defined by

Minimum $Z^* = 1200y_1 + 900y_2 + 1300y_3$

Subject to constraints $5y_1 + 7y_2 + 4y_3 \geq 15$, $3y_1 + 2y_2 + 10y_3 \geq 13.5$, $2y_1 + 4y_2 + 5y_3 \geq 10$ with $y_1, y_2, y_3 \geq 0$

Let $\overline{Z} = -Z^*$, then min. $Z^* = -(\max \overline{Z})$ and objective function becomes

Maximize $\bar{Z} = -1200y_1 - 900y_2 - 1300y_3$

Introducing surplus variables x_1, x_2, x_3 and artificial variables a_1, a_2 and a_3, the given objective function can be written as

$$\bar{Z} = -1200y_1 - 900y_2 - 1300y_3 - Ma_1 - a_2M - a_3M$$

With the above variables the LPP can be written as system of linear equations

$$\bar{Z} + 1200y_1 + 900y_2 + 1300y_2 + Ma_1 + Ma_2 + Ma_3 + 0x + 0x_2 + 0x_3 \ldots$$
$$5y_1 + 7y_2 + 4y_3 + a_1 - x_1 = 15$$
$$3y_1 + 2y_2 + 10y_3 + a_2 - x_2 = 13.5$$
$$2y_1 + 4y_2 + 5y_3 + a_3 - x_3 = 10$$

Now, the preliminary simplex table is

$\bar{Z}$	y_1	y_2	y_3	x_1	x_2	x_3	a_1	a_2	a_3
1	1200	900	1300	0	0	0	M	M	M
0	5	7	4	−1	0	0	1	0	0
0	3	2	10	0	−1	0	0	1	0
0	2	4	5	0	0	−1	0	0	1

Applying $R_1 \to R_2 - M(R_2 + R_3 + R_4)$

B	$\bar{Z}$	y_1	y_2	$y_3\downarrow$	x_1	x_2	x_3	a_1	a_2	a_3	b	Ratio
	1	1200−10M	900−13M	1300−19M	M	M	M	0	0	0	$-\frac{77}{2}M$	
a_1	0	5	7	4	−1	0	0	1	0	0	15	15/4
$\leftarrow a_2$	0	3	2	10	0	−1	0	0	1	0	27/2	$\frac{27/2}{10}$
a_3	0	2	4	5	0	0	−1	0	0	1	10	10/5

Basic variables : $a_1 = 15, a_2 = \frac{27}{2}, a_3 = 10$

Non-basic variables: $y_1 = y_2 = y_3 = x_1 = x_2 = x_3 = 0$

For these values, $Z = -\frac{77}{2}M$

Pivot column: column of y_3, pivot row: R_3, Pivot element: 10

Applying $R_1 \to R_1 + \left(\frac{19M - 1300}{10}\right)R_3, R_2 \to R_2 - \frac{2}{5}R_3, R_4 \to R_4 - \frac{1}{2}R_3$

The next simplex table is

B	$\bar{Z}$	y_1	$y_2\downarrow$	y_3	x_1	x_2	x_3	a_1	a_3	b	Ratio
	1	$810 - \frac{43M}{10}$	$640 - \frac{46M}{5}$	0	−M	$130 - \frac{9M}{10}$	−M	0	0	$\frac{-257M}{20} - 1755$	
a_1	0	19/5	31/5	0	−1	2/5	0	1	0	48/5	$\frac{48/5}{31/5} = 1.55$
y_3	0	3	2	10	0	−1	0	0	0	27/2	$\frac{27/2}{2} = 6.75$
$\leftarrow a_3$	0	1/2	3	0	0	1/2	−1	0	1	13/4	$\frac{13/4}{3} = 1.08$

Basic variables : $a_1 = \frac{48}{5}, y_3 = \frac{27}{20}, a_3 = \frac{13}{4}$

Non-basic variables: $y_1 = y_2 = x_2 = x_3 = a_2 = 0$

For these values, $\bar{Z} = -\frac{257M}{20} - 1755$

Since a_2 is the non-basic variables, so we remove the column of a_2.

Pivot column: column of y_2, pivot row: R_4, Pivot element: 3

Applying $R_1 \to R_1 + \frac{1}{3}\left(\frac{46M}{5} - 640\right)R_4, R_2 \to R_2 - \frac{31}{15}R_4, R_3 \to R_3 - \frac{2}{3}R_4$

The next simplex table is

B	Z	$y_1\downarrow$	y_2	y_3	x_1	x_2	x_3	a_1	b	Ratio
	1	$\frac{210}{3} - \frac{83M}{30}$	0	0	M	$\frac{70}{3} + \frac{19M}{30}$	$\frac{640}{3} - \frac{31M}{15}$	0	$\frac{-173M}{60} - \frac{7345}{3}$	
$\leftarrow a_1$	0	$\frac{83}{30}$	0	0	−1	$-\frac{19}{30}$	$\frac{31}{15}$	1	$\frac{173}{60}$	$\frac{173/60}{83/30} = 1.04$
y_3	0	$\frac{8}{3}$	0	10	0	$-\frac{4}{3}$	$\frac{2}{3}$	0	$\frac{34}{3}$	$\frac{34/3}{8/3} = 4.25$
y_2	0	$\frac{1}{2}$	3	0	0	$\frac{1}{2}$	−1	0	$\frac{13}{4}$	$\frac{13/4}{1/2} = 6.5$

Basic variables : $a_1 = \frac{173}{60}, y_3 = \frac{13}{40}, y_2 = \frac{13}{12}$

Non-basic variables: $y_1 = x_1 = x_2 = x_3 = 0$

For these values, $Z = -\frac{173M}{60} - \frac{7345}{3}$

Since a_1 is the non-basic variables, so we remove the column of a_1.

Pivot column: column of y_1, pivot row: R_2, Pivot element: $\frac{83}{80}$

Applying $R_1 \to R_1 + 30\left(\frac{83M}{30} - \frac{2110}{3}\right)R_2, R_3 \to R_3 - \frac{8}{83}R_2, R_4 \to R_4 - \frac{15}{83}R_2$

The next simplex table is

B	Z	y_1	y_2	y_3	x_1	x_2	$x_3\downarrow$	b	Ratio
	1	0	0	0	$\frac{21100}{83}$	$\frac{45900}{249}$	$\frac{-77700}{249}$	$\frac{-792150}{249}$	
$\leftarrow y_1$	0	$\frac{83}{30}$	0	0	−1	$-\frac{19}{30}$	$\frac{31}{15}$	$\frac{173}{60}$	$\frac{173/60}{31/15} = 1.39$
y_3	0	0	0	10	$\frac{80}{83}$	$\frac{-60}{83}$	$\frac{-110}{83}$	$\frac{710}{83}$	−
y_2	0	0	3	0	$\frac{15}{83}$	$\frac{51}{83}$	$\frac{-114}{83}$	$\frac{906}{332}$	

Basic variables : $\frac{83}{30}y_1 = \frac{173}{160}, 10y_3 = \frac{710}{83}, 3y_2 = \frac{906}{83}$

i.e., $y_1 = \frac{692}{166}, y_3 = \frac{71}{83}, y_2 = \frac{302}{83}$

Non-basic variables: $x_1 = x_2 = x_3 =$

For these values, $Z = \frac{-792150}{249}$

Pivot column: column of x_3, pivot row: R_2, Pivot element: $\frac{31}{15}$

Applying $R_1 \to R_1 + \frac{388500}{2573}R_2, R_3 \to R_3 + \frac{1650}{2573}R_2, R_4 \to R_4 + \frac{1770}{2573}R_2$

The next simplex table is

B	Z	y_1	y_2	y_3	x_1	x_2	x_3	b
	1	$\frac{388500}{2573}$	0	0	$\frac{3200}{31}$	$\frac{684750}{7719}$	0	$\frac{22196125}{7719}$
x_1	0	$\frac{83}{30}$	0	0	-1	$\frac{-19}{30}$	$\frac{31}{15}$	$\frac{173}{60}$
y_3	0	$\frac{1650}{2573}$	0	10	$\frac{10}{31}$	$\frac{35}{31}$	0	$\frac{54227}{5146}$
y_2	0	$\frac{1770}{2573}$	3	0	$\frac{-1305}{2573}$	$\frac{460}{2573}$	0	$\frac{12125}{2573}$

Since, there are no negative terms in first row so simplex process is completed.

From the last table,

$$y_1 = 0,\ 3y_2 = \frac{12125}{2573},\ 10y_3 = \frac{54227}{5146}$$

i.e., $y_1 = 0,\ y_2 = \frac{12125}{7719}$ and $y_3 = \frac{54227}{51460}$

$\therefore$ Minimum $Z = \frac{21196125}{7719}$ at $\left(0, \frac{12125}{7719}, \frac{54227}{51460}\right)$

and from the same table

$$x_1 = \frac{3200}{31},\ x_2 = \frac{684750}{7719},\ x_3 = 0$$

$\therefore$ Maximum $Z = \frac{21196125}{7719}$ at $x_1 = \frac{3200}{31},\ x_2 = \frac{684750}{7719},\ x_3 = 0$

Unit 4

Vector Algebra

Exercise 4.1

1. If $\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k})$, $\vec{b} = (2\hat{i} + \hat{j} - \hat{k})$ and $\vec{c} = (\hat{j} + \hat{k})$ find $[\vec{a}\ \vec{b}\ \vec{c}]$.

Solution

Given vectors are

$$\vec{a} = (\hat{i} - 2\hat{j} + 3\hat{k}) = (1, -2, 3)$$
$$\vec{b} = (2\hat{i} + \hat{j} - \hat{k}) = (2, 1, -1)$$
$$\vec{c} = (\hat{j} + \hat{k}) = (0, 1,)$$

Then $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 1(1 + 1) + 2(2 - 0) + 3(2 - 0)$$

$$= 2 + 4 + 6 = 12$$

2. Find the scalar triple product of $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$.

Solution

Given vectors are

$$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k} = (2, -3, 4)$$
$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k} = (1, 2, -1)$$
$$\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k} = (3, -1, 2)$$

Then, $\vec{a}\cdot\vec{b}\times\vec{c} = \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$

$= 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$

$= 2(4-1) + (2+3) + 4(-1-6)$

$= 6 + 15 - 28$

$= -7$

3. Find the volume of parallelopiped whose concurrent edges are given by:

i. $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + 2\hat{k}, \vec{c} = 2\hat{i} + \hat{j} - \hat{k}$

Solution

Given vectors are

$\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k} \quad = (2, -3, 1)$

$\vec{b} = \hat{i} - \hat{j} + 2\hat{k} \quad = (1, -1, 2)$

$\vec{c} = 2\hat{i} + \hat{j} - \hat{k} \quad = (2, 1, -1)$

The volume of parallelopiped

$= \vec{a}\cdot\vec{b}\times\vec{c}$

$= \begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix}$

$= 2\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$

$= 2(1-2) + 3(-1-4) + 1(1+2)$

$= -2 - 15 + 3$

$= -14$

$= 14$ cubic units

ii. $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = \hat{i} - \hat{j} + \hat{k}, \vec{c} = \hat{i} + 2\hat{j} - \hat{k}$

Solution

Given vectors are

$\vec{a} = \hat{i} + \hat{j} + \hat{k} \quad = (1, 1, 1)$

$\vec{b} = \hat{i} - \hat{j} + \hat{k} \quad = (1, -1, 1)$

$\vec{c} = \hat{i} + 2\hat{j} - \hat{k} \quad = (1, 2, -1)$

The volume of parallelopiped

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$

$= 1\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$

$= 1(1-2) - 1(-1-1) + 1(2+1)$

$= -1 + 2 + 3$

$= 4$ cubic units

iii. $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + 3\hat{k}$.

Solution

Given vectors are

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k} \quad = (1, 2, 3)$

$\vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k} \quad = (3, 4, -5)$

$\vec{c} = \hat{i} - 2\hat{j} + 3\hat{k} \quad = (1, -2, 3)$

The volume of parallelopiped

$= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & -5 \\ 1 & -2 & 3 \end{vmatrix}$

$= 1\begin{vmatrix} 4 & -5 \\ -2 & 3 \end{vmatrix} - 2\begin{vmatrix} 3 & -5 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix}$

$= 1(12-10) - 2(9+5) + 3(-6-4)$

$= 2 - 28 - 30$

$= 56$ cubic units

4. Show that four points with given position vectors are coplanar.

$-\hat{i} + 2\hat{j} - 4\hat{k},\ 2\hat{i} - \hat{j} + 3\hat{k},\ 6\hat{i} + 2\hat{j} - \hat{k},\ 12\hat{i} - \hat{j} - 3\hat{k}$

Solution

Let O be origin

Let $\vec{OA} = -\hat{i} + 2\hat{j} - 4\hat{k} \quad = (-1, 2, -4)$

$\vec{OB} = 2\hat{i} - \hat{j} + 3\hat{k} \quad (2, -1, 3)$

$\vec{OC} = 6\hat{i} + 2\hat{j} - \hat{k} \quad (6, 2, -1)$

$\vec{OD} = 12\hat{i} - \hat{j} - 3\hat{k} \quad (12, -1, -3)$

Then $\vec{AB} = \vec{OB} - \vec{OA} \quad = (2, -1, 3) - (1, 2, -4)$

$= (3, -3, 7)$

$\vec{BC} = \vec{OC} - \vec{OB} \quad = (6, 2, -1) - (2, -1, 3)$

$= (4, 3, -4)$

$\vec{CD} = \vec{OD} - \vec{OC} \quad = (12, -1, -3) - (6, 2, -1)$

$= (6, -3, -2)$

If given points are coplanar then

$[\vec{AB}\ \vec{BC}\ \vec{CD}] = 0$

L.H.S. $= [\vec{AB}\ \vec{BC}\ \vec{CD}]$

$= \vec{AB}\cdot(\vec{BC}\times\vec{CD})$

$= \begin{vmatrix} 3 & -3 & 7 \\ 4 & 3 & -4 \\ 6 & -3 & 2 \end{vmatrix}$

$= 3\begin{vmatrix} 3 & -4 \\ -3 & -2 \end{vmatrix} + 3\begin{vmatrix} 4 & -4 \\ 6 & -2 \end{vmatrix} + 7\begin{vmatrix} 4 & 3 \\ 6 & -3 \end{vmatrix}$

$= 3(-6-12) + 3(-8+24) + 7(-12-18)$

$= -54 + 48 - 210$

$\neq 0$

Hence, given points are not coplanar.

5. Find the value of λ such that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \lambda\hat{j} - 3\hat{k}$, $\vec{c} = 3\hat{i} - 4\hat{j} + 5\hat{k}$ are coplanar.

Solution

Given vectors are

$\vec{a} = 2\hat{i} - \hat{j} + \hat{k} = (2, -1, 1)$

$\vec{b} = \hat{i} + \lambda\hat{j} - 3\hat{k} = (1, \lambda, -3)$

$\vec{c} = 3\hat{i} - 4\hat{j} + 5\hat{k} = (3, -4, 5)$

If given vectors are coplanar then

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

or, $\begin{vmatrix} 2 & -1 & 1 \\ 1 & \lambda & -3 \\ 3 & -4 & 5 \end{vmatrix} = 0$

or, $2\begin{vmatrix} \lambda & -3 \\ -4 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & \lambda \\ 3 & -4 \end{vmatrix} = 0.$

or, $2(5\lambda - 12) + 1(5 + 9) + 1(-4 - 3\lambda) = 0$

or, $10\lambda - 24 + 14 - 4 - 3\lambda = 0$

or, $7\lambda = 24 - 10$

or, $7\lambda = 14$

or, $\lambda = 2$

$\therefore$ The value of $\lambda = 2$.

6. If the four points $2\hat{i} + 3\hat{j} - \hat{k}$, $\hat{i} - 2\hat{j} + 3\hat{k}$, $\lambda\hat{i} + 4\hat{j} - 2\hat{k}$ and $\hat{i} - 6\hat{j} + 6\hat{k}$ are coplanar find the value of λ.

Solution

Let O be origin

Let given vectors be

$\vec{OA} = 2\hat{i} + 3\hat{j} - \hat{k} = (2, 3, -1)$

$\vec{OB} = \hat{i} - 2\hat{j} + 3\hat{k} = (1, -2, 3)$

$\vec{OC} = \lambda\hat{i} + 4\hat{j} - 2\hat{k} = (\lambda, 4, -2)$

$\vec{OD} = \hat{i} - 6\hat{j} + 6\hat{k} = (1, -6, 6)$

Then,

$\vec{AB} = \vec{OB} - \vec{OA}$

$= (1, -2, 3) - (2, 3, -1)$

$= (-1, -5, 4)$

$\vec{BC} = \vec{OC} - \vec{OB}$

$= (\lambda, 4, -2) - (1, -2, 3)$

$= (\lambda - 1, 6, -5)$

$\vec{CD} = \vec{OD} - \vec{OC}$

$= (1, -6, 6) - (\lambda, 4, -2)$

$= (1 - \lambda, -10, 8)$

If given vectors are coplanar, then

$[\vec{AB}\ \vec{BC}\ \vec{CD}] = 0$

or, $\begin{vmatrix} -1 & -5 & 4 \\ \lambda-1 & 6 & -5 \\ 1-\lambda & -10 & 8 \end{vmatrix} = 0$

or, $-1\begin{vmatrix} 6 & -5 \\ -10 & 8 \end{vmatrix} + 5\begin{vmatrix} \lambda-1 & -5 \\ 1-\lambda & 8 \end{vmatrix} + 4\begin{vmatrix} \lambda-1 & 6 \\ 1-\lambda & -10 \end{vmatrix} = 0$

or, $-1(-48-50) + 5(\lambda - 1)\begin{vmatrix} 1 & -5 \\ -1 & -8 \end{vmatrix} + 4(\lambda - 1)\begin{vmatrix} 1 & 6 \\ -1 & -10 \end{vmatrix} = 0$

or, $98 + 5(\lambda - 1)(-8-5) + 4(\lambda - 1)(-10 + 6) = 0$

or, $98 + (\lambda - 1)(-65-16) = 0$

or, $-81(\lambda - 1) = -98$

or, $\lambda - 1 = \frac{98}{81}$

or, $\lambda = 1 + \frac{98}{81} = \frac{81 + 98}{81} = \frac{179}{81}$

7. For $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$, $\vec{c} = \hat{i} + 2\hat{j} - \hat{k}$, verify that $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$.

Solution

Given vectors are

$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k} = (2, -2, 1)$

$\vec{b} = 2\hat{i} + \hat{j} + \hat{k} = (2, 1, 1)$

$\vec{c} = \hat{i} + 2\hat{j} - \hat{k} = (1, 2, -1)$

Then,

$\vec{b} \times \vec{c} = (2, 1, 1) \times (1, 2, -1)$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$

$= \vec{i}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

$= \vec{i}(-1 - 2) - \vec{j}(-2 - 1) + \vec{k}(4 - 1)$

$= -3\vec{i} + 3\vec{j} + 3\vec{k}$

$= (-3, 3, 3)$

$$\text{L.H.S.} = \vec{a}\times(\vec{b}\times\vec{c})$$
$$= (2,-2,1)\times(-3,3,3)$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$$
$$= \vec{i}\begin{vmatrix} -2 & 1 \\ 3 & 3 \end{vmatrix} - \vec{j}\begin{vmatrix} 2 & 1 \\ -3 & 3 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & -2 \\ -3 & 3 \end{vmatrix}$$
$$= \vec{i}(-6-3) - \vec{j}(6+3) + \vec{k}(6-6)$$
$$= -9\vec{i} - 9\vec{j} + 0\vec{k}$$

$$\vec{a}\cdot\vec{b} = (2,-2,1)\cdot(2,1,1)$$
$$= 4-2+1$$
$$= 2+1 = 3$$

$$\vec{a}\cdot\vec{c} = (2,-2,1)\cdot(1,2,-1)$$
$$= 2-4-1$$
$$= 2-5 = -3$$

$$\text{R.H.S.} = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$
$$= -3(2\vec{i}+\vec{j}+\vec{k}) - 3(\vec{i}+2\vec{j}-\vec{k})$$
$$= -6\vec{i} - 3\vec{j} - 3\vec{k} - 3\vec{i} - 6\vec{j} + 3\vec{k}$$
$$= -9\vec{i} - 9\vec{j} + 0\vec{k}$$

Here, $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$. Hence, verified.

8. Show that $\hat{i}\times(\vec{a}\times\hat{i}) + \hat{j}\times(\vec{a}\times\hat{j}) + \hat{k}\times(\vec{a}\times\hat{k}) = 2\vec{a}$.

Solution

We have,

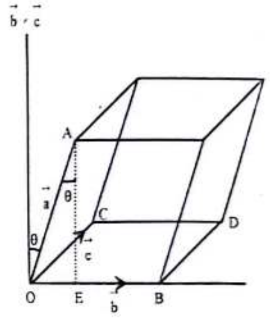
$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$$
$$\vec{i}\times(\vec{a}\times\vec{i}) = \vec{a}(\vec{i}\cdot\vec{i}) - \vec{i}(\vec{i}\cdot\vec{a})$$
$$\vec{j}\times(\vec{a}\times\vec{j}) = \vec{a}(\vec{j}\cdot\vec{j}) - \vec{j}(\vec{j}\cdot\vec{a})$$
$$\vec{k}\times(\vec{a}\times\vec{k}) = \vec{a}(\vec{k}\cdot\vec{k}) - \vec{k}(\vec{k}\cdot\vec{a})$$

$$\text{L.H.S.} = \vec{i}\times(\vec{a}\times\vec{i}) + \vec{j}\times(\vec{a}\times\vec{j}) + \vec{k}\times(\vec{a}\times\vec{k})$$
$$= \vec{a}(\vec{i}\cdot\vec{i}) - \vec{i}(\vec{i}\cdot\vec{a}) + \vec{a}(\vec{j}\cdot\vec{j}) - \vec{j}(\vec{j}\cdot\vec{a}) + \vec{a}(\vec{k}\cdot\vec{k}) - \vec{k}(\vec{k}\cdot\vec{a})$$
$$= 3\vec{a} - \{\vec{i}(\vec{i}\cdot\vec{a}) + \vec{j}(\vec{j}\cdot\vec{a}) + \vec{k}(\vec{k}\cdot\vec{a})\}$$
$$= 3\vec{a} - \vec{a}$$
$$= 2\vec{a}$$
$$= \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S.

9. If $\vec{a}, \vec{b}, \vec{c}$ are three non coplanar vectors then express $\vec{b}\times\vec{c}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Solution



Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$ be three edges of a parallelopiped. $\vec{b}\times\vec{c}$ is a perpendicular vectors with vector $\vec{b}$ and $\vec{c}$.

Let $\vec{a}$ makes an angle θ with $\vec{b}\times\vec{c}$.

$|\vec{b}\times\vec{c}|$ represents the area of parallelogram OBDC. Its adjacent sides are $\vec{OB} = \vec{b}$ and $\vec{OC} = \vec{c}$. Let us draw $\vec{AE}\perp\vec{OB}$. Let $\measuredangle OAE = \theta$.

In right angled triangle OAE, $\cos\theta = \dfrac{AE}{OA}$ or

$\vec{AE} = |\vec{a}|\cos\theta = \text{height}$

$$\text{Now, } \vec{a}\cdot\vec{b}\times\vec{c} = |\vec{a}|\,|\vec{b}\times\vec{c}|\cos\theta$$
$$= |\vec{b}\times\vec{c}|\cdot|\vec{a}|\cos\theta$$
$$= \text{Area of parallelogram} \times AE$$
$$= \text{Area of parallelogram} \times \text{height}$$
$$= \text{Volume of parallelopiped}$$

Hence, triple product of three vectors represents volume of parallelopied.

10. Using vector product of three vectors and vector sum condition of coplanarity, show that $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar if $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors. [Hint: Show that $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$]

Solution

We have to show that: $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

$$\text{L.H.S.} = \vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b})$$
$$= \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) + \vec{c}(\vec{b}\cdot\vec{a}) - \vec{a}(\vec{b}\cdot\vec{c}) + \vec{a}(\vec{c}\cdot\vec{b}) - \vec{b}(\vec{c}\cdot\vec{a})$$
$$= \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) + \vec{c}(\vec{a}\cdot\vec{b}) - \vec{a}(\vec{b}\cdot\vec{c}) + \vec{a}(\vec{b}\cdot\vec{c}) - \vec{b}(\vec{a}\cdot\vec{c})$$
$$= 0$$
$$= \text{R.H.S.}$$

Hence, $\vec{a}\times(\vec{b}\times\vec{c})$, $\vec{b}\times(\vec{c}\times\vec{a})$ and $\vec{c}\times(\vec{a}\times\vec{b})$ are coplanar.

Exercise 4.2

1. For the vectors $\vec{a} = -\hat{i} + 4\hat{j} - 3\hat{k}$, $\vec{b} = 3\hat{i} + 2\hat{j} - 5\hat{k}$, $\vec{c} = 3\hat{i} + 8\hat{j} - 5\hat{k}$, $\vec{d} = -3\hat{i} + 2\hat{j} + \hat{k}$, verify that:

   a. $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{c}[\vec{a}\ \vec{b}\ \vec{d}] - \vec{d}\ [\vec{a}\ \vec{b}\ \vec{c}]$

Solution

Given vectors are
$$\vec{a} = -\hat{i} + 4\hat{j} - 3\hat{k} \qquad = (-1, 4, -3)$$
$$\vec{b} = 3\hat{i} + 2\hat{j} - 5\hat{k} \qquad = (3, 2, -5)$$
$$\vec{c} = 3\hat{i} + 8\hat{j} - 5\hat{k} \qquad = (3, 8, -5)$$
$$\vec{d} = -3\hat{i} + 2\hat{j} + \hat{k} \qquad = (-3, 2, 1)$$

$$\vec{a} \times \vec{b} = (-1, 4, -3) \times (3, 2, -5)$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 4 & -3 \\ 3 & 2 & -5 \end{vmatrix}$$
$$= \vec{i}\begin{vmatrix} 4 & -3 \\ 2 & -5 \end{vmatrix} - \vec{j}\begin{vmatrix} -1 & -3 \\ 3 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} -1 & 4 \\ 3 & 2 \end{vmatrix}$$
$$= \vec{i}(-20 + 6) - \vec{j}(5 + 9) + \vec{k}(-2 - 12) = -14\vec{i} - 14\vec{j} - 14\vec{k}$$

$$\vec{c} \times \vec{d} = (3, 8, -5) \times (-3, 2, 1)$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 8 & -5 \\ -3 & 2 & 1 \end{vmatrix}$$
$$= \vec{i}\begin{vmatrix} 8 & -5 \\ 2 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 3 & -5 \\ -3 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 3 & 8 \\ -3 & 2 \end{vmatrix} = -18\vec{i} - 12\vec{j} + 30\vec{k}$$

$$[\vec{a}\ \vec{b}\ \vec{d}] = \vec{a} \cdot \vec{b} \times \vec{d}$$
$$= \begin{vmatrix} -1 & 4 & -3 \\ 3 & 2 & -5 \\ -3 & 2 & 1 \end{vmatrix} = -1\begin{vmatrix} 2 & -5 \\ 2 & 1 \end{vmatrix} - 4\begin{vmatrix} 3 & -5 \\ -3 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ -3 & 2 \end{vmatrix}$$
$$= 1 \times 12 - 4 \times (-12) - 3 \times 12 = -12 + 48 - 36 = 0$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a} \cdot \vec{b} \times \vec{c}$$
$$= \begin{vmatrix} -1 & 4 & -3 \\ 3 & 2 & -5 \\ 3 & 8 & -5 \end{vmatrix} = -1\begin{vmatrix} 2 & -5 \\ 8 & -5 \end{vmatrix} - 4\begin{vmatrix} 3 & -5 \\ 3 & -5 \end{vmatrix} - 3\begin{vmatrix} 3 & 2 \\ 3 & 8 \end{vmatrix}$$
$$= -1(-10 + 40) - 4(-15 + 15) - 3(24 - 6) = -30 - 0 - 54 = -84$$

L.H.S.
$$= (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$$
$$= (-14, -14, -14) \times (18, 12, 30)$$
$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -14 & -14 & -14 \\ 18 & 12 & 30 \end{vmatrix}$$

$$= (-14) \times 6 \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 2 & 5 \end{vmatrix}$$
$$= -84\left\{\vec{i}\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}\right\} = -84\ \{3\vec{i} - 2\vec{j} - \vec{k}\}$$

R.H.S.
$$= \vec{c}[\vec{a}\ \vec{b}\ \vec{d}] - \vec{d}\ [\vec{a}\ \vec{b}\ \vec{c}]$$
$$= 0 - (-84)\ (-3\vec{i} + 2\vec{j} + \vec{k})$$
$$= -84\ (3\vec{i} - 2\vec{j} - \vec{k})$$

∴ L.H.S. = R.H.S. proved.

   b. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})\ (\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})\ (\vec{b} \cdot \vec{c})$

Solution

$$\vec{a} \cdot \vec{c} = (-1, 4, -3) \cdot (3, 8, -5) = -3 + 32 + 15 = 44$$
$$\vec{b} \cdot \vec{d} = (3, 2, -5) \cdot (-3, 2, 1) = -9 + 4 - 5 = -10$$
$$\vec{a} \cdot \vec{d} = (-1, 4, -3) \cdot (-3, 2, 1) = 3 + 8 - 3 = 8$$
$$\vec{b} \cdot \vec{c} = (3, 2, -5) \cdot (3, 8, -5) = 9 + 16 + 25 = 50$$

L.H.S.
$$= (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$$
$$= (-14 - 14 - 14) \cdot (18 + 12 + 30)$$
$$= -84\ (1, 1, 1) \cdot (3, 2, 5) = -84\ (3 + 2 + 5) = -840$$

R.H.S.
$$= (\vec{a} \cdot \vec{c})\ (\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})\ (\vec{b} \cdot \vec{c})$$
$$= 44 \cdot (-10) - 8 \cdot 50 = -440 - 400 = -840$$

∴ L.H.S. = R.H.S. Hence, verified.

2. Show that $[\vec{a} \times \vec{b}\ \ \vec{c} \times \vec{d}\ \ \vec{e} \times \vec{f}] = [\vec{a}\ \vec{b}\ \vec{d}]\ [\vec{c}\ \vec{e}\ \vec{f}] - [\vec{a}\ \vec{b}\ \vec{c}]\ [\vec{d}\ \vec{e}\ \vec{f}]$.

Solution

L.H.S.
$$= [\vec{a} \times \vec{b}\ \ \vec{c} \times \vec{d}\ \ \vec{e} \times \vec{f}]$$
$$= [\vec{a} \times \vec{b}\ \ (\vec{c} \times \vec{d}) \times (\vec{e} \times \vec{f})]$$
$$= \vec{a} \times \vec{b}\ [(\vec{c}\ \vec{e}\ \ \vec{f})]\vec{d} - [(\vec{d}\ \ \vec{e}\ \ \vec{f})]\ \vec{c}]$$
$$= [\vec{a} \times \vec{b} \cdot \vec{d}]\ [\vec{c}\ \vec{e}\ \ \vec{f}] - [\vec{a} \cdot \vec{b} \times \vec{c}]\ [\vec{d}\ \ \vec{e}\ \ \vec{f}]$$
$$= [\vec{a}\ \vec{b}\ \vec{d}]\ [\vec{c}\ \vec{e}\ \vec{f}] - [\vec{a}\ \vec{b}\ \vec{c}]\ [\vec{d}\ \vec{e}\ \vec{f}] = \text{R.H.S.}$$

L.H.S. = R.H.S. proved

3. Prove that $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) + (\vec{a} \cdot \vec{b})\ (\vec{a} \cdot \vec{c}) = (\vec{a} \cdot \vec{a})\ (\vec{b} \cdot \vec{c})$

Solution

L.H.S.
$$= (\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) + (\vec{a} \cdot \vec{b})\ (\vec{a} \cdot \vec{c})$$
$$= \begin{vmatrix} \vec{a} \cdot \vec{a} & \vec{a} \cdot \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{c} \end{vmatrix} + (\vec{a} \cdot \vec{b})\ (\vec{a} \cdot \vec{c})$$
$$= (\vec{a} \cdot \vec{a})\ (\vec{b} \cdot \vec{c}) - (\vec{b} \cdot \vec{a})\ (\vec{a} \cdot \vec{c}) + (\vec{a} \cdot \vec{b})\ (\vec{a} \cdot \vec{c})$$

$= (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{c}) - (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c}) + (\vec{a}\cdot\vec{b})(\vec{a}\cdot\vec{c})$

$= \vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{c}) = \text{R.H.S.}$

$\therefore$ L.H.S. = R.H.S. proved

4. For any four vectors show that $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} + [\vec{c}\ \vec{a}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]$

Solution

Let $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors. Then,

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$ ... (1)

$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$ ... (2)

From equation (1) and (2), we get

$[\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$

or, $[\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

or, $[\vec{a}\ \vec{b}\ \vec{c}]\vec{d} = [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} - [\vec{c}\ \vec{a}\ \vec{d}]\vec{b} + [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}$

Hence, L.H.S. = R.H.S. proved

5. Prove that $\vec{a}\times[\vec{b}\times(\vec{c}\times\vec{d})] = (\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - (\vec{a}\cdot\vec{b})(\vec{c}\times\vec{d})$.

Solution

L.H.S. $= \vec{a}\times[\vec{b}\times(\vec{c}\times\vec{d})]$

$= \vec{a}\times[(\vec{b}\cdot\vec{d})\,\vec{c} - (\vec{b}\cdot\vec{c})\,\vec{d}]$

$= (\vec{b}\cdot\vec{d})(\vec{a}\times\vec{c}) - (\vec{b}\cdot\vec{c})(\vec{a}\times\vec{d}) = \text{M.H.S.}$

L.H.S. $= \vec{a}\times[\vec{b}\times(\vec{c}\times\vec{d})]$

$= \vec{a}\times[\vec{b}\times\vec{n}]$ $\quad [\because \text{ let } \vec{n} = \vec{c}\times\vec{d}]$

$= (\vec{a}\cdot\vec{n})\vec{b} - (\vec{a}\cdot\vec{b})\vec{n}$

$= [\vec{a}\cdot\vec{c}\times\vec{d}]\vec{b} - (\vec{a}\cdot\vec{b})(\vec{c}\times\vec{d})$

$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - (\vec{a}\cdot\vec{b})(\vec{c}\times\vec{d}) = \text{R.H.S.}$

$\therefore$ L.H.S. = M.H.S. = R.H.S. proved

6. Prove that $2(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = \begin{vmatrix} \vec{a} & \vec{b} & -\vec{c} & -\vec{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$.

Solution

We have to prove $2(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = \begin{vmatrix} \vec{a} & \vec{b} & -\vec{c} & -\vec{d} \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$

$(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{d}\ \vec{b}]\,\vec{a} - [\vec{c}\ \vec{d}\ \vec{a}]\vec{b}$ ... (1)

$\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{a}\ \vec{b}]\,\vec{d} - [\vec{d}\ \vec{a}\ \vec{b}]\vec{c}$ ... (2)

L.H.S. $= 2(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b}) = [\vec{c}\ \vec{d}\ \vec{b}]\,\vec{a} - [\vec{c}\ \vec{d}\ \vec{a}]\vec{b} + [\vec{c}\ \vec{a}\ \vec{b}]\,\vec{d} - [\vec{d}\ \vec{a}\ \vec{b}]\vec{c}$

$$= \begin{vmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\vec{a} - \begin{vmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \\ a_1 & a_2 & a_3 \end{vmatrix}\vec{b} - \begin{vmatrix} d_1 & d_2 & d_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\vec{c} + \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} c_1 & d_1 & b_1 \\ c_2 & d_2 & b_2 \\ c_3 & d_3 & b_3 \end{vmatrix}\vec{a} - \begin{vmatrix} c_1 & d_1 & a_1 \\ c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \end{vmatrix}\vec{b} - \begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}\vec{c} + \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}\vec{d}$$

$$= \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}\vec{a} - \begin{vmatrix} c_1 & d_1 & a_1 \\ c_2 & d_2 & a_2 \\ c_3 & d_3 & a_3 \end{vmatrix}\vec{b} - \begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}\vec{c} + \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\vec{d}$$

$$= \begin{vmatrix} \vec{a} & \vec{b} & -\vec{c} & -\vec{d} \\ a_1 & b_2 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = \text{R.H.S.}$$

$\therefore$ L.H.S. = R.H.S.

7. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar and $\vec{a}$ is not parallel to $\vec{b}$, prove that:

$$\begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}\vec{c} = \begin{vmatrix} \vec{c}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{c}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}\vec{a} + \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{c}\cdot\vec{a} \\ \vec{a}\cdot\vec{b} & \vec{c}\cdot\vec{b} \end{vmatrix}\vec{b}.$$

Solution

If $\vec{a}\ \vec{b}\ \vec{c}$ are coplanar, let m and n be any scalar. Then,

Let $\vec{c} = m\vec{a} + n\vec{b}$ ... (1)

Taking cross product with $\vec{b}$ on both sides

$\vec{c}\times\vec{b} = m\vec{a}\times\vec{b} + n\vec{b}\times\vec{b}$

or, $\vec{c}\times\vec{b} = m\vec{a}\times\vec{b}$

Taking dot product with $\vec{a}\times\vec{b}$ on both sides

$\vec{c}\times\vec{b}\cdot\vec{a}\times\vec{b} = m(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$

or, $\dfrac{\vec{c}\times\vec{b}\cdot\vec{a}\times\vec{b}}{\vec{a}\times\vec{b}\cdot\vec{a}\times\vec{b}} = m$ ... (2)

Taking cross product with $\vec{a}$ on both sides

$\vec{c}\times\vec{a} = m\vec{a}\times\vec{a} + n\vec{b}\times\vec{a}$

or, $\vec{c}\times\vec{a} = n\vec{b}\times\vec{a}$

or, $\vec{a}\times\vec{c} = n\vec{a}\times\vec{b}$

Taking dot product with $\vec{a}\times\vec{b}$ on both sides

$\vec{a}\times\vec{c}\cdot\vec{a}\times\vec{b} = n\vec{a}\times\vec{b}\cdot\vec{a}\times\vec{b}$

or, $\dfrac{\vec{a}\times\vec{c}\cdot\vec{a}\times\vec{b}}{\vec{a}\times\vec{b}\cdot\vec{a}\times\vec{b}} = n$

Then equation (1) becomes

$\vec{c} = \dfrac{\vec{c}\times\vec{b}\cdot\vec{a}\times\vec{b}}{\vec{a}\times\vec{b}\cdot\vec{a}\times\vec{b}}\vec{a} + \dfrac{\vec{a}\times\vec{c}\cdot\vec{a}\times\vec{b}}{\vec{a}\times\vec{b}\cdot\vec{a}\times\vec{b}}\vec{b}$

or, $(\vec{a}\times\vec{b}\cdot\vec{a}\times\vec{b})\vec{c}=(\vec{c}\times\vec{b}\cdot\vec{a}\times\vec{b})\vec{a}+(\vec{a}\times\vec{c}\cdot\vec{a}\times\vec{b})\vec{b}$

or, $\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{vmatrix}\vec{c}=\begin{vmatrix}\vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{vmatrix}\vec{a}+\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b}\end{vmatrix}\vec{b}$

or, $\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b}\end{vmatrix}\vec{c}=\begin{vmatrix}\vec{c}\cdot\vec{a} & \vec{a}\cdot\vec{b}\\ \vec{c}\cdot\vec{b} & \vec{b}\cdot\vec{b}\end{vmatrix}\vec{a}+\begin{vmatrix}\vec{a}\cdot\vec{a} & \vec{c}\cdot\vec{a}\\ \vec{a}\cdot\vec{b} & \vec{c}\cdot\vec{b}\end{vmatrix}\vec{b}$

Hence, proved

8. Find set of reciprocal system of vectors:

i. $\vec{a}=-\hat{i}+2\hat{j}+2\hat{k},\ \vec{b}=2\hat{i}+3\hat{j}+\hat{k},\ \vec{c}=\hat{i}-\hat{j}-2\hat{k}.$

Solution

Given vectors are

$\vec{a}=-\hat{i}+2\hat{j}+2\hat{k}\quad=(-1,2,2)$

$\vec{b}=2\hat{i}+3\hat{j}+\hat{k}\quad=(2,3,1)$

$\vec{c}=\hat{i}-\hat{j}-2\hat{k}.\quad=(1,-1,-2)$

Now, $[\vec{a}\ \vec{b}\ \vec{c}]=\vec{a}\cdot\vec{b}\times\vec{c}$

$$=\begin{vmatrix}-1 & 2 & 2\\ 2 & 3 & 1\\ 1 & -1 & -2\end{vmatrix}$$

$$=-1\begin{vmatrix}3 & 1\\ -1 & 2\end{vmatrix}-2\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}+2\begin{vmatrix}2 & 3\\ 1 & -1\end{vmatrix}$$

$$=-1(-6+1)-2(-4-1)+2(-2-3)=5+10-10=5$$

$\vec{a}\times\vec{b}=(1,2,2)\times(2,3,-1)$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 2 & 2\\ 2 & 3 & 1\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}2 & 2\\ 3 & 1\end{vmatrix}-\vec{j}\begin{vmatrix}-1 & 2\\ 2 & 1\end{vmatrix}+\vec{k}\begin{vmatrix}-1 & 2\\ 2 & 3\end{vmatrix}=-4\vec{i}+5\vec{j}-7\vec{k}$$

$\vec{b}\times\vec{c}=(2,3,-1)\times(1,-1,-2)$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & 1\\ 1 & -1 & -1\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}3 & 1\\ -1 & -1\end{vmatrix}-\vec{j}\begin{vmatrix}2 & 1\\ 1 & -2\end{vmatrix}+\vec{k}\begin{vmatrix}2 & 3\\ 1 & -1\end{vmatrix}$$

$$=\vec{i}(-6+1)-\vec{j}(-4-1)+\vec{k}(-2-3)=-5\vec{i}+5\vec{j}-5\vec{k}$$

$\vec{c}\times\vec{a}=(1,-1,-2)\times(-1,2,2)$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & -2\\ -1 & 2 & 2\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}-1 & -2\\ 2 & 2\end{vmatrix}-\vec{j}\begin{vmatrix}1 & -2\\ -1 & 2\end{vmatrix}+\vec{k}\begin{vmatrix}1 & -1\\ -1 & 2\end{vmatrix}$$

$$=\vec{i}(-2+4)-\vec{j}(2-2)+\vec{k}(2-1)=2\vec{i}+0\vec{j}+\vec{k}$$

The reciprocal vector of $\vec{a}=\vec{a}'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$$=\frac{1}{5}(-5\vec{i}+5\vec{j}-5\vec{k})=-\vec{i}+\vec{j}-\vec{k}$$

The reciprocal vector of $\vec{b}=\vec{b}'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$$=\frac{2\vec{i}+0\vec{j}+\vec{k}}{5}=\frac{2}{5}\vec{i}+\frac{1}{5}\vec{k}$$

The reciprocal vector of $\vec{c}=\vec{c}'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$$=\frac{-4\vec{i}+5\vec{j}-7\vec{k}}{5}=-\frac{4}{5}\vec{i}+\vec{j}-\frac{7}{5}\vec{k}$$

ii. $\vec{a}=2\hat{i}+3\hat{j}-\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}-3\hat{k},\ \vec{c}=3\hat{i}-4\hat{j}+2\hat{k}.$

Solution

Given vectors are

$\vec{a}=2\hat{i}+3\hat{j}-\hat{k}\quad=(2,3,-1)$

$\vec{b}=-\hat{i}+2\hat{j}-3\hat{k}\quad=(-1,2,-3)$

$\vec{c}=3\hat{i}-4\hat{j}+2\hat{k}\quad=(3,-4,2)$

Now, $\vec{a}\times\vec{b}=(2,3,-1)\times(-1,2,-3)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & -1\\ -1 & 2 & -3\end{vmatrix}$$

$$=\hat{i}\begin{vmatrix}3 & -1\\ 2 & -3\end{vmatrix}-\hat{j}\begin{vmatrix}2 & -1\\ -1 & -3\end{vmatrix}+\hat{k}\begin{vmatrix}2 & 3\\ -1 & 2\end{vmatrix}$$

$$=\hat{i}(-9+2)-\hat{j}(-6-1)+\hat{k}(4+3)$$

$$=-7\hat{i}+7\hat{j}+7\hat{k}=-7(\hat{i}-\hat{j}-\hat{k})$$

$\vec{b}\times\vec{c}=(-1,2,-3)\times(3,-4,2)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & -3\\ 3 & -4 & 2\end{vmatrix}$$

$$=\hat{i}\begin{vmatrix}2 & -3\\ -4 & 2\end{vmatrix}-\hat{j}\begin{vmatrix}-1 & -3\\ 3 & 2\end{vmatrix}+\hat{k}\begin{vmatrix}-1 & 2\\ 3 & -4\end{vmatrix}$$

$$=\hat{i}(4-12)-\hat{j}(-2+9)+\hat{k}(4-6)=-8\hat{i}-7\hat{j}-2\hat{k}$$

$\vec{c}\times\vec{a}=(3,-4,2)\times(2,3,-1)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -4 & 2\\ 2 & 3 & -1\end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} -4 & 2 \\ 3 & -1 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -4 \\ 2 & 3 \end{vmatrix}$$

$$= \hat{i}(4-6) - \hat{j}(-3-4) + \hat{k}(9+8) = -2\hat{i} + 7\hat{j} + 17\hat{k}$$

and $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

$$= \begin{vmatrix} 2 & 3 & -1 \\ -1 & 2 & -3 \\ 3 & -4 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & -3 \\ -4 & 2 \end{vmatrix} - 3\begin{vmatrix} -1 & -3 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 3 & -4 \end{vmatrix}$$

$$= 2(4-12) - 3(-2+9) - 1(4-6)$$

$$= -16 - 21 + 2 = -35$$

The reciprocal of $\vec{a} = \vec{a}' = \dfrac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]}$

$$= \frac{-8\hat{i} - 7\hat{j} - 2\hat{k}}{-35} = \frac{-8\hat{i} + 7\hat{j} + 2\hat{k}}{35}$$

The reciprocal of $\vec{b} = \vec{b}' = \dfrac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \dfrac{-2\hat{i} + 7\hat{j} + 17\hat{k}}{-35}$

The reciprocal of $\vec{c} = \vec{c}' = \dfrac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} = \dfrac{-7(\hat{i} - \hat{j} - \hat{k})}{-35} = \dfrac{\hat{i} - \hat{j} - \hat{k}}{5}$

9. Show that $(\vec{a} + \vec{b})\cdot\vec{a}' + (\vec{b} + \vec{c})\cdot\vec{b}' + (\vec{c} + \vec{a})\cdot\vec{c}' = 3$.

Solution

$\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors. $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are reciprocal of $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. Then,

$$\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\ \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

L.H.S. $= (\vec{a} + \vec{b})\cdot\vec{a}' + (\vec{b} + \vec{c})\cdot\vec{b}' + (\vec{c} + \vec{a})\cdot\vec{c}'$

$$= \vec{a}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{b}\cdot\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{b} = \frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{c}\,\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{c} + \frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]} + \vec{a}\,\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{b}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{b}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{c}\ \vec{c}\ \vec{a}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{c}\ \vec{a}\ \vec{b}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + \frac{[\vec{a}\ \vec{a}\ \vec{b}]}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$$= 1 + 0 + \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + 0 + \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{[\vec{a}\ \vec{b}\ \vec{c}]} + 0$$

$= 1 + 1 + 1 = 3 =$ R.H.S.

$\therefore$ L.H.S. = R.H.S. proved

Unit 5

Infinite Series

Exercise 5.1

1. List the first five terms of the sequence.

a. $a_n = \dfrac{2n}{n^2+1}$

Solution

Given, $a_n = \dfrac{2n}{n^2+1}$

If n = 1, then $a_1 = \dfrac{2}{1+1} = 1$

If n = 2, then $a_2 = \dfrac{4}{4+1} = \dfrac{4}{5}$

If n = 3, then $a_3 = \dfrac{6}{9+1} = \dfrac{6}{10} = \dfrac{3}{5}$

If n = 4, then $a_4 = \dfrac{8}{16+1} = \dfrac{8}{17}$

If n = 5, then $a_5 = \dfrac{10}{25+1} = \dfrac{10}{26} = \dfrac{5}{13}$

Then required sequence = $\left\{1, \dfrac{4}{5}, \dfrac{3}{5}, \dfrac{8}{17}, \dfrac{5}{13}\right\}$

b. $a_n = \dfrac{(-1)^{n-1}}{5^n}$

Solution

Given, $a_n = \dfrac{(-1)^{n-1}}{5^n}$

If n = 1, then $a_1 = \dfrac{(-1)^0}{5} = \dfrac{1}{5}$

If n = 2, then $a_2 = \dfrac{(-1)^1}{5^2} = \dfrac{-1}{25}$

If n = 3, then $a_3 = \frac{(-1)^{3-1}}{5^3} = \frac{1}{125}$

If n = 4, then $a_4 = \frac{(-1)^{4-1}}{5^4} = \frac{-1}{625}$

If n = 5, then $a_5 = \frac{(-1)^{5-1}}{5^5} = \frac{1}{3125}$

Then required sequence is $\left\{\frac{1}{5}, \frac{-1}{25}, \frac{1}{125}, \frac{-1}{625}, \frac{1}{3125}\right\}$.

c. $a_1 = 1, a_{n+1} = 5a_n - 3$

Solution

Given, $a_1 = 1$

$a_{n+1} = 5a_n - 3$

$a_2 = 5a_1 - 3 = 5 \times 1 - 3 = 2$

$a_3 = 5a_2 - 3 = 10 - 3 = 7$

$a_4 = 5a_3 - 3 = 35 - 3 = 32$

$a_5 = 5a_4 - 3 = 5 \times 32 - 3 = 157$

$\therefore$ {1, 2, 7, 32, 157}

d. $a_1 = 2, a_{n+1} = \frac{a_n}{1 + a_n}$

Solution

Given, $a_1 = 2, a_{n+1} = \frac{a_n}{1 + a_n}$

If n = 1, then $a_2 = \frac{a_1}{1 + a_1} = \frac{2}{1 + 2} = \frac{2}{3}$

If n= 2, then $a_3 = \frac{a_2}{1 + a_2} = \frac{\frac{2}{3}}{1 + \frac{2}{3}} = \frac{2}{5}$

If n = 3, then $a_4 = \frac{a_3}{1 + a_3} = \frac{\frac{2}{5}}{1 + \frac{2}{5}} = \frac{2}{7}$

If n = 4, then $a_5 = \frac{a_4}{1 + a_4} = \frac{\frac{2}{7}}{1 + \frac{2}{7}} = \frac{2}{9}$

Then required sequence = $\left\{2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \frac{2}{9}\right\}$

e. $a_n = \frac{(-1)^n n}{n! + 1}$

Solution

Given, $a_n = \frac{(-1)^n n}{n! + 1}$

If n = 1, then $a_1 = \frac{(-1)^n \cdot 1}{1! + 1} = \frac{-1}{2}$

If n = 2, then $a_2 = \frac{(-1)^2 \cdot 2}{2! + 1} = \frac{2}{3}$

If n = 3, then $a_3 = \frac{(-1)^3 \cdot 3}{3! + 1} = \frac{-3}{7}$

If n = 4, then $a_4 = \frac{(-1)^4 \cdot 4}{4! + 1} = \frac{4}{25}$

If n = 5, then $a_5 = \frac{(-1)^5 \cdot 5}{5! + 1} = \frac{-5}{121}$

Then required series is $\left\{\frac{-1}{2}, \frac{2}{3}, \frac{-3}{7}, \frac{4}{25}, \frac{-5}{121}\right\}$

2. Find the general term of the following series.

a. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

Solution

Given series is

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$

Let t_n be the general term of given series then, $t_n = \frac{1}{2^n}$.

b. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$

Solution

Given series is

$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots$

$= \frac{1}{1+1} + \frac{2}{2+1} + \frac{3}{3+1} + \frac{4}{4+1} + \dots$

Let t_n be the general term of given series then, $t_n = \frac{n}{n+1}$.

c. $\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$

Solution

Given series is

$\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$

Let t_n be the general term of given series then, $t_n = \frac{n^2}{n!}$.

d. $\frac{1}{2} + 1 + \frac{9}{8} + 1 + \frac{25}{32} + \dots$

Solution

Given series is

$\frac{1}{2} + 1 + \frac{9}{8} + 1 + \frac{25}{32} + \dots$

$= \frac{1^2}{2^1}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\frac{5^2}{2^5}+\ldots$

Let t_n be the general term of given series then, $t_n = \frac{n^2}{2^n}$.

3. Test the convergence of the following series.

a. $1+2+3+4+5+\ldots$

Solution

Given series is

$1+2+3+4+5+\ldots$

Let $u_n = n$.

Then $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} n = \infty$

$\therefore \lim\limits_{n\to\infty} u_n \neq 0$ so the series is not convergent

b. $1+\frac{5}{4}+\frac{6}{4}+\frac{7}{4}+\ldots$

Solution

Given series is

$1+\frac{5}{4}+\frac{6}{4}+\frac{7}{4}+\ldots$

$= \frac{4}{4}+\frac{5}{4}+\frac{6}{4}+\frac{7}{4}+\ldots$

$= \frac{3+1}{4}+\frac{3+2}{4}+\frac{3+3}{4}+\frac{3+4}{4}+\ldots$

Let $u_n = \frac{3+n}{4}$.

Then $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \frac{3+n}{4} = \infty$

$\therefore \lim\limits_{n\to\infty} u_n \neq 0$ so the given series is divergent.

c. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

Solution

Given series is

$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$

Let a_n be general positive term

$a_n = \frac{(-1)^{n-1}}{2^{n-1}}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 1_n = \frac{(-1)^{n-1}}{2^{n-1}} = \frac{(-1)^{\infty}}{2^{\infty}} = 0$

and given series is infinite geometric series and common ratio is $|r| < 1$, so given series is convergent.

d. $\log 2+\log\frac{4}{3}+\log\frac{5}{4}+\ldots$

Solution

Given series is

$\log 2+\log\frac{4}{3}+\log\frac{5}{4}+\ldots$

let $\log 2+\left[\log\frac{4}{3}+\log\frac{5}{4}+\ldots+\infty\right]$

Let a_n be general term then, $a_n = \log\frac{n+2}{n+1}$

Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \log\frac{n+2}{n+1}$

$= \lim\limits_{n\to\infty} \log\left(\frac{1+\frac{2}{n}}{1+\frac{1}{n}}\right) = \log\left(\frac{1+\frac{2}{\infty}}{1+\frac{1}{\infty}}\right)$

$= \log 1$

$= 0$

But $\log 2+\log\frac{4}{3}+\log\frac{5}{4}+\ldots\log\frac{n}{n-1}$

$= \log 2+\log 4-\log 3+\log 5-\log 4+\ldots+\log n-\log_{n-1}$

$= \log 2-\log 3+\log n$

Convergent as $n\to\infty$.

e. $\sum\left(1+\frac{1}{2n}\right)^n$

Solution

Given, $\sum\limits_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)^n$

Let $a_n = \left(1+\frac{1}{2n}\right)^n$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \left(1+\frac{1}{\infty}\right)^{\infty}$

$= 1$

$\neq 0$

$\therefore \lim\limits_{n\to\infty} a_n \neq 0$ so given series is divergent

f. $\sum\limits_{n=0}^{\infty}\frac{n}{\sqrt{n^2+1}}$

Solution

Given $\sum\limits_{n=0}^{\infty}\frac{n}{\sqrt{n^2+1}}$

Let $a_n = \dfrac{n}{\sqrt{n^2+1}}$

Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \dfrac{n}{n\sqrt{1+\frac{1}{n^2}}}$

$= \lim_{n\to\infty} \dfrac{1}{\sqrt{1+\frac{1}{n^2}}}$

$= 1$

$\lim_{n\to\infty} a_n \neq 0$ so given series is divergent.

g. $\sum_{n=0}^{\infty} \dfrac{3}{2n}$

Solution

Given, $\sum_{n=0}^{\infty} \dfrac{3}{2n}$

Let, $a_n = \dfrac{3}{2n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{3}{2n} = 0$

But $\sum_{n=0}^{\infty} \dfrac{1}{n}$ is divergent so given series $\sum_{n=0}^{\infty} \dfrac{3}{2n}$ is divergent.

h. $\sum_{n=1}^{\infty} \dfrac{n}{1+2^{-n}}$

Solution

Given $\sum_{n=1}^{\infty} \dfrac{n}{1+2^{-n}}$

Let $a_n = \dfrac{n}{1+2^{-n}}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{1+2^{-n}}$

$= \dfrac{\infty}{1+0}$

$= \infty$

$\therefore \lim_{n\to\infty} a_n = \infty$, so given series is divergent.

i. $\sum_{n=1}^{\infty} \dfrac{n^2}{5n^2+4}$

Solution

Given, $\sum_{n=1}^{\infty} \dfrac{n^2}{5n^2+4}$

Let $a_n = \dfrac{n^2}{5n^2+4}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n^2}{5n^2+4}$

$= \lim_{n\to\infty} \dfrac{n^2}{n^2\left(5+\frac{4}{n^2}\right)}$

$= \dfrac{1}{5}$

$\therefore \lim_{n\to\infty} a_n = \dfrac{1}{5} \neq 0$ so given series is not convergent.

j. $\sum_{n=2}^{\infty} (-1)^n \log(n)$

Solution

Given, $\sum_{n=2}^{\infty} (-1)^n \log(n)$

$\sum_{n=2}^{\infty} (-1)^n \log n = \log 2 - \log 3 + \log 4 - \log 5 \ldots$

let $a_n = (-1)^n \log n$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \log n$

$= \pm\infty$

$\therefore \lim_{n\to\infty} a_n \neq 0$ so given series is not convergent.

Exercise 5.2

1. Write the following numbers as the ratio of integers.

a. $2.3\overline{17} = 2.3171717\ldots$

Solution

Given, $2.3\overline{17} = 2.3171717\ldots$

$= 2.3 + 0.0171717 + \ldots$

$= \dfrac{23}{10} + 0.017 + 0.00017 + 0.00000.17 + \ldots$

$= \dfrac{23}{10} + \dfrac{17}{1000} + \dfrac{17}{100000} + \dfrac{17}{1000000} + \ldots$

$= \dfrac{23}{10} + \dfrac{17}{1000}\left(1 + \dfrac{1}{100} + \dfrac{1}{10000} + \ldots\right)$

$= \dfrac{23}{10} + \dfrac{17}{1000}\left(\dfrac{1}{1-\frac{1}{100}}\right)$

$= \dfrac{23}{10} + \dfrac{17}{1000} \times \dfrac{100}{99}$

$= \dfrac{23}{10} \times \dfrac{17}{990}$

$= \dfrac{2294}{990} = \dfrac{1147}{495}$

b. **0.3333**

Solution

$$\text{Given } 0.3333 = 0.3 + 0.03 + 0.003 + 0.0003 + \dots$$
$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots$$
$$= \frac{3}{10}\left(1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots\right)$$
$$= \frac{3}{10}\left(\frac{1}{1 - \frac{1}{10}}\right)$$
$$= \frac{3}{10} \times \frac{10}{9} = \frac{1}{3}$$

c. **0.234234234**

Solution

$$\text{Given } 0.234234234 = 0.234 + 0.000234 + 0.000000234 + \dots$$
$$= \frac{234}{1000} + \frac{234}{1000000} + \frac{234}{1000000000}$$
$$= \frac{234}{1000}\left(1 + \frac{1}{1000} + \frac{1}{1000000} + \dots\right)$$
$$= \frac{234}{1000} \times \left(\frac{1}{1 - \frac{1}{10}}\right)$$
$$= \frac{234}{1000} \times \frac{1000}{999}.$$
$$= \frac{234}{999} = \frac{26}{111}$$

2. **Determine whether the given geometric series is convergent or divergent**

a. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \dots$

Solution

Given series is $3 - 4 + \frac{16}{3} - \frac{64}{9} + \dots$

1^{st} term (a) = $t_1 = 3$

2^{nd} term (t_2) = -4

Common ratio (r) $= \frac{-4}{3} = -1.33 > 0$ $\qquad |r| > 1$

Hence, common ratio $|r| > 1$ so given series is divergent.

b. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \dots$

Solution

Given series is $4 + 3 + \frac{9}{4} + \frac{27}{16} + \dots$

1^{st} term (a) = $t_1 = 4$

2^{nd} term (t_2) = 3

Common ratio (r) $= \frac{3}{4} < 1$

Hence, common ratio $|r| < 1$ so given infinite geometric series is convergent.

c. **2 + 0.5 + 0.125 + 0.03125 + ...**

Solution

Given series is $2 + 0.5 + 0.125 + 0.03125 + \dots$

1^{st} term (a) = $t_1 = 2$

2^{nd} term (t_2) = 0.5

Common ratio (r) $= \frac{t_2}{t_1} = \frac{0.5}{2} = \frac{1}{4} < 1$

Hence, common ratio $|r| < 1$ so given infinite geometric series is convergent being $r < 1$.

Exercise 5.3

Test the convergent or divergent of the following series:

1. $\frac{2}{3^2} + \frac{3}{4^2} + \frac{4}{5^2} + \dots$

Solution

Given series is

$$\frac{2}{3^2} + \frac{3}{4^2} + \frac{4}{5^2} + \dots$$
$$= \frac{2}{(2+1)^2} + \frac{3}{(3+1)^2} + \frac{4}{(4+1)^2} + \dots$$

Let $u_n = \frac{n}{(n+1)^2} = \frac{n}{n^2\left(1 + \frac{1}{n}\right)^2} = \frac{1}{n\left(1 + \frac{1}{n}\right)^2}$

Let $v_n = \frac{1}{n}$ then $\Sigma V_n = \Sigma \frac{1}{n}$ is divergent by p-series having p = 1.

$$\text{Then, } \lim_{n \to \infty} \frac{u_n}{v_n} = \frac{\frac{n}{(n+1)^2}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{(n+1)^2}$$
$$= \lim_{n \to \infty} \frac{n^2}{n^2\left(1 + \frac{1}{n}\right)}$$
$$= 1 = \text{finite number}$$

Then Σu_n is divergent by using comparison test as Σv_n divergent.

2. $\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \dots$

Solution

Given series is

$$\frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \dots$$

Let $u_n = \frac{3 + (n-1) \cdot 2}{(n+1)^2}$

$$= \frac{2n+1}{(n+1)^2}$$

$$= \frac{(2n+1)}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$= \frac{\left(2+\frac{1}{n}\right)}{n\left(1+\frac{1}{n}\right)^2}$$

Let $v_n = \frac{1}{n}$

Then $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent by p-series.

Then, $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{\frac{(2n+1)}{(n+1)^2}}{\frac{1}{n}}$

$$= \lim_{n\to\infty} \frac{n^2\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)}$$

$$= \frac{2+0}{1+0}$$

$$= 2$$

$\therefore$ $u_n = 2v_n$

Since Σv_n is divergent then given series Σu_n is divergent.

3. $\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \dots$

Solution

Given series is

$$\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \dots$$

Let $u_n = \frac{n}{1+n\sqrt{n+1}}$

$$= \frac{n}{1+n^{3/2}\sqrt{1+\frac{1}{n}}}$$

$$= \frac{n}{n^{3/2}\left\{\frac{1}{n^{3/2}}+\sqrt{1+\frac{1}{n}}\right\}}$$

$$= \frac{1}{n^{1/2}\left(\sqrt{1+\frac{1}{n}}+\frac{1}{n^{3/2}}\right)}$$

Let $v_n = \frac{1}{n^{1/2}}$

Then $\Sigma v_n = \Sigma\frac{1}{n^{1/2}}$ is convergent by p-series having $p < 1$.

Then, $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{n^{1/2}\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^{3/2}}}\right)}}{\frac{1}{n^{1/2}}}$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}+\frac{1}{n^{3/2}}}}$$

$$= \frac{1}{\sqrt{1+0}}$$

$= 1$ = finite number

Then u_n is convergent by using comparison test.

4. $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots$

Solution

Given series is

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \dots$$

Let $u_n = \frac{1}{n(n+1)}$

$$= \frac{1}{n^2\left(1+\frac{1}{n}\right)}$$

Let $v_n = \frac{1}{n^2}$

Then $\Sigma v_n = \Sigma\frac{1}{n^2}$ is convergent by using p-series having $p > 1$.

Now, $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{n^2\left(1+\frac{1}{n}\right)}}{\frac{1}{n^2}}$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)}$$

$$= \frac{1}{1}$$

$$= 1$$

Since, Σv_n is convergent then Σu_n must be convergent. So given series is convergent.

5. $\sum\left[\frac{\sqrt{n}}{n^2+1}\right]$

Solution

Given series is $\sum\left[\frac{\sqrt{n}}{n^2+1}\right]$

$$\text{Let } u_n=\frac{\sqrt{n}}{n^2+1}=\frac{\sqrt{n}}{n^2\left(1+\frac{1}{n^2}\right)}=\frac{1}{n^{3/2}\left(1+\frac{1}{n^2}\right)}$$

Let $v_n=\frac{1}{n^{3/2}}$

Then $\Sigma v_n=\Sigma\frac{1}{n^{3/2}}$ is convergent by p-series having $p=\frac{3}{2}>1$.

$$\text{Now, }\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{\sqrt{n}}{n^2+1}}{\frac{1}{n^{3/2}}}=\lim_{n\to\infty}\frac{n^{3/2}\times\sqrt{n}}{n^2\left(1+\frac{1}{n^2}\right)}$$

$$=\lim_{n\to\infty}\frac{1}{1+\frac{1}{n^2}}$$

$$=1$$

Since Σv_n is convergent then Σu_n is convergent. So given series is convergent

6. $\sum\left[\frac{\sqrt{n+1}-1}{(n+2)^3-1}\right]$

Solution

Given series is $\sum\left[\frac{\sqrt{n+1}-1}{(n+2)^3-1}\right]$

$$\text{Let } u_n=\frac{\sqrt{n+1}-1}{(n+2)^3-1}=\frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right)}{n^3\left(\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right)}=\frac{\left(\sqrt{1+\frac{1}{n}}-\sqrt{\frac{1}{n}}\right)}{n^{5/2}\left(\left(1+\frac{1}{n}\right)^3-\frac{1}{n^3}\right)}$$

Let $v_n=\frac{1}{n^{5/2}}$. Then $\Sigma u_n=\Sigma\frac{1}{n^{5/2}}$ is convergent by using p-series. Now,

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{1}{n^{5/2}}\frac{\left(\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right)}{\left[\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}\right]}}{\frac{1}{n^{5/2}}}$$

$$=\lim_{n\to\infty}\frac{\sqrt{1+\frac{1}{n}-\frac{1}{\sqrt{n}}}}{\left(1+\frac{2}{n}\right)^3-\frac{1}{n^3}}$$

$$=\frac{1-0}{1-0}$$

$$=1$$

Since Σv_n is convergent then Σu_n is convergent. So given series $\sum\left[\frac{\sqrt{n+1}-1}{(n+2)^3-1}\right]$ is convergent.

7. $\Sigma[\sqrt{n^2+1}-n]$

Solution

Given series is $\Sigma[\sqrt{n^2+1}-n]$

$$\text{Let } u_n=\sqrt{n^2+1}-n$$

$$=\sqrt{n^2+1}-n\times\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}$$

$$=\frac{n^2+1-n^2}{\sqrt{n^2+1}+n}$$

$$=\frac{1}{n\left(\sqrt{1+\frac{1}{n^2}}+\frac{1}{n}\right)}$$

Let $v_n=\frac{1}{n}$. Then, $v_n=\Sigma\frac{1}{n}$ is divergent by using p-series.

$$\text{Then, }\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{1}{n\left(\sqrt{1+\frac{1}{n^2}}+\frac{1}{n}\right)}}{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}+n^2}}$$

$$=1$$

Since, Σv_n is divergent. Then Σu_n is divergent so the given series is divergent.

8. $\Sigma[\sqrt{n^4-1}-n^2]$

Solution

Given series is $\Sigma[\sqrt{n^4-1}-n^2]$

$$\text{Let } u_n=\sqrt{n^4-1}-n^2$$

$$=(\sqrt{n^4-1}-n^2)\times\frac{\sqrt{n^4-1}+n^2}{\sqrt{n^4-1}+n^2)}$$

$$=\frac{n^4-1-n^4}{\sqrt{n^4-1}+n^2}$$

$$= \frac{-1}{n^2\left(\sqrt{1-\frac{1}{n^4}}+1\right)}$$

Let $v_n = \frac{1}{n^2}$ then $\Sigma v_n = \Sigma\frac{1}{n^2}$ is convergent by using p-series.

$$\text{Now, } \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{-1}{n^2\left(\sqrt{1-\frac{1}{n^4}}+1\right)}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{-1}{\sqrt{1-\frac{1}{n^4}}+1}$$

$$= \frac{-1}{\sqrt{1-0}+1}$$

$$= \frac{-1}{2}$$

Since Σv_n is convergent then Σu_n is convergent. So given series is convergent.

9. $\Sigma[(n^3+1)^{1/3}-n]$

Solution

Given series is $\Sigma[(n^3+1)^{1/3}-n]$

Let $a_n = (n^3+1)^{1/3}-n$

$$= [(n^3+1)^{1/3}-n]\frac{[(n^3+1)^{1/2}+(n^3+1)^{1/3}n+n^3]}{(n^3+1)^{1/3}+(n^3+1)^{1/3}n+n^2}$$

$$= \frac{(n^3+1)-n^3}{(n^3+1)^{2/3}+(n^3+1)^{1/3}n+n^2}$$

$$= \frac{1}{n^2}\frac{(n^3+1-n^3)}{\left[\left(1+\frac{1}{n^3}\right)^{2/3}+\left(1+\frac{1}{n^3}\right)^{1/3}+1\right]}$$

Let $b_n = \frac{1}{n^2}$ then $\Sigma b_n = \Sigma\frac{1}{n^2}$ is convergent by using p-series $p > 1$.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2\left[\left(1+\frac{1}{n^3}\right)^{2/3}+\left(1+\frac{1}{n^3}\right)^{1/3}+1\right]}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty}\frac{1}{\left[\left(1+\frac{1}{n^3}\right)^{2/3}+\left(1+\frac{1}{n^3}\right)^{1/3}+1\right]}$$

$$= \frac{1}{1+1+1}$$

$$= \frac{1}{3}\text{ (finite value)}$$

Then Σa_n is convergent by using comparison test.

10. $\Sigma\frac{\sqrt{n+1}-\sqrt{n}}{n}$

Solution

Given series is $\Sigma\frac{\sqrt{n+1}-\sqrt{n}}{n}$

Let $u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$

$$= \frac{\sqrt{n+1}-\sqrt{n}}{n}\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{n^{3/2}\left(\sqrt{1+\frac{1}{n}}+1\right)}$$

Let $v_n = \frac{1}{n^{3/2}}$. Then $\Sigma v_n = \Sigma\frac{1}{n^{3/2}}$ is convergent by using p-series.

$$\text{Now, } \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n^{3/2}\left(\sqrt{1+\frac{1}{n}}+1\right)}}{n^{3/2}}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Since Σv_n is convergent then Σu_n is convergent. Then given series is convergent.

11. $\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$

Solution

Given $\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$

Let $u_n = \frac{1}{n^2+6n+13}$

$$= \frac{1}{n^2\left(1+\frac{6}{n}+\frac{13}{n^2}\right)}$$

Let $v_n = \frac{1}{n^2}$ then $\Sigma v_n = \Sigma\frac{1}{n^2}$ is convergent by using p-series.

$$\text{Now, } \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\frac{1}{n^2\left(1+\frac{1}{n}+\frac{13}{n^2}\right)}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}+\frac{13}{n^2}}$$

$$= \frac{1}{1}$$

$$= 1$$

Since, Σv_n is convergent then Σu is convergent. Then given series convergent.

12. $\sum_{n=1}^{\infty} \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$

Solution

Given, $\sum_{n=1}^{\infty} \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$

$$\text{Let } u_n = \frac{(2n-1)(n^2-1)}{(n+1)(n^2+4)^2}$$

$$= \frac{2n^3-2n-n^2+1}{(n+1)(n^4+8n^2+16)}$$

$$= \frac{2n^3-2n-n^2+1}{n^4+8n^2+n+16}$$

$$= \frac{n^3\left(2-\frac{1}{n}-\frac{2}{n^2}+\frac{1}{n^3}\right)}{n^4\left(1+\frac{8}{n^2}+\frac{1}{n^3}+\frac{16}{n^4}\right)}$$

$$= \frac{1}{n}\frac{\left(2-\frac{1}{n}-\frac{2}{n^2}+\frac{1}{n^3}\right)}{\left(1+\frac{8}{n^2}+\frac{1}{n^3}+\frac{16}{n^4}\right)}$$

Let $v_n = \frac{1}{n}$ then $\Sigma v_n = \Sigma\frac{1}{n}$ is divergent by using p-series.

$$\text{Now, } \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n}\frac{\left(2-\frac{1}{n}-\frac{2}{n^2}+\frac{1}{n^3}\right)}{\left(1+\frac{8}{n^2}+\frac{2}{n^3}+\frac{16}{n^4}\right)}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{2-\frac{1}{n}-\frac{2}{n^2}+\frac{1}{n^3}}{1+\frac{8}{n^2}+\frac{1}{n^3}+\frac{16}{n^4}}$$

$$= \frac{2}{1}$$

Since Σv_n is divergent then Σu_n is divergent so given series divergent.

13. $\Sigma \frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}$

Solution

Given, $\Sigma \frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}$

$$\text{Let } a_n = \frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}$$

$$= \frac{(n)^{1/3}}{(n)^{3/2}\sqrt{1+\frac{y}{n^2}+\frac{3}{n^3}}}$$

$$= \frac{1}{n^{7/6}\sqrt{1+\frac{y}{n^2}+\frac{3}{n^3}}}$$

Let $b_n = \frac{1}{n^{7/6}}$ then $\Sigma b_n = \Sigma\frac{1}{n^{7/6}}$ is convergent. Then p-series $r = \frac{7}{6} > 1$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{7/6}\sqrt{1+\frac{y}{n^2}+\frac{3}{n^3}}}}{\frac{1}{n^{7/6}}} = 1$$

$\therefore$ Σa_n is convergent by using comparison text.

14. $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^3}}$

Solution

Given, $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^3}}$

$$\text{Let } a_n = \frac{2n^2+3n}{\sqrt{5+n^3}}$$

$$= \frac{n^2\left(2+\frac{3}{n}\right)}{n^2\sqrt{n}\sqrt{\frac{5}{n^3}+1}}$$

$$= \frac{2+\frac{3}{n}}{\sqrt{n}\sqrt{\frac{5}{n^3}+1}}$$

Let $b_n = \frac{1}{\sqrt{n}}$ then $\Sigma b_n = \Sigma\frac{1}{\sqrt{n}}$ is divergent as p-series $p = \frac{1}{2} < 1$.

Then, $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2+\frac{3}{n}}{\sqrt{n}\sqrt{\frac{5}{n^3}+1}} \times \sqrt{n} = 2$

$\therefore$ Σa_n is divergent by using comparison test.

Exercise 5.4

1. **Test the convergence and divergence of the series by using D Alembert's ratio test.**

a. $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$

Solution

Given series is $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$

Let $u_n = \frac{1}{n!}$ then $u_{n+1} = \frac{1}{(n+1)!}$

Then, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}$

$= \lim_{n\to\infty}\frac{n!}{(n+1)!}$

$= \lim_{n\to\infty}\frac{n!}{(n+1)\,n!} = \lim_{n\to\infty}\frac{1}{n+1} = 0$

$\therefore$ $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} < 1$, so the given series is convergent.

b. $\frac{1}{2}+\frac{4}{2^2}+\frac{9}{2^3}+\ldots$

Solution

Given series is $\frac{1}{2}+\frac{4}{2^2}+\frac{9}{2^3}+\ldots$

Let $u_n = \frac{n^2}{2^n}$ then $u_{n+1} = \frac{(n+1)^2}{2^{n+1}}$

Then, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{(n+1)^2}{2^n+1}}{\frac{n^2}{2^n}}$

$= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\times\frac{2^n}{2^{n+1}}$

$= \lim_{n\to\infty}\frac{(n^2+2n+1)}{n^2}\times\frac{1}{2}$

$= \lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\cdot\frac{1}{2} = \frac{1}{2} < 1$

So that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$, then given series is convergent.

c. $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots$

Solution

Given series is $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\ldots$

Let $u_n = \frac{1+(n-1)\cdot 2}{n!} = \frac{2n-1}{n!}$

Then $u_{n+1} = \frac{2(n+1)-1}{(n+1)!} = \frac{2n+1}{(n+1)!}$

Then, $\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{2n+1}{(n+1)!}}{\frac{2n-1}{n!}}$

$= \lim_{n\to\infty}\frac{(2n+1)}{(n+1)!}\times\frac{n!}{(2n-1)!}$

$= \lim_{n\to\infty}\frac{n\left(2+\frac{1}{n}\right)\cdot n!}{(n+1)\,n!\,n\left(2-\frac{1}{n}\right)}$

$= \lim_{n\to\infty}\frac{\left(2+\frac{1}{n}\right)}{(n+1)\left(2-\frac{1}{n}\right)}$

$= \frac{2}{\infty}$

$= 0$

$\therefore$ $\lim_{n\to\infty}\frac{u_{n+1}}{v_n} = 0 < 1$, then by applying D' Alembert ratio test, given series is convergent.

d. $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\ldots$

Solution

Given series is $\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot 2}{3\cdot 5}\right)^2+\left(\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)^2+\ldots$

Let the general term $a_n = \left[\frac{1\cdot 2\cdot 3\ldots n}{3\cdot 5\cdot 7\ldots(2n+1)}\right]^2 = \left[\frac{n!}{3\cdot 5\cdot 7\ldots(2n+1)}\right]^2$ and

$(n+1)^{th}$ term $(a_n+1) = \left[\frac{1\cdot 2\cdot 3\ldots n\cdot(n+1)}{3\cdot 5\cdot 7\ldots(2n+1)(2n+3)}\right]^2$

$= \left[\frac{1\cdot 2\cdot 3\ldots(n+1)}{3\cdot 5\cdot 7\ldots(2n+3)}\right]^2$

$= \left[\frac{(n+1)!}{3\cdot 5\cdot 7\ldots(2n+3)}\right]^2$

$$\text{Now, } \frac{a_{n+1}}{a_n} = \frac{[(n+1)!]^2}{[3\cdot5\cdot7\ldots(2n+1)]^2\,(2n+3)^2} \times \frac{[3\cdot5\cdot7\ldots(2n+1)]^2}{(n!)^2}$$

$$= \frac{[1\cdot2\cdot3\ldots n]^2\,(n+1)^2}{(n!)^2\,(2n+3)^2}$$

$$= \frac{n^2\left(1+\frac{1}{n}\right)^2}{n^2\left(2+\frac{3}{n}\right)^2}$$

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} < 1$$

∴ The given series is convergent.

e. $\sum \frac{1}{(2n+1)!}$

Solution

Given $\sum \frac{1}{(2n+1)!}$

Let $a_n = \frac{1}{(2n+1)!}$ then $a_{n+1} = \frac{1}{(2n+3)!}$

$$\text{Now, } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{(2n+3)!}}{\frac{1}{(2n+1)!}}$$

$$= \lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!}$$

$$= \lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)\,(2n+2)\,(2n+1)!}$$

$$= \lim_{n\to\infty}\frac{1}{n^2\left(2+\frac{3}{n}\right)\left(2+\frac{2}{n}\right)}$$

$$= 0$$

Therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1$, so the given series is convergent by using D' Alembert ratio test.

f. $\sum \frac{n^2}{3^n}$

Solution

Given series is $\sum \frac{n^2}{3^n}$

$a_n = \frac{n^2}{3^n}$ then $a_{n+1} = \frac{(n+1)^2}{3^{n+1}}$

$$\text{Now, } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\times\frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty}\frac{n^2\left(1+\frac{2}{n}\right)^2}{n^2}\cdot\frac{1}{3}$$

$$= \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^2\cdot\frac{1}{3}$$

$$= \frac{1}{3} < 1$$

Therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{3} < 1$. So, the given series is convergent.

g. $\frac{(n+1)(n+2)}{n!}$

Solution

Given series is $\sum \frac{(n+1)(n+2)}{n!}$

Let $a_n = \frac{(n+1)(n+2)}{n!}$, then $a_{n+1} = \frac{(n+2)\,(n+3)}{(n+1)!}$

$$\text{Now, } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+2)\,(n+3)}{(n+1)!}\times\frac{n!}{(n+1)\,(n+2)}$$

$$= \lim_{n\to\infty}\frac{(n+3)\,n!}{(n+1)^2\,n!}$$

$$= \lim_{n\to\infty}\frac{n\left(1+\frac{3}{n}\right)}{n^2\left(1+\frac{1}{n}\right)^2}$$

$$= 0$$

Therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1$. So the given series is convergent by using A' Alembert ratio test.

h. $\sum \frac{n(n+1)^2}{n!}$

Solution

Given series is $\sum \frac{n(n+1)^2}{n!}$

Let $a_n = \frac{n\,(n+1)^2}{n!}$ and $a_{n+1} = \frac{(n+1)\,(n+2)^2}{(n+1)!}$

$$\text{Now, } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)\,(n+2)^2}{(n+1)!}}{\frac{n(n+1)^2}{n!}}$$

$$= \lim_{n\to\infty}\frac{(n+1)\,(n+2)^2}{(n+1)!}\times\frac{n!}{n(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{2}{n}\right)^2}{(n+1)\,n!}\cdot\frac{n!}{n(n+1)}$$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{2}{n}\right)^2}{n^3\left(1+\frac{1}{n}\right)^2}$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^2}{n\left(1+\frac{1}{n}\right)^2}$$

$$= 0$$

Therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1$. So the given series is convergent by using D. Alembert ratio test.

i. $\frac{(2n)!}{n!\,n!}$

Solution

Given series is $\sum \frac{(2n)!}{n!\,n!}$

Let $a_n = \frac{(2n)!}{n!\,n!}$ and $a_{n+1} = \frac{(2n+2)!}{(n+1)!\,(n+1)!}$

Then, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+2)!}{(n+1)!\,(n+1)!}\times\frac{n!\,n!}{(2n)!}$

$$= \lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n)!\,n!\,n!}{(n+1)\,n!\,(n+1)\,n!\,(2n)!}$$

$$= \lim_{n\to\infty}\frac{n^2\left(2+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$= \frac{\left(2+\frac{1}{\infty}\right)\left(2+\frac{1}{\infty}\right)}{\left(1+\frac{1}{\infty}\right)\left(1+\frac{1}{\infty}\right)}$$

$$= \frac{4}{1}$$

$$= 4 > 1$$

$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 4 > 1$, so given series is divergent.

j. $\sum_{n=1}^{\infty}\frac{2n}{n^2+2}$

Solution

Given series is $\sum_{n=1}^{\infty}\frac{2n}{n^2+2}$

Let $a_n = \frac{2n}{n^2+2}$ and $a_{n+1} = \frac{2(n+1)}{(n+1)^2+2} = \frac{2n+2}{n^2+2n+3}$

Now, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2n+2}{n^2+2n+3}\times\frac{n^2+2}{2n}$

$$= \lim_{n\to\infty}\frac{2\cdot n^3\left(1+\frac{2}{n^2}\right)\left(2+\frac{2}{n}\right)}{n^3\left(1+\frac{2}{n}+\frac{3}{n^2}\right)\cdot 2}$$

$$= 2$$

Therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 2 > 1$. So that the given series is divergent by using D. Alembert ratio test.

k. $\sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$

Solution

Given series is $\sum_{n=1}^{\infty}\frac{2^n+5}{3^n}$

Let $a_n = \frac{2^n+5}{3^n}$ and $a_{n+1} = \frac{2^{n+1}+5}{3^{n+1}}$

Now, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2^{n+1}+5}{3^{n+1}}\times\frac{3^n}{2^n+5}$

$$= \lim_{n\to\infty}\frac{2^n\cdot 2+5}{3^n\cdot 3}\,\frac{3^n}{2^n+5}$$

$$= \lim_{n\to\infty}\frac{2^n\left(2+\frac{5}{2^n}\right)}{3\cdot 2^n\left(1+\frac{5}{2^n}\right)}$$

$$= \frac{2+0}{1+0}$$

$$= 2 > 1$$

Therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 2 > 1$. So, the given series is divergent.

l. $\sum\sqrt{\frac{n^2+a}{2^n+a}}$

Solution

Given series is $\sum\sqrt{\frac{n^2+a}{2^n+a}}$

Let $a_n = \sqrt{\frac{n^2+a}{2^n+a}}$ and $a_{n+1} = \sqrt{\frac{(n+1)^2+a}{2^{n+1}+a}} = \sqrt{\frac{n^2+2n+1+a}{2^n\cdot 2+a}}$

Then, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\sqrt{\frac{n^2+2n+1+a}{2^n\cdot 2+a}\times\frac{2^n+a}{n^2+a}}$

$$= \lim_{n\to\infty}\sqrt{\frac{n^2\left(1+\frac{2}{n}+\frac{1+a}{n^2}\right)\cdot 2^n\left(1+\frac{a}{2^n}\right)}{2^n\left(2+\frac{a}{2^n}\right)n^2\left(1+\frac{a}{n^2}\right)}}$$

$$= \frac{(1+0+0)\cdot(1+0)}{(2+0)(1+0)} = \frac{1}{2} < 1$$

Hence, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \frac{1}{2} < 1.$

So, that given series is convergent by using Cauchy root test.

2. Test the convergence of divergence of the series given below.

a. $\frac{1}{(\log 2)^2}+\frac{1}{(\log 3)^3}+\frac{1}{(\log 4)^4}+\ldots$

Solution

Given series is $\frac{1}{(\log 2)^2}+\frac{1}{(\log 3)^3}+\frac{1}{(\log 4)^4}+\ldots$

Let $a_n = \frac{1}{\{\log(n+1)\}^{n+1}}$

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left[\frac{1}{\{\log(n+1)\}^{n+1}}\right]^{1/n}$$

$$= \lim_{n\to\infty}\frac{1}{\{\log(n+1)\}^{\left(1+\frac{1}{n}\right)}}$$

$$= \frac{1}{\log\infty} = 0 < 1.$$ So given series is convergent.

b. $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots$

Solution

Given series is $1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots$

Let a_n be general term. Then,

$a_n = \frac{1}{n^n}$

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n}$$

$$= \lim_{n\to\infty}\frac{1}{n}$$

$$= 0 < 1$$

$\therefore$ $\lim\limits_{n\to\infty}(a_n)^{1/n} = 0 < 1$, so that given series is convergent by using Cauchy root test.

c. $\left(\frac{1}{4}\right)+\left(\frac{2}{7}\right)^2+\left(\frac{3}{10}\right)^3+\left(\frac{4}{13}\right)^4+\ldots$

Solution

Given series is $\left(\frac{1}{4}\right)+\left(\frac{2}{7}\right)^2+\left(\frac{3}{10}\right)^3+\left(\frac{4}{13}\right)^4+\ldots$

Let $a_n = \left(\frac{n}{3n+1}\right)^n$

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left\{\left(\frac{n}{3n+1}\right)^n\right\}^{1/n}$$

$$= \lim_{n\to\infty}\frac{n}{3n+1}$$

$$= \frac{1}{3} < 1$$

$\therefore$ $\lim\limits_{n\to\infty}(a_n)^{1/n} = \frac{1}{3} < 1$. So given series is convergent by using Cauchy root test.

d. $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots$

Solution

Given series $= \frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots$

$$= \frac{1}{10}+\frac{1}{(10)^2}+\frac{1}{(10)^3}+\ldots$$

Let $(a_n) = \left(\frac{1}{10}\right)^n$

Then $\lim\limits_{n\to\infty}(a_n)^{1/n} = \lim\limits_{n\to\infty}\left\{\left(\frac{1}{10}\right)^n\right\}^{\frac{1}{n}}$

$$= \lim_{n\to\infty}\frac{1}{10}$$

$$= \frac{1}{10} < 1$$

$\therefore$ $\lim\limits_{n\to\infty}(a_n)^{1/n} = \frac{1}{10} < 1$. So given series is convergent by using Cauchy root test.

e. $\sum\limits_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

Solution

Given series is $\sum\limits_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

Let $a_n = \left(\frac{n}{n+1}\right)^{n^2}$

Then $\lim\limits_{n\to\infty}(a_n)^{1/n} = \lim\limits_{n\to\infty}\left\{\left(\frac{n}{n+1}\right)^{n^2}\right\}^{1/n}$

$$= \lim_{n \to \infty} \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n \to \infty} e^{\log\left(\frac{n}{n+1}\right)^n}$$

$$= \lim_{n \to \infty} e^{n \log\left(\frac{n}{n+1}\right)}$$

$$= \lim_{n \to \infty} e^{\left(\frac{\log \frac{n}{n+1}}{\frac{1}{n}}\right)}$$

$$= \lim_{n \to \infty} e^{\frac{\frac{n+1}{n} \cdot \frac{(n+1)-n}{(n+1)^2}}{-\frac{1}{n^2}}}$$

$$= \lim_{n \to \infty} e^{-\frac{n+1}{n} \cdot \frac{1}{(n+1)^2} \cdot n^2}$$

$$= \lim_{n \to \infty} e^{-\frac{n}{n+1}}$$

$$= \lim_{n \to \infty} e^{\frac{-1}{1+\frac{1}{n}}}$$

$$= e^{-1} = \frac{1}{e} < 1$$

$\therefore \lim_{n \to \infty} (a_n)^{1/n} = \frac{1}{e} < 1$, so that given series is convergent by using Cauchy root test.

f. $\sum \left(\frac{n - \ell n\, n}{2n}\right)^n$

Solution

Given series is $\sum \left(\frac{n - \ell n\, n}{2n}\right)^n$

Let $a_n = \left(\frac{n - \ell n\, n}{2n}\right)^n$

$$\therefore \lim_{n \to \infty} (a_n)^{1/n} = \lim_{n \to \infty} \left\{\left(\frac{n - \ell nn}{2n}\right)^n\right\}^{1/n}$$

$$= \lim_{n \to \infty} \left(\frac{n - \ell nn}{2n}\right)$$

$$= \lim_{n \to \infty} \frac{1 - \frac{1}{n}}{2}$$

$$= \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{2n}\right) = \frac{1}{2} < 1$$

$\therefore \lim_{n \to \infty} (a_n)^{1/n} = \frac{1}{2} < 1$, so that the given series is convergent by using Cauchy root test.

g. $\sum_{n=1}^{\infty} \frac{n^{n^2}}{\left(n + \frac{1}{5}\right)^{n^2}}$

Solution

Let $a_n = \frac{n^{n^2}}{\left(n + \frac{1}{5}\right)^{n^2}}$

Then, $\lim_{n \to \infty} (a_n)^{1/n^2} = \lim_{n \to \infty} \left(\frac{n}{n^2 + \frac{1}{5}}\right)^{n^2 \cdot \frac{1}{n^2}}$

$$= \lim_{n \to \infty} \frac{n}{n^2 + \frac{1}{5}}$$

$$= \lim_{n \to \infty} \frac{n}{n^2\left(1 + \frac{1}{5n^2}\right)}$$

$$= \lim_{n \to \infty} \frac{1}{n\left(1 + \frac{1}{5n^2}\right)} = 0 < 1$$

$\therefore \lim_{n \to \infty} (a_n)^{1/n^2} = 0 < 1$.

So given series Σa_n is convergent by using Cauchy root test.

h. $\sum 8\left(1 + \frac{1}{n}\right)^n$

Solution

Given series $\sum 8\left(1 + \frac{1}{n}\right)^n = 8 \sum \left(1 + \frac{1}{n}\right)^n$

Let $a_n = \left(1 + \frac{1}{n}\right)^n$

Then, $\lim_{n \to \infty} (a_n)^{1/n} = \lim_{n \to \infty} \left\{\left(1 + \frac{1}{n}\right)^n\right\}^{1/n}$

$$= \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1$$

$\therefore \lim_{n \to \infty} (a_n)^{1/n} = 1$ and term $\frac{1}{n}$ has come so that given series is divergent.

i. $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Solution

Given series is $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$

Let $a_n = \left(\frac{2n+3}{3n+2}\right)^n$

Then, $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left\{\left(\frac{2n+3}{3n+2}\right)^n\right\}^{1/n}$

$= \lim_{n\to\infty} \frac{2n+3}{3n+2}$

$= \lim_{n\to\infty} \frac{2+\frac{3}{n}}{3+\frac{2}{n}}$

$= \frac{2+\frac{3}{\infty}}{3+\frac{2}{\infty}}$

$= \frac{2}{3} < 1$

$\therefore \lim_{n\to\infty} (a_n)^{1/n} = \frac{2}{3} < 1$, so that the given series is convergent by using Cauchy root test.

j. $\sum_{n=1}^{\infty} \frac{n^{100}\,100^n}{n!}$

Solution

Given $\sum_{n=1}^{\infty} \frac{n^{100}\,100^n}{n!}$

Let $a_n = \frac{n^{100}\,100^n}{n!}$

Then $a_{n+1} = \frac{(n+1)^{100}\,(100)^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^{100}(100)^{n+1}}{(n+1)!}}{\frac{n^{100}(100)^n}{n!}}$

$= \lim_{n\to\infty} \frac{n!\,(n+1)^{100}(100)^{n+1}}{(n+1)!\,n^{100}(100)^n}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{100} \frac{n!}{(n+1)n!}\,100$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{100} \frac{100}{n+1}$

$= (1+0)^{100} \frac{100}{\infty+1}$

$= 0$

< 1

Hence, given series is convergent by using D. Alembert ratio test.

Exercise 5.5

1. **Test the convergence and absolute convergence of the following infinite series.**

a. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

Solution

Given series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

It is an alternative series. Let $a_n = \frac{1}{n}$ then

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$

and each term is numerically less than preceding term. So that given series is convergent by using Leibniz theorem. Also,

$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$

$= \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent according to p-series.

Hence, given series is convergent but not absolutely convergent. So it is conditional convergent.

b. $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots$

Solution

Given series is $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \ldots$

It is an alternative series.

Let $a_n = \frac{1}{1+(n-1)4} = \frac{1}{4n+3}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{4n+3} = \frac{1}{\infty} = 0$

Since each term is numerically less than preceding so given series is convergent by using Leibniz theorem.

Also for absolute convergence

$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \ldots$

$= \sum_{n=1}^{\infty} \frac{1}{4n-3}$

$= \sum_{n=1}^{\infty} \frac{1}{n\left(4-\frac{3}{n}\right)}$ is divergent.

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent so that given series is not absolutely convergent.

So this series is conditional convergent.

c. $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Solution

Given series is $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

It is an alternative series and let $a_n = \frac{1}{n^2}$

$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^2} = 0$

and each term is numerically less than preceding term. So given series is convergent by using Leibniz theorem

For absolute convergent

$$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$\therefore \sum \frac{1}{n^2}$ is convergent by using p-series. So that given series is convergent and absolutely convergent both.

d. $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots$

Solution

Given series is $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots$

It is an alternative series and let $a_n = \frac{1}{n!}$

Then, $\lim_{n \to \infty} \frac{1}{n!} = \lim_{n \to \infty} \frac{1}{n(n-1)\dots 3 \cdot 2 \cdot 1} = 0$

and each term is numerically less than preceding term. So the given series is convergent by using Leibniz theorem.

For absolute convergence

$$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$< 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

Since $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent then given series is both convergent and absolutely convergent.

e. $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \dots$

Solution

Given series is $1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} \dots$

It is an alternative series and let $a_n = \frac{1}{n\sqrt{n}}$.

$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n\sqrt{n}} = \lim_{n \to \infty} \frac{1}{n^{3/2}} = 0$

And each term is numerically less than preceding term so given series is convergent by using Leibniz theorem.

For absolute convergence

$$\sum_{n=1}^{\infty} |a_n| = 1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \dots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \text{ is convergent by p-series.}$$

Hence given series is both convergent and absolutely convergent.

f. $\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) + \dots$

Solution

Given series is $\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) + \dots$

It is an alternative series. Let $a_n = \left(\frac{1}{2} - \frac{1}{\log(n+1)}\right)$

$$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \left(\frac{1}{2} - \frac{1}{\log(n+1)}\right)$$

$$= \frac{1}{2} - \frac{1}{\log \infty}$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

$\therefore \lim_{n \to \infty} a_n$ is not zero. So given series is divergent.

g. $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} + \frac{1}{\sqrt{5}+1} + \dots$

Solution

Given series is $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} + \frac{1}{\sqrt{5}+1} + \dots$

It is an alternative series. Let $a_n = \dfrac{1}{\sqrt{n+1}+1}$.

Then, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}+1} = \dfrac{1}{\infty} = 0$

Since each term is numerically less than preceding term so given series is convergent by Leibniz theorem.

For absolute convergence

$$\sum_{n=1}^{\infty} |a_n| = \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} + \frac{1}{\sqrt{5}+1} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+1}$$

$$= \lim_{n\to\infty} \sum \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}\right)}$$

Let $b_n = \dfrac{1}{\sqrt{n}}$ then, $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$

$$= \lim_{n\to\infty} \frac{\dfrac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\frac{1}{\sqrt{n}}\right)}}{\dfrac{2}{\sqrt{n}}}$$

$= 1 \neq 0$. So, given series is divergent.

Since $\Sigma \dfrac{1}{\sqrt{n}}$ is divergent so $\sum\limits_{n=1}^{\infty} |a_n|$ is divergent.

Hence, given series is convergent but not absolutely convergent.

h. $\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dfrac{\sin 4x}{4} + \ldots$

Solution

Given series is $\dfrac{\sin x}{1} - \dfrac{\sin 2x}{2} + \dfrac{\sin 3x}{3} - \dfrac{\sin 4x}{4} + \ldots$

It is an alternative series. Let $a_n = \dfrac{\sin nx}{n}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\sin nx}{n} = 0$ [$\because$ value of $\sin nx = (-1, 1]$]

Since each term is numerically less than preceding term so given series is convergent by using Leibniz theorem.

For absolute convergence

$$\Sigma |a_n| = \frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Let $b_n = \dfrac{1}{n}$ then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{\frac{\sin nx}{n}}{\frac{1}{n}} = \sin n x$

$|\sin nx| \leq n$, since $\Sigma \dfrac{1}{n}$ is divergent by using p-series. So given series Σa_n is divergent as comparison test.

i. $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{1+n}{n^2}$

Solution

Given series is $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \dfrac{1+n}{n^2}$

$$= \frac{2}{1^2} - \frac{3}{2^2} + \frac{4}{3^2} - \frac{5}{4^2} + \ldots$$

It is an alternative series and let $a_n = \dfrac{1+n}{n^2}$.

$$\lim_{n\to\infty} = \lim_{n\to\infty} \frac{(1+n)}{n^2} = \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n^2} = \lim_{n\to\infty} \frac{1}{n}\left(1+\frac{1}{n}\right) = 0$$

and each term is numerically less than preceding term. So, given series is convergent.

Absolute convergent

$$\sum_{n=1}^{\infty} |a_n| = \frac{2}{1^2} + \frac{3}{2^2} + \frac{4}{3^2} + \frac{5}{4^2} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1+n}{n^2}$$

$$= \sum_{n=1}^{\infty} \frac{n\left(1+\frac{1}{n}\right)}{n^2} = \sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)}{n}$$

Let $b_n = \dfrac{1}{n}$ then $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is divergent by p-series. Then,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{\left(1+\frac{1}{n}\right)}{n}}{\dfrac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right) = 1$$

Since Σb_n is divergent so $\Sigma |a_n|$ is also divergent.

Hence given series is convergent but not absolutely convergent (conditionally convergent).

j. $\sum_{n=1}^{\infty}\left(\frac{3+n}{5+n}\right)(-1)^{n+1}$

Solution

Given series is $\sum_{n=1}^{\infty}\left(\frac{3+n}{5+n}\right)(-1)^{n+1} = \frac{-4}{6}+\frac{5}{7}-\frac{6}{8}+\frac{7}{9}+\dots$

and $a_n = \frac{3+n}{5+n}$.

It is an alternative series.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3+n}{5+n} = \lim_{n\to\infty}\frac{n\left(\frac{3}{n}+1\right)}{n\left(\frac{5}{n}+1\right)} = 1$$

Since $\lim_{n\to\infty} a_n \neq 0$, so given series is not convergent.

Hence, given series is divergent.

k. $\Sigma(-1)^{n+1}\frac{1}{n2^n}$

Solution

Given series is $\Sigma(-1)^{n+1}\frac{1}{n2^n}$

$$=\frac{1}{2}-\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\dots$$

It is an alternative series.

Let $a_n = \frac{1}{n\cdot 2^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n\,2^n} = 0$$

Since each term is numerically less than preceding term. So given series is convergent by using Leibniz theorem

For absolute convergent

$$\sum_{n=1}^{\infty}|a_n| = \frac{1}{2}+\frac{1}{2\cdot 2^2}+\frac{1}{3\cdot 2^3}+\frac{1}{4\cdot 2^4}+\dots$$

$$=\sum_{n=1}^{\infty}\frac{1}{n\cdot 2^n}$$

Let $b_n = \frac{1}{n}$ then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n\,2^n}}{\frac{2}{n}}$$

$$=\lim_{n\to\infty}\frac{1}{2^n} = 0$$

l. $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$

Solution

Given series is $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$

It is an alternative series.

Let $a_n = (\sqrt{n+1}-\sqrt{n})$

$$= (\sqrt{n+1}-\sqrt{n})\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$=\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0.$$

Each term is numerically less than preceding term. So given series is convergent by using Leibniz theorem.

For absolute convergence

$$\Sigma|a_n| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\frac{1}{n}\right)}$$

Let $b_n = \frac{1}{\sqrt{n}}$ is divergent by using p-series.

$$\therefore \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}}{\frac{1}{\sqrt{n}}}$$

$$=\frac{1}{2}$$

Then Σa_n is divergent by using comparison test.

m. $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

Solution

Given series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

$$=-1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{5}}+\dots$$

It is an alternative series.

Let $a_n = \frac{1}{\sqrt{n}}$

Then, $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = \frac{1}{\sqrt{\infty}} = 0$

Since each term is numerically less than preceding term. So given series convergent by using Leibniz theorem.

For absolute convergence

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}}\right| = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\dots$$

Let $a_n = \frac{1}{\sqrt{n}}$

Then $\Sigma a_n = \Sigma\frac{1}{\sqrt{n}} = \Sigma\frac{1}{n^{1/2}}$ is divergent by using p-series having $p = \frac{1}{2}$.

Hence, given series is convergent and not absolutely convergent.

2. **For the given series, find the value of x such that the series converges.**

a.

Solution

Given series $= \sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots$

This is an alternative series. Let a_n be the general positive term. Then,

$a_n = \frac{x^n}{n}$ and $a_{n+1} = \frac{x^{n+1}}{n+1}$

Now, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{x^{n+1}}{n+1}\times\frac{n}{x^n}$

$= \lim\limits_{n\to\infty}\frac{x\,n}{n\left(1+\frac{1}{n}\right)} = x$

According to D. Alembert ratio test

Given series is convergent if $|x| < 1$, divergent if $|x| > 1$ and further test is needed for $|x| = 1$.

If $x = 1$ then given series becomes

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$ and $a_n = \frac{1}{n}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{n} = 0$

According to Leibniz theorem, given series is convergent at $x = 1$.

If $x = -1$ then given series becomes

$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}+\dots$

$= -\sum\limits_{n=1}^{\infty}\frac{1}{n}$ is divergent

Since, $\Sigma\frac{1}{n}$ is divergent by using p-series. Hence, given series is divergent at $x = 1$.

Then value of x is $-1 < x \le 1$.

b. $\sum\limits_{n=2}^{\infty}\frac{(-1)^{n-1}x^n}{n(n-1)}$

Solution

Given series $= \sum\limits_{n=2}^{\infty}\frac{(-1)^{n-1}x^n}{n(n-1)}$

$= \frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{12}-\frac{x^5}{20}+\dots$

It is an alternative series. Let a_n be the general term. Then.

$a_n = \frac{x^{n+1}}{n(n+1)}$ and $a_{n+1} = \frac{x^{n+2}}{(n+1)(n+2)}$

Now, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\frac{x^{n+2}}{(n+1)(n+2)}\times\frac{n(n+1)}{x^{n+1}}$

$= \lim\limits_{n\to\infty}\frac{n\cdot x}{n\left(1+\frac{2}{n}\right)}$

$= x$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If $x = 1$, then given series becomes

$\frac{1}{2}-\frac{1}{6}+\frac{1}{12}-\frac{1}{20}+\dots$

Then $a_n = \frac{1}{n(n+1)}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\frac{1}{n(n+1)} = 0$

Each term is numerically less than preceding term. So given series is convergent by using Leibniz theorem at $x = 1$.

If $x = -1$, then given series becomes

$-\frac{1}{2}-\frac{1}{6}-\frac{1}{12}-\frac{1}{20}+\dots$

Let $a_n = \frac{1}{n(n+1)} = \frac{1}{n^2\left(1+\frac{1}{n}\right)}$

Let $b_n = \frac{1}{n^2}$ then $\Sigma b_n = \Sigma\frac{1}{n^2}$ is convergent by p-series.

Now, $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = \lim\limits_{n\to\infty}\frac{\frac{1}{n^2\left(1+\frac{1}{n}\right)}}{\frac{1}{n^2}} = 1$

Hence the given series $\Sigma - a_n = -\Sigma\frac{1}{n(n+1)}$ is convergent.

The possible value of x is $-1 \le x \le 1$.

c. $\sum\limits_{n=1} \dfrac{\quad}{n+1}$

Solution

Given series $= \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n x^n}{n+1} = \dfrac{-x}{2} + \dfrac{x^2}{3} - \dfrac{x^3}{4} + \dfrac{x^4}{5} + \ldots$

It is an alternative series. Let a_n be the general positive term. Then,

$a_n = \dfrac{x^n}{n+1}$ and $a_{n+1} = \dfrac{x^{n+1}}{n+2}$

Now, $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{x^{n+1}}{n+2} \times \dfrac{n+1}{x^n}$

$$= \lim_{n\to\infty} \frac{x \cdot n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If x = 1

Then given series becomes

$\dfrac{-1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \dfrac{1}{5} + \ldots$. It is an alternative series.

Let $a_n = \dfrac{1}{n+1}$ then $\lim\limits_{n\to\infty} a_n = \dfrac{1}{n+1} = 0$.

Each term is numerically less than preceding term. So given series is convergent by using Leibniz theorem.

If x = –1

Then given series becomes

$\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \ldots$

Let $a_n = \dfrac{1}{n+1}$, then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{n+1} = 0$

But $a_n = \dfrac{1}{n+1} = \dfrac{1}{n\left(1+\frac{1}{n}\right)}$ let $b_n = \dfrac{1}{n}$

Since $\Sigma b_n = \Sigma \dfrac{1}{n}$ is divergent by p-series.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\dfrac{1}{n\left(1+\frac{1}{n}\right)}}{\dfrac{1}{n}} = 1$$

$\therefore$ $\Sigma a_n = \Sigma \dfrac{1}{(n+1)}$ is divergent.

Then the possible value of x is $-1 < x \le 1$ or $x \in (-1, 1]$.

d. $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$

Solution

Given series $= \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$

$= \dfrac{1}{0!} + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

$= 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

Let $a_n = \dfrac{x^{n-1}}{(n-1)!}$ then $a_{n+1} = \dfrac{x^n}{n!}$

Then, $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{\frac{x^n}{n!}}{\frac{x^{n-1}}{(n-1)!}}$

$= \lim\limits_{n\to\infty} \dfrac{x}{n}$

$= \dfrac{x}{\infty}$

$= 0 < 1$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$

or, $-1 < \dfrac{x}{\infty} < 1$

or, $-\infty < x < \infty$

$\therefore$ The possible value of $x \in (-\infty, \infty)$.

e. $\Sigma\, n!\, x^n$

Solution

Given series $= \Sigma\, n!\, x^n$

$= 1!x + 2!x^2 + 3!x^3 + 4!x + 5!x^5 + \ldots$

Let $a_n = n!x^n$ and $a_{n+1} = (n+1)!\, x^{n+1}$

$\therefore$ $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{(n+1)!\, x^{n+1}}{n!\, x^n} = \lim\limits_{n\to\infty} \dfrac{(n+1)\, n!\, x}{n!} = (n+1)x = \infty$

Hence, this series is convergent if $x = 0$ only case.

f. $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

Solution

Given series $= x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$

It is an alternative series. Let a_n be the general term of positive term

$a_n = \dfrac{x^n}{n}$, then $a_{n+1} = \dfrac{x^{n+1}}{n+1}$

Then, $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{x^{n+1}}{n+1} \times \dfrac{n}{x^n}$

$$= \lim_{n\to\infty} \frac{x\,n}{n\left(1+\frac{1}{n}\right)}$$

$= x$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If $x = 1$ then given series becomes

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, it is an alternative series.

Let $a_n = \frac{1}{n}$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$

Each term is numerically less than preceding term. So given series is convergent by using Leibniz theorem

If $x = -1$ then given series becomes

$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\ldots$

$= -\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\right)$

$= -\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent according to p-series.

$\therefore$ Given series is divergent if $x = -1$.

The possible value of x is $-1 < x \le 1$ or $x \in (-1, 1]$.

g. $\frac{x^2}{2}+\frac{2x^3}{3}+\frac{3x^4}{4}+\frac{4x^5}{5}+\ldots$

Solution

Given series $= \frac{x^2}{2}+\frac{2x^3}{3}+\frac{3x^4}{4}+\frac{4x^5}{5}+\ldots$

Let $a_n = \frac{nx^{n+1}}{n+1}$, then $a_{n+1} = \frac{(n+1)\,x^{n+2}}{n+2}$

$$\text{Now, } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{(n+1)\,x^{n+2}}{(n+2)}}{\frac{n\,x^{n+1}}{n+1}}$$

$$= \lim_{n\to\infty}\frac{(n+1)^2\,x^{n+2}}{(n+2)\,n\,x^{n+1}}$$

$$= \lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)^2 x}{n^2\left(1+\frac{2}{n}\right)}$$

$= x$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If $x = 1$, then given series becomes

$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\ldots$

Let $a_n = \frac{n}{n+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{n}{n\left(1+\frac{1}{n}\right)} = 1$$

$\therefore \lim_{n\to\infty} a_n \ne 0$, so given series is not convergent at $x = 1$.

If $x = -1$ then given series becomes

$\frac{1}{2}-\frac{2}{3}+\frac{3}{4}-\frac{4}{5}+\ldots$

Let $a_n = \frac{n}{n+1}$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{n}{n\left(1+\frac{1}{n}\right)} = 1$

$\therefore \lim_{n\to\infty} a_n \ne 0$, so given series is not convergent if $x = -1$.

The possible value of $x\ -1 < x < 1$.

h. $\Sigma\frac{x^n}{3^n n^2}$ for $x > 0$

Solution

Given series $= \Sigma\frac{x^n}{3^n n^2}$

$= \frac{x}{3\cdot 1^2}+\frac{x^2}{3^2\cdot 2^2}+\frac{x^3}{3^3\cdot 3^2}+\frac{x^4}{3^4\cdot 4^2}+\ldots$

Let $a_n = \frac{x^n}{3^n n^2}$ and $a_{n+1} = \frac{x^{n+1}}{3^{n+1}(n+1)^2}$

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{x^{n+1}}{3^{n+1}(n+1)^2}\times\frac{3^n n^2}{x^n}$$

$$= \lim_{n\to\infty}\frac{x\cdot n^2}{3n^2\left(1+\frac{1}{n}\right)^2}$$

$= \frac{x}{3}$

According to D' Alembert ratio test given series is convergent if $\left|\frac{x}{3}\right| < 1$ and divergent if $\left|\frac{x}{3}\right| > 1$ and further test is needed if $\left|\frac{x}{3}\right| = 1$.

If $\frac{x}{3} = 1$ then given series becomes

$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$

$= \Sigma\frac{1}{n^2}$ is convergent by using p-series having $p = 2 > 1$.

If $\frac{x}{3} = -1$ then given series becomes

$-\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\frac{1}{5^2}+\ldots$

It is alternative series. Let general positive term be, $a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n^2} = 0$$

Each term is numerically less than preceding term. Hence given series is convergent by using Leibniz theorem if $\frac{x}{3} = -1$.

The possible value is

$$-1 \le \frac{x}{3} \le 1$$

or, $-3 \le x \le 3$

or, $x \in [-3, 3]$

i. $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Solution

Given series $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

Let $a_n = \frac{x^{n-1}}{(n-1)!}$ then $a_{n+1} = \frac{x^n}{n!}$

$$\text{Now, } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{x^n}{n!}}{\frac{x^{n-1}}{(n-1)!}}$$

$$= \lim_{n\to\infty}\frac{x^n}{n!}\times\frac{(n-1)!}{x^{n-1}}$$

$$= \lim_{n\to\infty}\frac{x}{n}$$

$$= \frac{x}{\infty}$$

According to D' Alembert ratio test, given series is convergent if

$$\left|\frac{x}{\infty}\right| < 1$$

or, $-1 < \frac{x}{8} < 1$

or, $-\infty < x < \infty$

j. $\Sigma\sqrt{\frac{n+1}{n^3+1}}\,x^n$

Solution

Given series $= \Sigma\sqrt{\frac{n+1}{n^3+1}}\,x^n$

$$= x + \sqrt{\frac{2}{9}}x^2 + \sqrt{\frac{4}{28}}x^3 + \sqrt{\frac{5}{65}}x^4 + \dots$$

$$= x + \frac{\sqrt{2}}{3}x^2 + \frac{1}{\sqrt{7}}x^3 + \frac{1}{\sqrt{13}}x^4 + \dots$$

Let $a_n = \sqrt{\frac{n+1}{n^3+1}}\,x^n$ then $a_{n+1} = \sqrt{\frac{n+2}{n^3+3n^2+3n+1+1}}\,x^{n+1}$

$$a_{n+1} = \sqrt{\frac{n+2}{n^3+3n^2+3n+2}}\,x^{n+1}$$

$$\text{Then } \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\sqrt{\frac{n+2}{n^3+3n^2+3n+2}}\,x^{n+1}}{\sqrt{\frac{n+1}{n^3+1}}\,x^n}$$

$$= \lim_{n\to\infty} x.\frac{\frac{1}{n^2}\sqrt{\frac{1+\frac{2}{n}}{\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}\right)}}}{\frac{1}{n^2}\sqrt{\frac{1+\frac{1}{n}}{1+\frac{1}{n^3}}}}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If $x = 1$, then given series becomes

$$1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{13}} + \dots$$

$$\text{Let } a_n = \sqrt{\frac{n+2}{n^3+3n^2+3n+2}} = \frac{1}{n}\sqrt{\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^2}\right)}}$$

Let $b_n = \frac{1}{n}$ then $\Sigma b_n = \Sigma\frac{1}{n}$ is divergent by using D' Alembert ratio rest.

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n}\sqrt{\frac{\left(1+\frac{2}{n}\right)}{\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{2}{n^3}\right)}}}{\frac{1}{n}} = 1$$

$\therefore$ Σa_n is divergent by using comparison ratio test at $x = 1$.

If $x = -1$ then given series becomes

$$-1 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{13}} + \dots$$

It is an alternative series

$$\text{Let } a_n = \sqrt[\frac{1}{n^2}]{\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)}}$$

$$\text{then } \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[\frac{1}{n^2}]{\frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)}} = 0$$

Hence given series is convergent by using Leibniz theorem at $x = -1$. Then possible value of $x = -1 \le x < 1$.

k. $\Sigma \frac{n^2-1}{n^2+1} x^n$

Solution

$$\text{Given series } = \Sigma \frac{n^2-1}{n^2+1} x^n$$

$$= \frac{0}{2}x + \frac{3}{5}x^2 + \frac{8}{10}x^3 + \frac{15}{17}x^4 + \ldots$$

$$\text{Let } a_n = \frac{n^2-1}{n^2+1}x^n \text{ and } a_{n+1} = \frac{n^2+2n}{n^2+2n+2}x^{n+1}$$

$$\text{Now, } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n^2+2n}{n^2+2n+2}\cdot x^{n+1}}{\frac{n^2-1}{n^2+1}\cdot x^n}$$

$$= \lim_{n\to\infty} \frac{x\, n^2\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)} \times \frac{n^2\left(1+\frac{1}{n^2}\right)}{n^2\left(1-\frac{1}{n^2}\right)}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If = 1, then given series becomes

$$\frac{3}{5}+\frac{8}{10}+\frac{15}{17}+\ldots$$

$$\text{Let } a_n = \frac{n^2-1}{n^2+1}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2\left(1-\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = 1 \ne 0$$

So given series is not convergent at $x = 1$

If $x = -1$, then given series becomes

$$\frac{3}{5}-\frac{8}{10}+\frac{15}{17}+\ldots$$

It is an alternative series but each term is not less than numerically preceding term So given series is divergent at $x = -1$. So possible value of x is $-1 < x < 1$.

l. $\Sigma(-1)^{n-1}\frac{x^n}{\sqrt{n}}$

Solution

$$\text{Given series } = \Sigma(-1)^{n-1}\frac{x^n}{\sqrt{n}}$$

$$= \frac{x}{\sqrt{1}} - \frac{x^2}{\sqrt{2}} + \frac{x^3}{\sqrt{3}} + \frac{x^4}{\sqrt{4}} + \frac{x^5}{\sqrt{5}} - \frac{x^6}{\sqrt{6}} + \ldots$$

It is an alternative series.

Let a_n be the general positive term.

$$a_n = \frac{-x^n}{\sqrt{n}} \text{ then } a_{n+1} = \frac{x^{n+1}}{\sqrt{n+1}}$$

$$\text{Now, } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{\sqrt{n+1}} \times \frac{\sqrt{n}}{x^n}$$

$$= \lim_{n\to\infty} \frac{x}{\sqrt{n}\sqrt{1+\frac{1}{n}}} \times \sqrt{n}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If x = 1

Given series becomes

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \ldots$$

It is an alternative series, each term is numerically less than preceding term and general positive term $a_n = \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

Hence given series is convergent by using Leibniz theorem at $x = 1$.

If x = –1

Then given series becomes

$$-1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

$$= -\left\{1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots\right\}$$

$$= -\Sigma\frac{1}{\sqrt{n}} \text{ is divergent by p-series having } p = \frac{1}{2} < 1.$$

$\therefore$ Given series is divergent at $x = -1$.

The possible value of x is $-1 < x \le 1$.

3. **Show that the series $\frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$ converges absolutely.**

Solution

$$\text{Given series } = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots$$

For absolute convergence

$$\Sigma\left|\frac{(-1)^{n+1}\sin nx}{n^2}\right| = \frac{\sin x}{1^2}+\frac{\sin 2x}{2^2}+\frac{\sin 3x}{3^2}+\ldots$$

Let $a_n = \frac{\sin nx}{n^2} \le \frac{1}{n^2}$

Since $\Sigma\frac{1}{n^2}$ is convergent by using p-series as p = 2, so that $\Sigma a_n = \Sigma\frac{\sin nx}{n^2}$ is convergent by using comparison test.

Hence given series $\Sigma\left|\frac{(-1)^{n+1}\sin nx}{n^2}\right| = \frac{\sin x}{1^2}+\frac{\sin 2x}{2^2}+\frac{\sin 3x}{3^2}+\ldots$ is absolutely convergent.

4. **Test the convergence and absolute convergence of the series $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\log n)^2}$.**

Solution

Given series is $\sum_{n=2}^{\infty}\frac{(-1)^n}{n(\log n)^2}$

It is an alternative series.

Let $a_n = \frac{1}{n(\log n)^2}$

Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n(\log n)^2} = \frac{1}{\infty} = 0$

Hence, given series is convergent by using Leibniz test because each term is numerically less than preceding term.

For absolute convergence

$$\sum\left|\frac{(-1)^n}{n(\log n)^2}\right| = \frac{1}{2(\log 2)^2}+\frac{1}{3(\log 3)^2}+\frac{1}{4(\log 4)^2}+\ldots+\frac{1}{n(\log n)^2}+\frac{1}{(n+1)\log(n+1)^2}$$

Let $a_n = \frac{1}{(n+1)\log(n+1)^2} = \frac{1}{2(n+1)\log(n+1)}$ and $a_{n+1} = \frac{1}{(n+2)\log(n+2)^2} = \frac{1}{2(n+2)\log(n+a)}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{\frac{1}{2(n+2)\log(n+2)}}{\frac{1}{2(n+1)\log(n+1)}}$$

$$= \lim_{n\to\infty}\frac{(n+1)\log(n+1)}{(n+2)\log(n+2)} \quad \left[\frac{\infty}{\infty}\right]$$

$$= \lim_{n\to\infty}\frac{\frac{n+1}{n+1}+\log(n+1)\cdot 1}{\frac{n+2}{n+2}+\log(n+2)}$$

$$= \lim_{n\to\infty}\frac{1+\log(n+1)}{1+\log(n+2)}$$

$$= \lim_{n\to\infty}\frac{0+\frac{1}{n+1}}{0+\frac{1}{n+2}}$$

$$= \lim_{n\to\infty}\frac{n+2}{n+1}$$

$= 1$

$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$, so it is not convergent. So the given series is not absolutely convergent.

5. **Show that the series $\sum_{n=0}^{\infty}\frac{n!}{100^n}x^n$ converges at $x = 0$ only.**

Solution

Given series is $\sum_{n=0}^{\infty}\frac{n!}{100^n}x^n$

Let $a_n = \frac{n!}{(100)^n}x^n$ and $a_{n+1} = \frac{(n+1)!\,x^{n+1}}{(100)^{n+1}}$

Then, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)!\,x^{n+1}}{(100)^{n+1}}\times\frac{(100)^n}{n!\,x^n}$

$$= \lim_{n\to\infty}\frac{(n+1)\cdot n!\cdot x^n\cdot x}{100\,(100)^n}\times\frac{(100)^n}{n!\,x^n}$$

$$= \lim_{n\to\infty}(n+1)\,x$$

$= 0$ if $x = 0$

< 1

$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 0 < 1$ if $x = 0$. The given series is convergent only when $x = 0$.

6. **Find the interval of convergence, centre of convergence and radius of convergence of the following series.**

a. $\Sigma\frac{x^n}{n^2}$

Solution

Given series $= \Sigma\frac{x^n}{n^2}$

$$= \frac{x}{1}+\frac{x^2}{2^2}+\frac{x^3}{2^3}+\frac{x^4}{4^2}+\ldots$$

Let $a_n = \frac{x^n}{n^2}$, $a_{n+1} = \frac{x^{n+1}}{(n+1)^2} = \frac{x^{n+1}}{n^2+2n+1}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{x^{n+1}}{n^2+2n+1}\times\frac{n^2}{x^n}$$

$$= \lim_{n\to\infty}\frac{x\,n^2}{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}$$

$= x$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

Now, x = 1

Then given series becomes

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

$= \Sigma \frac{1}{n^2}$ is convergent by p-series having $p = 2 > 1$.

Hence, given series is convergent at $x = 1$.

At x = –1

Then the given series becomes

$$-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \dots$$

It is an alternative series. Each term is numerically less than preceding term. Let its general positive term be $a_n = \frac{1}{n^2}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

Hence given series is convergent at $x = -1$.

$\therefore$ The possible value of $x = -1 \le x \le 1$.

The interval of convergence = [–1, 1]

Centered convergence $= \frac{-1+1}{2} = 0$

Radius of convergence $= \frac{1-(-1)}{2} = \frac{2}{2} = 1$

b. $\sum \frac{x^n}{n(n+1)}$

Solution

Given series $= \sum \frac{x^n}{n(n+1)}$

$$= \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \frac{x^4}{4\cdot 5} + \frac{x^5}{5\cdot 6} + \dots$$

Let general term $a_n = \frac{x^n}{n(n+1)}$, $a_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$

$$\text{Now, } \lim_{n\to\infty} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)(n+2)} \times \frac{n(n+1)}{x^n}$$

$$= \lim_{n\to\infty} \frac{x\,n}{n\left(1+\frac{2}{n}\right)}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If x = 1

Given series becomes

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \dots$$

Let general term $a_n = \frac{1}{n(n+1)} = \frac{1}{n^2\left(1+\frac{1}{n}\right)}$

Let $b_n = \frac{1}{n^2}$. Then $\Sigma \frac{1}{n^2}$ is convergent by using p-series.

$$\text{Now, } \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2\left(1+\frac{1}{n}\right)}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = 1$$

Hence, Σa_n is convergent by using comparison test. Hence, given series is convergent at $x = 1$.

If x = –1

Given series becomes

$$-\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} - \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} - \frac{1}{5\cdot 6} + \dots$$

It is an alternative series. Each term is numerically less than preceding term. Let its general positive term $a_n = \frac{1}{n(n+1)}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n(n+1)} = 0$$

Hence given series is convergent at $x = -1$ by using Leibniz theorem.

The possible value of $x = -1 \le x \le 1$.

Interval of convergence = [–1, 1]

Centre of convergence $= \frac{-1+1}{2} = 0$

Radius of convergence $= \frac{1+1}{2} = 1$

c. $\sum \frac{x^n}{(2n-1)(2n+1)}$

Solution

Given series $= \sum \frac{x^n}{(2n-1)(2n+1)} = \frac{x}{1\cdot 3} + \frac{x^2}{3\cdot 5} + \frac{x^3}{5\cdot 7} + \frac{x^4}{7\cdot 9} + \dots$

Let $a_n = \frac{x^n}{(2n-1)(2n+1)}$ $a_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+3)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+1)(2n+3)} \times \frac{(2n-1)(2n+1)}{x^n}$$

$$\lim_{n\to\infty} \frac{x\,n\left(2-\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)} = x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If x = 1

Then given series becomes

$\overline{1\cdot 3} + 3\cdot 5 + 5\cdot 7 + 7\cdot 9$

Let $a_n = \frac{1}{(2n-1)(2n+1)} = \frac{1}{n^2\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

Let $b_n = \frac{1}{n^2}$ and $\Sigma\frac{1}{n^2}$ is convergent by p-series.

Now, $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^2\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{\frac{1}{n^2}} = \frac{1}{4}$

Hence Σa_n is convergent by using comparison test. Hence given series is convergent at $x = 1$.

If $x = -1$

Then given series becomes

$$-\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}-\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\ldots$$

It is an alternative series. Each term is numerically less than preceding term. Let a_n be general positive term

$$a_n = \frac{1}{(2n-1)(2n+1)}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{(2n-1)(2n+1)} = 0$$

Hence given series is convergent by using Leibniz theorem at $x = 1$.

The possible value of $x = -1 \le x \le 1$.

Interval of convergence = $[-1, 1]$

Centre of convergence = $\frac{-1+1}{2} = 0$

Radius of convergence = $\frac{1-(-1)}{2} = 1$

d. $\sum\frac{x^n}{(n+1)\sqrt{n+2}}$

Solution

Given series $= \sum\frac{x^n}{(n+1)\sqrt{n+2}}$

$$= \frac{x}{2\sqrt{4}}+\frac{x^2}{3\sqrt{5}}+\frac{x^3}{4\sqrt{6}}+\frac{x^4}{5\sqrt{7}}+\ldots$$

Let $a_n = \frac{x^n}{(n+1)\sqrt{n+2}}$ $a_{n+1} = \frac{x^{n+1}}{(n+2)\sqrt{n+3}}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{x^{n+1}}{(n+2)\sqrt{n+3}}\times\frac{(n+1)\sqrt{n+2}}{x^n}$$

$$= \lim_{n\to\infty} x.\frac{n^{3/2}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}}{n^{3/2}\left(1+\frac{2}{n}\right)\sqrt{1+\frac{3}{n}}}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If $x = 1$

Then given series becomes

$$= \frac{1}{2\sqrt{4}}+\frac{1^2}{3\sqrt{5}}+\frac{1^3}{4\sqrt{6}}+\frac{1^4}{5\sqrt{7}}+\ldots$$

Let $a_n = \frac{1}{(n+1)\sqrt{n+2}} = \frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}}$

Let $b_n = \frac{1}{n^{3/2}}$, then $\Sigma\frac{1}{n^{3/2}}$ is convergent by p-series having $p = \frac{3}{2} > 1$.

$$\text{Now, } \lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{n^{3/2}\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}}}{\frac{1}{n^{3/2}}}$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)\sqrt{1+\frac{2}{n}}} = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{\infty}\right)\sqrt{1+\frac{2}{\infty}}} = 1$$

Since $\Sigma b_n = \Sigma\frac{1}{n^{3/2}}$ is convergent then Σa_n is convergent by comparison test. Hence given series is convergent at $x = 1$.

At $x = -1$

Then given series becomes

$$-\frac{1}{2\sqrt{4}}+\frac{1^2}{3\sqrt{5}}-\frac{1^3}{4\sqrt{6}}+\frac{1^4}{5\sqrt{7}}+\ldots$$

It is an alternative series. Each term is numerically less than preceding term. Let a_n be general positive term.

$$a_n = \frac{1}{(n+1)\sqrt{n+2}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{(n+1)\sqrt{n+2}} = 0$$

$\therefore$ Given series is convergent by using Leibniz theorem at $x = -1$.

The possible value of $x = -1 \le x \le 1$.

Interval of convergence = $[-1, 1]$

Radius of convergence = $\frac{1+1}{2} = 1$

Centre of convergence = $\frac{-1+1}{2} = 0$

e. $x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots$

Solution

Given series $=x-\frac{x^2}{2^2}+\frac{x^3}{3^2}-\frac{x^4}{4^2}+\ldots$

Let general positive term be $a_n=\frac{x^n}{n^2}$ $a_{n+1}=\frac{x^{n+1}}{(n+1)^2}$

Now, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{x^{n+1}}{(n+1)^2}\times\frac{n^2}{x^n}=\lim\limits_{n\to\infty}\frac{x\cdot n^2}{n^2\left(1+\frac{1}{n}\right)^2}=x$

According to D' Alembert ratio test, given series is convergent if $|x|<1$ and divergent if $|x|>1$. Also further test is needed for $|x|=1$.

If x = 1

Then given series becomes

$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\ldots$

It is an alternative series. Each term is numerically less than preceding term. Let a_n be general positive term

$a_n=\frac{1}{n^2}$

$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}\frac{1}{n^2}=\frac{1}{\infty}=0$

Hence, given series is convergent by using Leibniz theorem at x = 1.

If x = –1

Then given series becomes

$-1-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\ldots$

$=-\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\right)$

$=-\Sigma\frac{1}{n^2}$ is convergent by p-series having $p=2>1$.

Hence, given series is convergent at $x=-1$.

The possible value of $x=-1\le x\le 1$.

Interval of convergence = $[-1, 1]$

Centre of convergence $=\frac{1-1}{2}=0$

Radius of convergence $=\frac{1+1}{2}=1$

f. $\sum\limits_{n=1}^{\infty}\frac{n^2}{2^n}x^n$

Solution

Given series $=\sum\limits_{n=1}^{\infty}\frac{n^2}{2^n}x^n=\frac{1}{2}x+\frac{4}{2^2}x^2+\frac{9}{2^3}x^3+\frac{16}{2^4}x^4+\frac{25}{2^5}x^5+\ldots$

Let $a_n=\frac{n^2}{2^n}x^n$, $a_{n+1}=\frac{(n+1)^2x^{n+1}}{2^{n+1}}$

Then, $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim\limits_{n\to\infty}\frac{(n+1)^2x^{n+1}}{2^{n+1}}\times\frac{2^n}{n^2x^n}=\lim\limits_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)^2x}{2n^2}=\frac{x}{2}$

According to D' Alembert ratio test, given series is convergent if $\left|\frac{x}{2}\right|<1$ and divergent if $\left|\frac{x}{2}\right|>1$. Also further test is needed for $\left|\frac{x}{2}\right|=1$.

If $\frac{x}{2}=1$

Then given series becomes

$1+4+9+16+25+\ldots$

$=\Sigma n^2$

Let $a_n=n^2$ and $\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}n^2=\infty\ne 0$.

So given series is not convergent at $\frac{x}{2}=1$.

If $\frac{x}{2}=-1$

Then given series becomes

$-1+4-9+16-25+\ldots$

It is an alternative series. Let a general positive term $a_n=n^2$.

$\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}n^2=\infty\ne 0$.

Hence given series is not convergent at $\frac{x}{2}=-1$.

The possible value of x

$=-1<\frac{x}{2}<1$

or, $-2<x<2$

Interval of convergence = $(-2, 2)$

Centre of convergence $=\frac{-2+2}{2}=0$

Radius of convergence $=\frac{2+2}{2}=2$

g. $\sum\limits_{n=1}^{\infty}\frac{(n+1)(x-4)^n}{10^n}$

Solution

Given series $=\sum\limits_{n=1}^{\infty}\frac{(n+1)(x-4)^n}{10^n}$

$=\frac{2(x-4)^1}{10}+\frac{3(x-4)^2}{10^2}+\frac{4(x-4)^3}{10^3}+\frac{5(x-4)^4}{10^4}+\ldots$

Let a_n be general given series

$a_n = \frac{(n+1)(x-4)^n}{10^n}$, $a_{n+1} = \frac{(n+2)(x-4)^{n+1}}{10^{n+1}}$

Now, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+2)(x-4)^{n+1}}{10^{n+1}} \times \frac{10^n}{(n+1)(x-4)^n}$

$$= \lim_{n\to\infty} \frac{n\left(1+\frac{2}{n}\right)(x-4)}{10\,n\left(1+\frac{1}{n}\right)}$$

$$= \frac{x-4}{10}$$

According to D' Alembert ratio test, given series is convergent if $\left|\frac{x-4}{10}\right| < 1$ and divergent if $\left|\frac{x-4}{10}\right| > 1$. Also further test is needed for $\left|\frac{x-4}{10}\right| = 1$.

Let $\frac{x-4}{10} = 1$. Then given series becomes

$2 + 3 + 4 + 5 + \ldots$

Let $a_n = (n+1)$ then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (n+1) = \pm\infty$.

Hence, given series is divergent at $\frac{x-4}{10} = 1$.

Let $\frac{x-4}{10} = -1$. Then given series becomes

$-2 + 3 - 4 + 5 + \ldots$

Let $a_n (-1)^n (n+1)$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n (n+1) = \infty$

Hence given series is divergent at $\frac{x-4}{10} = -1$

The possible value of x

$= \ -1 < \frac{x-4}{10} < 1.$

or, $-6 < x < 14$

Interval of convergence = $(-6, 14)$

Centre of convergence = $\frac{-6+14}{2} = 4$

Radius of convergence = $\frac{16+6}{2} = 10$

h. $\sum_{n=1}^{\infty} \frac{n^2}{2^{3n}} (x+4)^n$

Solution

Given series $= \sum_{n=1}^{\infty} \frac{n^2}{2^{3n}} (x+4)^n$

$= \frac{1}{2^3}(x+4) + \frac{4}{2^6}(x+4)^2 + \frac{9}{2^9}(x+4)^3 + \frac{16}{2^{12}}(x+4)^4$

Let $a_n = \frac{n^2}{2^{3n}}(x+4)^n$ and $a_{n+1} = \frac{(n+1)^2 (x+4)^{n+1}}{2^{3(n+1)}}$

Now, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2 (x+4)^{n+1}}{2^{3(n+1)}} \times \frac{2^{3n}}{n^2 (x+4)^n}$

$$= \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)^2 (x+4)}{2^3 n^2}$$

$$= \frac{x+4}{8}$$

According to D' Alembert ratio test, given series is convergent if $\left|\frac{x+4}{8}\right| < 1$ and divergent if $\left|\frac{x+4}{8}\right| > 1$. Also further test is needed for $\left|\frac{x+4}{8}\right| = 1$.

Let $\left(\frac{x+4}{8}\right) = 1$

Then given series becomes

$1^2 + 2^2 + 3^2 + 4^2 + \ldots$

Let $a_n = n^2$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^2 = \infty$.

Hence, given series is convergent at $\left(\frac{x+4}{8}\right) = 1$.

Let $\left(\frac{x+4}{8}\right) = -1$

The given series becomes

$-1 + 4 - 9 + 16 - 25 + \ldots$

Let $a_n = (-1)^n n^2$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n n^2 = \pm\infty$.

Hence, given series is divergent at $\left(\frac{x+4}{8}\right) = -1$.

The possible value of x

$= \ -1 < \frac{x+4}{8} < 1$

or, $-8 < x + 4 < 8$

or, $-12 < x < 4$

Interval of convergence $(-12, 4)$

Centre of convergence = $\frac{+4-12}{2} = -4$

Radius of convergence = $\frac{4+12}{2} = 8$

i. $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!} (x-4)^n$

Solution

Given series $= \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n!} (x-4)^n$

$$= \frac{-3(x-4)}{1!} + \frac{9(x-4)^2}{2!} - \frac{27(x-4)^3}{3!} + \frac{81(x-4)^4}{4!}$$

It is an alternative series.

Let $a_n = \frac{(-1)^n 3^n (x-4)^n}{n!}$ $a_{n+1} = \frac{(-1)^{n+1} 3^{n+1} (x-4)^{n+1}}{(n+1)!}$

$$\text{Now, } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(-1)^{n+1} 3^{n+1} (x-4)^{n+1}}{(n+1)!} \times \frac{n!}{(-1)^n 3^n (x-4)^n}$$

$$= \lim_{n\to\infty} \frac{(-1)\,3\,(x-4)}{(n+1)\,n!} n!$$

$$= \frac{12-3x}{\infty}$$

According to D' Alembert ratio test, given series is convergent if $\left|\frac{12-3x}{\infty}\right| < 1$

or, $\left|\frac{3x-12}{\infty}\right| < 1$

or, $-1 < \frac{3x-12}{\infty} < 1$

or, $-\infty < x < 8$

Centre of convergence $= (-\infty, \infty)$

j. $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right) x^n$

Solution

$$\text{Given series } = \sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right) x^n$$

$$= \frac{1}{2}x + \frac{2}{5}x^2 + \frac{3}{10}x^3 + \frac{4}{17}x^4 + \dots$$

Let $a_n = \frac{n}{(n^2+1)} x^n$ and $a_{n+1} = \frac{(n+1)}{\{(n+1)^2+1\}} x^{n+1}$

$$\text{Now, } \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\,x^{n+1}}{(n^2+2n+2)} \times \frac{(n^2+1)}{n x^n}$$

$$= \lim_{n\to\infty} \frac{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n^2}\right) x^{n+1}}{n^3\left(1+\frac{2}{n}+\frac{2}{n^2}\right) x^n}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If $x = 1$

Then given series becomes

$$\frac{1}{2} + \frac{2}{5} + \frac{3}{10} + \dots$$

Let $a_n = \frac{n}{n^2+1} = \frac{n}{n^2\left(1+\frac{1}{n}\right)} = \frac{1}{n\left(1+\frac{1}{n}\right)}$

Let $b_n = \frac{1}{n}$, $\Sigma b_n = \Sigma \frac{1}{n}$ is divergent by p-series $p = 1$.

$$\text{Then, } \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\left(1+\frac{1}{n}\right)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

Σa_n is divergent by using comparison test. Hence given series is divergent at $x = 1$.

∴ The possible value of $x = -1 \le x < 1$.

If $x = -1$

Then given series becomes

$$-\frac{1}{2} + \frac{2}{5} - \frac{3}{10} + \dots$$

It is an alternative, so given series

$a_n = \frac{n}{n^2+1}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$$

Interval of convergence $= [-1, 1)$

Centre of convergence $= \frac{-1+1}{2} = 0$

Radius of convergence $= \frac{1+1}{2} = 1$

k. $\sum_{n=1}^{\infty} \frac{(3x+4)^n}{\sqrt{3n+4}}$

Solution

$$\text{Given series } = \sum_{n=1}^{\infty} \frac{(3x+4)^n}{\sqrt{3n+4}}$$

$a_n = \frac{(3x+4)^n}{\sqrt{3n+4}}$ and $a_{n+1} = \frac{(3x+4)^{n+1}}{\sqrt{3n+7}}$

$$\text{Then, } \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(3x+4)^{n+1}}{\sqrt{3n+7}} \times \frac{\sqrt{3n+4}}{((3x+4)^n}\right|$$

$$= \lim_{n\to\infty} \left|(3x-4)^{n+1-n} \frac{\sqrt{n}\sqrt{3+\frac{4}{n}}}{\sqrt{n}\sqrt{3+\frac{7}{n}}}\right|$$

$$= \left|(3x+4)\frac{\sqrt{3+\frac{1}{\infty}}}{\sqrt{3+\frac{1}{\infty}}}\right|$$

$$= |3x+4|$$

Thus, by the absolute ratio text $\sum_{n=0}^{\infty}\frac{(3x+4)^n}{\sqrt{3n+4}}$ converges if $|3x+4|<1$ and diverges if $|3x+4|>1$. The test fails if $|3x+4|=1$ when $|3x+4|=1$ i.e., $(3x+4)=\pm 1$, $|3x+4|=1$.

At $(3x+4)=1$ given series becomes

$$\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{3+4}}+\frac{1}{\sqrt{3\cdot 2+4}}+\ldots+\frac{1}{\sqrt{3n+4}}+\ldots$$

Its general term $a_n=\frac{1}{\sqrt{3n+4}}$, which is divergent by P-test.

At $3x+4=-1$, the series becomes

$$\frac{1}{\sqrt{4}}-\frac{1}{\sqrt{3+4}}+\frac{1}{\sqrt{3\cdot 2+4}}-\frac{1}{\sqrt{3\cdot 3+4}}+\ldots$$

Which is an alternating series and each term is less than preceding term and

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{\sqrt{3n+4}}=0$$

Thus, by Leibniz test at $3x+4=-1$, the series is convergent.

Therefore, we get the series convergent at $-1\le 3x+4<1$, otherwise divergent.

Hence, interval of convergence is

$-1\le 3x+4<1$

or, $-1-4\le 3x<1-4$

or, $-5\le 3x<-3$

i.e., $-\frac{5}{3}\le x<-1$

$\therefore$ Required interval of convergence is $\left[-\frac{5}{3},-1\right)$.

Centre of convergence at $\left(\frac{-\frac{5}{3}-1}{2}\right)=-\frac{4}{3}$

Radius of convergence at $\left(\frac{-\frac{5}{3}+1}{2}\right)=\frac{1}{3}$

l. $\sum_{n=1}^{\infty}\frac{(x-5)^n}{n5^n}$

Solution

Given series $=\sum_{n=1}^{\infty}\frac{(x-5)^n}{n5^n}$

$$=\frac{(x-5)}{5}+\frac{(x-5)^2}{2\cdot 5^2}+\frac{(x-5)^3}{3\cdot 5^3}+\frac{(x-5)^4}{4\cdot 5^4}+\ldots$$

Let $a_n=\frac{(x-5)^n}{n\cdot 5^n}$ and $a_{n+1}=\frac{(x-5)^{n+1}}{(n+1)\cdot 5^{n+1}}$

Now, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(x-5)^{n+1}}{(n+1)\cdot 5^{n+1}}\times\frac{n\cdot 5^n}{(x-5)^n}$

$$=\lim_{n\to\infty}\frac{(x-5)\cdot n\cdot 5^n}{n\left(1+\frac{1}{n}\right)5^n\cdot 5}$$

$$=\frac{x-5}{5}$$

According to D' Alembert ratio test, given series is convergent if $\left|\frac{x-5}{5}\right|<1$ and divergent if $\left|\frac{x-5}{5}\right|>1$. Also further test is needed for $\left|\frac{x-5}{5}\right|=1$.

Let $\frac{x-5}{5}=1$

Then given series becomes

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$$

$=\sum\frac{1}{n}$ is divergent by p-series.

Hence given series is divergent at $\frac{x-5}{5}=1$.

Let $\frac{x-5}{5}=-1$

Then given series becomes

$$-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}\ldots$$

It is an alternative series. Each term is numerically less than preceding term. Let general term $a_n=\frac{1}{n}$ then

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n}=0$$

Hence given series is convergent according to Leibniz theorem. Hence, given series is convergent at $\frac{x-5}{5}=-1$.

The possible interval

$=-1\le\frac{x-5}{5}<1$

or, $-5\le x-5<5$

or, $0\le x<10$

Interval of convergence $=[0, 10)$

Centre of convergence $=\frac{0+10}{2}=5$

Radius of convergence $=\frac{10-0}{2}=5$

m. $\sum_{n=0}^{\infty} \frac{10^{n+1}}{3^{2n}} x^n$

Solution

Given series $= \sum_{n=0}^{\infty} \frac{10^{n+1}}{3^{2n}} x^n$

$$= \frac{(10)^2}{9} x + \frac{(10)^3}{9^2} x^2 + \frac{(10)^4}{9^3} x^3 + \frac{(10)^5}{9^4} x^4 + \ldots$$

Let $a_n = \frac{(10)^{n+1}}{9^n} x^n$ $a_{n+1} = \frac{(10)^{n+2} x^{n+1}}{9^{n+1}}$

Now, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{(10)^{n+2} x^{n+1}}{9^{n+1}} \times \frac{9^n}{(10)^{n+1} \times x^n}$

$$= \frac{10x}{9}$$

$$= \frac{10x}{9}$$

According to D' Alembert ratio test, given series is convergent if $\left|\frac{10x}{9}\right| < 1$ and divergent if $\left|\frac{10x}{9}\right| > 1$. Also further test is needed for $\left|\frac{10x}{9}\right| = 1$.

Let $\frac{10x}{9} = 1$, then given series becomes

$10 + 10 + 10 + \ldots\ldots$

$= \Sigma 10n$ is divergent

Hence given series is divergent if $\frac{10x}{9} = 1$.

Let $\frac{10x}{9} = -1$, then given series becomes

$-10 + 10 - 10 + 10 - 10 + \ldots$

$\Sigma(-1)^n 10$

It is an oscilate series.

Hence, given series is not convergent at $\frac{10x}{9} = -1$.

Now, possible values of x

$-1 < \frac{10x}{9} < 1$

or, $\frac{-9}{10} < x < \frac{9}{10}$

Interval of convergence $= \left(\frac{-9}{10}, \frac{9}{10}\right)$

Centre of convergence $= \frac{-9+9}{10} = 0$

Radius of convergence $= \frac{\frac{9}{10} + \frac{9}{10}}{2} = \frac{9}{10}$

n. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+3)}$

Solution

Given series $= \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+3)}$

$$= \frac{-x}{1 \cdot 4} + \frac{x^2}{2 \cdot 5} - \frac{x^3}{3 \cdot 6} + \frac{x^4}{4 \cdot 7} \ldots$$

It is an alternative series.

Let a_n be general positive term.

$a_n = \frac{x^n}{n(n+3)}$ and $a_{n+1} = \frac{x^{n+1}}{(n+1)(n+4)}$

Now, $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\frac{x^{n+1}}{(n+1)(n+4)}}{\frac{x^n}{n(n+3)}}$

$$= \lim_{n \to \infty} \frac{x}{(n+1)(n+4)} n(n+3)$$

$$= \lim_{n \to \infty} x \frac{n^2\left(1+\frac{3}{n}\right)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)}$$

$$= x$$

According to D' Alembert ratio test, given series is convergent if $|x| < 1$ and divergent if $|x| > 1$. Also further test is needed for $|x| = 1$.

If x = 1

Then equation becomes

$\sum \frac{(-1)^n}{n(n+3)} \quad = -\frac{1}{1.4} + \frac{1}{2.5} - \frac{1}{3.6} + \frac{1}{4.7} + \ldots$

It is an alternative series. Each term is numerically less than preceding term/

Let a_n be general positive term. Then

$a_n = \frac{1}{n(n+3)}$

$\therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n(n+3)} = 0$

Hence, given series is convergent by using Leibniz theorem.

If x = –1

Then equation becomes

$\frac{1}{1.4} + \frac{1}{2.5} + \frac{1}{3.6} + \frac{1}{4.7} + \ldots$

Let general term $a_n = \frac{1}{n(n+3)} = \frac{1}{n^2\left(1+\frac{3}{n}\right)}$

Let $b_n = \frac{1}{n^2}$ then $\Sigma b_n = \Sigma \frac{1}{n^2}$ is convergent by p-series.

$$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^2\left(1+\frac{3}{n}\right)}}{\frac{1}{n^2}} = 1$$

Hence, $\Sigma a_n = \sum \frac{1}{n^2\left(1+\frac{3}{n}\right)}$ is convergent by using comparison test at $x = -1$

The possible value of x is $-1 \le x \le 1$.

The convergence interval = $[-1, 1]$

Centre of convergence $= \frac{-1+1}{2} = 0$

Radius of convergence $= \frac{1+1}{2} = 1$

Unit 6

Two Dimensional Geometry

Exercise 6.1

1. **Transform the equation**

a. **$x^2 - 3y^2 + 4x + 6y = 0$ to the parallel axes with origin through the point (–2, 1).**

Solution

We have given equation

$$x^2 - 3y^2 + 4x + 6y = 0 \quad \ldots(1)$$

Which is the equation of a curve with reference to origin (0, 0). Since the origin is transferred to (h, k) = (–2, 1). So the transformation equation are

$$x = x' + h = x' - 2 = x - 2$$

$$y = y' + k = y' + 1 = y + 1$$

Substituting in equation (1), we get

$$(x-2)^2 - 3(y+1)^2 + 4(x-2) + 6(y+1) = 0$$

or, $x^2 - 4x + 4 - 3y^2 - 6y - 3 + 4x - 8 + 6y + 6 = 0$

or, $x^2 - 3y^2 - 1 = 0$

This is the required equation.

b. **$x^2 + 3y^2 + 3x - 40 = 0$ with origin through the point (4, –1).**

Solution

We have given equation

$$x^2 + 3y^2 + 3x - 40 = 0 \quad \ldots(1)$$

Which is the equation of a curve with reference to origin O(0, 0). Since the origin is transferred to (h, k) = (4, –1). So the transformation equation are

$$x = x' + h = x; + 4 = x + 4$$

$$y = y' + k = y' - 1 = y - 1$$

Substituting in equation (1), we get

$$(x+4)^2 + 3(y-1)^2 + 3(x+4) - 40 = 0$$

or, $x^2 + 8x + 16 + 3y^2 - 6y + 3 + 3x + 12 - 40 = 0$

or, $x^2 + 3y^2 + 11x - 6y - 9 = 0$

This is the required equation.

c. $2x^2 + y^2 - 4x + 4y + 3 = 0$ **with origin through the point (1, –2).**

Solution

We have given equation

$$2x^2 + y^2 - 4x + 4y + 3 = 0 \quad ...(1)$$

Which is the equation of a curve with reference to origin O(0, 0). Since the origin transferred to (h, k) = (1, –2). So the transformation equation are

$$x = x' + h = x' + 1 = x + 1$$
$$y = y' + k = y' - 2 = y - 2$$

Substituting in equation (1), we get

$$2(x+1)^2 + (y-2)^2 - 4(x+1) + 4(y-2) + 3 = 0$$

or, $2x^2 + 4x + 2 + y^2 - 4y + 4 - 4x - 4 + 4y - 8 + 3 = 0$

or, $2x^2 + y^2 = 3$

This is the required equation.

2. **Show that the equation** $x^2 + y^2 - 10x - 6y + 30 = 0$ **can be transferred to** $x^2 + y^2 = k^2$ **and find value of** k.

Solution

We have given equation

$$x^2 + y^2 - 10x - 6y + 30 = 0 \quad ...(1)$$

Which is the equation of the curve with reference to origin O(0, 0). Since the origin transferred to (h, k), so the transformation equations are

$$x = x' + h;\ h = y' + k$$

Substituting in (1), we get

$$(x' + h)^2 + (y' + k)^2 - 10(x' + h) - 6(y' + k) + 30 = 0$$

or, $(x')^2 + 2hx' + h^2 + (y')^2 + 2ky' + k^2 - 10x' - 10h - 6y' - 6k + 30 = 0$

or, $(x')^2 + (y')^2 + (2h - 10)x' + (2k - 6)y' - 10h - 6k + h^2 + k^2 + 30 = 0 \quad ...(2)$

The given transformed equation is

$$x^2 + y^2 = k^2$$

To eliminate linear terms from (2), we must have

$$2h - 10 = 0 \Rightarrow h = 5$$
$$2k - 6 = 0 \Rightarrow k = 3$$

∴ New origin (h, k) = (5, 3).

Substituting value of (h, k) = (5, 3) in equation (2), then

$$(x')^2 + (y')^2 - 50 - 18 + 25 + 9 + 30 = 0$$

or, $(x')^2 + (y')^2 = 4$

Locus of (x', y') is

$$x^2 + y^2 = 4 \quad ...(3)$$

Which is in the form of $x^2 + y^2 = k^2$, where $k^2 = 4$, $k = 2$ (radius of circle)

3. **Find the angle of rotation of axes so that the equation** $x + 2y + 3 = 0$ **may be reduced to the form** $x = c$ **and also find the value of** c.

Solution

The given equation is

$$x + 2y + 3 = 0 \quad ...(1)$$

Let θ be the angle between the old and new axes. So that the transformation equations are

$$x = x'\cos\theta - y'\sin\theta = x\cos\theta - y\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta = x\sin\theta + y\cos\theta$$

Substituting in equation (1), we get

$$x\cos\theta - y\sin\theta + 2(x\sin\theta + y\cos\theta) + 3 = 0$$

or, $x(\cos\theta + 2\sin\theta) + y(2\cos\theta - \sin\theta) + 3 = 0 \quad ...(2)$

For equation (2) becomes x = c (constant)

We must have coefficient of y = 0

$$2\cos\theta - \sin\theta = 0$$

or, $\sin\theta = 2\cos\theta$

or, $\tan\theta = 2$

or, $\theta = \tan^{-1}(2)$

Since $\tan\theta = 2$, $\sin\theta = \dfrac{2}{\sqrt{5}}$, $\cos\theta = \dfrac{1}{\sqrt{5}}$

Substituting the value of sinθ and cosθ in equation (2), we get

$$x\left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}}\right) + y \cdot 0 + 3 = 0$$

or, $x\dfrac{5}{\sqrt{5}} = -3$

or, $x = \dfrac{-3}{\sqrt{5}}$

∴ $x = c$ (constant), where $c = \dfrac{-3}{\sqrt{5}}$.

4. **Find the angle of rotation of axes so that the equation** $9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10$ **is reduced to** $3x^2 + 5y^2 = 5$.

Solution

Given equation is

$$9x^2 - 2\sqrt{3}\,xy + 7y^2 = 10 \quad ...(1)$$

which is reduced to $3x^2 + 5y^2 = 5 \quad ...(2)$

In order to eliminate the product term xy, we need to use axis rotation. Let θ be the angle of rotation of new axes to original axes, so that the transformation equations are

$$x = x'\cos\theta - y\sin\theta = x\cos\theta - y\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta = x\sin\theta + y\cos\theta$$

Substituting in (1), we get

$$9(x\cos\theta - y\sin\theta)^2 + 2\sqrt{3}(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 7(x\sin\theta + y\cos\theta)^2 = 0$$

or, $(9\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + 7\sin^2\theta)x^2 + [-4\sin\theta\cos\theta + 2\sqrt{3}(\cos\theta - \sin^2\theta)] + 9(\sin^2\theta - 2\sqrt{3}\sin\theta\cos\theta + 7\cos^2\theta)y^2 = 10$

To eliminate product term xy from (2), we have

Coefficient of xy = 0

$$-4\sin\theta\cos\theta + 2\sqrt{3}\cos^2\theta - \sin^2\theta = 0$$

or, $-\sin 2\theta + \sqrt{3}\cos 2\theta = 0$

or, $\sin 2\theta = \sqrt{3}\cos 2\theta$

or, $\tan 2\theta = \sqrt{3} = \tan\frac{\pi}{3}$

or, $2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$

$\therefore$ Angle of rotation is $\theta = \frac{\pi}{6}$.

5. **Through what angle must the axes be turned so that product term xy from the equation $11x^2 + 4xy + 14y^2 = 5$ is removed.**

Solution

Given equation is

$$11x^2 + 4xy + 14y^2 = 5 \quad \ldots(1)$$

In order to eliminate the product term xy, we need to use axis rotation. Let θ be the angle of rotation of new axes to original axes, so that the transformation equation are

$$x = x'\cos\theta - y\sin\theta = x\cos\theta - y\sin\theta$$

$$y = y'\sin\theta + y'\cos\theta = x\sin\theta + y\cos\theta$$

Substituting in (1), we get

$11(x\cos\theta - y\sin\theta)^2 + 4(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta) + 14(x\sin\theta + y\cos\theta)^2 = 5$

or, $11(x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta) + 4x^2\sin\theta\cos\theta + 4xy\cos^2\theta - 4xy\sin^2\theta - 4y^2\sin\theta\cos\theta + 14(x^2\sin^2\theta + 2xy\sin\theta\cos\theta + y^2\cos^2\theta = 5 \quad \ldots(2)$

To eliminate product term xy from (2), we have

Coefficient of $xy = 0$

$-22\sin\theta\cos\theta + 4(\cos^2\theta - \sin^2\theta) + 28\sin\theta\cos\theta = 0$

or, $4\cos 2\theta + 6\sin\theta\cos\theta = 0$

or, $4\cos 2\theta + 3\sin 2\theta = 0$

or, $4\cos 2\theta = -2\sin 2\theta$

or, $\tan 2\theta = \frac{-4}{3}$

or, $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{-4}{3}$

or, $3\tan\theta = -2 + 2\tan^2\theta$

or, $2\tan^2\theta - 3\tan\theta - 2 = 0$

or, $2\tan^2\theta - 4\tan\theta + \tan\theta - 2 = 0$

or, $2\tan\theta(\tan\theta - 2) + 1(\tan\theta - 2) = 0$

or, $(\tan\theta - 2)(2\tan\theta + 1) = 0$

or, $\tan\theta = 2, \tan\theta = \frac{-1}{2}$

$\therefore$ $\theta = \tan^{-1}(2), \tan^{-1}\left(\frac{-1}{2}\right)$

6. **What does the equation of pair of straight lines $7x^2 + 4xy + 4y^2 = 0$ become when the axes are the bisectors of the angle between them?**

Solution

Given equation of pair of straight line is

$$7x^2 + 4xy + 4y^2 = 0 \quad \ldots(1)$$

Comparing equation (1) with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 7, h = 2, b = 4$

We know, the bisector of the angles between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is

$h(x^2 - y^2) = (a - b)xy$

$2(x^2 - y^2) = (7 - 4)xy$

or, $2x^2 - 3xy - 2y^2 = 0$

or, $2x^2 - 4xy + xy - 2y^2 = 0$

or, $2x(x - 2y) + y(x - 2y) = 0$

or, $(2x + y)(x - 2y) = 0$

or, Either $2x + y = 0 \quad \ldots(1)$ slope $m_1 = -2$

or, $x - 2y = 0 \quad \ldots(2)$ slope of $m_2 = \frac{1}{2}$

Taking positive sign

$\tan\theta = \frac{1}{2}$

So, $\sin\theta = \frac{1}{\sqrt{5}}$ $\left[\because \tan\theta = \frac{1}{2} = \frac{p}{b} \text{ so } h = \sqrt{5}\right]$

$\cos\theta = \frac{2}{\sqrt{5}}$

Then transformation equation are

$x = x\cos\theta - y\sin\theta = \frac{2x}{\sqrt{5}} - \frac{y}{\sqrt{5}} = \frac{2x - y}{\sqrt{5}}$

$y = x\sin\theta + y\cos\theta = \frac{x}{\sqrt{5}} + \frac{2y}{\sqrt{5}} = \frac{x + 2y}{\sqrt{5}}$

Substituting these values in equation (1), we get

$$7\left(\frac{2x-y}{\sqrt{5}}\right)^2 + 4\left(\frac{2x-y}{\sqrt{5}}\right)\left(\frac{x+2y}{\sqrt{5}}\right) + 4\left(\frac{x+2y}{\sqrt{5}}\right)^2 = 0$$

or, $7(4x^2 - 4xy + y^2) + 4(2x^2 + 4xy - xy - 2y^2) + 4(x^2 + 4xy + 4y^2) = 0$

or, $28x^2 - 28xy + 7y^2 + 8x^2 + 12xy - 8y^2 + 4x^2 + 16xy + 16y^2 = 0$

or, $40x^2 + 15y^2 = 0$

or, $8x^2 + 3y^2 = 0$

7. **Show that the rectangular hyperbola $x^2 - y^2 = a^2$ can be reduced to $xy = c^2$ by the rotation of axes. Also find the values of angle of rotation and the constant c.**

Solution

The given equation is

$$x^2 - y^2 = a^2 \quad \ldots(1)$$

Let the angle between the original axes and the new axes be θ, so that after substituting

$x = x\cos\theta - y\sin\theta$ and $y = x\sin\theta + y\cos\theta$

Then equation (1) becomes

$(x\cos\theta - y\sin\theta)^2 - (x\sin\theta + y\cos\theta)^2 = a^2$

or, $x^2\cos^2\theta - 2xy\sin\theta\cos\theta + y^2\sin^2\theta - x^2\sin^2\theta - 2xy\sin\theta\cos\theta - y^2\cos^2\theta = a^2$

or, $(\cos^2\theta - \sin^2\theta)x^2 - 4xy\sin\theta\cos\theta - (\cos^2\theta - \sin^2\theta)y^2 = a^2$

or, $\cos 2\theta\, x^2 - 2\sin 2\theta\, xy - \cos 2\theta\, y^2 = a^2 \quad \ldots(2)$

This must be identical with $xy = c^2$ provided $\cos 2\theta = 0 \Rightarrow \theta = \pm\frac{\pi}{4}$

Taking $\theta = -\frac{\pi}{4}$ (neglecting +ve θ)

We have, $2xy = a^2$ (from equation 2)

Hence, the equation is

$xy = \frac{a^2}{2}$

$\therefore$ $xy = c^2$, where $c^2 = \frac{a^2}{2}$.

8. **What does the equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ becomes when the origin is transferred to (2, 3) and the new axes are turned through 45° to original axes.**

Hint: *Apply two transformations in turn first transfer origin using $x = x - h$ and $y = y + k$ and then transfer the resulting equation by using axes transformation equations.*

$x = x\cos\theta - y\sin\theta,\ y = x\sin\theta + y\cos\theta$ *with* $\theta = 45°$.

Solution

The given equation is

$$3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0 \quad \ldots(1)$$

Which is equation of a curve with reference to origin (0, 0). Since the origin is transferred to (h, k) = (2, 3), so the transformation equation are

$x = x + 2$

$y = y + 3$

Substituting in (1), we get

$3(x + 2)^2 + 2(x + 2)(y + 3) + 3(y + 3)^2 - 18(x + 2) - 22(y + 3) + 50 = 0$

or, $3(x^2 + 4x + 4) + 2(xy + 3x + 2y + 6) + 3(y^2 + 6y + 9) - 18x - 36 - 22y - 66 + 50 = 0$

or, $3x^2 + 12x + 12 + 2xy + 6x + 4y + 12 + 3y^2 + 18y + 27 - 18x - 22y - 52 = 0$

or, $3x^2 + 2xy + 3y^2 = 1 \quad \ldots(2)$

Which is equation of curve with reference to (h, k) = (2, 3)

Now the angle of rotation of axes is $\theta = 45°$.

Therefore transformation equation are

$x = x\cos\theta - y\sin\theta = x\cos 45° - y\sin 45°$

i.e., $x = \frac{x - y}{\sqrt{2}}$

$y = x\sin\theta + y\cos\theta = x\sin 45° + y\cos 45°$

$\therefore$ $y = \frac{x + y}{\sqrt{2}}$

Substituting these values in equation (2), we get

$$3\left(\frac{x-y}{\sqrt{2}}\right)^2 + 2\left(\frac{x-y}{\sqrt{2}}\right)\left(\frac{x+y}{\sqrt{2}}\right) + 3\left(\frac{x+y}{\sqrt{2}}\right)^2 = 1$$

or, $\frac{3}{2}(x^2 - 2xy + y^2) + \frac{2}{2}(x^2 - y^2) + \frac{3}{2}(x^2 + 2xy + y^2) = 1$

or, $8x^2 + 4y^2 = 2$

or, $4x^2 + 2y^2 = 1$

Which is new equation such that the origin transferred to (2, 3) and the new axes are turned through 45° to original axes.

Exercise 6.2

1. **Find the eccentricity, foci, directrix, major axis, minor axis and latus rectum of following ellipses.**

   a. $9x^2 + 25y^2 = 225$

Solution

The given equation of ellipse is

$$9x^2 + 25y^2 = 225 \text{ i.e., } \frac{x^2}{25} + \frac{y^2}{9} = 1 \quad \ldots(1)$$

Comparing equation (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 25$ and $b^2 = 9$, i.e., $a = 5$, $b = 3$, $a > b$

For eccentricity, we have

$b^2 = a^2(1 - e^2)$

$9 = 25(1 - e^2)$

or, $1 - e^2 = \frac{9}{25}$

or, $e^2 = 1 - \frac{9}{25} = \frac{16}{25}$

or, $e = \frac{4}{5}$

$\therefore$ Coordinate of foci $= (\pm ae, 0) = \left(\pm 5 \cdot \frac{4}{5}, 0\right) = (\pm 4, 0)$

Equation of directrix

$x = \pm\frac{a}{e} \Rightarrow x = \pm\frac{5}{\frac{4}{5}} \Rightarrow x = \pm\frac{25}{4}$

Major axis: $y = 0$ (x-axis), minor axis is $x = 0$ (y-axis), length of major axis $= 2a = 10$, length of minor axis $= 2b = 6$, length of latus rectum $= \frac{2b^2}{a} = \frac{2\times 9}{5} = \frac{18}{5}$.

b. $25x^2 + 4y^2 = 100$

Solution

The given equation of ellipse is

$25x^2 + 4y^2 = 100$

$$\frac{x^2}{4} + \frac{y^2}{25} = 1 \quad \ldots(1)$$

Comparing equation (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 4$ and $b^2 = 25$, i.e., $a = 2$, $b = 5$ and $b > a$

For eccentricity, we have

$a^2 = b^2(1 - e^2)$

or, $4 = 25(1 - e^2)$

or, $1 - e^2 = \frac{4}{25}$

or, $e^2 = 1 - \frac{4}{25} = \frac{21}{25}$

$\therefore$ Eccentricity (e) $= \frac{\sqrt{21}}{5}$

Coordinate of foci $= (0, \pm be) = (0, \pm\sqrt{21})$

Equation of directrix $y = \pm\frac{b}{e}$, i.e., $y = \pm\frac{5}{\frac{\sqrt{21}}{5}}$, i.e., $y = \pm\frac{25}{\sqrt{21}}$

Major axis is x = (y-axis), length of major axis = 2b = 10

Minor axis is y = 0 (x-axis), length of minor axis = 2a = 4

Length of latus rectum $= \frac{2a^2}{b} = \frac{2\times 2^2}{5} = \frac{8}{5}$

c. $3x^2 + 4y^2 = 36$

Solution

The given equation of ellipse is

$3x^2 + 4y^2 = 36$

or, $\frac{x^2}{12} + \frac{y^2}{9} = 1$ ...(1)

Comparing equation (1) with

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 12, b^2 = 9$

$a = 2\sqrt{3}$, $b = 3$; $a > b$

For eccentricity

$b^2 = a^2(1 - e^2)$

or, $9 = 12(1 - e^2)$

or, $1 - e^2 = \frac{9}{12}$

or, $e^2 = 1 - \frac{3}{4}$

or, $e^2 = \frac{1}{4}$

or, $e = \frac{1}{2}$

$\therefore$ Eccentricity (e) $= \frac{1}{2}$

Centre (h, k) = (0, 0)

Foci $= (\pm ae, 0) = \left(\pm 2\sqrt{3}\cdot\frac{1}{2}, 0\right) = (\pm\sqrt{3}, 0)$

Equation of directrix, $x = \pm\frac{a}{e} \Rightarrow x = \pm\frac{2\sqrt{3}}{1/2} \Rightarrow x = \pm 4\sqrt{3}$.

Major axis is y = 0 (x-axis)

Minor axis is x = 0 (y-axis)

Length of major axis $= 2a = 2\times 2\sqrt{3} = 4\sqrt{3}$

Length of minor axis $= 2b = 2\times 3 = 6$

1. **Find the centre, eccentricity and foci of the ellipses.**

a. $\frac{(x-1)^2}{16} + \frac{(y-2)^2}{4} = 1$

Solution

Given equation of the ellipse is

$\frac{(x-1)^2}{16} + \frac{(y-2)^2}{4} = 1$ ...(1)

Comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$(h, k) = (1, 2)$, $a = 4$, $b = 2$

Since a > b, the major axis is parallel to x-axis

Centre (h, k) = (1, 2)

For eccentricity, we have

$b^2 = a^2(1 - e^2)$

or, $4 = 16(1 - e^2)$

or, $1 - e^2 = \frac{1}{4}$

or, $e^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ Eccentricity (e) $= \frac{\sqrt{3}}{2}$

Foci $= (h \pm ae, k)$

$= \left(1 \pm \frac{4\sqrt{3}}{2}, 2\right)$

$= (1 \pm 2\sqrt{3}, 2)$

$\therefore$ Centre (h, k) = (1, 2), eccentricity $= \frac{\sqrt{3}}{2}$, foci $= (1 \pm 2\sqrt{3}, 2)$.

b. $x^2 + 4y^2 - 4x + 24y + 24 = 0$

Solution

The given equation of ellipse is

$x^2 + 4y^2 - 4x + 24y + 24 = 0$

or, $(x^2 - 4x + 4) + 4(y^2 + 6y + 9) + 24 = 4 + 4\times 9$

or, $(x-2)^2 + 4(y+3)^2 = 16$

or, $\frac{(x-2)^2}{4^2} + \frac{(y+3)^2}{2^2} = 1$ ...(1)

Comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$(h, k) = (2, -3)$, $a = 4$, $b = 2$

Since a > b, major axis parallel to x-axis.

Centre (h, k) = (2, –3)

For eccentricity (e), we have

$b^2 = a^2(1 - e^2)$

or, $4 = 16(1 - e^2)$

or, $1 - e^2 = \frac{1}{4}$

or, $e^2 = 1 - \frac{1}{4} = \frac{3}{4}$

∴ $e = \frac{\sqrt{3}}{2}$

Foci $= (h \pm ae, k)$

$= \left(2 \pm 4 \cdot \frac{\sqrt{3}}{2}, -3\right)$

$= (2 \pm 2\sqrt{3}, -3)$

c. $\mathbf{3x^2 - 12x + 4y^2 - 8y + 4 = 0}$

Solution

The given equation of ellipse is

$3x^2 - 12x + 4y^2 - 8y + 4 = 0$

or, $3(x^2 - 4x + 4) + 4(4^2 - 2y + 1) + 4 = 3 \times 4 + 4 \times 1$

or, $\frac{3(x-2)^2}{12} + \frac{4(y-1)^2}{12} = 1$

or, $\frac{(x-2)^2}{2^2} + \frac{(y-1)^2}{(\sqrt{3})^2} = 1$ ...(1)

Comparing equation (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$h = 2, k = 1, a = 2, b = 3$

Since a > b, the major axis is parallel to x-axis.

Centre $(h, k) = (2, 1)$

For eccentricity (e), we have

$b^2 = a^2(1 - e^2)$

or, $3 = 4(1 - e^2)$

or, $1 - e^2 = \frac{3}{4}$

or, $e^2 = 1 - \frac{3}{4} = \frac{1}{4}$

∴ $e = \frac{1}{2}$

Foci $= (h \pm ae, k)$

$= \left(2 \pm 2 \times \frac{1}{2}, 1\right)$

$= (2 \pm 1, 1)$

$= (3, 1)$ and $(1, 1)$

3. **Find the centre, vertex and eccentricity, foci, length of major and minor axes of the ellipses.**

a. $\frac{x^2}{4} + \frac{y^2}{9} = 1$

Solution

The given equation of ellipse is

$\frac{x^2}{4} + \frac{y^2}{9} = 1$ ...(1)

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a = 2, b = 3, b > a$ (major axis parallel to y-axis)

Centre $(h, k) = (0, 0)$

Vertices $= (0, \pm b) = (0, \pm 3)$

For eccentricity, we have

$a^2 = b^2(1 - e^2)$

or, $4 = 9(1 - e^2)$

or, $1 - e^2 = \frac{4}{9}$

or, $e^2 = 1 - \frac{4}{9} = \frac{5}{9}$

∴ $e = \frac{\sqrt{5}}{3}$

Foci $= (0, \pm be)$

$= \left(0, \pm 3\frac{\sqrt{5}}{3}\right)$

$= (0, \pm 5)$

Length of major axis $= 2b = 6$

Length of minor axis $= 2a = 4$

b. $\frac{x^2}{144} + \frac{y^2}{169} = 1$

Solution

The given equation of ellipse is

$\frac{x^2}{144} + \frac{y^2}{169} = 1$

i.e. $\frac{x^2}{12^2} + \frac{y^2}{13^2} = 1$ ...(1)

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a = 12, b = 13$ and $b > a$, so major axis is parallel to y-axis.

Centre $(h, k) = (0, 0)$

Vertices $= (0, \pm b) = (0, \pm 13)$

For eccentricity, we have

$a^2 = b^2(1 - e^2)$

or, $144 = 169(1 - e^2)$

or, $1 - e^2 = \frac{144}{169}$

or, $e^2 = 1 - \frac{144}{169} = \frac{25}{169}$

∴ $e = \frac{5}{13}$

Foci $= (0, \pm be) = \left(0, \pm 13\frac{\sqrt{5}}{13}\right) = (0, \pm 5)$

Length of major axis $= 2b = 26$

Length of minor axis $= 2a = 24$

c. $3x^2 + 2y^2 = 18$

Solution

The given equation of ellipse is

$3x^2 + 2y^2 = 18$

i.e. $\frac{x^2}{6} + \frac{y^2}{9} = 1$

Comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 6$, $b^2 = 9$, i.e., $a = \sqrt{6}$, $b = 3$, $b > a$

Centre (h, k) = (0, 0)

Vertices = $(0, \pm b) = (0, \pm 3)$

For eccentricity, we have

$a^2 = b^2(1 - e^2)$

or, $6 = 9(1 - e^2)$

or, $1 - e^2 = \frac{2}{3}$

or, $e^2 = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore$ $e = \frac{1}{\sqrt{3}}$

Foci $= (0, \pm be) = \left(0, \pm 3\frac{1}{\sqrt{3}}\right) = (0, \pm\sqrt{3})$

Length of major axis = 2b = 6

Length of minor axis = $2a = 2\sqrt{6}$

d. $\frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1$

Solution

The given equation of ellipse is

$\frac{(x-3)^2}{9} + \frac{(y-5)^2}{25} = 1$ ...(1)

Comparing equation (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

h = 3, k = 5, a = 3, b = 5

Centre (h, k) = (3, 5)

Vertices = $(h, k \pm b) = (3, 5 \pm 5) = (3, 0)$ and (3, 10)

For eccentricity, we have

$a^2 = b^2(1 - e^2)$

or, $9 = 25(1 - e^2)$

or, $1 - e^2 = \frac{9}{25}$

or, $e^2 = 1 - \frac{9}{25} = \frac{25-9}{25} = \frac{16}{25}$

$\therefore$ $e = \frac{4}{5}$

Foci = $(h, k \pm be) = (3, 5 \pm 4) = (3, 1)$ and (3, 9)

Length of major axis = 2b = 10

Length of minor axis = 2a = 6

e. $\frac{(x+5)^2}{9} + \frac{(y-1)^2}{4} = 1$

Solution

The given equation of ellipse is

$\frac{(x+5)^2}{9} + \frac{(y-1)^2}{4} = 1$ ...(1)

Comparing equation (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

h = –5, k = 1, a = 3, b = 2, a > b

Centre (h, k) = (–5, 1)

Vertices = $(h \pm a, k) = (-5 \pm 3, 1) = (-8, 1)$ and (–2, 1)

For eccentricity, we have

$b^2 = a^2(1 - e^2)$

or, $4 = 9(1 - e^2)$

or, $1 - e^2 = \frac{4}{9}$

or, $e^2 = 1 - \frac{4}{9} = \frac{5}{3}$

$\therefore$ $e = \frac{\sqrt{5}}{3}$

Foci $= (h \pm ae, k) = (-5 \pm \sqrt{5}, 1)$

Length of major axis = 2a = 6

Length of minor axis = 2b = 4

f. $x^2 + 4y - 6x - 8y = 3$

Solution

The given equation is

$x^2 + 4y - 6x - 8y = 3$ ...(1)

Which does not represent ellipse.

g. $9x^2 + 4y^2 + 10y + 18x + 73 = 0$

Solution

Given equation of ellipse is

$9x^2 + 4y^2 + 10y + 18x + 73 = 0$

or, $9(x^2 + 2x) + 4(y^2 + 10y) = -73$

or, $9(x^2 + 2x + 1) + 4(y^2 + 10y + 25) = -73 + 9 + 100$

or, $9(x+1)^2 + 4(y+5)^2 = 36$

or, $\frac{(x+1)^2}{4} + \frac{(y+5)^2}{9} = 1$ ...(1)

Comparing equation (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

h = –1, k = –5, a = 2, b = 3

Centre (h, k) = (–1, –5)

For eccentricity, we have

$a^2 = b^2(1 - e^2)$

or, $4 = 9(1 - e^2)$

or, $1 - e^2 = \frac{4}{9}$

or, $e^2 = 1 - \frac{4}{9} = \frac{5}{9}$

$\therefore$ $e = \frac{\sqrt{5}}{3}$

Foci $= (h, k \pm be) = (-1, -5 \pm \sqrt{5})$

Vertices $= (h, k \pm b) = (-1, -5 \pm 3) = (-1, -8)$ and $(-1, -2)$

Length of major axis $= 2b = 6$

Length of minor axis $= 2a = 4$

4. Find the equation of the ellipse in the following cases.

a. passing through the points (-3,1) and (-2,2)

Solution

Let the ellipse has axes along the coordinate axes. So, the centre of ellipse is origin. Therefore, the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots(1)$$

The equation (1) is passing through (–3, 1) so

$$\frac{(-3)^2}{a^2} + \frac{1^2}{b^2} = 1$$

or, $\frac{1}{b^2} = 1 - \frac{9}{a^2}$ ...(2)

Again, the ellipse (1) passes through (–2, 2) so

$$\frac{(-2)^2}{a^2} + \frac{(2)^2}{b^2} = 1$$

or, $\frac{4}{a^2} + 4\left(1 - \frac{9}{a^2}\right) = 1$

or, $\frac{4}{a^2} + 4 - \frac{36}{a^2} = 1$

or, $\frac{-32}{a^2} = -3$

or, $a^2 = \frac{32}{3}$

Substituting the value of a^2 in equation (2)

$$\frac{1}{b^2} = 1 - \frac{9}{\frac{32}{3}} = 1 - \frac{27}{32} = \frac{5}{32}$$

or, $b^2 = \frac{32}{5}$

Now, equation (i) becomes

$$\frac{x^2}{\frac{32}{3}} + \frac{y^2}{\frac{32}{5}} = 1 \Rightarrow 3x^2 + 5y^2 = 32$$

Which is the required equation.

or, $\frac{4x^2}{81}+\frac{4y^2}{45}=1$

or, $20x^2+36y^2=405$

Which is required equation of ellipse.

d. focus (0,3), equation of directrix $x+7=0$, $e=\frac{1}{3}$

Solution

Let p(x, y) be any point on the ellipse and focus $(\alpha, \beta) = (0, 3)$, directrix $x+7=0$ and eccentricity $(e)=\frac{1}{3}$.

Then, using general equation of conic section

$$(x-\alpha)^2+(y-\beta)^2=e^2\left(\frac{(ax+by+c)}{\sqrt{a^2+b^2}}\right)^2$$

or, $(x-0)^2+(y-3)^2=\frac{1}{9}\frac{(x+7)^2}{1}$

or, $x^2+y^2-6y+9=\frac{x^2+14x+49}{9}$

or, $9x^2+9y^2-54+81=x^2+14x+49$

or, $8x^2+9y^2-14x-54y+32=0$

e. focus (2,5), equation of directrix $x+y=1$, $e=\frac{2}{3}$

Solution

Let p(x, y) be any point of the ellipse with focus $(\alpha, \beta) = (2, 5)$, equation of directrix $x+y=1$ and eccentricity $(e)=\frac{2}{3}$.

Then, using general equation of conic section

$$(x-\alpha)^2+(y-\beta)^2=e^2\left(\frac{(ax+by+c)}{\sqrt{a^2+b^2}}\right)^2$$

or, $(x-2)^2+(y-5)^2=\frac{4}{9}\left(\frac{x+y-1}{\sqrt{2}}\right)^2$

or, $x^2-4x+4+y^2-10y+25=\frac{2}{9}(x^2+y+2+1+2xy-2x-2y)$

or, $9x^2-36x+9y^2-90y+261=2x^2+2y^2+4xy-4x-4y+2$

or, $7x^2+7y^2-4xy-32x-86y+259=0$

f. focus (–1, 1), eccentricity $\frac{1}{2}$ and directrix: $x-y+3=0$.

Solution

Let p(x, y) be any point of the ellipse with focus $(\alpha, \beta) = (-1, 1)$, equation of directrix $x-y+3=0$ and eccentricity $(e)=\frac{1}{2}$.

Then, using general equation of conic section

$$(x-\alpha)^2+(y-\beta)^2=e^2\left(\frac{(ax+by+c)}{\sqrt{a^2+b^2}}\right)^2$$

or, $(x+1)^2+(y-1)^2=\frac{1}{4}\left(\frac{x-y+3}{\sqrt{2}}\right)^2$

or, $x^2+2x+1+y^2-2y+1=\frac{1}{8}(x^2+y^2+9-2xy+6x-6y)$

or, $8x^2+8y^2+16x-16y+16=x^2+y^2-2xy+6x-6y+9$

or, $7x^2+7y^2+2xy+10x-10y+7=0$

or, Which is the required equation of ellipse.

5. Find the equation of ellipse whose foci are at (-2,4) and (4,4), length of major axis is 10. Also find the eccentricity.

Solution

The foci of the ellipse are (–2, 4) and (4, 4)

y-coordinate is same, so major axis y = 4 (parallel to x-axis)

Centre of ellipse $(h, k)=\left(\frac{-2+4}{2}, \frac{4+4}{2}\right)=(1,4)$.

Major axis $2a=10 \Rightarrow a=5$

Now, we have

$h+ae=4$

or, $1+5\cdot e=4$

or, $5e=3$

or, $e=\frac{3}{5}$

Also, $b^2=a^2(1-e^2)$

or, $b^2=25\left(1-\frac{9}{25}\right)$

$b^2=16$

Hence required equation of ellipse is

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

or, $\frac{(x-1)^2}{25}+\frac{(y-4)^2}{16}=1$

or, $16(x^2-2x+1)+25(y^2-8y+16)=400$

or, $16x^2+25y^2-32x-200y+16=0$

6. The end points of the major and minor axes of ellipse are (1,1), (3,4), (1,7) and (-1,4). Sketch the ellipse.

Solution

Given that the end points of major and minor axes of ellipse are A(1, 1), B93, 4), C(1, 7) and D(–1, 4).

Therefore, $2a=|-1-3|=4 \Rightarrow a=2$

$2b=|7-1|=6 \Rightarrow b=3$

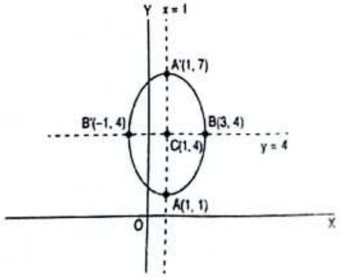
Here, b > a, major axis is parallel to y-axis

Centre of ellipse $(h, k)=\left(\frac{1+1}{2}, \frac{1+7}{2}\right)=(1, 4)$

Therefore equation of ellipse is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

or, $\frac{(x-1)^2}{4}+\frac{(y-4)^2}{9}=1$



7. **Show the $2x^2+y^2=3x$ represents an ellipse. Find its centre, eccentricity and co-ordinates of foci.**

Solution

Given equation is

$2x^2+y^2=3x$

or, $2\left(x^2-\frac{3}{2}x\right)+y^2=0$

or, $2\left\{x^2-2\cdot\frac{3}{4}x+\left(\frac{3}{4}\right)^2\right\}+y^2=2\times\left(\frac{3}{4}\right)^2$

or, $\frac{2\left(x-\frac{3}{4}\right)^2}{\frac{9}{8}}+\frac{(y-0)^2}{\frac{9}{8}}=1$

or, $\frac{\left(x-\frac{3}{4}\right)^2}{\frac{9}{16}}+\frac{(y-0)^2}{\frac{9}{8}}=1$

Which is the form of $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

So, given equation represents an ellipse.

We have,

$(h, k)=\left(\frac{3}{4}, 0\right)$

$a^2=\frac{9}{16}\Rightarrow a=\frac{3}{4}$

$b^2=\frac{9}{8}\Rightarrow b=\frac{3}{2\sqrt{2}}$ $(b>a)$

Centre $(h, k)=\left(\frac{3}{4}, 0\right)$

For eccentricity, we have

$a^2=b^2(1-e^2)$

or, $\frac{9}{16}=\frac{9}{8}(1-e^2)$

or, $1-e^2=\frac{1}{2}$

or, $e^2=1-\frac{1}{2}=\frac{1}{2}$

$\therefore$ $e=\frac{1}{\sqrt{2}}$

Foci $=(h, k\pm be)=\left(\frac{3}{4}, 0\pm\frac{3}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\right)=\left(\frac{3}{4}, \pm\frac{3}{4}\right)$.

8. **A point P moves in such a way that the sum of its distances from S and S_1 is 10 and $SS_1=8$. Find the equation of locus and show that it is an ellipse. Also find its eccentricity and latus rectum.**

Solution

Let p(x, y) be any point on the plane such that $PS+PS_1=10$ and $SS_1=8$

Let origin be mid point of SS_1, then

$OS=OS_1=\frac{8}{2}=4$

Then, we have S(4, 0) and S'(–4, 0)

$PS=\sqrt{(x-4)^2+(y-0)^2}$

$PS'=\sqrt{(x+4)^2+(y-0)^2}$

Given that

$PS+PS'=10$

or, $\sqrt{(x-4)^2+y^2}+\sqrt{(x+4)^2+y^2}=10$

or, $\sqrt{(x+4)^2+y^2}=10-\sqrt{(x-4)^2+y^2}$

Squaring on both sides, we get

$x^2+8x+16+y^2=100-20\sqrt{(x-4)^2+y^2}+x^2-8x+16+y^2$

or, $16x=100-20\sqrt{(x-4)^2+y^2}$

or, $4x=25-5\sqrt{(x-4)^2+y^2}$

or, $4x-25=-5\sqrt{(x-4)^2+y^2}$

Squaring both sides, we get

$(4x-25)^2=(-5\sqrt{(x-4)^2+y^2})^2$

or, $16x^2-200x+625=25(x^2-8x+16+y^2)$

or, $16x^2-200x+625=25x^2-200x+400+25y^2$

or, $9x^2+25y^2=225$

or, $\frac{x^2}{25}+\frac{y^2}{9}=1$

or, $\frac{x^2}{5^2}+\frac{y^2}{3^2}=1$

Which is in the form of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

So locus of the point p represents ellipse.

Here, we have a = 5, b = 3

For eccentricity

$b^2 = a^2(1-e^2)$

or, $9 = 25(1-e^2)$

or, $1-e^2 = \frac{9}{25}$

or, $e^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$\therefore \quad e = \frac{4}{5}$

Length of latus rectum $= \frac{2b^2}{a} = \frac{18}{5}$.

9. For any parameter t, show the equations $x = a\frac{1-t^2}{1+t^2}$, $y = b\frac{2t}{1+t^2}$ satisfy the equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

We have the parametric equation

$x = a\frac{1-t^2}{1+t^2}, y = b\frac{2t}{1+t^2}$

Substituting x and y in $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, we get

$\frac{1}{a^2}\cdot a^2\left(\frac{1-t^2}{1+t^2}\right)^2 + \frac{1}{b^2}\cdot b^2\left(\frac{2t}{1+t^2}\right)^2 = 1$

or, $\frac{1-2t^2+t^4}{(1+t^2)^2} + \frac{4t^2}{(1+t^2)^2} = 1$

or, $\frac{t^4-2t^2+1+4t^2}{(1+t^2)^2} = 1$

or, $\frac{(t^2)^2+2t^2+1}{(1+t^2)^2} = 1$

or, $\frac{(t^2+1)^2}{(1+t^2)^2} = 1$

$\therefore \quad 1 = 1$ (verified)

Which shows that the parametric equations $x = a\frac{1-t^2}{1+t^2}$, $y = b\frac{2t}{1+t^2}$ satisfy the equation of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

Exercise 6.3

1. Find the equation of the tangents to the ellipse $4x^2 + 3y^2 = 5$ which is parallel to the line $y = 3x + 7$.

Solution

Given equation of ellipse is

$4x^2 + 3y^2 = 5$

i.e. $\frac{x^2}{\frac{5}{4}} + \frac{y^2}{\frac{5}{3}} = 1$ ...(1)

Comparing equation (1) with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$a^2 = \frac{5}{4}, b^2 = \frac{5}{3}$

Any line parallel to $y = 3x + 7$ is

$y = 3x + K$ ...(2)

Comparing equation (2) with $y = mx + C$, we get

$m = 3, C = K$

We know, condition for the line $y = mx + C$ to be the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is

$C^2 = a^2m^2 + b^2$

or, $K^2 = \frac{5}{4}\times 9 + \frac{5}{3} = \frac{155}{12}$

or, $K = \pm\sqrt{\frac{155}{12}}$

Now equation (2) becomes

$y = 3x \pm \sqrt{\frac{155}{12}}$

or, $3x - y \pm \sqrt{\frac{155}{12}} = 0$

Which are the required equation.

2. Find the equation of the tangents to the ellipse $\frac{x^2}{4}+\frac{y^2}{9} = 1$ which is parallel to the line $x = y + 4$.

Solution

Given equation of the ellipse is

$\frac{x^2}{4}+\frac{y^2}{9} = 1$ ...(1)

Comparing equation (1) with $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, we get

$a^2 = 4, b^2 = 9$

Given equation of straight line is

$x = y + 4$

i.e., $y = x - 4$

Any line parallel to given straight line is

$y = x + K$ ...(2)

We know, the condition for the line $y = mx + c$ to be tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ is

$c^2 = a^2m^2 + b^2$

or, $K^2 = 4\cdot 1^2 + 9$

or, $K^2 = 13$

or, $K = \pm\sqrt{13}$

Now, equation (2) becomes

$y = x \pm \sqrt{13}$

i.e. $x - y \pm \sqrt{13} = 0$

Which are the required equation of tangent to the given ellipse.

3. Find the condition that the line $lx + my + n = 0$ may touch the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Also find the point of contact.

Solution

We have equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots(1)$$

The given line is $\ell x + my + n = 0$ ...(2)

We know that the equation of tangent at (x_1, y_1) to equation (1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

or, $\dfrac{x_1}{a^2}x + \dfrac{y_1}{b^2}y - 1 = 0$ ...(3)

If equation (2) is tangent to equation (1), then it must be identical with equation (3), for this the coefficient of x, y and constants must be in proportion.

$$\frac{x_1/a^2}{l} = \frac{y_1/b_2}{m} = \frac{-1}{n}$$

$\therefore$ $x_1 = \dfrac{-a^2 l}{n}, y_1 = \dfrac{-b^2 m}{n}$

Thus, point of contact is $(x_1, y_1) = \left(\dfrac{-a^2 l}{n}, \dfrac{-b^2 m}{n}\right)$

The point of contact $(x_1, y_1) = \left(\dfrac{-a^2 l}{n}, \dfrac{-b^2 m}{n}\right)$ also lies on the ellipse (1)

$$\frac{1}{a^2}\left(\frac{-a^2 l}{n}\right)^2 + \frac{1}{b^2}\left(\frac{-b^2 m}{n}\right)^2 = 1$$

or, $\dfrac{a^2 l^2}{n^2} + \dfrac{b^2 m^2}{n^2} = 1$

$\therefore$ $n^2 = a^2 l^2 + b^2 m^2$

Which is the required condition.

4. Find the value of λ for the straight line $y = x + \lambda$ touches the ellipse $2x^2 + 3y^2 = 6$.

Solution

Given equation of ellipse is $2x^2 + 3y^2 = 6$

or, $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$ ...(1)

Comparing equation (1) with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we get

$a^2 = 3, b^2 = 2$

Given equation of straight line is

$y = x + \lambda$ ...(2)

Comparing equation (2) with $y = mx + c$, we get

$m = 1, c = \lambda$

We know that the condition for the line $y = mx + c$ to be the tangent to the ellipse (1) is

$c^2 = a^2m^2 + b^2$

or, $\lambda^2 = 3 + 2 = 5$

or, $\lambda = \pm\sqrt{5}$

5. Show that the line $3x + 4y + \sqrt{7} = 0$ touch the ellipse $3x^2 + 4y^2 = 1$. Also find the point of contact.

Solution

Given equation of ellipse is

$3x^2 + 4y^2 = 1$

or, $\dfrac{x^2}{\frac{1}{3}} + \dfrac{y^2}{\frac{1}{4}} = 1$ ...(1)

Comparing equation (1) with $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we get

$a^2 = \dfrac{1}{3}, b^2 = \dfrac{1}{4}$

Given equation of straight line is

$3x + 4y + \sqrt{7} = 0$ ...(2)

Also, we know the equation of tangent at (x_1, y_1) to

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

So equation of tangent at (x_1, y_1) to (1) is

$3xx_1 + 4yy_1 = 1$

i.e. $3x_1 \cdot x + 4y_1 \cdot y - 1 = 0$ ...(3)

If (2) is tangent to (1), then it must be identical with (3), for this the coefficients of x, y and constants must be in proportion.

$$\frac{3x_1}{3} = \frac{4y_1}{4} = \frac{-1}{\sqrt{7}}$$

$\therefore$ $x_1 = \dfrac{-1}{\sqrt{7}}, y_1 = \dfrac{-1}{\sqrt{7}}$

Point of contact $(x_1, y_1) = \left(\dfrac{-1}{\sqrt{7}}, \dfrac{-1}{\sqrt{7}}\right)$

The point of contact (x_1, y_1) also lies on the ellipse (1). So,

$$3 \cdot \left(\frac{1}{\sqrt{7}}\right)^2 + 4 \cdot \left(\frac{1}{\sqrt{7}}\right)^2 = 1$$

or, $\dfrac{3 + 4}{7} = 1$ or $1 = 1$ (True)

Hence, the line $3x + 4y + \sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$ and point of contact is $\left(\dfrac{-1}{\sqrt{7}}, \dfrac{-1}{\sqrt{7}}\right)$.

6. Find value of c so that the line $2x - 3y = c$ is normal to the ellipse $9x^2 + 116y^2 = 144$.

Solution

Given equation of ellipse is

$9x^2 + 116y^2 = 144$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$ ...(1)

Comparing equation (1) with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$a^2 = 16, b^2 = 9$

Equation of normal to the ellipse (1) is

$2x - 3y - c = 0$ ...(2)

Slope of normal (m) $= \frac{2}{3}$, constant $= -c$

We know condition for the line $y = mx + c$ to be normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$c = \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2m^2}}$

or, $-c = \pm \frac{\frac{2}{3}(16-9)}{\sqrt{16 + 9\frac{4}{9}}} = \frac{\pm\frac{2}{3}\times 7}{\sqrt{20}}$

or, $c = \pm\frac{7}{3\sqrt{5}}$

7. Find the locus of a point from which the two tangents drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes angle θ_1 and θ_2 with major axis such that $\tan\theta_1 + \tan\theta_2 = 2c$.

Solution

We know that for all values of m, the line

$y = mx + \sqrt{a^2m^2 + b^2}$ ...(1)

is always tangent to an ellipse.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ...(2)

Let (h, k) be a point from which two tangents represented by (1) are drawn to the ellipse (2).

$\therefore$ $k = mh + \sqrt{a^2m^2 + b^2}$

or, $k - mh = \sqrt{a^2m^2 + b^2}$

Squaring both sides,

or, $(k - mh)^2 = (\sqrt{a^2m^2 + b^2})^2$

or, $k^2 - 2mhk + m^2h^2 = a^2m^2 + b^2$

$\therefore$ $(h^2 - a^2)m^2 - 2hkm + (k^2 - b^2) = 0$ ...(3)

Equation (3) is quadratic in m, so m has two roots $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ such that

$\tan\theta_1 + \tan\theta_2 = m_1 + m_2 = \frac{2hk}{h^2 - a^2}$ $[\because m_1 + m_2 = \frac{-B}{A}]$

or, $2c = \frac{2hk}{h^2 - a^2}$

or, $c(h^2 - a^2) = 2hk$

Thus, the locus of point (h, k) is

$c(x^2 - a^2) = 2xy$

$\therefore$ $c(x^2 - a^2) - 2xy = 0$

Exercise 6.4

1. Find the equation of hyperbola in the following information.

a. foci (±5, 0) and vertex (±2, 0)

Solution

Given foci F(± 5, 0) and vertices A(± 2, 0)

We have foci F_1 (–5, 0) and F_2(5, 0), where y-coordinate is same and y = 0 so x-axis is transverse axis and centre (h, k) $= \left(\frac{-5+5}{2}, \frac{0+0}{2}\right) = (0, 0)$

Also, vertices $(\pm a, 0) = (\pm 2, 0) \Rightarrow a = 2$

Foci $(\pm ae, 0) = (\pm 5, 0) \Rightarrow ae = 5$

We know,

$b^2 = a^2(e^2 - 1) = (ae)^2 - a^2 = 25 - 4 = 21$

$\therefore$ Required equation of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ $\frac{x^2}{4} - \frac{y^2}{21} = 1$

b. foci (±7, 0) and eccentricity is $\frac{7}{4}$

Solution

Given foci F (±7, 0) so transverse axis is y = 0 (x-axis) and centre (h, k) = (0, 0).

Now, ae = 7

Given, $e = \frac{7}{4}$

So, $a\cdot\frac{7}{4} = 7 \Rightarrow a = 4$

Now, $b^2 = a^2(e^2 - 1) = (ae)^2 - a^2 = 7^2 - 4^2 = 33$

Required equation of hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\therefore$ $\frac{x^2}{16} - \frac{y^2}{33} = 1$

c. **conjugate axis is of length 5 and distance the foci is 13**

Solution

Let the hyperbola has axes along the coordinate axes. Then.

Required equation of hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

Where x-axis is transverse and y-axis is conjugate axis.

Given that, length of conjugate axis $2b = 5 \Rightarrow b = \frac{5}{2}$

Distance between the foci $2ae = 13 \Rightarrow ae = \frac{13}{2}$

Now, we know

$b^2 = a^2(e^2-1) = (ae)^2 - a^2$

or, $a^2 = (ae)^2 - b^2 = \left(\frac{13}{2}\right)^2 - \left(\frac{5}{2}\right)^2 = \frac{144}{4}$

Now equation (1) becomes

$$\frac{x^2}{144/4}-\frac{y^2}{25/4}=1$$

$\therefore$ $25x^2 - 144y^2 = 900$

d. **length of latus rectum is 8 and eccentricity is $\frac{3}{\sqrt{5}}$**

Solution

Let required equation of ellipse be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

We have,

Length of latus rectum $\frac{2b^2}{a} = 8 \Rightarrow b^2 = 4a$

Eccentricity $(e) = \frac{3}{\sqrt{5}}$

or, $\sqrt{1+\frac{b^2}{a^2}} = \frac{3}{\sqrt{5}}$

or, $1 + \frac{4a}{a^2} = \frac{9}{5}$

or, $\frac{4}{a} = \frac{9}{5} - 1 = \frac{4}{5}$

$\therefore$ $a = 5$

Then, $b^2 = 4 \times 5 = 20$

Now equation (1) becomes

$$\frac{x^2}{25}-\frac{y^2}{20}=1$$

e. **foci are at (8, 3) and (0, 3) eccentricity $= \frac{4}{3}$**

Solution

Given foci are $F_1(8, 3)$ and $F_2(0, 3)$

Here, y-coordinate is same and $y = 3$

So transverse axis of hyperbola is $y = 3$ and centre $(h, k) = \left(\frac{8+0}{2}, \frac{3+3}{2}\right) = (4,3)$.

Required equation of hyperbola be

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \quad \text{...(1)}$$

We have eccentricity $(e) = \frac{4}{3}$

Distance between foci

$2ae = 8 \Rightarrow ae = 4$

$\therefore$ $a = 3;\ e = \frac{4}{3}$

We know,

$b^2 = a^2(e^2-1) = (ae)^2 - a^2 = 4^2 - 3^2 = 7$

Now, equation (1) becomes

$$\frac{(x-4)^2}{9}-\frac{(y-3)^2}{7}=1$$

or, $7(x^2 - 8x + 16) - 9(y^2 - 6y + 9) = 63$

or, $7x^2 - 9y^2 - 56x + 54y - 32 = 0$

f. **focus (6,0), directrix $4x - 3y = 6$ and $e = \frac{5}{4}$**

Solution

Let $p(x, y)$ be any points on the hyperbola and focus $(\alpha, \beta) = (6, 0)$ and directrix $4x - 3y - 6 = 0$ and eccentricity $e = \frac{5}{4}$. Then using general equation of conic section

$$(x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(ax+by+c)^2}{(\sqrt{a^2+b^2})^2}$$

or, $(x-6)^2 + (y-0)^2 = \frac{25}{16}\frac{(4x-3y-6)^2}{(\sqrt{(4)^2+(-3)^2})^2}$

or, $x^2 - 12x + 36 + y^2 = \frac{25}{16}\frac{(16x^2+9y^2+36-24xy-48x+36y)}{25}$

or, $16x^2 - 192x + 576 + 16y^2 = 16x^2 + 9y^2 + 36 - 24xy - 48x + 36y$

or, $7y^2 + 24xy - 144x - 36y + 540 = 0$

Which is required equation of hyperbola.

g. **Centre of origin passing through (2,1) and (4,3)**

Solution

We know standard equation of equation of hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \text{...(1)}$$

Since (1) is passing through (2, 1)

$$\frac{4}{a^2}-\frac{1}{b^2}=1 \Rightarrow \frac{1}{b^2}=\frac{4}{a^2}-1$$

Also equation (1) passes through (4, 3)

$$\frac{16}{a^2}-\frac{9}{b^2}=1$$

or, $\frac{16}{a^2}-9\cdot\frac{1}{b^2}=1$

or, $\frac{16}{a^2}-9\left(\frac{4}{a^2}-1\right)=1$ [∵ Using equation (2)]

or, $\frac{-20}{a^2}=-8 \Rightarrow a^2=\frac{5}{2}$

Substitute value of a^2 in equation (2)

$$\frac{1}{b^2}=\frac{4}{5/2}-1 \Rightarrow b^2=\frac{5}{3}$$

Now, equation (1) becomes

$$\frac{x^2}{5/2}-\frac{y^2}{5/3}=1$$

∴ $2x^2-3y^2=5$

h. distance between foci is 16 and eccentricity is $\sqrt{2}$.

Solution

We have, standard equation of hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad \ldots(1)$$

Given, distance between two foci $=16 \Rightarrow 2ae=16 \Rightarrow ae=8$

Given that, $e=\sqrt{2}$ so $a=4\sqrt{2}$

Also, we have

$b^2=a^2(e^2-1)=(ae)^2-a^2=(8)^2-(4\sqrt{2})^2 \Rightarrow b^2=32$

Now, equation (1) becomes

$$\frac{x^2}{32}-\frac{y^2}{32}=1$$

∴ $x^2-y^2=32$

2. Find the co-ordinates of vertices, eccentricity and foci for the following hyperbola.

a. $3x^2-2y^2=1$

Solution

Given equation of hyperbola is

$3x^2-2y^2=1$

or, $\frac{x^2}{1/3}-\frac{y^2}{1/2}=1$

Comparing with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get

$a^2=\frac{1}{3}, b^2=\frac{1}{2}$

For eccentricity

$b^2=a^2(e^2-1)$

$\frac{1}{2}=\frac{1}{3}(e^2-1)$

or, $e^2-1=\frac{3}{2}$

or, $e^2=\frac{3}{2}+1=\frac{5}{2}$

∴ $e=\sqrt{\frac{5}{2}}$

Vertices $V(\pm a, 0)=V\left(\pm\frac{1}{\sqrt{3}}, 0\right)$

Foci $F(\pm ae, 0)=F\left(\pm\frac{1}{\sqrt{3}}\cdot\sqrt{\frac{5}{2}}, 0\right)=\left(\pm\sqrt{\frac{5}{6}}, 0\right)$

b. $9x^2-16y^2-18x-64y-199=0$

Solution

Given equation of hyperbola is

$9x^2-16y^2-18x-64y-199=0$

or, $9(x^2-2x)-16(y^2+4y)=199$

or, $9(x^2-2x+1)-16(y^2+4y+4)=199+9-64$

or, $9(x-1)^2-16(y+2)^2=144$

or, $\frac{(x-1)^2}{4^2}-\frac{(y+2)^2}{3^2}=1 \quad \ldots(1)$

Comparing equation (1) with $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

We get, $h=1, k=-2, a=4, b=3$

For eccentricity

$b^2=a^2(e^2-1)$

or, $9=16(e^2-1)$

or, $e^2-1=\frac{9}{16}$

or, $e^2=\frac{9}{16}+1=\frac{25}{16}$

∴ $e=\frac{5}{4}$

Vertices $V(h\pm a, k)=(1\pm 4, -2)=(5, -2)$ and $(-3, -2)$

Foci $F(h\pm ae, k)=\left(1\pm 4\cdot\frac{5}{4}, -2\right)=(1\pm 5, -2)=(6, -2)$ and $(-4, -2)$

c. $3x^2-6y^2=-18$

Solution

Given equation of hyperbola is

$3x^2-6y^2=-18$

or, $\frac{x^2}{6}-\frac{y^2}{3}=-1 \quad \ldots(1)$

Comparing equation (1) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, we get

$a^2 = 6, b^2 = 3$

For eccentricity, we have

$a^2 = b^2(e^2 - 1)$

or, $6 = 3(e^2 - 1)$

or, $e^2 - 1 = 2$

or, $e^2 = 3$

$\therefore$ $e = \sqrt{3}$

Vertices = V(0, ±b) = $(0, \pm\sqrt{3})$

Foci = F(0, ± be) = $(0, \pm\sqrt{3}\cdot\sqrt{3}) = (0, \pm 3)$

d. $5x^2 - 4y^2 + 20x + 8y - 4 = 0$

Solution

Given equation is

$5x^2 - 4y^2 + 20x + 8y - 4 = 0$

or, $5(x^2 + 4x) - 4(y^2 - 2y) = 4$

or, $5(x^2 + 4x + 4) - 4(y^2 - 2y + 1) = 4 + 20 - 4$

or, $5(x + 2)^2 - 4(y - 1)^2 = 20$

or, $\frac{(x+2)^2}{4} - \frac{(y-1)^2}{5} = 1$

Comparing equation (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get

$h = -2, k = 1, a^2 = 4, b^2 = 5$

For eccentricity

$b^2 = a^2(e^2 - 1)$

or, $5 = 4(e^2 - 1)$

or, $e^2 - 1 = \frac{5}{4}$

or, $e^2 = \frac{5}{4} + 1 = \frac{9}{4}$

$\therefore$ $e = \frac{3}{2}$

Vertices (V) = (h ± a, k) = (−2 ± 2, 1) = (−4, 1) and (0, 1)

Foci (F) = (h ± ae, k) = (−2 ± 3, 1) = (−5, 1) and (1, 1)

e. $4(y + 3)^2 - 9(x - 2)^2 = 1$

Solution

Given equation of hyperbola is

$4(y + 3)^2 - 9(x - 2)^2 = 1$

or, $\frac{(x-2)^2}{1/9} - \frac{(y+3)^2}{1/4} = -1$ ...(1)

Comparing equation (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$, we get

$h = 2, k = -3, a^2 = \frac{1}{9}, b^2 = \frac{1}{4}$

For eccentricity

$a^2 = b^2(e^2 - 1)$

$\frac{1}{9} = \frac{1}{4}(e^2 - 1)$

$e^2 - 1 = \frac{4}{9}$

$e^2 = \frac{4}{9} + 1 = \frac{13}{9}$

$e = \frac{\sqrt{13}}{3}$

Vertices (V) = (h, k ± b) = $\left(2 - 3 \pm \frac{1}{2}\right)$

Foci (F) = (h, k ± be) $= \left(2, -3 \pm \frac{1}{2}\cdot\frac{\sqrt{13}}{3}\right)$

$= \left(2, -3 \pm \frac{\sqrt{13}}{6}\right)$

f. $9x^2 - 16y^2 - 18x - 32y - 151 = 0$

Solution

Given equation of hyperbola is

$9x^2 - 16y^2 - 18x - 32y - 151 = 0$

or, $9(x^2 - 2x) - 16(y^2 + 2y) = 151$

or, $9(x^2 - 2x + 1) - 16(y^2 + 2y + 1) = 151 + 9 - 16$

or, $9(x - 1)^2 - 16(y + 1)^2 = 144$

or, $\frac{(x-1)^2}{16} - \frac{(y+1)^2}{9} = 1$ ...(1)

Comparing equation (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get

$h = 1, k = -1, a = 4, b = 3$

For eccentricity

$b^2 = a^2(e^2 - 1)$

or, $9 = 16(e^2 - 1)$

or, $e^2 - 1 = \frac{9}{16}$

or, $e^2 = \frac{9}{16} + 1 = \frac{25}{16}$

$\therefore$ $e = \frac{5}{4}$

Vertices (V) = (h ± a, k) = (1 ± 4, −1) = (−3, −1) and (5, −1)

Foci (F) = (h ± ae, k) = (1 ± 5, 1) = (−4, −1) and (6, −1)

g. $4x^2 - y^2 + 4y - 8 = 0$

Solution

Given equation of hyperbola is

$4x^2 - y^2 + 4y - 8 = 0$

or, $4x^2 - (y^2 - 4y) = 8$

or, $4x^2 - (y^2 - 4y + 4) = 8 - 4$

or, $4x^2 - (y - 2)^2 = 4$

or, $\frac{x^2}{1^2} - \frac{(y-2)^2}{2^2} = 1$

Comparing equation (1) with $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$, we get

$h = 0, k = 2, a = 1, b = 2$

For eccentricity

$b^2 = a^2(e^2 - 1)$

or, $4 = e^2 - 1$

or, $e^2 = 5$

$\therefore$ $e = \sqrt{5}$

Vertices (V) = $(h \pm a, k) = (0 \pm 1, 2) = (\pm 1, 2)$

Foci (F) = $(h \pm ae, k) = (0 \pm \sqrt{5}, 2) = (\pm\sqrt{5}, 2)$

3. Find the value of λ when the line $y = 2x + \lambda$ is tangent to hyperbola $3x^2 - y^2 = 3$.

Solution

Given equation of hyperbola is

$3x^2 - y^2 = 3$

i.e. $\frac{x^2}{1} - \frac{y^2}{3} = 1$ ... (1)

Comparing equation (1) with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get

$a^2 = 1, b^2 = 3$

Given equation of straight line is

$y = 2x + \lambda$ ... (2)

Comparing equation (2) with $y = mx + c$, we get

$m = 2, c = \lambda$

When the line (2) is tangent to hyperbola (1) then we have

$c^2 = a^2m^2 - b^2$

or, $\lambda^2 = 1 \cdot 2^2 - 3$

or, $\lambda^2 = 1$

$\therefore$ $\lambda = \pm 1$

4. Show that the line $lx + my + n = 0$ touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $a^2l^2 - b^2m^2 = n^2$.

Solution

We have equation of hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(1)

We know, equation of tangent to the hyperbola (1) at (x_1, y_1) is

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$

or, $\frac{x_1}{a^2}x + \frac{y_1}{b^2}y - 1 = 0$ ...(2)

The given equation of straight line which touches the hyperbola (1) is

$lx + my + n = 0$ ...(3)

If (3) is tangent to (1), then it must be identical with (2), for this the coefficient of x, y and constants must be in proportion.

i.e. $\frac{\frac{x_1}{a^2}}{l} = \frac{\frac{y_1}{b^2}}{m} = \frac{-1}{n}$

or, $\frac{x_1}{a^2l} = \frac{y_1}{b^2m} = \frac{-1}{n}$

or, $x_1 = -\frac{a^2l}{n}, y_1 = -\frac{b^2m}{n}$

$\therefore$ The point of contact $(x_1, y_1) = \left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$

The point of contact (x_1, y_1) must satisfy the equation (1).

$\therefore$ $\frac{1}{a^2}\left(\frac{-a^2l}{n}\right)^2 - \frac{1}{b^2}\left(\frac{-b^2m}{n}\right)^2 = 1$

or, $\frac{a^2l^2}{n^2} - \frac{b^2m^2}{n^2} = 1$

or, $n^2 = a^2l^2 - b^2m^2$

Which is the required condition.

5. Find the centre and eccentricity of the hyperbola $x^2 - 4y^2 - 2x + 24y - 37 = 0$.

Solution

Given equation of hyperbola is

$x^2 - 4y^2 - 2x + 24y - 37 = 0$

or, $x^2 - 2x - 4(y^2 - 6y) - 37 = 0$

or, $x^2 - 2x + 1 - 4(y^2 - 6y + 9) - 37 = 1 - 4 \times 9$

or, $(x-1)^2 - 4(y-3)^2 = 2$

or, $\frac{(x-1)^2}{2} - \frac{(y-3)^2}{\frac{1}{2}} = 1$ ...(1)

Comparing equation (1) with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$h = 1, k = 3, a^2 = 2, b^2 = \frac{1}{2}$

Centre (h, k) = (1, 3)

For eccentricity, we have

$b^2 = a^2(e^2 - 1)$

or, $\frac{1}{2} = 2(e^2 - 1)$

or, $\frac{1}{4} = e^2 - 1$

or, $e^2 = 1 + \frac{1}{4} = \frac{5}{4}$

$\therefore$ $e = \frac{\sqrt{5}}{2}$

6. **Show that the eccentricity of a hyperbola whose transverse axis is $2a$ and axes are axes of coordinates that passes through point (h, k) is $\sqrt{\left(\frac{h^2+k^2-a^2}{h^2-a^2}\right)}$.**

Solution

Let $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be the equation of hyperbola which passes through (h, k). So,

$\frac{h^2}{a^2}-\frac{k^2}{b^2}=1$

or, $\frac{h^2}{a^2}-1=\frac{k^2}{b^2}$

or, $\frac{h^2-a^2}{a^2}=\frac{k^2}{b^2}$

or, $b^2=\frac{a^2k^2}{h^2-a^2}$

For eccentricity, we have

$e=\sqrt{1+\frac{b^2}{a^2}}$ ...(1)

Here, $\frac{b^2}{a^2}=\frac{a^2k^2/h^2-a^2}{a^2}=\frac{k^2}{h^2-a^2}$

Now from (1)

$e=\sqrt{1+\frac{k^2}{h^2-a^2}}=\sqrt{\frac{h^2-a^2+k^2}{h^2-a^2}}$

$\therefore$ $e=\sqrt{\frac{h^2+k^2-a^2}{h^2-a^2}}$

This completes the proof.

7. **Find equation of tangent to the hyperbola $4x^2-9x^2=1$ which is parallel to the line $4y=5x+7$.**

Solution

Given equation of hyperbola is

$4x^2-9y^2=1$

or, $\frac{x^2}{1/4}-\frac{y^2}{1/9}=1$ ...(1)

Comparing with $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, we get $a^2=\frac{1}{4}$, $b^2=\frac{1}{9}$

Given equation of straight line is

$4y=5x+7$

i.e., $y=\frac{5}{4}x+\frac{7}{4}$ ...(2)

Any line parallel to (2) is

$y=\frac{5}{4}x+k$ ...(3)

Comparing equation (3) with $y=mx+c$, we get

$m=\frac{5}{4}$, $c=k$

We know, condition for $y=mx+c$ to be tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$c^2=a^2m^2-b^2$

or, $k^2=\frac{1}{4}\left(\frac{5}{4}\right)^2-\frac{1}{9}=\frac{25}{64}-\frac{1}{9}=\frac{161}{576}$

or, $k=\pm\frac{\sqrt{161}}{24}$

Now equation (3) becomes

$y=\frac{5}{4}x\pm\frac{\sqrt{161}}{24}$

or, $24y=30x\pm\sqrt{161}$

or, $24y-30x=\pm\sqrt{161}$

Which is required equation of tangent.

8. **Show that the circle described on the line joining the foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as a diameter passes through foci of its conjugate hyperbola.**

Solution

Let e_1 be the eccentricity of standard equation of hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1)

and e_2 be the eccentricity of conjugate hyperbola of (1)

i.e., $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ ...(2)

Then, $e_1=\sqrt{1+\frac{b^2}{a^2}}$ and $e_2=\sqrt{1+\frac{a^2}{b^2}}$

Also foci of (1) are $A'(-ae_1, 0)$ and $A(ae_1, 0)$

Foci of (2) are $B'(0, -be_2)$ and $B(0, be_2)$

Now, equation of circle whose ends of diameters are $A'(-ae_1, 0)$ and $A(ae_1, 0)$ is

$(x+ae_1)(x-ae_1)+(y-0)(y-0)=0$

or, $x^2-a^2e_1^2+y^2=0$ ...(3)

We need to show that the circle (3) passes through $(0, \pm be_2)$

So put $x=0$ and $y=\pm be_2$ in equation (3)

or, $(0)^2-a^2e_1^2+(\pm be_2)^2=0$

or, $b^2e_2^2-a^2e_1^2=0$

or, $b^2\left(1+\frac{a^2}{b^2}\right)-a^2\left(1+\frac{b^2}{a^2}\right)$

or, $b^2+a^2-(a^2+b^2)=0$

or, $0=0$ (True)

Hence, the circle described on the line joining the foci of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ as a diameter passes through foci of its conjugate hyperbola.

9. Find the point of intersection of tangents at points 't_1' and 't_2' on the hyperbola $xy = c^2$.

Solution

Let $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ be two points of hyperbola $xy = c^2$.

We have equation of tangent to the hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ is given by

$$x + t^2y = 2ct \qquad \text{...(1)}$$

Equation of tangent to the hyperbola at $P\left(ct_1, \frac{c}{t_1}\right)$ is

$$x + t_1^2y = 2ct_1 \qquad \text{...(2)}$$

Equation of tangent to the hyperbola at $Q\left(ct_2, \frac{c}{t_2}\right)$ is

$$x + t_2^2y = 2ct_2 \qquad \text{...(3)}$$

Solving equation (2) and (3)

$$\begin{array}{l} x + t_1^2y = 2ct_1 \\ x + t_2^2y = 2ct_2 \\ - \quad - \qquad - \\ \hline (t_1^2 - t_2^2)y = 2c(t_1 - t_2) \end{array}$$

or, $(t_1 - t_2)(t_1 + t_2)y = 2c(t_1 - t_2)$

or, $y = \dfrac{2c}{t_1 + t_2}$

Substituting value of $y = \dfrac{2c}{t_1 + t_2}$ in equation (2), we get

$x = 2ct_1 - t^2 \cdot \dfrac{2c}{t_1 + t_2}$

or, $x = 2ct_1\left(\dfrac{t_1 + t_2 - t_1}{t_1 + t_2}\right)$

or, $x = \dfrac{2ct_1t_2}{t_1 + t_2}$

$\therefore$ Point of intersection of tangent to the hyperbola $xy = c^2$ is

$$(x, y) = \left(\frac{2ct_1t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$$

10. Find the equation of a hyperbola whose asymptotes are $2x - y - 3 = 0$ and $3x + y - 7 = 0$ and passes through (1, 1).

Solution

The given equation of asymptotes are

$2x - y - 3 = 0$

and $3x + y - 7 = 0$

Then their combined form is

$$(2x - y - 3)(3x + y - 7) = 0 \qquad \text{...(1)}$$

Since the equation of asymptotes differ from the equation of hyperbola by a constant so the equation of hyperbola is

$$(2x - y - 3)(3x + y - 7) = c \qquad \text{...(2)}$$

where c is constant.

The point (1, 1) lies on the curve (2), then

$(+2 - 1 - 3)(+3 + 1 - 7) = c$

or, $(-2)(-3) = c \Rightarrow c = 6$

Thus the equation of hyperbola is

$(2x - y - 3)(3x + y - 7) = 6$

11. If the normal at $\left(ct, \frac{c}{t}\right)$ of the hyperbola $xy = c^2$ meets the hyperbola again at $\left(ct_1, \frac{c}{t_1}\right)$, prove that $t^3t_1 = -1$.

Solution

We know equation of normal to the hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ is

$xt^2 - y = ct^3 - \dfrac{c}{t}$

i.e., $t^3x - ty = ct^4 - c$

Which is passing through other point $\left(ct_1, \frac{c}{t_1}\right)$ so

$t^3 \cdot ct_1 - t \cdot \dfrac{c}{t_1} = ct^4 - c$

or, $t^3t_1 - \dfrac{t}{t_1} = t^4 - 1$

or, $t^3t_1^2 - t = t^4t_1 - t_1$

or, $t^3t_1^2 - t^4t_1 + t_1 - t = 0$

or, $t^3t_1(t_1 - t) + 1(t_1 - t) = 0$

or, $(t_1 - t)(t^3t_1 + 1) = 0$

or, $t^3t_1 + 1 = 0$ [$\because t_1 \neq t$]

or, $t^3t_1 = -1$ proved

Exercise 6.5

1. Show that the equations $16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$ represents a parabola. Also, determine its equation of axis, vertex, latus rectum, focus, and directrix.

Solution

The given equation of conic is

$$16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0 \qquad \text{...(1)}$$

Comparing equation (1) with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 16, h = -12, b = 9, g = -52, f = -86, c = 44$

$$\begin{aligned} \Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 16 \cdot 9 \cdot 44 + 2(-86)(-52)(-12) - 16(-86)^2 - 9(-52)^2 - 44(-12)^2 \\ &= 6336 - 107328 - 118336 - 24336 - 6336 \\ &= -25000 \neq 0 \end{aligned}$$

and $h^2 - ab = (-12)^2 - 16 \times 9 = 0$

Since $\Delta \neq 0$ and $h^2 - ab = 0$, the given curve represent parabola.

a. Standard form

Equation (1) can be written as

$(4x)^2 - 2 \cdot 4x\ 3y + (3y)^2 = 104x + 172y - 44$

or, $(4x - 3y)^2 = 104x + 172y - 44$

or, $(4x - 3y + \lambda)^2 = \lambda^2 + 8\lambda x - 6\lambda y + 104x + 172y - 4y$

where λ is a constant to be determined from equation.

or, $(4x - 3y + \lambda)^2 = (8\lambda + 104)x + (-6\lambda + 172)y + \lambda^2 - 44$ ...(2)

We choose λ such that the lines

$4x - 3y + \lambda = 0$ ...(3)

and $(8\lambda + 104)x + (-6\lambda + 172)y + \lambda^2 - 44 = 0$

are perpendicular to each other.

Then,

Slope of (2) × Slope of (3) = –1

or, $m_1 \cdot m_2 = -1$

or, $\frac{4}{3} \times \left(-\frac{8\lambda + 104}{-6\lambda + 172}\right) = -1$

or, $4(8\lambda + 104) = 3(-6\lambda + 172)$

or, $32\lambda + 416 = -18\lambda + 516$

or, $50\lambda = 100$

$\therefore$ $\lambda = 2$

Substituting value of λ in equation (2), we get

$(4x - 3y + 2)^2 = 120x + 160y - 40$

or, $(4x - 3y + 2)^2 = 40(2x + 4y - 1)$

Which can be written as

$$\left(\frac{4x - 3y + 2}{\sqrt{4^2 + (-3)^2}}\right)^2 \cdot \{4^2 + (-3)^2\} = 40\,\frac{(3x + 4y - 1)}{\sqrt{3^2 + 4^2}} \cdot \sqrt{3^2 + 4^2}$$

or, $\left(\frac{4x - 3y + 2}{5}\right)^2 = 4 \cdot 2\,\frac{(3x + 4y - 1)}{5}$ ...(5)

Which is in the form $y^2 = 4aX$

Where $X = \frac{3x + 4y - 1}{5}$, $y = \frac{4x - 3y + 2}{2}$, $a = 2$

b. Equation of axis and tangent

Equation of axis of the parabola (5) is

$y = 0$

$\frac{4x - 3y + 2}{5} = 0$

or, $4x - 3y + 2 = 0$ ...(6)

Also, equation of tangent of the parabola (5) at vertex is

$X = 0$

or, $\frac{3x + 4y - 1}{5} = 0$

$\therefore$ $3x + 4y - 1 = 0$ ...(7)

c. **Vertex:** Since the point of intersection of equation axis of parabola and equation of tangent at vertex is vertex.

So, solving equation (6) and (7), we get

$x = \frac{1}{5}, y = \frac{2}{5}$

$\therefore$ Vertex $(h,k) = \left(\frac{-1}{5}, \frac{2}{5}\right)$

d. Length of latus rectum = 4a = 8

e. Equation of latus rectum

$X = a$

$\frac{3x + 4y - 1}{5} = 2$

or, $3x + 4y - 1 = 10$

or, $3x + 4y = 11$ ...(8)

Since focus is the point of intersection of axis (6) and equation of latus rectum (8) so solving, we get

$x = 1, y = 2$

Focus = F(1, 2)

f. Equation of directrix

$X = -a$

$\frac{3x + 4y - 1}{5} = -2$

or, $3x - 4y + 9 = 0$ ...(9)

2. Show that the equation $16x^2 - 24xy + 9y^2 - 80x - 140y + 100 = 0$ represents a parabola. Also, determine its equation of axis, vertex, latus rectum and focus.

Solution

The given equation of the conic is

$16x^2 - 24xy + 9y^2 - 80x - 140y + 100 = 0$ ...(1)

Comparing equation (1) with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 16, b = 9, c = 100, h = -12, g = -40, f = -70$

Then,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 16 \times 9 \times 100 + 2(-70)(-40) - (-12) - 16(-70)^2 - 9(-40)^2 - 100(-12)^2$

$= -16000 \neq 0$

Also, $h^2 - ab = (-12)^2 - 16 \times 9 = 0$

Hence, $\Delta \neq 0$ and $h^2 - ab = 0$, the given conic represents parabola.

a. Standard form

Equation (1) can be written as

$(4x - 3y)^2 = 80x + 140y - 100$

or, $(4x - 3y + \lambda)^2 = \lambda^2 + 8\lambda x - 6\lambda y + 80x + 140y - 100$

or, $(4x - 3y + \lambda)^2 = \lambda^2 + 8\lambda x - 6\lambda y + 80x + 140y - 100$

or, $(4x - 3y + \lambda)^2 = (8\lambda + 80)x + (-6\lambda + 140)y + \lambda^2 - 100$ ...(2)

where, the scalar λ is chosen in such a way that the straight lines

$4x - 3y + \lambda = 0$ ...(3)

$(8\lambda + 80)x + (-6\lambda + 140)y + \lambda^2 - 100 = 0$ ...(4)

are perpendicular to each other. Then,

$m_1 \cdot m_2 = -1$

or, $\frac{4}{3}\left[-\left(\frac{8\lambda + 80}{-6\lambda + 140}\right)\right] = -1$

or, $4(8\lambda + 80) = 3(-6\lambda + 140)$

or, $32\lambda + 320 = -18\lambda + 420$

or, $50\lambda = 100$

$\therefore$ $\lambda = 2$

Substituting the value of λ in equation (2), we get

$(4x - 3y + 2)^2 = 96x + 128y - 96$

which can be written as

$$\left(\frac{4x - 3y + 2}{\sqrt{(4)^2 + (-3)^2}}\right)^2 \{4^2 + (-3)^2\} = \frac{96x + 128y - 96}{\sqrt{(96)^2 + (128)^2}} \sqrt{(96)^2 + (128)^2}$$

or, $\left(\frac{4x - 3y + 2}{5}\right)^2 \times 25 = \left(\frac{96x + 128y - 96}{160}\right) 160$

or, $\left(\frac{4x - 3y + 2}{5}\right)^2 = 4\,\frac{8}{5}\left(\frac{96x + 128y - 96}{160}\right)$ ...(5)

which is in the form of $Y^2 = 4aX$

where, $Y = \frac{4x - 3y + 2}{5}$

$X = \frac{96x + 128y - 96}{160}$, $a = \frac{8}{5}$

b. Equation of axis and tangent

Equation of axis of parabola (5) is

$Y = 0$

$\frac{4x - 3y + 2}{5} = 0$

or, $4x - 3y + 2 = 0$ ...(6)

Also, equation of tangent of the parabola (5) at vertex is

$X = 0$

$\frac{96x + 128y - 96}{160} = 0$

or, $96x + 128y - 96 = 0$ ...(7)

c. Vertex: Since the point of intersection of equation of axis and equation of tangent at the vertex is vertex. So solving equation (6) and (7) we get

$x = \pm\frac{1}{25}, y = \pm\frac{18}{25}$

$\therefore$ Vertex $= \left(\pm\frac{1}{25}, \pm\frac{18}{25}\right)$

d. Length of latus rectum: $4a = \frac{32}{5}$

Equation of latus rectum is

$X = a$

$\frac{96x + 128y - 96}{160} = \frac{8}{5}$

$96x + 128y - 352 = 0$ ...(8)

d. Since focus is the point of intersection of axis (6) and equation of latus rectum is (8).

e. So, solving (6) and (8), we get

Focus $= (1,2)$

3. **Determine the nature of the conic-section represented by $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$. Also, determine foci and eccentricity.**

Solution

The given conic section is

S: $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$ ...(1)

Comparing equation (1) with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 1, b = 1, c = -6, h = 2, g = -1, f = 1$

Now,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 1 \times 1 \times (-6) + 2 \times 1 \times (-1) \times (-2) - 1 \cdot 1^2 - 1 \cdot (-1)^2 - (-6)^2 (-2)^2$

$= 20 \neq 0$

Now, $h^2 - ab = (-2)^2 - 1 = 4 - 1 = 3 > 0$

The given conic section represent hyperbola.

a. For centre

Solving $\frac{\partial s}{\partial x} = 0, \frac{\partial s}{\partial y} = 0$

From (1)

$\frac{\partial s}{\partial x} = 2x + 4y - 2 = 0$

$\frac{\partial s}{\partial y} = 4x + 2y + 2 = 0$

Solving, we get $x = -1, y = +1$

$\therefore$ The coordinates of centre $(\alpha, \beta) = (-1, 1)$

b. Equation of conic referred to centre as origin

$x^2 + 4xy + y^2 + c_1 = 0$

where, $c_1 = g\alpha + f\beta + c$

or, $c_1 = (-1) \times (-1) + 1 \times (-1) + (-6)$

$c_1 = -4$

and $x = X + \alpha = X - 1$

$y = Y + \beta = Y + 1$

Equation of conic with referred to origin (centre) at $(\alpha, \beta) = (-1, 1)$ is

$X^2 + 4xy + y^2 - 4 = 0$

or, $\frac{1}{4}X^2 + xy + \frac{1}{4}y^2 = 1$

which is of the form $AX^2 + 2HXY + BY^2 = 1$

where, $A = \frac{1}{4}, H = \frac{1}{2}, B = \frac{1}{4}$

$H^2 - AB = \left(\frac{1}{2}\right)^2 - \frac{1}{4}\cdot\frac{1}{4} = \frac{3}{16}$

$A + B = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

c. **For the length and equations of axes**

The length of semi axes is given by the roots of the equations

$(H^2 - AB)r^4 + (A + B)r^2 - 1 = 0$

or, $\frac{3}{16}r^4 + \frac{1}{2}r^2 - 1 = 0$

or, $3r^4 + 8r^2 - 16 = 0$

or, $r^2 = \frac{4}{3}, -4$ then $r_1^2 = \frac{4}{3}, r_2^2 = 4$

Length of semi transverse axis $r_1 = \frac{2}{\sqrt{3}}$

Length of semi conjugate axis $r_2 = 2$

d. **Eccentricity**

We have, $r_2^2 = r_1^2(e^2 - 1)$

or, $4 = \frac{4}{3}(e^2 - 1)$

or, $e^2 - 1 = 3$

or, $e^2 = 4$

$\therefore$ $e = 2$

e. **The coordinate foci** $= (\alpha + er_1\cos\theta, \beta + er_1\sin\theta)$

Now equation of transverse axis is

$\left(A - \frac{1}{r_1^2}\right)X + AY = 0$

$\left(\frac{1}{4} - \frac{3}{4}\right)X + \frac{1}{2}Y = 0$

or, $\frac{-1}{2}X + \frac{1}{2}Y = 0$

or, $-X + Y = 0$

$X = x - \alpha = x + 1, Y = y - \beta = y - 1$, then

$-(x + 1) + (y - 1) = 0$

or, $-x + y - 2 = 0$

or, $x - y + 2 = 0$

Which is transverse axis.

Slope (m) $= \tan\theta = 1, \theta = 45°$

Then $\sin\theta = \frac{1}{\sqrt{2}}, \cos\theta = \frac{1}{\sqrt{2}}$

Then foci $= (\alpha \pm er_1\cos\theta, \beta \pm er_2\sin\theta)$

foci $= \left(-1 \pm 2\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}, 1 \pm 2\cdot\frac{2}{\sqrt{3}}\cdot\frac{1}{\sqrt{2}}\right)$

$\therefore$ foci $= \left(-1 \pm \frac{2\sqrt{2}}{\sqrt{3}}, 1 \pm \frac{2\sqrt{2}}{\sqrt{3}}\right)$

4. **Determine the nature of the conic section represented by $5x^2 + 6xy + 5y^2 + 18x - 2y - 3 = 0$. Also, determine its eccentricity, major axis and foci.**

Solution

The given conic section is

$S := 5x^2 + 6xy + 5y^2 + 18x - 2y - 3 = 0$ ...(1)

Comparing equation (i) with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$a = 5, b = 5, c = -3, h = 3, g = 9, f = -1$

Now,

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 5 \times 5 \times (-3) + 2(-1)\,9 \times 3 - 5(-1)^2 - 5(9)^2 + 3(3)^2$

$= -75 - 54 - 5 - 405 + 27$

$= -512 \neq 0$

and, $h^2 - ab = (3)^2 - 5 \times 5 = -16 < 0$

So, the given conic section represents ellipse.

a. **For coordinate of centre**

Solving $\frac{\partial s}{\partial x} = 0$ and $\frac{\partial s}{\partial y} = 0$

From (i) $\frac{\partial s}{\partial x} = 10x + 6y + 18 = 0$ and $\frac{\partial s}{\partial y} = 6x + 10y - 2 = 0$

Solving, $x = -3, y = 2$

The centre of ellipse is $(\alpha, \beta) = (-3, 2)$

b. **Transferring the origin from (0, 0) to $(\alpha, \beta) = (-3,2)$**

Equation (1) can be reduced to

$5x^2 + 6xy + 5y^2 + c_1 = 0$ ...(2)

where, $c_1 = 9\alpha + f\beta + c$

$c_1 = 9(-3) + (-1)2 - 3$

$c_1 = -32$

Equation (2) becomes

$5x^2 + 6xy + 5y^2 - 32 = 0$

or, $5x^2 + 6xy + 5y^2 = 32$

or, $\frac{5}{32}x^2 + \frac{6}{32}xy + \frac{5}{32}y^2 = 1$ ...(3)

Which is of the form $AX^2 + 2HXY + BY^2 = 1$

Where, $A = \frac{A}{32}, B = \frac{5}{32}, H = \frac{3}{32}, X = x + 3, Y = y - 2$

c. **Length and equation of axes**

The length of axes are given by the roots of the equation

$(H^2 - AB)r^4 + (A + B)r^2 - 1 = 0$

or, $\left[\left(\frac{3}{32}\right)^2 - \frac{5}{32}\cdot\frac{5}{32}\right]r^4 + \left(\frac{5}{32} + \frac{5}{32}\right)r^2 - 1 = 0$

or, $\frac{-16}{1024}r^4 + \frac{10}{32}r^2 - 1 = 0$

or, $-16r^4 + 320r^2 - 1024 = 0$

or, $r^2 = 4, 16$

Here, $r_1^2 = 16, r_2^2 = 4$

Length of major axis = 4
Length of minor axis = 2
Equation of major axis is

$$\left(A-\frac{1}{r_1^2}\right)X+HY=0$$

or, $\left(\frac{5}{32}-\frac{1}{16}\right)X+\frac{3}{32}Y=0$

or, $\frac{3}{32}X+\frac{3}{32}Y=0$

or, $X+Y=0$

or, $x+3+y-2=0$

or, $x+y+1=0$

Slope of major axis $\tan\theta=-1$

then $\sin\theta=\frac{-1}{\sqrt{2}}$, $\cos\theta=\frac{1}{\sqrt{2}}$

d. **Eccentricity:** The eccentricity of ellipse is given by

$r_2^2=r_1^2(1-e^2)$

or, $4=16(1-e^2)$

or, $\frac{1}{4}=1-e^2$

or, $e^2=1-\frac{1}{4}$

or, $e^2=\frac{3}{4}$

$\therefore$ $e=\frac{\sqrt{3}}{2}$

e. **Foci:** The foci of ellipse are given by

$=(\alpha\pm er_1\cos\theta,\ \beta\pm er_1\sin\theta)$

$=\left(-3\pm\frac{\sqrt{3}}{2}\cdot\frac{32}{3}\cdot\frac{1}{\sqrt{2}},\ 2\pm\frac{2}{\sqrt{7}}\cdot\frac{32}{3}\cdot\left(\frac{-1}{\sqrt{2}}\right)\right)$

$=\left(-3\pm\frac{32\sqrt{2}}{3\sqrt{7}},\ 2\pm\left(\frac{-32\sqrt{2}}{3\sqrt{7}}\right)\right)$

5. **Determine the nature of conic sections and find the product of lengths semi axes of the conic $5x^2+6xy+5y^2+12x+4y-4=0$.**

Solution

Given conic is

$$S:=5x^2+6xy+5y^2+12x+4y-4=0 \qquad \ldots(1)$$

Comparing with $ax^2+2hxy+by^2+2gx+2fy+c=0$, we get

$a=5, b=5, c=-4, h=3, g=6, f=2$

Now, $\Delta=abc+2fgh-af^2-bg^2-ch^2$

$=5\times5\times(-4)+2\times2\times6\times3-5\times2^2-5\times6^2+4\times3^2$

$=-192\neq0$

Also, $h^2-ab=3^2-5\times5=-16<0$

So, the given conic is an ellipse.

For the coordinate of centre

a. Solving $\frac{\partial s}{\partial x}=0$ and $\frac{\partial s}{\partial y}=0$

From (1)

$\frac{\partial s}{\partial x}=10x+6y+12=0$ and $\frac{\partial s}{\partial y}=6x+10y+4=0$

Solving, we get

$x=\frac{-3}{2}, y=\frac{1}{2}$

$\therefore$ The centre of the ellipse is $(\alpha,\beta)=\left(-\frac{3}{2},\frac{1}{2}\right)$

b. **Transferring the origin from (0, 0) to $(\alpha,\beta)=\left(-\frac{3}{2},\frac{1}{2}\right)$**

Equation (1) can be reduced to

$5x^2+6xy+5y^2+c_1=0 \qquad \ldots(2)$

where, $c_1=g\alpha+f\beta+c$

$c_1=6\left(-\frac{3}{2}\right)+2\times\frac{1}{2}+(-4)$

$c_1=-12$

Equation (2) becomes

$5X^2+6XY+5Y^2-12=0$

or, $\frac{5}{12}X^2+\frac{6}{12}XY+\frac{5}{12}Y^2=1 \qquad \ldots(3)$

Which is of the form $AX^2+BXY+CY^2=1$

where, $A=\frac{5}{12}, B=\frac{5}{12}, H=\frac{3}{12}, X=x+\frac{3}{2}, Y=y-\frac{1}{2}$

c. **Length and equation of axes**

The length of axes are given by the roots of the equation

$(H^2-AB)r^4+(A+B)r^2-1=0$

$\left[\left(\frac{3}{12}\right)^2-\frac{5}{12}\cdot\frac{5}{12}\right]r^4+\left(\frac{5}{12}+\frac{5}{12}\right)r^2-1=0$

or, $\frac{-16}{144}r^4+\frac{10}{12}r^2-1=0$

or, $-16r^4+120r^2-144=0$

or, $r^2=\frac{3}{2}, 6$

$r_1^2=6, r_2^2=\frac{3}{2}$

Length of semi major axis $=\sqrt{6}$

Length of semi minor axis $=\sqrt{\frac{3}{2}}$

Product $=\sqrt{6}\times\frac{\sqrt{3}}{\sqrt{2}}=3$

Also, equation of major axis

$\left(A - \frac{1}{r_1^2}\right) X + HY = 0$

or, $\left(\frac{5}{12} - \frac{1}{6}\right) X + \frac{3}{12} Y = 0$

or, $\frac{3}{12} X + \frac{3}{12} Y = 0$

or, $X + Y = 0$

or, $x + \frac{3}{2} + y - \frac{1}{2} = 0$

or, $x + y + 1 = 0$

Slope of major axis $m = -1$, $\tan\theta = -1$

Then, $\sin\theta = \frac{1}{\sqrt{2}}$, $\cos\theta = \frac{1}{\sqrt{2}}$

d. **Eccentricity:** The eccentricity of the ellipse is given by:

$r_2^2 = r_1^2(1 - e^2)$

or, $\frac{3}{2} = 6(1 - e^2)$

or, $1 - e^2 = \frac{1}{4}$

or, $e^2 = 1 - \frac{1}{4} = \frac{3}{4}$

∴ $e = \frac{\sqrt{3}}{2}$

e. **Foci:** The foci of the ellipse are given by

$= (\alpha + er_1 \cos\theta, \beta \pm er_1 \sin\theta)$

$= \left(-\frac{3}{2} \pm \frac{\sqrt{3}}{2} \cdot 6 \cdot \frac{1}{\sqrt{2}}, \pm \frac{\sqrt{3}}{2} \cdot 6 \cdot \left(\frac{-1}{\sqrt{2}}\right)\right)$

$= \left(-\frac{3}{2} \pm \frac{3\sqrt{3}}{\sqrt{2}}, \frac{1}{2} \pm \left(\frac{-3\sqrt{3}}{\sqrt{2}}\right)\right)$

6. Describe and sketch the polar conic sections.

i. $r = \dfrac{10}{1 + \cos\theta}$

Solution

The given conic is $r = \frac{10}{1 + \cos\theta}$

Comparing with $r = \frac{ed}{1 + e\cos\theta}$, we get

$e = 1$ (parabola)

$ed = 10 \Rightarrow d = 10$

i.e., Equation of directrix lies 10 units to the right of the pole.

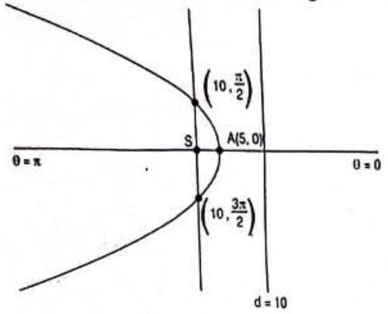
θ	0	π/2	π	3π/2
r	5	10	∞	10
(r, θ)	(5, 0)	(10, π/2)	(∞, π)	(10, 3π/2)

Focus: S(0, 0)

Vertex: A(5, 0)

Directrix: d = 10 to the right of the pole

The parabola with above information is sketched in figure.



ii. $r = \dfrac{10}{3 + 2\cos\theta}$

Solution

The given conic is $r = \frac{10}{3 + 2\cos\theta} = \frac{10/3}{1 + \frac{2}{3}\cos\theta}$

Comparing with $r = \frac{ed}{1 + e\cos\theta}$, we get

$e = \frac{2}{3} < 1$ (ellipse)

$ed = \frac{10}{3} \Rightarrow d = 5$

i.e., Equation of directrix lies 5 units to the right of the pole.

θ	0	π/2	π	3π/2
r	2	10/3	10	10/3
(r, θ)	(2, 0)	(10/3, π/2)	(10, π)	(10/3, 3π/2)

Vertices = A(2, 0) A'(10, π)

Centre = C(6, π)

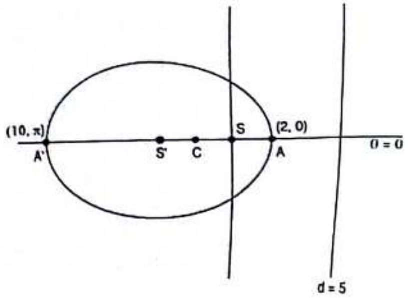
Foci = S(0, 0) S'(8, π)

Equation of directrix d = 5, d' = 8 + 5 = 13

Length of major axis = SA + SA' = 2 + 10 = 12

Length of semi major axis = $\frac{12}{2} = 6$

The parabola with above information is sketched in figure.



iii. $r = \dfrac{4}{1 + \cos\theta}$

Solution

The given conic is $r = \dfrac{4}{1 + \cos\theta}$

Comparing with $r = \dfrac{ed}{1 + e\cos\theta}$, we get

$e = 1$ (parabola)

$ed = 4 \Rightarrow d = 4$

i.e., Equation of directrix lies 4 units to the right of the pole.

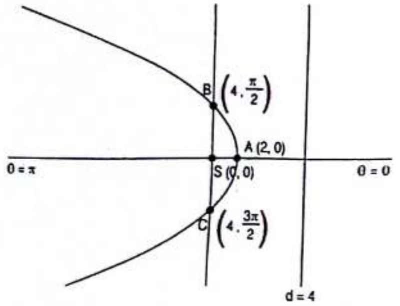
θ	0	π/2	π	3π/2
r	2	4	∞	4
(r, θ)	(2, 0)	(4, π/2)	(π, ∞)	(4, 3π/2)

Focus: S(0, 0)

Vertex: A(2, 0)

Directrix: d = 4 to the right of the pole.

The parabola with above information is sketched in figure.



iv. $r = \dfrac{6}{3 + \sin\theta}$

Solution

The given conic is $r = \dfrac{6}{3 + \sin\theta} = \dfrac{2}{1 + \frac{1}{3}\sin\theta}$

Comparing with $r = \dfrac{ed}{1 + e\cos\theta}$, we get

$e = \frac{1}{3} < 1$ (ellipse)

$ed = 2 \Rightarrow d = 6$

i.e., Equation of directrix lies 6 units to the right of the pole.

θ	0	π/2	π	3π/2
r	2	2/3	2	3
(r, θ)	(2, 0)	(2/3, π/2)	(2, π)	(3, 3π/2)

Vertices: A $\left(\frac{2}{3}, \frac{\pi}{2}\right)$ A' $\left(3, \frac{3\pi}{2}\right)$

Centre: C $\left(\frac{4}{3}, \frac{3\pi}{2}\right)$

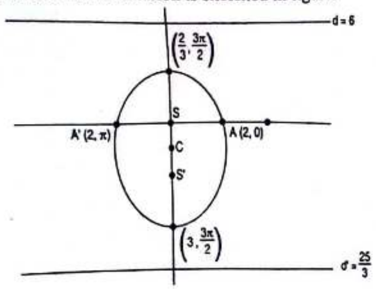
Foci: S (0, 0) S' $\left(\frac{7}{3}, \frac{3\pi}{2}\right)$

Equation of directrix d = 6, $d' = \frac{25}{3}$

Length of major axis = SA + SA' $= \frac{2}{3} + 3 = \frac{11}{3}$

Length of semi major axis $= \frac{11}{6}$

The parabola with above information is sketched in figure.



v. $r = \dfrac{7}{2 + 5\sin\theta}$

Solution

The given conic is $r = \dfrac{7}{2 + 5\sin\theta} = \dfrac{7/2}{1 + \frac{5}{2}\sin\theta}$

Comparing with $r = \dfrac{ed}{1 + e\cos\theta}$, we get

$e = \dfrac{5}{2}$ (hyperbola)

$ed = \dfrac{7}{2} \Rightarrow d = \dfrac{7}{5}$

i.e., Equation of directrix lies $\frac{7}{5}$ units above the pole.

θ	0	$\pi/2$	π	$3\pi/2$
r	7/2	1	7/2	−7/3
(r, θ)	(7/2, 0)	(1, π/2)	(7/2, π)	(7/3, 3π/2)

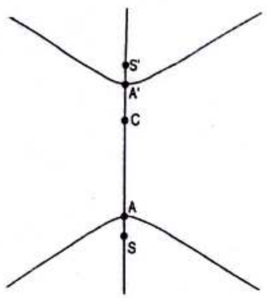
Vertices $A\left(1, \frac{\pi}{2}\right)$, $A' = \left(\frac{7}{3}, \frac{\pi}{2}\right)$

Foci = S(0, 0), $S'\left(\frac{10}{3}, \frac{\pi}{2}\right)$

Centre = $\left(\frac{5}{3}, \frac{\pi}{2}\right)$

Directrix, $d = \frac{7}{5}$, $d' = \frac{24}{15}$

The parabola with above information is sketched in figure.



Unit 7

Three Dimensional Geometry

Exercise 7.1

1. Find point of intersection of the line $x - 2y + 4z + 4 = 0, x + y + z - 8 = 0$ with plane $x - y + 2z + 1 = 0$.

Solution

First we change the given line

$$x - 2y + 4z + 4 = 0,\ x + y + z - 8 = 0 \quad \dots (i)$$

in the symmetrical. For this, we need the

(i) direction ratios of a line

(ii) co-ordinate of any point on it.

Let l, m, n be the direction cosines of a line (i). Since the line (i) lies on the both planes $x + 2y + 3z + 4 = 0$ and $x + y + z + 1 = 0$. So it is perpendicular to the normal of both the planes. So we have

$$l - 2m + 4n = 0 \quad \dots (ii)$$

$$l + m + n = 0 \quad \dots (iii)$$

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{l}{-2-4} = \frac{m}{4-1} = \frac{n}{1+2}$$

or, $\dfrac{l}{-6} = \dfrac{m}{3} = \dfrac{n}{3}$

or, $\dfrac{l}{-2} = \dfrac{m}{1} = \dfrac{n}{1} = k$ (say)

Thus, the direction ratios of the line (i) are −2, 1, 1.

To find the co-ordinates of any point on the line (i)

Suppose the line (i) meets the plane $z = 0$ at any point $(x_1, y_1, 0)$. Then,

$$x_1 - 2y_1 + 4 = 0 \quad \dots (iv)$$

$$x_1 + y_1 - 8 = 0 \quad \dots (v)$$

Solving (iv) and (v), we get

$$\frac{x_1}{16-4} = \frac{y_1}{4+8} = \frac{1}{1+2}$$

or, $\frac{x_1}{12} = \frac{y_1}{12} = \frac{1}{3}$

$\therefore$ $x_1 = 4$ and $y_1 = 4$

Thus the line (i) meets the plane $z = 0$ at point (4, 4, 0).

So the equation of line in symmetrical form is

$$\frac{x-4}{-2} = \frac{y-4}{1} = \frac{z-0}{1}$$

Let $\frac{x-4}{-2} = \frac{y-4}{1} = \frac{z-0}{1} = r$ (say) .... (vi)

Then, the co-ordinates of any point on this line are

$(-2r+4,\ r+4, r)$ .....(*).

If the line (i) meets the plane $x - y + 2z + 1 = 0$ at point (*), then

$-2r + 4 - (r+4) + 2(r) + 1 = 0$

or, $-2r + 4 - r - 4 + 2r + 1 = 0$

or, $-r + 1 = 0$

$\therefore$ $r = 1$

Substituting the value of r in (*), we get (2, 5, 1).

Hence, the point of intersection is (2, 5, 1).

2. Find the coordinates of the point of intersection of line $\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-2}{-2}$ with the plane $3x + 4y + 5z = 5$.

Solution

The given line is

$$\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-2}{-2} = r \text{ (say)} \quad \text{.... (i)}$$

Then, the co-ordinates of any point on this line are

$(r-1, 3r-3, -2r+2)$ .... (*)

If the line (i) meets the plane $3x + 4y + 5z - 5 = 0$ at point (*), then

$3(r-1) + 4(3r-3) + 5(-2r+2) - 5 = 0$

or, $3r - 3 + 12r - 12 - 10r + 10 - 5 = 0$

or, $r = 2$

Now, substituting the value of r in (*), we get

$(2-1, 6-3, -4+2)$

i.e., (1, 3, −2)

Hence, the line (i) meets the plane $3x + 4y + 5z - 5 = 0$ at point (1, 3, −2).

3. Find k so that the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ may be perpendicular to each other.

Solution:

The given lines are

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2} \quad \text{.... (i)}$$

and $$\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5} \quad \text{.... (ii)}$$

The direction ratios of line (i) and (ii) are $-3, 2k, 2$ and $3k, 1, -5$ respectively.

If the line (i) and (ii) are perpendicular to each other, then applying the condition of perpendicularity, we have

$-3.3k + 2k.1 + 2.(-5) = 0$

$-9k + 2k - 10 = 0$

or, $-7k = 10$

or, $k = -\frac{10}{7}$.

4. Find the point where the line joining (2, 1, 3) and (4, −2, 5) cuts the plane $2x + y - z - 3 = 0$.

Solution:

The equation of the line passing through (2, 1, 3) and (4, −2, 5) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

i.e., $\frac{x-2}{4-2} = \frac{y-1}{-2-1} = \frac{z-3}{5-3}$

i.e., $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z-3}{2} = r$ (say) .... (i)

Then, the co-ordinate of any point on this line are

$(2r+2, -3r+1, 2r+3)$ .... (*)

Suppose the line (i) cuts the plane $2x + y - z - 3 = 0$ at point (*), then

$2(2r+2) - 3r + 1 - (2r+3) - 3 = 0$

or, $4r + 4 - 3r + 1 - 2r - 3 - 3 = 0$

or, $-r - 1 = 0$

or, $r = -1$

Thus, the required point of intersection of line (i) and plane (ii) is

$(2r+1, -3r+1, 2r+3)$ i.e., (0, 4, 1).

5. Find the point where the line joining (1, 2, 3) and (3, −1, 4) cuts the plane $3x + 2y + z - 2 = 0$.

Solution:

The equation of straight line joining the points (1, 2, 3) and (3, −1, 4) is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

or, $\frac{x-1}{3-1} = \frac{y-2}{-1-2} = \frac{z-3}{4-3}$

or, $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z-3}{1} = r$ (say) .... (i)

Then, the co-ordinate of any point on this line are

$(2r+1, 2-3r, r+3)$ .... (*)

If the line (i) meets the plane $3x + 2y + z - 2 = 0$ at point (*), then

$3.(2r+1) + 2.(2-3r) + r + 3 - 2 = 0$

or, $6r + 3 + 4 - 6r + r + 3 - 2 = 0$

or, $r = -8$

Substituting the value of r in (*), we get (−15, 26, −5)

Thus, the line joining the points (1, 2, 3) and (3, −1, 4) cuts the plane $3x + 2y + z - 2 = 0$ at point (−15, 26, −5).

6. Find the distance from the point (–1, –5, –10) to the point where the line $\frac{x-2}{2}=\frac{y+1}{4}=\frac{z-2}{12}$ meets the plane $x-y+z=5$.

Solution:

The given line is

$$\frac{x-2}{2}=\frac{y+1}{4}=\frac{z-2}{12}=r\text{ (say)} \qquad \ldots\text{(i)}$$

Then, the co-ordinate of any point on this line are

$$(2r+2,\ 4r-1,\ 12r+2) \qquad \ldots(*)$$

If the line (i) meets the plane $x-y+z=5$ at point (*), then

$2r+2-(4r-1)+12r+2=5$

or, $2r+2-4r+1+12r+2=5$

or, $10r+5=5$

or, $r=0$

Substituting the value of r in (*), we get

i.e., $(2,-1,2)$

Now, the distance from the point $(-1,-5,-10)$ to $(2,-1,2)$

$=\sqrt{(2+1)^2+(-1+5)+(2+10)^2}$

$=\sqrt{3^2+4^2+12^2}$

$=\sqrt{9+16+144}=\sqrt{169}=13$

7. Find the two points on the line $\frac{x-2}{1}=\frac{y+3}{2}=\frac{z+5}{2}$ on either side of $(2,-3,-5)$ and at a distance 3 from it.

Solution:

The given line is

$$\frac{x-2}{1}=\frac{y+3}{2}=\frac{z+5}{2}=r\text{ (say)} \qquad \ldots\text{(i)}$$

Then, the co-ordinates of any point on this line are

$(r+2,\ 2r-3,\ 2r-5)$ .... (*)

If the distance of the point (*) from $(2,-3,-5)$ is 3. Then

$3=\sqrt{(r+2-2)^2+(2r-3+3)^2+(2r-5+5)^2}$

or, $r^2+4r^2+4r^2=9$

or, $9r^2=9$

or, $r=\pm 1$

Substituting the value of r in (*), we get

$(1+2,\ 2-3,\ 2-5)$ and $[-1+2,\ 2.(-1)-3,\ 2.(-1)-5]$

i.e., $(3,-1,-3)$ and $(1,-5,-7)$.

8. Find the distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured along a line with direction cosines proportional to 2, 3, –6.

Solution:

The equation of the line through $(1,-2,3)$ and parallel to the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-6}$ is

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-6}=r\text{ (say)} \qquad \ldots\text{(i)}$$

Then, the co-ordinates of any point on this line are

$$(2r+1,\ 3r-2,\ -6r+3) \qquad \ldots(*)$$

If the line (i) meets the plane $x-y+z=5$ at point (*), then

$2r+1-(3r-2)-6r+3=5$

or, $2r+1-3r+2-6r+3=5$

or, $7r=1$

$\therefore$ $r=\frac{1}{7}$

Substituting the value of r in (*), we get

$$\left(2.\frac{1}{7}+1,\ 3.\frac{1}{7}-2,\ -6.\frac{1}{7}+3\right)$$

i.e., $\left(\frac{2+7}{7},\frac{3-14}{7},\frac{-6+21}{7}\right)$

i.e., $\left(\frac{9}{7},-\frac{11}{7},\frac{15}{7}\right)$

Now, the distance of the point $\left(\frac{9}{7},-\frac{11}{7},\frac{15}{7}\right)$ from the point $(1,-2,3)$ is

$$d=\sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{-11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$=\sqrt{\left(\frac{9-7}{7}\right)^2+\left(\frac{-11+14}{7}\right)^2+\left(\frac{15-21}{7}\right)^2}$$

$$=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2}$$

$$=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=\sqrt{\frac{49}{49}}=1$$

9. Find the equation to the line through (–1, 3, 2) and perpendicular to the plane $x+2y+2z=3$. Also find length of perpendicular and coordinate of foot of perpendicular.

Solution:

The equation of line through $(-1,3,2)$ and perpendicular to the plane $x+2y+2z=3$ is $\frac{x+1}{1}=\frac{y-3}{2}=\frac{z-2}{2}$.

Let $\frac{x+1}{1}=\frac{y-3}{2}=\frac{z-2}{2}=r$ (say) $\qquad \ldots$(i)

Then, the co-ordinates of any point on this line are

$(r-1,\ 2r+3,\ 2r+2)$ $\qquad \ldots(*)$

Suppose line (i) meets the plane $x+2y+2z=3$ at point (*).

Then,

$r-1+2(2r+3)+2(2r+2)=3$

or, $r-1+4r+6+4r+4=3$

or, $9r+9=3$

or, $r=-\frac{2}{3}$

Hence, the coordinates of foot of the perpendicular is $\left(-\frac{5}{3}, \frac{5}{3}, \frac{2}{3}\right)$

Now, length of the perpendicular is

$$= \sqrt{\left(-\frac{5}{3}+1\right)^2 + \left(\frac{5}{3}-3\right)^2 + \left(\frac{2}{3}-2\right)^2}$$

$$= \sqrt{\left(\frac{-5+3}{3}\right)^2 + \left(\frac{5-9}{3}\right)^2 + \left(\frac{2-6}{3}\right)^2}$$

$$= \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{-4}{3}\right)^2 + \left(-\frac{4}{3}\right)^2}$$

$$= \sqrt{\frac{4}{9} + \frac{16}{9} + \frac{16}{9}}$$

$$= \sqrt{\frac{36}{9}}$$

$$= \sqrt{4}$$

$$= 2$$

10. Find the length of the perpendicular of the point (1, 1, 1) from the line $\frac{x-2}{3} = \frac{y+2}{2} = \frac{z}{-1}$. Also, find the coordinate of the foot of the perpendicular and equation of the perpendicular.

Solution:

The given line is

$$\frac{x-2}{3} = \frac{y+2}{2} = \frac{z}{-1} = r \text{ (say)} \quad \ldots\ldots \text{(i)}$$

Then, the co-ordinates of any point on this line are

$(3r+2, 2r-2, -r)$ ..... (*).

Also, the direction ratios of the line (i) are 3, 2, –1.

Let Q be the foot of the perpendicular from the point $P(1, 1, 1)$ on the line (i). Then, the co-ordinates of Q is of the form $(3r+2, 2r-2, -r)$. The direction ratios of PQ are

$3r+2-1, 2r-2-1, -r-1$.

i.e., $3r+1, 2r-3, -r-1$

Since, PQ is perpendicular to the line (i). So we have

$(3r+3).3 + (2r-3).2 + (-r-1).(-1) = 0$

or, $9r+9+4r-6+r+1=0$

or, $14r+4=0$

or, $r = \frac{2}{7}$

Hence, the coordinates of Q is $\left(\frac{20}{7}, -\frac{10}{7}, -\frac{2}{7}\right)$

Now, length of PQ

$$= \sqrt{\left(\frac{20}{7}-1\right)^2 + \left(\frac{-10}{7}-1\right)^2 + \left(\frac{-2}{7}-1\right)^2}$$

$$= \sqrt{\left(\frac{20-7}{7}\right)^2 + \left(\frac{-10-7}{7}\right)^2 + \left(\frac{-2-7}{7}\right)^2}$$

$$= \sqrt{\left(\frac{13}{7}\right)^2 + \left(\frac{-17}{7}\right)^2 + \left(-\frac{9}{7}\right)^2}$$

$$= \sqrt{\frac{169}{49} + \frac{289}{49} + \frac{81}{49}}$$

$$= \sqrt{\frac{539}{49}}$$

$$= \sqrt{11}$$

Again, the equation of the line PQ is

$$\frac{x-1}{20-7} = \frac{y-1}{-10-7} = \frac{z-1}{-2-7}$$

$$\frac{x-1}{13} = \frac{y-1}{-17} = \frac{z-1}{-9}.$$

11. Reduce the equation of the line

$x+2y+z=3$, $6x+8y+3z=13$ in the symmetrical form.

Solution:

To change the given line

$$x+2y+z=3,\ 6x+8y+3z=13 \quad \ldots \text{(i)}$$

in the symmetrical form we need

(i) direction ratios of a line

(ii) co-ordinate of any point on it.

Let l, m, n be the direction cosines of a line (i). Since the line (i) lies on the both planes $x+2y+z=3$ and $6x+8y+3z=13$. So it is perpendicular to the normal of both the planes. So we have

$$l+2m+n=0 \quad \ldots \text{(ii)}$$

$$6l+8m+3n=0 \quad \ldots \text{(iii)}$$

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{l}{6-8} = \frac{m}{6-3} = \frac{n}{8-12}$$

or, $\frac{l}{-2} = \frac{m}{3} = \frac{n}{-4}$

or, $\frac{l}{2} = \frac{m}{-3} = \frac{n}{4} = k$ (say)

Thus, the direction ratios of the line (i) are 2, –3, 4.

To find the co-ordinates of any point on the line (i)

Suppose the line (i) meets the plane $z=0$ at any point $(x_1, y_1, 0)$. Then,

$$x_1 + 2y_1 - 3 = 0 \quad \ldots \text{(iv)}$$

$$6x_1 + 8y_1 - 13 = 0 \quad \ldots \text{(v)}$$

Solving (iv) and (v), we get

$$\frac{x_1}{-26+24} = \frac{y_1}{-18+13} = \frac{1}{8-12}$$

or, $\frac{x_1}{-2} = \frac{y_1}{-5} = \frac{1}{-4}$

$\therefore \quad x_1 = \frac{1}{2}$

and $y_1 = \frac{5}{4}$

Thus the line (i) meets the plane $z = 0$ at point $\left(\frac{1}{2}, \frac{5}{4}, 0\right)$

Hence, the equation of line in symmetrical form is

$$\frac{x - \frac{1}{2}}{2} = \frac{y - \frac{5}{4}}{-3} = \frac{z - 0}{4}$$

or, $\frac{2x - 1}{4} = \frac{4y - 5}{-12} = \frac{z}{4}$.

12. Find the co-ordinate of the foot of the perpendicular from the origin on the straight line given by $x + 2y + 3z + 4 = 0$, $x + y + z + 1 = 0$.

Solution:

First we change the given line

$x + 2y + 3z + 4 = 0, \; x + y + z + 1 = 0$ .... (i)

in the symmetrical. For this, we need the

(i) direction ratios of a line

(ii) co-ordinate of any point on it.

Let l, m, n be the direction cosines of a line (i). Since the line (i) lies on the both planes $x + 2y + 3z + 4 = 0$ and $x + y + z + 1 = 0$. So it is perpendicular to the normal of both the planes. So we have

$l + 2m + 3n = 0$ .... (ii)

$l + m + n = 0$ .... (iii)

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{l}{2-3} = \frac{m}{3-1} = \frac{n}{1-2}$$

or, $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

or, $\frac{l}{1} = \frac{m}{-2} = \frac{n}{1} = k$ (say)

Thus, the direction ratios of the line (i) are $1, -2, 1$.

To find the co-ordinates of any point on the line (i)

Suppose the line (i) meets the plane $z = 0$ at any point $(x_1, y_1, 0)$. Then,

$x_1 + 2y_1 + 4 = 0$ .... (iv)

$x_1 + y_1 + 1 = 0$ .... (v)

Solving (iv) and (v), we get

$$\frac{x_1}{2-4} = \frac{y_1}{4-1} = \frac{1}{1-2}$$

or, $\frac{x_1}{-2} = \frac{y_1}{3} = \frac{1}{-1}$

$\therefore \quad x_1 = 2$ and $y_1 = -3$

Thus the line (i) meets the plane $z = 0$ at point $(2, -3, 0)$.

So the equation of line in symmetrical form is

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z-0}{1}$$

Let $\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z-0}{1} = r$ (say) .... (vi)

Then, the co-ordinates of any point on this line are

$(r + 2, -2r - 3, r)$ ..... (*).

Also, the direction ratios of the line (vi) are $1, -2, 1$.

Let Q be the foot of the perpendicular from $O(0, 0, 0)$ on the given line. Then, the co-ordinates of Q is of the form $(r + 2, -2r - 3, r)$. The direction ratios of OQ are $r + 2 - 0, -2r - 3 - 0, r - 0$.

i.e., $r + 2, -2r - 3, r$.

Since, OQ is perpendicular to the line (i). So we have

$(r + 2).\, 1 + (-2r - 3).\, (-2) + r\,.\, 1 = 0$

or, $r + 2 + 4r + 6 + r = 0$

or, $6r + 8 = 0$.

or, $r = -\frac{4}{3}$

Substituting the value of r in (*), we get $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$.

Hence, the coordinates of foot of the perpendicular from origin on the line (i) is $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$.

13. Find the equation of the line through the point (1, 2, 3) parallel to the line $x - y + 2z = 5, 3x + y + z = 6$.

Solution:

The equation of the line through the point $(1, 2, 3)$ is

$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z-3}{c}$ .... (i)

where a, b, c are the direction ratios of the line (i).

If the line (i) is parallel to the line $x - y + 2z = 5$, $3x + y + z = 6$

then, the line (i) is parallel to plane $x - y + 2z = 5$ and $3x + y + z = 6$. So the normal to the plane is perpendicular to the line (i). Hence, applying the condition of perpendicularity, we get

$a - b + 2c = 0$ .... (i)

$3a + b + c = 0$ .... (iii)

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{-1-2} = \frac{b}{6-1} = \frac{c}{1+3}$$

or, $\frac{a}{-3} = \frac{b}{5} = \frac{c}{4}$

or, $\frac{a}{-3} = \frac{b}{5} = \frac{c}{4} = k$ (say)

$\therefore \quad a = -3k, b = 5k, c = 4k$

Substituting the values of a, b, c in equation (i), we get

$$\frac{x-1}{-3k} = \frac{y-2}{5k} = \frac{z-3}{4k}$$

or, $\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$

which is the required equation of the line.

14. Find the angle between the lines in which the plane $x - y + z = 5$ is cut by the planes $2x + y - z = 3$ and $2x + 2y + 3z - 1 = 0$.

Solution:

Let a_1, b_1, c_1 and a_2, b_2, c_2 be the direction ratios of line of intersection with the planes $x - y + z = 5$, $2x + y - z = 3$ and $x - y + z = 5$, $2x + 2y + 3z - 1 = 0$ respectively. Then, the line of intersection will be perpendicular to normal of both the planes. Hence, applying the condition of perpendicularity, we get

$$a_1 - b_1 + c_1 = 0 \quad \ldots \text{(iii)}$$

$$2a_1 + b_1 - c_1 = 0 \quad \ldots \text{(iv)}$$

Solving (iii) and (iv) by cross multiplication, we get

$$\frac{a_1}{1-1} = \frac{b_1}{2+1} = \frac{c_1}{1+2}$$

or, $\frac{a_1}{0} = \frac{b_1}{3} = \frac{c_1}{3}$

Thus, the direction ratios of line (i) are 0, 3, 3.

Again, applying the condition of perpendicularity, we get

$$a_1 - b_1 + c_1 = 0 \quad \ldots \text{(v)}$$

$$2a_2 + 2b_2 + 3c_2 = 0 \quad \ldots \text{(vi)}$$

Solving (v) and (vi) by cross multiplication, we get

$$\frac{a_2}{-3-2} = \frac{b_2}{2-3} = \frac{c_2}{2+2}$$

or, $\frac{a_2}{-5} = \frac{b_2}{-1} = \frac{c_2}{4}$

or, $\frac{a_2}{5} = \frac{b_2}{1} = \frac{c_2}{-4}$

Thus, the direction ratios of line (ii) are 5, 1, – 4.

Let θ be the angle between the line of intersection, then

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{0.5 + 3.1 + 3.(-4)}{\sqrt{0^2 + 3^2 + 3^2}.\sqrt{5^2 + 1^2 + (-4)^2}}$$

$$= \frac{0 + 3 - 12}{\sqrt{18}\sqrt{42}} = \frac{-9}{3\sqrt{2}\sqrt{42}} = \frac{-3}{2\sqrt{21}}$$

$\therefore \quad \theta = \cos^{-1}\left(\frac{-3}{2\sqrt{21}}\right)$.

15. Prove that the lines $x = -2y + 7$, $z = 3y + 10$ and $x = 5y - 1$, $z = 3y - 6$ are perpendicular to each other.

Solution:

The given first line is

$x = -2y + 7, z = 3y + 10$

or, $\frac{x-7}{-2} = y, \frac{z-10}{3} = y$

or, $\frac{x-7}{-2} = \frac{y-0}{1} = \frac{z-10}{3} \quad \ldots \text{(i)}$

Thus, the direction ratios of the line (i) are –2, 1, 3.

Also, the given second line is

$x = 5y - 1, z = 3y - 6$

or, $\frac{x+1}{5} = y, \frac{z+6}{3} = y$

or, $\frac{x+1}{5} = \frac{y-0}{1} = \frac{z-6}{3} \quad \ldots \text{(ii)}$

Thus, the direction ratios of line (ii) are 5, 1, 3.

If the line (i) and (ii) are perpendicular to each other, then

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

or, $(-2)(5) + (1)(1) + (3)(3) = 0$

or, $-10 + 1 + 9 = 0$

or, $-10 + 10 = 0$

or, $0 = 0$, which is true.

Hence, the lines are coplanar.

16. Prove that the lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$

Solution:

The given first line is

$x = ay + b, z = cy + d$

or, $\frac{x-b}{a} = y, \frac{z-d}{c} = y$

or, $\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c} \quad \ldots \text{(i)}$

Thus, the direction ratios of the line (i) are a, 1, c.

Also, the given second line is

$x = a'y + b', z = c'y + d'$

or, $\frac{x-b'}{a'} = y, \frac{z-d'}{c'} = y$

or, $\frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'} \quad \ldots \text{(ii)}$

Thus, the direction ratios of line (ii) are a', 1, c'.

If the line (i) and (ii) are perpendicular to each other, then

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore \quad aa' + 1.1 + cc' = 0$

or, $\quad aa' + 1 + cc' = 0$

17. Find a, b, c, d so that the lines $x = ay + b, z = cy + d$ may pass through the given points (3, 2, – 4), (5, 4, – 6) and hence show that (3, 2, – 4), (5, 4, – 6) and (9, 8 – 10) are collinear.

Solution:

The given line is

$x = ay + b, z = cy + d$

or, $\frac{x-b}{a} = y, \frac{z-d}{c} = y$

or, $\frac{x-b}{a} = \frac{y}{1} = \frac{z-d}{c} \quad \ldots \text{(i)}$

If the line (i) passes through the points (3, 2, – 4), then

$$\frac{3-b}{a}=\frac{2}{1}=\frac{-4-d}{c}$$

Taking first two ratios, we get

$$\frac{3-b}{a}=\frac{2}{1}$$

or, $2a+b=3$ ....(ii)

Taking last two ratios, we get

$$\frac{-4-d}{c}=\frac{2}{1}$$

or, $2c+d=-4$ ....(iii)

If the line (i) passes through the points (5, 4, – 6), then

$$\frac{5-b}{a}=\frac{4}{1}=\frac{-6-d}{c}$$

Taking first two ratios, we get

$$\frac{5-b}{a}=\frac{4}{1}$$

or, $4a+b=5$ ....(iv)

Taking last two ratios, we get

$$\frac{-6-d}{c}=\frac{4}{1}$$

or, $4c+d=-6$ ....(v)

Solving (ii) and (iv), we get

∴ $a=1$ and $b=1$

Solving (iii) and (v), we get

$c=-1$ and $d=-2$

Thus, $a=1$, $b=1$, $c=-1$ and $d=-2$

Substituting the values of a, b, c, d in equation (i), we get

$$\frac{x-1}{1}=\frac{y}{1}=\frac{z+2}{-1} \quad \text{....(vi)}$$

which is the equation of straight line passing through the given point.

If the points (3, 2, – 4), (5, 4, – 6) and (9, 8, – 10) are collinear then the point (9, 8, – 10) must satisfy equation (vi).

i.e., $\frac{9-1}{1}=\frac{8}{1}=\frac{-10+2}{-1}$

i.e., $\frac{8}{1}=\frac{8}{1}=\frac{-8}{-1}$

i.e., $8=8=8$

Hence, the points (3, 2, – 4), (5, 4, – 6) and (9, 8, – 10) are collinear.

Exercise 7.2

1. Prove that the line $\frac{x+1}{-2}=\frac{y+2}{3}=\frac{z+5}{4}$ lies in the plane $x+2y-z=0$.

Solution:

The given line and plane are

$$\frac{x+1}{-2}=\frac{y+2}{3}=\frac{z+5}{4} \quad \text{.... (i)}$$

and $x+2y-z=0$ .... (ii) respectively.

The line (i) passes through (–1, –2, –5) and the direction ratios of the line (i) are –2, 3, 4. Also the direction ratios of line normal to the plane (ii) are 1, 2, –1.

If the line (i) lies in the plane (ii), then

$a_1a_2+b_1b_2+c_1c_2=0$ and

$ax_1+by_1+cz_1+d=0$

Now,

$a_1a_2+b_1b_2+c_1c_2=0$

or, $1(-2)+2(3)-1(4)=0$

or, $-2+6-4=0$

or, $0=0$

and the equation (ii) becomes

$ax_1+by_1+cz_1+d=0$

or, $1(-1)+2(-2)-1(-5)=0$

or, $-1-4+5=0$

or, $0=0$

Hence, the line (i) lies in the plane (ii).

2. Prove that the line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is parallel to the plane $2x+y-2z=0$.

Solution:

The given line and plane are

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} \quad \text{....(i)}$$

and $2x+y-2z=0$ ....(ii) respectively.

The direction ratios of the line (i) and line normal to the plane (ii) are 3, 4, 5 and 2, 1, – 2 respectively.

If the line (i) and plane (ii) are parallel, then the line normal to the plane (ii) is perpendicular to the line (i).

Hence, applying the condition of perpendicularity, we get

$a_1a_2+b_1b_2+c_1c_2=0$

or, $3.2+4.1+5.(-2)=0$

or, $6+4-10=0$

or, $10-10=0$

or, $0=0$ which is true.

Hence, line (i) and plane (ii) are parallel.

3. **Find the equation of the plane through the points (1, 0, −1), (3, 2, 2) and parallel to the line $x-1=\frac{1-y}{2}=\frac{z-2}{3}$.**

Solution:

The given line is

$$\frac{x-1}{1}=\frac{1-y}{2}=\frac{z-2}{3}$$

or, $$\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3} \quad \ldots (i)$$

The equation of plane through the point (1, 0, − 1) is

$$a(x-1)+b(y-0)+c(z+1)=0 \quad \ldots (ii)$$

where a, b, c are the direction ratios of the line normal to the plane (ii).

If the plane (ii) passes through the point (3, 2, 2), then

$$a(3-1)+b(2-0)+c(2+1)=0$$

or, $$2a+2b+3c=0 \quad \ldots (iii)$$

If the plane (ii) is parallel to the line (i), then the line normal to the plane (ii) will be perpendicular to the line (i). Hence, applying the condition of perpendicularity, we get

$$a-2b+3c=0 \quad \ldots (iv)$$

Solving (iii) and (iv) by cross multiplication, we get

$$\frac{a}{6+6}=\frac{b}{3-6}=\frac{c}{-4-2}$$

or, $$\frac{a}{12}=\frac{b}{-3}=\frac{c}{-6}$$

or, $$\frac{a}{4}=\frac{b}{-1}=\frac{c}{-2}=k \text{ (say)}$$

$\therefore$ $a=4k, b=-k, c=-2k$

Substituting the values of a, b, c in equation (ii), we get

$$4k(x-1)-k(y-0)-2k(z+1)=0$$

or, $4x-1-y-2z-2=0$

or, $4x-y-2z=6$

which is the required equation of the plane.

4. **Find the equation of line through the point (1, −2, 3 and parallel to the planes $2x+3y+4z=5$ and $3x+4y+5z=6$.**

Solution:

The equation of the line through the point (1, − 2, 3) is

$$\frac{x-1}{a}=\frac{y+2}{b}=\frac{z-3}{c} \quad \ldots (i)$$

where a, b, c are the direction ratios of the line (i).

If the line (i) is parallel to the planes $2x+3y+4z=5$ and $3x+4y+5z=6$, then the line normal to the planes $2x+3y+4z=5$ and $3x+4y+5z=6$ will be perpendicular to the line (i). Hence applying the condition of perpendicularity, we get

$$2a+3b+4c=0 \quad \ldots (ii)$$

$$3a+4b+5c=0 \quad \ldots (iii)$$

Solving (ii) and (iii) by cross multiplication, we get

$$\frac{a}{15-16}=\frac{b}{12-10}=\frac{c}{8-9}$$

or, $$\frac{a}{-1}=\frac{b}{2}=\frac{c}{-1}$$

or, $$\frac{a}{1}=\frac{b}{-2}=\frac{c}{1}=k \text{ (say)}$$

$\therefore$ $a=k, b=-2k, c=k$

Substituting the values of a, b, c in equation (i), we get

$$\frac{x-1}{k}=\frac{y+2}{-2k}=\frac{z-3}{k}$$

or, $$\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-3}{1}$$

which is the required equation of the line.

5. **Show that the lines $\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3}$ and $\frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$ are coplanar. Also, find the equation of the plane containing them.**

Solution:

The given lines are

$$\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3} \quad \ldots (i)$$

and $$\frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1} \quad \ldots (ii)$$

The given lines (i) and (ii) will be coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

or, $$\begin{vmatrix} -1-3 & -1-5 & -1+7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix}=0$$

or, $$\begin{vmatrix} -4 & -6 & 6 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix}=0$$

or, $-4(-3+15)-6(-12+2)+6(10-12)=0$

or, $-48+60-12=0$

or, $0=0$ which is true.

Hence, the given lines are coplanar.

The equation of plane containing the line (i) and (ii) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

or, $\begin{vmatrix} x-3 & y-5 & z+7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0$

or, $(x-3)(-3+15) + (y-5)(-12+2) + (z+7)(10-12) = 0$

or, $12(x-3) - 10(y-5) - 2(z+7) = 0$

or, $12x - 36 - 10y + 50 - 2z - 14 = 0$

or, $6x - 5y - z = 0$

which is the required equation of the plane.

6. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Also, find the equation of the plane containing them.

Solution:

The given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{.... (i)}$$

and $$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \quad \text{.... (ii)}$$

The given lines (i) and (ii) will be coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or, $\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

or, $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

or, $1(15-16) + 1(12-10) + 1(8-9) = 0$

or, $-1 + 2 - 1 = 0$

or, $0 = 0$ which is true.

Hence, the given lines are coplanar.

The equation of plane containing the line (i) and (ii) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or, $\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$

or, $(x-1)(15-16) + (y-2)(12-10) + (z-3)(8-9) = 0$

or, $-(x-1) + 2(y-2) - (z-3) = 0$

or, $x - 2y + z = 0$,

which is the required equation of the plane.

7. Prove that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{2} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Find also their point of intersection and the plane through them.

Solution

Here, let $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = r$ and $\frac{x-2}{2} = \frac{y-4}{3} = \frac{z-6}{5} = r'$

$\therefore$ The coordinates of any point on these lines are $(3r-1, 5r-3, 7r-5)$ and $(2r'+2, 3r'+4, 4r'+6)$

$\therefore$ $3r - 1 = 2r' + 2$

$3r - 2r' = 3$ ...(1)

and $5r - 3 = 3r' + 4$

$5r - 3r' = 7$ ...(2)

Multiplying equation (1) by 3 and equation (2) by 2, we get

$9r - 6r' = 9$

$10r - 6r' = 14$

$-\quad +\quad -$

$-r = -5$

$\therefore$ $r = 5$

From equation (1)

$3 \times 5 - 2r' = 3$

or, $2r' = 15 - 3$

$\therefore$ $r' = 6$

The point of intersection is (14, 22, 30).

Now, equation of plane through them is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

or, $(x+1)(20-21) - (y+3)(12-14) + (z+5)(9-10) = 0$

or, $-x - 1 + 2y + 6 - z - 5 = 0$

or, $-x + 2y - z = 0$

$\therefore$ $x - 2y + z = 0$ is required equation of the plane.

8. Show that the lines $\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2}$ and $3x + 2y + z - 2 = 0 = x - 3y + 2z - 13$ are coplanar and find the equation to the plane in which they lie.

Solution:

Let the given lines are

$$\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2} \quad \text{.... (i)}$$

and $3x + 2y + z - 2 = 0 = x - 3y + 2z - 13$ .... (ii)

Let $\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2} = r$ (say)

Then, the co-ordinate of any point on this line are $(3r-5, r-4, -2r+7)$

The lines (i) and (ii) will be coplanar if they intersect at a point. Since line (ii) is obtained by the intersection of two planes

$3x + 2y + z - 2 =$ and $x - 3y + 2z - 13 = 0$.

If the equation of given lines (i) and (ii) intersect, then there must be a common p... between the line (i) and the plane

$3x + 2y + z - 2 = 0.$

So the point $(3r - 5, r - 4, -2r + 7)$ lies on the plane

$3x + 2y + z - 2 = 0.$

or, $3(3r - 5) + 2(r - 4) + (-2r + 7) - 2 = 0$

or, $9r - 15 + 2r - 8 - 2r + 7 - 2 = 0$

or, $9r - 18 = 0$

$\therefore$ $r = 2$

Hence, the line (i) meets the plane $3x + 2y + z - 2 = 0$ at

$(6 - 5, 2 - 4, -4 + 7) = (1, -2, 3)$

The point $(1, -2, 3)$ clearly satisfies the another plane of line (ii) i.e., $x - 3y + 2z - 13 = 0$.

Thus, the two lines (i) and (ii) intersect at point $(1, -2, 3)$.

Hence, they are coplanar.

The equation of plane containing the line (ii) is

$(3x + 2y + z - 2) + k(x - 3y + 2z - 13) = 0$ .... (iii)

As lines (i) and (ii) are coplanar, this plane must contain (i) i.e., the point $(-5, -4, 7)$ in (i) must satisfy (iii).

$\therefore$ $(3(-5) + 2(-4) + 7 - 2) + k(-5 - 3(-4) + 2(7) - 13) = 0$

or, $-18 + 8k = 0$

or, $8k = 18$

or, $k = \frac{9}{4}$

The equation of plane containing the given lines from (iii) is

$(3x + 2y + z - 2) + \frac{9}{4}(x - 3y + 2z - 13) = 0$

or, $4(3x + 2y + z - 2) + 9(x - 3y + 2z - 13) = 0$

or, $12x + 8y + 4z - 8 + 9x - 27y + 18z - 117 = 0$

or, $21x - 19y + 22z - 125 = 0$

Exercise 7.3

1. **Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$. Also find the equation of the shortest distance.**

Solution:

The given lines are

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ .... (i)

and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ .... (ii)

The equation of plane containing the line (i) and parallel to line (ii) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

or, $(x - 1)(15 - 16) + (y - 2)(12 - 10) + (z - 3)(8 - 9) = 0$

or, $-1(x - 1) + 2(y - 2) - 1(z - 3) = 0$

or, $-x + 1 + 2y - 4 + z + 3 = 0$

or, $x - 2y + z = 0$ .... (iii)

Also, any point on line (ii) is $(2, 4, 5)$.

Hence, length of shortest distance = perpendicular distance from $(2, 4, 5)$ to the plane (iii).

$$= \left|\frac{2 - 2.4 + 5}{\sqrt{1^2 + (-2)^2 + 1^2}}\right|$$

$$= \left|\frac{7 - 8}{\sqrt{1 + 4 + 1}}\right|$$

$$= \left|\frac{-1}{\sqrt{1 + 4 + 1}}\right| = \frac{1}{\sqrt{6}}$$

Equation of the line of S.D.

The equation of plane containing the line (i) and perpendicular to the plane (iii) is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

or, $(x - 1)(3 + 8) + (y - 2)(4 - 2) + (z - 3)(-4 - 3) = 0$

or, $11(x - 1) + 2(y - 2) - 7(z - 3) = 0$

or, $11x - 11 + 2y - 4 - 7z + 21 = 0$

or, $11x + 2y - 7z + 6 = 0$ .... (iv)

Again, the equation of plane containing line (ii) and perpendicular to the plane (iii) is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 0$$

or, $(x - 2)(4 + 10) + (y - 4)(5 - 3) + (z - 5)(-6 - 4) = 0$

or, $14(x - 2) + 2(y - 4) - 10(z - 5) = 0$

or, $14x - 28 + 2y - 8 - 10z + 50 = 0$

or, $14x + 2y - 10z + 14 = 0$

or, $7x + y - 5z + 2 = 0$ .... (v)

Thus, the equation of lines of shortest distance is

$11x + 2y - 7z - 6 = 0 = 7x + y - 5z + 2.$

2. **Find the shortest distance between the line $\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2}$ and $\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1}$. Also find the equations of shortest distance.**

Solution:

The given lines are

$$\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2} \quad \cdots\cdots (i)$$

and $$\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1} \quad \cdots\cdots (ii)$$

Now, the equation of the plane containing line (i) and parallel to line (ii) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or. $$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix} = 0$$

or. $(x-3)(3-2) + (y-6)(-8+4) + z(-4+12) = 0$

or. $(x-3) - 4(y-6) + 8z = 0$

or. $x - 3 - 4y + 24 + 8z = 0$

or. $x - 4y + 8z + 21 = 0 \quad \cdots\cdots (iii)$

Also, any point on line (ii) is $(-2, 0, 7)$.

Now,

Length of shortest distance = perpendicular distance from the point $(-2, 0, 7)$ to the plane (iii)

$$= \left|\frac{-2-4.0+8.7+21}{\sqrt{1^2+(-4)^2+8^2}}\right|$$

$$= \left|\frac{-2+56+21}{\sqrt{1+16+64}}\right|$$

$$= \left|\frac{75}{\sqrt{81}}\right|$$

$$= \frac{25}{3}$$

Equation of the line of S.D.

The equation of plane containing line (i) and perpendicular to plane (iii) is

$$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 3 & 2 \\ 1 & -4 & 8 \end{vmatrix} = 0$$

or, $(x+3)(24+8) + (y-6)(2-16) + z(16-3) = 0$

or, $32(x+3) - 14(y-6) + 13z = 0$

or, $32x + 96 - 14y + 84 + 13z = 0$

or. $32x - 14y + 13z + 180 = 0 \quad \cdots\cdots (iv)$

Also, the equation of plane containing line (ii) and perpendicular to the plane (iii) is

$$\begin{vmatrix} x+2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -4 & 8 \end{vmatrix} = 0$$

or. $(x+2)(8+4) + y(1+32) + (z-7)(16-1) = 0$

or. $12(x+2) + 33y + 15(z-7) = 0$

or. $12x + 24 + 33y + 15z - 105 = 0$

or. $12x + 33y + 15z - 81 = 0 \cdots\cdots (v)$

Hence, the required equation of line of shortest distance is

$32x - 14y + 13z + 180 = 0 = 12x + 33y + 15z - 81.$

3. **Find the magnitude and the equation of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.**

Solution:

The given lines are

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \quad \cdots\cdots (i)$$

and $$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \cdots\cdots (ii)$$

Now, the equation of the plane containing line (i) and parallel to line (ii) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = 0$$

or, $(x-8)(80-56) + (y+9)(21+15) + (z-10)(24+48) = 0$

or, $24(x-8) + 36(y+6) + 72(z-10) = 0$

or, $24x - 192 + 36y + 216 + 72z - 720 = 0$

or, $24x + 36y + 72z - 696 = 0$

or, $2x + 3y + 6z - 58 = 0 \quad \cdots\cdots (iii)$

Also, any point on line (ii) is $(15, 29, 5)$.

Now,

Length of shortest distance = perpendicular distance from the point $(15, 29, 5)$ to the plane (iii)

$$= \left|\frac{2.15+3.29+6.5-58}{\sqrt{2^2+3^2+6^2}}\right|$$

$$= \left|\frac{30+87+30-58}{\sqrt{4+9+36}}\right|$$

$$= \left|\frac{89}{\sqrt{49}}\right|$$

$$= \frac{89}{7}$$

Equation of the line of S.D.

The equation of plane containing line (i) and perpendicular to plane (iii) is

$$\begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

or, $(x-8)(-96-21)+(y+9)(14-18)+(z-10)(9+32)=0$

or, $24(x-8)+36(y+6)+72(z-10)=0$

or, $24x-192+36y+216+72z-720=0$

or, $24x+36y+72z-696=0$

or, $2x+3y+6z-58=0$ .... (iv)

Also, the equation of plane containing line (ii) and perpendicular to the plane (iii) is

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

or, $(x-15)(48+15)+(y-29)(-10-18)+(z-5)(9-16)=0$

or, $63(x-15)-28(y-29)-7(z-5)=0$

or, $63x-945-28y+812-7z+35=0$

or, $63x-28y-7z-98=0$

or, $9x-4y-z-14=0$ .... (v)

Hence, the required equation of line of shortest distance is

$$2x+3y+6z-58=0=9x-4y-z-14.$$

4. Find the shortest distance between the lines $\frac{x+3}{-4}=\frac{y-6}{6}=\frac{z}{2}$ and $\frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}$. Also find the equation of shortest distance.

Solution:

The given lines are

$$\frac{x+3}{-4}=\frac{y-6}{6}=\frac{z}{2} \quad .... \text{(i)}$$

and $$\frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1} \quad .... \text{(ii)}$$

Now, the equation of the plane containing line (i) and parallel to line (ii) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or, $$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 6 & 2 \\ -4 & 1 & 1 \end{vmatrix} = 0$$

or, $(x-3)(6-2)+(y-6)(-8+4)+z(-4+24)=0$

or, $4(x-3)-4(y-6)+20z=0$

or, $4x-12-4y+24+20z=0$

or, $4x-4y+20z+12=0$

or, $x-y+5z+3=0$ .... (iii)

Also, any point on line (ii) is $(-2, 0, 7)$.

Now,

Length of shortest distance = perpendicular distance from the point $(-2, 0, 7)$ to the plane (iii)

$$= \left|\frac{-2-0+5.7+3}{\sqrt{1^2+(-1)^2+5^2}}\right|$$

$$= \left|\frac{-2+35+3}{\sqrt{1+1+25}}\right|$$

$$= \left|\frac{36}{\sqrt{27}}\right|$$

$$= \left|\frac{36}{3\sqrt{3}}\right|$$

$$= \left|\frac{12}{\sqrt{3}}\right|$$

Equation of the line of S.D.

The equation of plane containing line (i) and perpendicular to plane (iii) is

$$\begin{vmatrix} x+3 & y-6 & z \\ -4 & 6 & 2 \\ 1 & -1 & 5 \end{vmatrix} = 0$$

or, $(x+3)(30+2)+(y-6)(2+20)+z(4-6)=0$

or, $32(x+3)+22(y-6)-2z=0$

or, $32x+96+22y-132-2z=0$

or, $32x+22y-2z-36=0$

or, $16x+11y-z-18=0$ .... (iv)

Also, the equation of plane containing line (ii) and perpendicular to the plane (iii) is

$$\begin{vmatrix} x+2 & y & z-7 \\ -4 & 1 & 1 \\ 1 & -1 & 5 \end{vmatrix} = 0$$

or, $(x+2)(5+1)+y(1+20)+(z-7)(4-1)=0$

or, $6(x+2)+21y+3(z-7)=0$

or, $6x+12+21y+3z-21=0$

or, $6x+21y+3z-9=0$

or, $2x+7y+z-3=0$ .... (v)

Hence, the required equation of line of shortest distance is

$$16x+11y-z-18=0=2x+7y+z-3.$$

5. Find the shortest distance between the line $\frac{x+1}{2}=\frac{y-3}{4}=\frac{z-2}{5}$ and $\frac{x-2}{3}=\frac{y+3}{-2}=\frac{z-4}{1}$. Also find the equation of the line of shortest distance.

Solution:

The given lines are

$\frac{x+1}{2}=\frac{y-3}{4}=\frac{z-2}{5}$ .... (i)

and $\frac{x-2}{3}=\frac{y+3}{-2}=\frac{z-4}{1}$ .... (ii)

The equation of plane containing the line (i) and parallel to line (ii) is

$$\begin{vmatrix} x+1 & y-3 & z-2 \\ 2 & 4 & 5 \\ 3 & -2 & 1 \end{vmatrix}=0$$

or, $(x+1)(4+10)+(y-3)(15-2)+(z-2)(-4-12)=0$

or, $14(x+1)+13(y-3)-16(z-2)=0$

or, $14x+14+13y-39-16z+32=0$

or, $14x+13y-16z+7=0$ .... (iii)

Also, any point on line (ii) is $(2,-3,4)$.

Hence, length of shortest distance = perpendicular distance from $(2,-3,4)$) to the plane (iii).

$$=\left|\frac{14.2+13.(-3)-16.4+7}{\sqrt{(14)^2+(13)^2+(-16)^2}}\right|=\left|\frac{28-39-64+7}{3\sqrt{69}}\right|=\frac{68}{3\sqrt{69}}$$

Then equation of plane containing the line (i) and perpendicular to the plane (iii) is

$$\begin{vmatrix} x+1 & y-3 & z-2 \\ 2 & 4 & 5 \\ 14 & 13 & -16 \end{vmatrix}=0$$

or, $(x+1)(-64-65)+(y-3)(70+32)+(z-2)(26-56)=0$

or, $(x+1)(-129)+(y-3)(102)+(z-2)(-30)=0$

or, $-129x-129+102y-306-30z+60=0$

or, $-129x+102y-30z-375=0$

or, $129x-102y+30z+375=0$ .... (iv)

Again, the equation of plane containing line (ii) and perpendicular to the plane (iii) is

$$\begin{vmatrix} x-2 & y+3 & z-4 \\ 3 & -2 & 1 \\ 14 & 13 & -16 \end{vmatrix}=0$$

or, $(x-2)(32-13)+(y+3)(14+48)+(z-4)(39+28)=0$

or, $(x-2)(19)+(y+3)(62)+(z-4)(67)=0$

or, $19x-38+62y+186+67z-268=0$

or, $19x+62y+67z-120=0$ .... (v)

Thus, the equation of line of shortest distance is

$$129x-102y+30z+375=0=19x+62y+67z-120$$

6. Find the length of the perpendicular between the skew lines, $x+a=2y=-12z$ and $x=y+2a=6z-6a$.

Solution:

The given lines are

$x+a=2y=-12z$

or, $\frac{x+a}{12}=\frac{y}{6}=\frac{z}{-1}$ .... (i)

and $x=y+2a=6z-6a$

or, $\frac{x}{6}=\frac{y+2a}{6}=\frac{z-a}{1}$ .... (ii)

Now, the equation of the plane containing the first line and parallel to the second line is

$$\begin{vmatrix} x+a & y & z \\ 12 & 6 & -1 \\ 6 & 6 & 1 \end{vmatrix}=0$$

or, $(x+a)(6+6)-y(12+6)+z(72-36)=0$

or, $12x+12a-18y+36z=0$

or, $2x-3y+6z+2a=0$ .... (iii)

Also, $(0,-2a,a)$ is any point on the line (ii).

Now,

Length of S.D. = Perpendicular distance of the point $(0,-2a,a)$ from the plane (iii).

$$=\left|\frac{2.0-3.(-2a)+6a+2a}{\sqrt{2^2+(-3)^2+6^2}}\right|$$

$$=\frac{14a}{\sqrt{49}}$$

$$=2a$$

7. Find the length and equation of the S.D. between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$; $2x-3y+27=0=2y-z+20$.

Solution:

The given lines are

$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ .... (i)

and $2x-3y+27=0=2y-z+20$ .... (ii)

The equation of plane through the line (ii) is

$(2x-3y+27)+\lambda(2y-z+20)=0$

or, $2x+(2\lambda-3)y-\lambda z+(20\lambda+27)=0$ .... (iii)

If the plane (iii) is parallel to line (i), then the line normal to plane (iii) will be perpendicular to the line (i). Hence, applying the condition of perpendicularity, we get

$2.3+(2\lambda-3).(-1)+(-\lambda).1=0$

or, $6-2\lambda+3-\lambda=0$

or, $-3\lambda=-9$

or, $\lambda=3$

Thus the equation of plane through line (ii) and parallel to line (i) is

$2x-3y+27+3(2y-z+20)=0$

or, $2x-3y+27+6y-3z+60=0$

or, $2x+3y-3z+87=0$ .... (iv)

Also, any point on line (i) is $(3,8,3)$.

Now, length of SD = perpendicular distance from $(3,8,3)$ to plane (iv)

$$\sqrt{2^2+3^2+(-3)^2}\ \Big|$$

$$= \left|\frac{6+24-9+87}{\sqrt{22}}\right|$$

$$= \left|\frac{108}{\sqrt{22}}\right|$$

$$= \frac{108}{\sqrt{22}}$$

Equation of the line of S.D.

The equation of plane containing the line (i) and perpendicular to the plane (iv) is

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 3 & -3 \end{vmatrix} = 0$$

or, $(x-3)(3-3)+(y-8)(2+9)+(z-3)(9+2)=0$

or, $(x-3).0+11(y-8)+11(z-3)=0$

or, $11(y-8+z-3)=0$

or, $y+z-11=0$ .... (v)

Again, the equation of plane containing line (ii) is

$(2x-3y+27)+\lambda(2y-z+20)=0$

or, $2x+(2\lambda-3)y-\lambda z+(20\lambda+27)=0$

If this plane is perpendicular to the plane (iv), then

$2.2+(2\lambda-3).3+(-\lambda).(-3)=0$

or, $4+6\lambda-9+3\lambda=0$

or, $9\lambda=5$

or, $\lambda=\frac{5}{9}$

Hence, the equation of plane containing line (ii) and perpendicular to plane (iv) is

$(2x-3y+27)+\frac{5}{9}(2y-z+20)=0$

or, $18x-27y+243+10y-5z+100=0$

or, $18x-17y-5z+343=0$ .... (vi)

Thus, the equation of shortest distance is $y+z-11=0$, $18x-17y-5z+343=0$

Exercise 7.4

1. Find the equation of the sphere with centre (1, –2, 4) and radius 4.

Solution:

The equation of the sphere having its centre at (1, – 2, 4) and radius 4 is

$(x-1)^2+(y+2)^2+(z-4)^2=4^2$

or, $x^2-2x+1+y^2+4y+4+z^2-8z+16=16$

or, $x^2+y^2+z^2-2x+4y-8z+5=0$

which is the required equation of the sphere.

2. Find the centre and the radius of the sphere $x^2+y^2+z^2+2x-4y-6z+5=0$.

Solution:

The given equation of sphere is

$x^2+y^2+z^2+2x-4y-6z+5=0$ .... (i)

Comparing equation of sphere (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get

$u=1$, $v=-2$, $w=-3$ and $d=5$

We know that,

Centre of sphere $=(-u,-v,-w)=(-1,2,3)$

and radius $=\sqrt{u^2+v^2+w^2-d}=\sqrt{1+4+9-5}=3$

3. Find the equation of the sphere through the points (0, 0, 0), (– *a*, *b*, *c*), (*a*, – *b*, *c*) and (*a*, *b*, – *c*).

Solution:

Let the equation of sphere passing through origin be

$x^2+y^2+z^2+2ux+2vy+2wz=0$ .... (i)

Since the sphere (i) passes through the points

$(-a,b,c)$ $(a,-b,c)$ and $(a,b,-c)$. So, we have

For $(-a,b,c)$

$a^2+b^2+c^2-2ua+2vb+2wc=0$

or, $-2ua+2vb+2wc=-(a^2+b^2+c^2)$ .... (ii)

For $(a,-b,c)$

$(a^2+b^2+c^2+2ua-2vb+2wc=0$

or, $2ua-2vb+2wc=-(a^2+b^2+c^2)$ .... (iii)

For $(a,b,-c)$

$(a^2+b^2+c^2+2ua+2vb-2wc=0$

or, $2ua+2vb-2wc=-(a^2+b^2+c^2)$ .... (iv)

Now, adding (ii) and (iii), we get

$w=-\frac{(a^2+b^2+c^2)}{2c}$

Adding (iii) and (iv), we get

$u=-\frac{(a^2+b^2+c^2)}{2a}$

Adding (ii) and (iv), we get

$v=-\frac{(a^2+b^2+c^2)}{2b}$

Substituting the value of u, v, w in (i), we get

$x^2+y^2+z^2-\frac{(a^2+b^2+c^2)x}{a}-\frac{(a^2+b^2+c^2)y}{b}-\frac{(a^2+b^2+c^2)z}{c}=0$

or, $x^2+y^2+z^2-(a^2+b^2+c^2)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$

or, $\frac{x^2+y^2+z^2}{a^2+b^2+c^2}=\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$

which is the required equation of sphere.

4. Find the equation of the sphere which passes through the points (1, ... (1, −5, 2) and (1, −3, 0) and whose centre lies on the plane $x + y + z = 0$

Solution:

Let the equation of sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Since the sphere (i) passes through the point (1, − 4, 3) (1, − 5, 2) and (1, ... we have

For (1, −4, 3):

$1 + 16 + 9 + 2u - 8v + 6w + d = 0$

or, $2u - 8v + 6w + d + 26 = 0$ ....(ii)

For (1, −5, 2)

$1 + 25 + 4 + 2u - 10v + 4w + d = 0$

or, $30 + 2u - 10v + 4w + d = 0$ ....(iii)

For (1, −3, 0)

$1 + 9 + 0 + 2u - 6v + 0 + d = 0$

or, $10 + 2u - 6v + d = 0$ ....(iv)

As centre of sphere (i) lies on the plane $x + y + z = 0$

$\therefore$ $-u - v - w = 0$

or, $u + v + w = 0$ ....(v)

Subtracting (iii) from (ii), we get

$-4 + 2v + 2w = 0$

or, $v + w = 2$ .... (vi)

Subtracting (iv) from (iii) we get

$20 - 4v + 4w = 0$

or, $v - w = 5$ ....(vii)

Solving equation (vi) and (vii), we get

$2v = 7$ and $2w = -3$

Substituting the value of v and w in equation (v), we get

$2u = -4.$

Substituting the value of u and v in equation (iv), we get

$d = 15.$

Thus, $2u = -4$, $2v = 7$, $2w = -3$ and $d = 15$.

Now, substituting the values of u, v, w and d in (i), we get

$x^2 + y^2 + z^2 - 4x + 7y - 3z + 15 = 0$

which is the required equation of the sphere.

5. Find the equation of the sphere described by the join of (2, −3, 4) and (−5, 6, −7) as diameter.

Solution:

The equation of the sphere described on the join of (2, −3, 4) and (−5, 6, −7) as diameter is

$(x - 2)(x + 5) + (y + 3)(y - 6) + (z - 4)(z + 7) = 0$

or, $x^2 + 3x - 10 + y^2 - 3y - 18 + z^2 + 3z - 28 = 0$

or, $x^2 + y^2 + z^2 + 3x - 3y - 3z - 56 = 0$

which is the required equation of the sphere.

6. Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $x + 3y + 4z = 2$ and (i) the origin (2) the point (2, 2, 3).

Solution:

The given circle is

$x^2 + y^2 + z^2 = 9$, $x + 3y + 4z = 2$ ....(i)

The equation of the sphere through the circle (i) is

$x^2 + y^2 + z^2 - 9 + k(x + 3y + 4z - 2) = 0$ ....(ii)

a) Since sphere (i) passes through origin, so

$0 + 0 + 0 - 9 - 2k = 0$

or, $k = -\frac{9}{2}$

Substituting the value of k in equation (ii), we get

$x^2 + y^2 + z^2 - 9 - \frac{9}{2}(x + 3y + 4z - 2) = 0$

or, $2(x^2 + y^2 + z^2 - 9) - 9(x + 3y + 4z - 2) = 0$

or, $2x^2 + 2y^2 + 2z^2 - 9x - 27y - 36z = 0$

or, $2(x^2 + y^2 + z^2) - 9(x + 3y + 4z) = 0$

which is the required equation of the sphere.

b) Since the sphere (i) passes through the point (2, 2, 3).

So we have,

$4 + 4 + 9 - 9 + 2k + 6k + 12k - 2k = 0$

$\therefore$ $k = -\frac{4}{9}$

Substituting the value of k in equation (ii), we get

$x^2 + y^2 + z^2 - 9 - \frac{9}{4}(x + 3y + 4z - 2) = 0$

$9(x^2 + y^2 + z^2) - 4(x + 3y + 4z) - 73 = 0$

which is the required equation of the sphere.

7. Show that the plane $2x - y + 2z = 14$ touches the sphere $x^2 + y^2 + z^2 - 4x + 2y - 4 = 0$. Find the point of contact.

Solution:

The given sphere is

$x^2 + y^2 + z^2 - 4x + 2y - 4 = 0$ .... (i)

Comparing equation (i) with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$u = -2$, $v = 1$, $w = 0$, $d = -4$

Thus, Centre of sphere (i) $= (-u, -v, -w) = (2, -1, 0)$

radius of sphere (i) $= \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{(2)^2 + (-1)^2 + 0^2 + 4}$

$= \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

The given plane is

$2x - y + 2z = 14$

or, $2x - y + 2z - 14 = 0$ ....(ii)

If the plane (ii) touches the sphere (i), then

Radius of sphere = Perpendicular distance from centre of the sphere to the plane

$$= \left|\frac{2.2 - 1(-1) + 2.0 - 14}{\sqrt{(2)^2 + (-1)^2 + (2)^2}}\right|$$

$$= \left|\frac{4 + 1 - 14}{\sqrt{9}}\right|$$

$$= \left|\frac{-9}{\sqrt{9}}\right|$$

$$= \frac{9}{3}$$

$$= 3$$

Hence, the plane (ii) touches the sphere (i).

For point of contact:

The equation of the line through the centre (2, – 1, 0) and perpendicular to the plane (ii) is

$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-0}{2} = r \text{ (say)} \quad \cdots\cdots \text{(iii)}$$

Any point on line (iii) are $(2r + 2, -r - 1, 2r)$.

This point lies on the plane (ii) if

$2.(2r + 2) + r + 1 + 2.2r = 14$

or, $4r + 4 + r + 1 + 4r = 14$

or, $9r = 14 - 5$

or, $9r = 9$

or, $r = 1$

Thus, the required point of contact is obtained by putting $r = 1$ in $(2r + 2, -r - 1, 2r)$ i.e., $(4, -2, 2)$.

8. Obtain the equation of the sphere having the circle $x^2 + y^2 + z^2 + 8y - 2z - 4 = 0, x + 2y + z = 2$ as the great circle.

Solution:

The equation of sphere through the circle

$x^2 + y^2 + z^2 + 8y - 2z - 4 = 0, x + 2y + z = 2$ is

$x^2 + y^2 + z^2 + 8y - 2z - 4 + \lambda (x + 2y + z - 2) = 0$

or, $x^2 + y^2 + z^2 + \lambda x + (2\lambda + 8)y + (\lambda - 2)z - (2\lambda + 4) = 0 \quad \cdots\cdots$ (i)

Comparing equation (i) with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$$u = \frac{\lambda}{2}, v = \frac{2\lambda + 8}{2} = \lambda + 4, w = \frac{\lambda - 2}{2}$$

Thus, Center of sphere $= (-u, -v, -w) = \left(-\frac{\lambda}{2}, -\lambda - 4, \frac{2-\lambda}{2}\right)$

Since the circle is great circle, so the centre of sphere (i) must lie on the plane of the circle $x + 2y + z = 2$. So we have

$$-\frac{\lambda}{2} + 2(-\lambda - 4) + \frac{2-\lambda}{2} = 2$$

or, $-\lambda - 4\lambda - 16 + 2 - \lambda = 4$

or, $-6\lambda = 4 + 16 - 2$

or, $-6\lambda = 18$

$\lambda = -3$

Substituting the value of λ in equation (i), we get

$x^2 + y^2 + z^2 - 3x + 2y - 5z + 2 = 0$

which is the required equation of the sphere.

9. Find the equation of sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, 2x + 3y + 4z = 8$ is a great circle.

Solution:

The given circle is

$$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0, 2x + 3y + 4z - 8 = 0 \quad \cdots\cdots \text{(i)}$$

The equation of sphere through the circle (i) is

$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda (2x + 3y + 4z - 8) = 0$,

where λ is any scalar.

or, $x^2 + y^2 + z^2 + 2\lambda x + (3\lambda + 7) y + (4\lambda - 2) z + (2 - 8\lambda) = 0 \quad \cdots\cdots$ (ii)

Comparing equation (ii) with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$$u = \lambda, v = \frac{3\lambda + 7}{2}, w = 2\lambda - 1, d = 2 - 8\lambda$$

Thus, centre of sphere (ii) $= (-u, -v, -w) = \left(-\lambda, -\frac{3\lambda + 7}{2}, 1 - 2\lambda\right)$

If the circle (i) is great circle of the sphere (ii), then it's centre must lie in the plane of the circle $2x + 3y + 4z - 8 = 0$

or, $-2\lambda + 3\left(-\frac{3\lambda + 7}{2}\right) + 4.(1 - 2\lambda) - 8 = 0$

or, $-4\lambda - 9\lambda - 21 + 8 - 16\lambda - 16 = 0$

or, $-29\lambda - 29 = 0$

or, $\lambda = -1$

Substituting the value of λ in equation (ii), we get

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$$

which is the required equation of the sphere.

10. Obtain the equation of sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z = 3$ as a great circle.

Solution:

The equation of sphere through the given circle

$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0, x + y + z - 3 = 0$ is

$x^2 + y^2 + z^2 + 10y - 4z - 8 + \lambda (x + y + z - 3) = 0 \ldots\ldots (*)$

or, $x^2 + y^2 + z^2 + \lambda x + (\lambda + 10) y + (\lambda - 4) z - (3\lambda + 8) = 0 \cdots\cdots$ (i)

Comparing equation (i) with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get $u = \frac{\lambda}{2}, v = \frac{\lambda + 10}{2}, w = \frac{\lambda - 4}{2}$

Thus, Centre of sphere (i) $= (-u, -v, -w) = \left(-\frac{\lambda}{2}, -\frac{\lambda + 10}{2}, \frac{4 - \lambda}{2}\right)$

If the given circle is great circle of the sphere (i), then the centre of the sphere (i) must lie in the plane of the circle $x + y + z = 3$. So we have

$$-\frac{\lambda}{2}-\frac{\lambda+10}{2}+\frac{4-\lambda}{2}=3$$

or, $-\lambda-\lambda-10+4-\lambda=6$

or, $-3\lambda=12$

or, $\lambda=-4$

Substituting the value of λ in equation (*), we get

$x^2+y^2+z^2+10y-4z-8-4(x+y+z-3)=0$

or, $x^2+y^2+z^2-4x+6y-8z+4=0$

which is the required equation of the sphere.

11. Find the equation of the sphere through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touches the plane $z=0$.

Solution:

The equation of the sphere through the circle

$x^2+y^2+z^2=1$, $2x+4y+5z=6$ is

$$x^2+y^2+z^2-1+\lambda(2x+4y+5z-6)=0 \quad \cdots\cdots (i)$$

i.e., $$x^2+y^2+z^2+2\lambda x+4\lambda y+5\lambda z-(1+6\lambda)=0 \quad \cdots\cdots (*)$$

Comparing equation (*) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get $u=\lambda$, $v=2\lambda$, $w=\frac{5\lambda}{2}$, $d=-(1+6\lambda)$

Thus, centre $=(-u,-v,-w)=\left(-\lambda,-2\lambda,-\frac{5\lambda}{2}\right)$

and radius $(r)=\sqrt{u^2+v^2+w^2-d}$

$$=\sqrt{(-\lambda)^2+(-2\lambda)^2+\left(-\frac{5\lambda}{2}\right)^2+(1+6\lambda)}$$

$$=\sqrt{\lambda^2+4\lambda^2+\frac{25}{4}\lambda^2+1+6\lambda}$$

$$=\sqrt{\frac{4\lambda^2+16\lambda^2+25\lambda^2+4+24\lambda}{4}}=\frac{1}{2}\sqrt{45\lambda^2+24\lambda+4}$$

If the sphere (i) touches the plane $z=0$, then

Radius of sphere = Perpendicular distance from centre of the sphere to the plane $z=0$

$$=\pm\frac{0+0+\left(-\frac{5\lambda}{2}\right)}{\sqrt{0+0+1}}=\pm\frac{5\lambda}{2}$$

Thus, $\frac{1}{2}\sqrt{45\lambda^2+24\lambda+4}=\pm\frac{5\lambda}{2}$

or, $\sqrt{45\lambda^2+24\lambda+4}=\pm 5\lambda$

or, $45\lambda^2+24\lambda+4=25\lambda^2$ (squaring both sides)

or, $20\lambda^2+24\lambda+4=0$

or, $5\lambda^2+6\lambda+1=0$

or, $5\lambda^2+5\lambda+\lambda+1=0$

or, $5\lambda(\lambda+1)+1(\lambda+1)=0$

or, $(5\lambda+1)(\lambda+1)=0$

$\lambda=-\frac{1}{5},-1$

Substituting the value of $\lambda=-1$ in equation (i), we get

$x^2+y^2+z^2-1-2x-4y-5z+6=0$

or, $x^2+y^2+z^2-2x-4y-5z+5=0$

Substituting the value of $\lambda=-\frac{1}{5}$ in equation (i), we get

$$x^2+y^2+z^2-1-\frac{1}{5}(2x+4y+5z-6)=0$$

or, $5x^2+5y^2+5z^2-2x-4y-5z+6-5=0$

or, $5x^2+5y^2+5z^2-2x-4y-5z+1=0$

Thus, the required equations of spheres are

$x^2+y^2+z^2-2x-4y-5z+5=0$ and $5x^2+5y^2+5z^2-2x-4y-5z+1=0$.

12. A sphere S has points (0,1, 0), (3, – 5, 2) as opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle.

Solution:

The equation of the sphere whose end points of the diameter are (0, 1, 0) and (3, – 5, 2) is

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

or, $(x-0)(x-3)+(y-1)(y+5)+(z-0)(z-2)=0$

or, $x(x-3)+y^2+5y-y-5+z(z-2)=0$

or, $x^2-3x+y^2+4y-5+z^2-2z=0$

or, $$x^2+y^2+z^2-3x+4y-2z-5=0 \quad \cdots\cdots (i)$$

The given plane is $5x-2y+4z+7=0$ $\cdots\cdots$ (ii)

The equation of the sphere passing through the intersection of (i) and (ii) is

$x^2+y^2+z^2-3x+4y-2z-5+\lambda(5x-2y+4z+7)=0$

or, $x^2+y^2+z^2+(5\lambda-3)x+(4-2\lambda)y+(4\lambda-2)z+$

$$(7\lambda-5)=0 \quad \cdots\cdots (iii)$$

Comparing equation (iii) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get

$u=\frac{5\lambda-3}{2}$, $v=2-\lambda$, $w=2\lambda-1$, $d=7\lambda-5$

The centre of the sphere (iii) is $(-u,-v,-w)=\left(\frac{3-5\lambda}{2},\lambda-2,1-2\lambda\right)$. The intersection of equation (i) and (ii) will be great circle of the sphere (iii) if the centre of sphere (iii) lies on the plane (ii).

i.e., $5\left(\frac{3-5\lambda}{2}\right)-2(\lambda-2)+4(1-2\lambda)+7=0$

or, $\frac{15-25\lambda}{2}-2\lambda+4+4-8\lambda+7=0$

or, $15-25\lambda-4\lambda+8+8-16\lambda+14=0$

or, $45-45\lambda=0$

or, $45\lambda=45$

$\therefore$ $\lambda=1$

Thus the required equation of sphere is obtained by putting $\lambda = 1$ in equation (iii)

i.e., $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$

13. Show that the equation of the sphere through the circle $x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$, $x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$ and having its centre on the plane $4x - 5y - z = 3$ is $x^2 + y^2 + z^2 + 7x + 9x - 11z - 1 = 0$.

Solution:

The given circle are

$S_1 \equiv x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0$ .... (i)

and $S_2 \equiv x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$ .... (ii)

Now, subtracting S_2 from S_1, we get

$3x + 4y - 5z - 3 = 0$ .... (iii)

The circle represented by (i) and (ii) is the same as the circle given by (i) and (iii).

The equation of sphere through the circle given by (i) and (iii) is

$x^2 + y^2 + z^2 - 2x + 3y + 4z + 8 + k(3x + 4y - 5z - 3) = 0$

or, $x^2 + y^2 + z^2 + x(-2 + 3k) + y(-3 + 4k) + z(4 - 5k) + (8 - 3k) = 0$ .... (iv)

Comparing (iv) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$u = \frac{3k-2}{2}, v = \frac{4k-3}{2}, w = \frac{4-5k}{2}$

$\therefore$ Centre of sphere (iv) $= (-u, -v, -w) = \left(\frac{2-3k}{2}, \frac{3-4k}{2}, \frac{5k-4}{2}\right)$

Since centre of sphere (iv) lies on the plane $4x - 5y - z = 3$. So, we have

$\frac{8-12k}{2} - \frac{15-20k}{2} - \frac{-4+5k}{2} = 3$

or, $k = 3$

Substituting the value of k in equation (iv), we get $x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0$

which is the required equation of the sphere.

14. Find the centre and radius of the circle

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$, $x - 2y + 3z = 3$.

Solution:

The given sphere is

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$ .... (i)

Comparing equation (i) with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$u = -4, v = 2, w = 4, d = -45$

Thus, Centre of sphere $= (-u, -v, -w) = (4, -2, -4)$

and radius of sphere $R = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{(-4)^2 + (2)^2 + (4)^2 + 45}$

$= \sqrt{16 + 4 + 16 + 45}$

$= \sqrt{81} = 9$

The equation of the line through the centre of the sphere and perpendicular to the given plane $x - 2y + 3z = 3$ .... (ii) is

$\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z+4}{3} = r$ (say) .... (iii)

Any point on line (iii) is $(r + 4, -2r - 2, 3r - 4)$

This point lie in the plane (ii) if

$r + 4 - 2(-2r - 2) + 3(3r - 4) = 3$

or, $r + 4 + 4r + 4 + 9r - 12 = 3$

or, $14r = 7$

or, $r = \frac{1}{2}$

Thus the centre of the circle $= \left(\frac{1}{2} + 4, -\frac{2}{2} - 2, \frac{3}{2} - 4\right) = \left(\frac{9}{2}, -3, -\frac{5}{2}\right)$

Also, length of perpendicular from the centre of the sphere to the plane (ii) is

$p = \left|\frac{4 - 2.(-2) + 3(-4) - 3}{\sqrt{(1)^2 + (-2)^2 + (3)^2}}\right|$

$= \left|\frac{4 + 4 - 12 - 3}{\sqrt{1 + 4 + 9}}\right|$

$= \left|\frac{-7}{\sqrt{14}}\right|$

$= \frac{7}{\sqrt{14}}$

$= \sqrt{\frac{7}{2}}$

Now, radius of the circle $= \sqrt{R^2 - p^2}$

$= \sqrt{9^2 - \left(\frac{7}{2}\right)}$

$= \sqrt{\frac{162-7}{2}}$

$= \sqrt{\frac{155}{2}}$

15. Find the centre and radius of the circle in which the sphere $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$ is cut off by the plane $x - 2y + 2z = 3$.

Solution:

The given sphere is

$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$ .... (i)

Comparing equation (i) with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get

$u = -4, v = 2, w = 4, d = -45$

Thus, Centre of sphere $= (-u, -v, -w) = (4, -2, -4)$

and radius of sphere $R = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{(-4)^2 + (2)^2 + (4)^2 + 45}$

$= \sqrt{16 + 4 + 16 + 45} = \sqrt{81} = 9$

The equation of the line through the centre of the sphere and perpendicular to the given plane $x - 2y + 2z = 3$ .... (ii) is

$\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z+4}{2} = r$ (say) .... (iii)

Any point on line (iii) is $(r + 4, -2r - 2, 2r - 4)$

This point lie in the plane (ii) if

$r + 4 - 2(-2r - 2) + 2(2r - 4) = 3$

or, $r + 4 + 4r + 4 + 4r - 8 = 3$

or, $9r = 3$

or, $r = \frac{1}{3}$

Thus the centre of the circle $= \left(\frac{1}{3} + 4, -\frac{2}{3} - 2, \frac{2}{3} - 4\right) = \left(\frac{13}{3}, -\frac{8}{3}, -\frac{10}{3}\right)$

Also, length of perpendicular from the centre of the sphere to the plane (ii) is

$$p = \left|\frac{4 - 2.(-2) + 2(-4) - 3}{\sqrt{(1)^2 + (-2)^2 + (2)^2}}\right|$$

$$= \left|\frac{4 + 4 - 8 - 3}{\sqrt{1 + 4 + 4}}\right|$$

$$= \left|\frac{-3}{\sqrt{9}}\right|$$

$$= \left|\frac{3}{3}\right|$$

$$= 1$$

Now, radius of the circle $= \sqrt{R^2 - p^2}$

$= \sqrt{9^2 - 1^2}$

$= \sqrt{80}$

$= 4\sqrt{5}$

16. Prove that the circles

$x^2 + y^2 + z^2 - y + 2z = 0$, $x - y + z - 2 = 0$ and $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0$, $2x - y + 4z - 1 = 0$ lies on the same sphere and find its equation.

Solution:

The given circles are

$x^2 + y^2 + z^2 - y + 2z = 0$, $x - y + z - 2 = 0$ .... (i)

and

$x^2 + y^2 + z^2 + x - 3y + z - 5 = 0$, $2x - y + 4z - 1 = 0$ .... (ii)

The equation of sphere through the circle (i) is

$x^2 + y^2 + z^2 - y + 2z + k_1(x - y + z - 2) = 0$

or, $x^2 + y^2 + z^2 + k_1x + (-1 - k_1)y + (2 + k_1)z - 2k_1 = 0$ .... (iii)

The equation of sphere through the circle (ii) is

$x^2 + y^2 + z^2 + x - 3y + z - 5 + k_2(2x - y + 4z - 1) = 0$

or, $x^2 + y^2 + z^2 + (1 + 2k_2)x + (-3 - k_2)y + (1 + 4k_2)z - 5 - k_2 = 0$ .... (iv)

The circles (i) and (ii) will lie on the same sphere if (iii) and (iv) represents the same sphere for some values of k_1 and k_2.

Equating the coefficients of like terms in (iii) and (iv), we get

$k_1 = 1 + 2k_2$ .... (v)

$-1 - k_1 = -3 - k_2$ .... (vi)

$2 + k_1 = 1 + 4k_2$ .... (vii)

$-2k_1 = -5 - k_2$ .... (viii)

From (v), we have $k_2 = 1$

From (viii), we have $k_1 = 3$

The values of k_1 and k_2 also satisfies the equation (vi) and (vii).

Hence, the two circles (i) and (ii) lies on the same sphere.

Now, substituting the value of k_1 in equation (iv), we get

$x^2 + y^2 + z^2 + x - 3y + z - 5 + k_2(2x - y + 4z - 1) = 0$

or, $x^2 + y^2 + z^2 + x - 3y + z - 5 + 1(2x - y + 4z - 1) = 0$

or, $x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$

which is the required equation of the sphere.

17. Obtain the equation of the sphere which passes through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ is a circle of radius 3.

Solution:

The given equation of circle is

$x^2 + y^2 = 4, z = 0$

or, $x^2 + y^2 + z^2 - 4 = 0 = z$ .... (i)

The equation of sphere through the circle (i) is

$x^2 + y^2 + z^2 - 4 + kz = 0$ .... (ii)

Comparing the equation of sphere (ii) with

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$, we get $u = 0$, $v = 0$, $w = \frac{k}{2}$ and $d = -4$

Now, centre of sphere $= (-u, -v, -w) = (0, 0, -k/2)$

and radius $(R) = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{\frac{k^2}{4} + 4} = \sqrt{\frac{k^2 + 16}{4}}$

Now, The length of the perpendicular from $(0, 0, -k/2)$ to the plane

$x + 2y + 2z = 0$ is

$$p = \left|\frac{0 + 0 - k}{\sqrt{1 + 4 + 4}}\right| = k/3$$

Given that radius of circle (i) is 3.

We know that,

$r = \sqrt{R^2 - p^2}$

or, $3 = \sqrt{\frac{k^2 + 16}{4} - \frac{k^2}{9}}$

or, $9 = \frac{5k^2 + 144}{36}$

or, $k = \pm 6$.

Substituting the value of k in equation (ii), we get $x^2 + y^2 + z^2 \pm 6z - 4 = 0$ which are the required equation of the spheres.

Exercise 7.5

1. **Find the equations of the tangent planes to the following sphere**

   i. $x^2+y^2+z^2=49$ **at the point (6, –3, 2)**

Solution:

The given sphere is

$$x^2+y^2+z^2=49 \quad \ldots(i)$$

The equation of the tangent plane to the sphere (i) at point (6, – 3, 2) is

$$6x-3y+2z=49 \; [\because \alpha x+\beta y+\gamma z=r^2]$$

which is the required equation of the tangent plane.

   ii. $3(x^2+y2+z^2)-2x-3y-4z-22=0$ **at the point (1, 2, 3).**

Solution:

The given sphere is

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0$$

or, $$x^2+y^2+z^2-\frac{2}{3}x-y-\frac{4}{3}z-\frac{22}{3}=0 \quad \ldots(i)$$

Comparing (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get $u=-\frac{1}{3}, v=-\frac{1}{2}, w=-\frac{2}{3}, d=-\frac{11}{3}$

Now, the equation of tangent plane to the sphere (i) at point (1, 2, 3) is

$$x(1)+y(2)+z(3)-\frac{1}{3}(x+1)-\frac{1}{2}(y+2)-\frac{2}{3}(z+3)-\frac{11}{3}=0$$

or, $6x+12y+18z-2x-2-3y-6-4z-12-22=0$

or, $4x+9y-14z-42=0$

which is the required equation of the tangent plane.

2. **Find the value of c for which the plane $x+y+z=c\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$.**

Solution:

The given equation of sphere and plane are

$$x^2+y^2+z^2-2x-2y-2z-6=0 \quad \ldots(i)$$

and $$x+y+z=c\sqrt{3} \quad \ldots(ii)$$

Comparing the equation of the sphere (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get $u=-1, v=-1, w=-1, d=-6$

Thus, centre of sphere (i) $=(-u,-v,-w)=(1, 1, 1)$

and radius $(r)=\sqrt{u^2+v^2+w^2-d}$

$=\sqrt{(-1)^2+(-1)^2+1^2-(-6)}$

$=\sqrt{1+1+1+6}$

$\therefore r=3$

If the plane (ii) touches the sphere (i) then, the length of the perpendicular from centre of the sphere (i) is equal to the radius of the sphere.

i.e. $\pm\frac{1+1+1-c\sqrt{3}}{\sqrt{1+1+1}}=3$

or. $\pm\frac{3-c\sqrt{3}}{\sqrt{3}}=3$

$\therefore c=\sqrt{3}\pm 3.$

3. **Show that the plane $2x-y-2z=4$ touches the sphere $x^2+y^2+z^2=24$. Also, find the point of contact.**

Solution

The given sphere is $x^2+y^2+z^2-4x+2y-4=0 \quad \ldots(1)$

Comparing equation (1) with $x^2+y^2+z^2+2hx+2hy+2wz+d=0$, we get

$u=-2, v=1, w=0, d=-4$

Thus, centre of sphere (1) $=(-u_1, v, -w)=(2, -1, 0)$

Radius of sphere (1) $=\sqrt{u^2+v^2+w^2+d}$

$=\sqrt{(-2)^2+1^2+0^2+4}$

$=\sqrt{4+1+4}=3$ units

The given plane is

$$2x-y+2z-14=0 \quad \ldots(2)$$

If the plane (2) touches the spheres (1) then

Radius of sphere = Perpendicular distance from the centre of the sphere to the plane (2)

$$=\left|\frac{2(2)-(-1)+2(0)-14}{\sqrt{2^2+(-1)^2+0^2}}\right|$$

$$=\left|\frac{4+1-14}{\sqrt{9}}\right|=\left|\frac{-9}{\sqrt{9}}\right|=3$$

Hence, the plane (2) touches the sphere (1)

For point of intersection

The equation of the line through the centre (2, –1, 0) and perpendicular to the plane (2) is $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z-0}{2}=r$ (say) $\quad \ldots(3)$

Any point on the line (3) are $(2r+2, -r-1, 2r)$

This point lies on the plane (2) if

$2(2r+2)+(r+1)+2(2r)=14$

or, $4r+4+r+1+4r=14$

or, $9r=14-5$

or, $r=1$

Thus, the required point of contact is obtained by putting $r=1$ in $(2r+2, -r-1, 2r)$ i.e., (4, –2, 2).

4. **Find the equations of the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$.**

Solution:

Let the given sphere is

$x^2+y^2+z^2-4x+2y-6z+5=0$

or, $$x^2+y^2+z^2+2(-2)x+2.1.y+2(-3)z+5=0 \quad \ldots(i)$$

Comparing equation (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get, $u=-2, v=1, w=-3, d=5$

So centre of sphere (i) $= (-u, -v, -w) = (2, -1, 3)$

and radius of sphere (i) $= \sqrt{u^2+v^2+w^2-d}$

$= \sqrt{(-2)^2+(1)^2+(-3)^2-5}$

$= \sqrt{4+1+9-5}$

$= \sqrt{9}$

$= 3$

Also the given plane is $2x+2y-z=0$ .... (ii)

Any plane parallel to the plane (ii) is

$2x+2y-z+k=0$ .... (iii)

Since the plane (iii) is tangent to the sphere (i) then the radius of the sphere is equal to the perpendicular distance from centre of sphere (i) to the plane (iii)

radius $= \pm\dfrac{2\times 2+(-1)\times 2-3+k}{\sqrt{(2)^2+(2)^2+(-1)^2}}$

or, $3 = \pm\dfrac{4-2-3+k}{\sqrt{4+4+1}}$

or, $3 = \pm\dfrac{-1+k}{3}$

or, $9 = \pm(-1+k)$

Taking positive sign, we have

or, $9 = -1+k$

or, $k = 10$

Taking negative sign, we have

or, $9 = 1-k$

or, $k = -8$

Substituting the value of k in equation (iii), we get

$2x+2y-z+10=0$ and $2x+2y-z-8=0$

which are the required equation of the tangent planes.

5. Find the equations of the tangent planes to the sphere $x^2+y^2+z^2+2x-4y+6z-7=0$ which intersect in the line $6x-3y-23=0=3z+2$

Solution:

The given equation of sphere and line are

$x^2+y^2+z^2+2x-4y+6z-7=0$ .... (i)

$6x-3y-23=0=3z+2$ .... (ii) respectively.

Comparing equation (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get $u=1$, $v=-2$, $w=3$ and $d=-7$

Thus, centre of sphere $= (-u, -v, -w) = (-1, 2, -3)$

and radius $= \sqrt{u^2+v^2+w^2-d} = \sqrt{1+4+9+7} = \sqrt{21}$.

Now, the equation of plane intersecting the line (ii) is

$6x-3y-23+k(3z+2)=0$

or, $6x-3y+3kz-23+2k=0$ .... (iii)

Then, the length of the perpendicular from centre of sphere (i) to the plane (iii) is

$$p = \left|\frac{-6-6-9k-23+2k}{\sqrt{36+9+9k^2}}\right| = \left|\frac{-35-7k}{\sqrt{45+9k^2}}\right|$$

If the plane (iii) is tangent plane to sphere (i), then the length of the perpendicular from center of sphere (i) to plane (iii) is equal to the radius of sphere (i).

$\therefore$ $\dfrac{-35-7k}{\sqrt{45+9k^2}} = \sqrt{21}$

or, $(35+7k)^2 = 21(45+9k^2)$

or, $1225+490k+49k^2 = 945+189k^2$

or, $14k^2-49k-28=0$

or, $2k^2-8k+k-4=0$

or, $(2k+1)(k-4)=0$

$\therefore$ $k = 4, -\dfrac{1}{2}$.

Substituting value of $k=4$ in equation (iii), we get $2x-y+4z-5=0$

Substituting value of $k=-\dfrac{1}{2}$ in equation (iii), we get

$6x-3y-\dfrac{3}{2}z-24=0$

or, $12x-6y-3z-48=0$

or, $4x-2y-z-16=0$

Thus, the required equation of tangent planes are

$2x-y+4z-5=0$ and $4x-2y-z-16=0$.

6. Find the equation of the sphere which touches the sphere $x^2+y^2+z^2+2x-6y+1=0$ at $(1, 2, -2)$ and passes through the origin.

Solution:

The given equation of sphere is

$x^2+y^2+z^2+2x-6y+1=0$ .... (i)

Comparing equation of sphere (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get $u=1$, $v=-3$, $w=0$ and $d=1$

The equation of tangent plane at $(1, 2, -2)$ to the sphere (i) is

$x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)\,w(z+\gamma)+d=0$

i.e., $x+2y-2z+1(x+1)-3(y+2)+0(z-2)+1=0$

or, $x+2y-2z+x+1-3y-6+1=0$

or, $2x-y-2z-4=0$

Now, the equation of sphere touching sphere (i) at $(1, 2, -2)$ is

$x^2+y^2+z^2+2x-6y+1+k(2x-y-2z-4)=0$ .... (ii)

Since sphere (ii) passes through $(0, 0, 0)$. So we have

$1-4k=0$

or, $k=\dfrac{1}{4}$

Substituting the value of k in equation (iii), we get

$(x^2+y^2+z^2+2x-6y+1)+\dfrac{1}{4}(2x-y-2z-4)=0$

or, $4(x^2+y^2+z^2+2x-6y+1)+(2x-y-2z-4)=0$

or, $4x^2+4y^2+4z^2+8x-24y+4+2x-y-2z-4=0$

or, $4x^2+4y^2+4z^2+10x-25y-2z=0$

which is the required equation of the sphere.

7. **Obtain the equations of the tangent planes to the sphere $x^2+y^2+z^2+6x-2z+1=0$ which pass through the line $3(16-x)=3z=2y+30$.**

Solution:

The given equation of sphere is

$$x^2+y^2+z^2+6x-2z+1=0 \quad \dots (i)$$

Comparing sphere (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, we get $u=3$, $v=0$, $w=-1$ and $d=1$

Thus, centre of sphere (i) $=(-u,-v,-w)=(-3,0,1)$

and radius $=\sqrt{u^2+v^2+w^2-d}=\sqrt{9+0+1-1}=3$

Also, the given equation of line is

$3(16-x)=3z=2y+30$

or, $16-x-z=2y+30-3z \dots (ii)$

The equation of plane through line (ii) is

$16-x-z+k(2y-3z+30)=0$

or, $-x+2ky-z(1+3k)+16+30k=0 \dots (iii)$

If plane (iii) is tangent plane to sphere (i), then the length of the perpendicular from centre of the sphere (i) to the plane (iii) is equal to the radius sphere (i).

$\therefore$ $\left|\dfrac{3-1-3k+16+30k}{\sqrt{1+4k^2+1+9k^2+6k}}\right|=3$

or, $\dfrac{18+27k}{\sqrt{13k^2+6k+2}}=3$

or, $(6+9k)^2=13k^2+6k+2$

or, $36+81k^2+108k=13k^2+6k+2$

or, $68k^2+102k+34=0$

or, $2k^2+3k+1=0$

or, $k=-1,-\dfrac{1}{2}$

Substituting the value of $k=-1$ in equation (iii), we get $x+2y-2z+14=0$

Substituting the value of $k=-\dfrac{1}{2}$ in equation (iii), we get $2x+2y-z-2=0$

Thus, the required equation of tangent planes are

$x+2y-2z+14=0$ and $2x+2y-z-2=0$.

8. **Find the equations to the tangent planes to the sphere $x^2+y^2+z^2-2x+4y-6z+10=0$ which pass through the line $\dfrac{x+3}{14}=\dfrac{y+1}{-3}=\dfrac{z-5}{4}$.**

Solution:

The given equation of sphere is

$$x^2+y^2+z^2-2x+4y-6z+10=0 \quad \dots (i)$$

Comparing (i) with

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

we get,

$u=-1$, $v=2$, $w=-3$, $d=10$

So, centre of the sphere $(-u,-v,-w)=(1,-2,3)$

and radius of the sphere $=\sqrt{u^2+v^2+w^2-d}$

$=\sqrt{1+4+9-10}$

$=\sqrt{4}$

$=2$

Also, the given line is

$$\frac{x+3}{14}=\frac{y+1}{-3}=\frac{z-5}{4} \quad \dots (ii)$$

Now, $\dfrac{x+3}{14}=\dfrac{y+1}{-3}$

$\Rightarrow$ $-3x-9=14y+14$

$\Rightarrow$ $3x+14y+23=0$

And $\dfrac{y+1}{-3}=\dfrac{z-5}{4}$

$\Rightarrow$ $4y+4=-3z+15$

$\Rightarrow$ $4y+3z-11=0$

The equation of the plane through the line (ii) is

$(3x+14y+23)+k(4y+3z-11)=0$

$$\Rightarrow \quad 3x+(14+4k)y+3kz+23-11k=0 \quad \dots (iii)$$

If the plane (iii) is tangent plane to the sphere (i), then the length of the perpendicular from centre of the sphere (i) to the plane (ii) is equal to the radius of the sphere (i).

i.e. $\pm\dfrac{3.1+(14+4k)(-2)+3k(3)+23-11k}{\sqrt{3^2+(14+4k)^2+(3k)^2}}=2$

$\Rightarrow$ $\pm\dfrac{-10k-2}{\sqrt{25k^2+112k+205}}=2$

$\Rightarrow$ $\pm\dfrac{10k+2}{\sqrt{25k^2+112k+205}}=2$

$\Rightarrow$ $\pm\dfrac{5k+1}{\sqrt{25k^2+112k+205}}=1$

Squaring both sides, we get

$(5k+1)^2=25k^2+112k+205$

$\Rightarrow$ $25k^2+10k+1=25k^2+112k+205$

$\Rightarrow$ $-102k=204$

$\Rightarrow$ $k=-2$

Substituting the value of $k=-2$ in (iii), we get

$x+2y-2z+15=0$

Exercise 7.6

1. **Find the equation of the cone with vertex at the origin and which passes through the curve of intersection of $x^2+y^2+2z^2=1$ and $x-y+2z=7$**

Solution:

The given equations are

$$x^2+y^2+2z^2=1 \quad \dots (i)$$

and $$x-y+2z=7 \quad \dots (ii)$$

As we know that the equation of the cone with vertex at the origin must be homogeneous. So the required equation of the cone can be obtained by making (i) homogeneous with the help of (ii). Thus, we have

$$x^2 + y^2 + 2z^2 = \left(\frac{x - y + 2z}{7}\right)^2$$

or, $49(x^2 + y^2 + 2z^2) = (x - y + 2z)^2$

which is the required equation of the cone.

2. Find the equation of the cone with vertex (α, β, γ) and base $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$

Solution:

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\text{(i)}$$

Suppose line (i) meets the plane $z = 0$ at point $(x_1, y_1, 0)$.

Then,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$\therefore$ $x_1 = \alpha - \frac{l\gamma}{n}, y_1 = \beta - \frac{m\gamma}{n}$

If $(x_1, y_1, 0)$ lies on the given conic $y^2 = 4ax$, then

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$$

or, $b^2x_1^2 + a^2y_1^2 = a^2b^2$

or, $b^2\left(\alpha - \frac{l\gamma}{n}\right)^2 + a^2\left(\beta - \frac{m\gamma}{n}\right)^2 = a^2b^2 \ldots$ (ii)

Eliminating l, m, n from (i) and (ii), we get

$$b^2\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2 + a^2\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = a^2b^2$$

or, $b^2[\alpha z - \alpha\gamma - x\gamma + \alpha\gamma]^2 + a^2[\beta z - \beta\gamma - y\gamma + \beta\gamma]^2 = a^2b^2(z-\gamma)^2$

or, $b^2(\alpha z - \gamma x)^2 + a^2(\beta z - y\gamma)^2 = a^2b^2(z-\gamma)^2$

which is the required equation of the cone.

3. Find the equation of cone whose vertex is (α, β, γ) and base (guiding curve) is $ax^2 + by^2 = 1, z = 0$.

Solution:

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\text{(i)}$$

Suppose line (i) meets the plane $z = 0$ at point $(x_1, y_1, 0)$.

Then,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$\therefore$ $x_1 = \alpha - \frac{l\gamma}{n}, y_1 = \beta - \frac{m\gamma}{n}$

If $(x_1, y_1, 0)$ lies on the given conic $ax^2 + by^2 = 1$, then

or, $ax_1^2 + by_1^2 = 1$

or, $a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 = 1 \ldots$ (ii)

Eliminating l, m, n from (i) and (ii), we get

$$a\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2 + b\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = 1$$

or, $a[\alpha z - \alpha\gamma - x\gamma + \alpha\gamma]^2 + b[\beta z - \beta\gamma - y\gamma + \beta\gamma]^2 = (z-\gamma)^2$

or, $a(\alpha z - \gamma x)^2 + b(\gamma z - y\gamma)^2 = (z-\gamma)^2$

which is the required equation of the cone.

4. Find the equation of cone whose vertex is (α, β, γ) and guiding curve is $y^2 = 4ax, z = 0$.

Solution:

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\text{(i)}$$

Suppose line (i) meets the plane $z = 0$ at point $(x_1, y_1, 0)$. Then,

$$\frac{x_1-\alpha}{l} = \frac{y_1-\beta}{m} = \frac{-\gamma}{n}$$

$\therefore$ $x_1 = \alpha - \frac{l\gamma}{n}, y_1 = \beta - \frac{m\gamma}{n}$

If $(x_1, y_1, 0)$ lies on the given conic $y^2 = 4ax$, then $y_1^2 = 4ax_1$

$$\left(\beta - \frac{m\gamma}{n}\right)^2 = 4a\left(\alpha - \frac{l\gamma}{n}\right) \quad \ldots\text{(ii)}$$

Eliminating l, m, n from (i) and (ii), we get

$$\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 = 4a\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$

or, $[\beta z - \beta\gamma - y\gamma + \beta\gamma]^2 = 4a[\alpha z - \alpha\gamma - x\gamma + \alpha\gamma]$

or, $(\beta z - y\gamma)^2 = 4a(\alpha z - x\gamma)(z - \gamma)$

which is the required equation of the cone.

5. Find the equation of cone whose vertex is $(1, 1, 0)$ and whose guiding curve is $y = 0, x^2 + z^2 = 4$.

Solution:

Any line through $(1, 1, 0)$ is

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-0}{n} \quad \ldots\text{(i)}$$

Suppose line (i) meets the plane $y = 0$ at point $(x_1, 0, z_1)$. Then,

$$\frac{x_1-1}{l} = \frac{-1}{m} = \frac{z_1}{n}$$

$\therefore$ $x_1 = 1 - \frac{l}{m}, z_1 = -\frac{n}{m}$

If $(x_1, 0, z_1)$ lies on the given conic $x^2 + z^2 = 4$, then

or, $x_1^2 + z_1^2 = 4$

or, $\left(1 - \frac{l}{m}\right)^2 + \left(-\frac{n}{m}\right)^2 = 4 \ldots$ (ii)

Eliminating l, m, n from (i) and (ii), we get

$$\left[1-\left(\frac{x-1}{y-1}\right)\right]^2+\left[-\left(\frac{z}{y-1}\right)\right]^2=4$$

or, $(y-1)^2-(x-1)^2+z^2=4(y-1)^2$

or, $-3(y-1)^2-(x-1)^2+z^2=0$

or, $3(y-1)^2+(x-1)^2-z^2=0$

which is the required equation of the cone.

6. Find equation of cone which pass through (α, β, γ) and intersect along the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$.

Solution:

Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots.(i)$$

Suppose line (i) meets the plane $z=0$ at point $(x_1, y_1, 0)$.

Then,

$$\frac{x_1-\alpha}{l}=\frac{y_1-\beta}{m}=\frac{-\gamma}{n}$$

$\therefore \quad x_1=\alpha-\frac{l\gamma}{n}, y_1=\beta-\frac{m\gamma}{n}$

If $(x_1, y_1, 0)$ lies on the given conic $y^2=4ax$, then

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

or, $b^2x_1^2+a^2y_1^2=a^2b^2$

or, $b^2\left(\alpha-\frac{l\gamma}{n}\right)^2+a^2\left(\beta-\frac{m\gamma}{n}\right)^2=a^2b^2 \ldots.(ii)$

Eliminating l, m, n from (i) and (ii), we get

$$b^2\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2+a^2\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2=a^2b^2$$

or, $b^2[\alpha z-\alpha\gamma-x\gamma+\alpha\gamma]^2+a^2[\beta z-\beta\gamma-y\gamma+\beta\gamma]^2=a^2b^2(z-\gamma)^2$

or, $b^2(\alpha z-\gamma x)^2+a^2(\beta z-y\gamma)^2=a^2b^2(z-\gamma)^2$

which is the required equation of the cone.

7. Find equation of right circular cone whose vertex is at origin and axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ with semi vertical angle 30°.

Solution:

Given that axis of a right circular cone is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

The direction cosines of axis of right circular cone are proportional to 1, 2, 3.

The vertex of the right circular cone is $O(0, 0, 0)$. Let $P(x, y, z)$ be any point on the right circular cone. Then, the direction ratios of OP are x, y, z. Since the semi-vertical angle is 30°, so

$$\cos 30^\circ=\frac{1.(x)+2(y)+3(z)}{\sqrt{1^2+2^2+3^2}\sqrt{x^2+y^2+z^2}}$$

or, $\frac{\sqrt{3}}{2}=\frac{1}{\sqrt{14}}\cdot\frac{x+2y+3z}{\sqrt{x^2+y^2+z^2}}$

Squaring both sides, we get $42(x^2+y^2+z^2)=4(x+2y+3z)^2$

which is the required equation of the cone.

8. Find the equation to the right circular cone whose vertex is at origin, the axis along x-axis and semi vertical angle α.

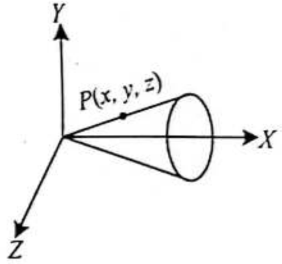
Solution:

Let $P(x, y, z)$ be any point on the surface of the cone. Then, the direction ratios of OP are $x-0, y-0, z-0$ i.e., x, y, z.

Also, the direction cosines of x-axis are 1, 0, 0. Since semi-vertical angle is α, So

$$\cos\alpha=\frac{x.1+y.0+z.0}{\sqrt{x^2+y^2+z^2}}$$

or, $\cos\alpha=\frac{x}{\sqrt{x^2+y^2+z^2}}$



Squaring both sides, we get

$$\cos^2\alpha=\frac{x^2}{x^2+y^2+z^2}$$

or, $\frac{1}{\sec^2\alpha}=\frac{x^2}{x^2+y^2+z^2}$

or, $x^2+y^2+z^2=x^2\sec^2\alpha$

or, $x^2+y^2+z^2=x^2(1+\tan^2\alpha)$

or, $x^2+y^2+z^2=x^2+x^2\tan^2\alpha$

or, $y^2+z^2=x^2\tan^2\alpha$

9. The equation of axis to a cone whose vertex is at origin is $x=-2y=z$. If its semi vertical angle is $\frac{\pi}{2}$, find its equation.

Solution:

Given that axis of a right circular cone is

$$x=-2y=z$$

i.e., $\frac{x}{1}=\frac{y}{\frac{-1}{2}}=\frac{z}{1}$

The direction cosines of axis of right circular cone are proportional to $1, -\frac{1}{2}, 1$.

The vertex of the right circular cone is $O(0, 0, 0)$. Let $P(x, y, z)$ be any point on the right circular cone. Then, the direction ratios of OP are x, y, z. Since the semi-vertical angle is 90°, so

$$\cos 90^\circ=\frac{1(x)+\frac{-1}{2}(y)+1(z)}{\sqrt{1^2+\left(\frac{-1}{2}\right)^2+1^2}\sqrt{x^2+y^2+z^2}}$$

or, $1=\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{2x-y+2z}{\sqrt{x^2+y^2+z^2}}$

or, $1=\frac{1}{3}\cdot\frac{2x-y+2z}{\sqrt{x^2+y^2+z^2}}$

Squaring both sides, we get $9(x^2+y^2+z^2)=(2x-y+2z)^2$

which is the required equation of the cone.

10. Find the equation of right circular cone with vertex at (2, –3, 5) axis makes equal angle with coordinate axes and semi vertical angle is 30°.

Solution:

Suppose the axis of a right cone with vertex at O makes an angle α with the co-ordinate axes.

Then, the direction cosines of axis of a right cone are

$l = \cos\alpha,\ m = \cos\alpha,\ n = \cos\alpha$

But we know that

$l^2 + m^2 + n^2 = 1$

or, $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$

or, $3\cos^2\alpha = 1$

$\therefore$ $\cos\alpha = \dfrac{1}{\sqrt{3}}$

i.e., $l = m = n = \dfrac{1}{\sqrt{3}}$

Thus, the direction cosines of axis of right cone are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$

The direction cosines of axis of right circular cone are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

The vertex of the right circular cone is $O(2, -3, 5)$. Let $P(x, y, z)$ be any point on the right circular cone. Then, the direction ratios of OP are $x - 2,\ y + 3,\ z + 5$. Since the semi-vertical angle is 30°, so

$$\cos 30° = \frac{\frac{1}{\sqrt{3}}(x-2) + \frac{1}{\sqrt{3}}(y+3) + \frac{1}{\sqrt{3}}(z+5)}{\sqrt{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}\ \sqrt{(x-2)^2 + (y+3)^2 + (z+5)^2}}$$

or, $\dfrac{\sqrt{3}}{2} = \dfrac{1}{\sqrt{3}} \cdot \dfrac{(x-2) + (y+3) + (z+5)}{\sqrt{(x-2)^2 + (y+3)^2 + (z+5)^2}}$

which is the required equation of the cone.

11. Find the equation of a cone which has vertex at origin and passes through the curve $ax^2 + by^2 = 2z,\ \ell x + my + nz = p$.

Solution:

The given equations are

$ax^2 + by^2 = 2z$ ....(i)

and $lx + my + nz = p$ ....(ii)

As we know that the equation of the cone with vertex at the origin must be homogeneous. So the required equation of the cone can be obtained by making (i) homogeneous with the help of (ii). Thus, we have

$$ax^2 + by^2 = 2z\left(\frac{lx + my + nz}{p}\right)$$

or, $p(ax^2 + by^2) = 2z(lx + my + nz)$

which is the required equation of the cone.

Exercise 7.7

1. Find the equation of a cylinder whose generating lines have direction cosines (ℓ, m, n) and which passes through the circle $x^2 + y^2 = a^2, y = 0$.

Solution:

Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder. Then the equations of the generator through $P(\alpha, \beta, \gamma)$ with direction cosines l, m, n is $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n}$

If this meets the plane $z = 0$, then

$\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{-\gamma}{n}$

$\therefore$ $x = \alpha - \dfrac{l\gamma}{n},\ y = \beta - \dfrac{m\gamma}{n}$

If this point $(x, y, 0)$ lies on the circle $x^2 + y^2 = a^2$, then

$$\left(\alpha - \frac{l\gamma}{n}\right)^2 + \left(\beta - \frac{m\gamma}{n}\right)^2 = a^2$$

Thus, the locus of $P(\alpha, \beta, \gamma)$ is $(y - z)^2 = 4a(x - z)$

which is the required equation of the cylinder.

2. Find the equation of a cylinder whose generators are parallel to the line $\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z}{3}$ and guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 0$.

Solution:

The given line is

$\dfrac{x}{1} = \dfrac{y}{-2} = \dfrac{z}{3}$ ....(i)

and the guiding curve is the ellipse

$x^2 + 2y^2 = 1,\ z = 0$ .....(ii)

Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder. Then the equations of the generator through $P(\alpha, \beta, \gamma)$ and parallel to (i) are

$\dfrac{x-\alpha}{1} = \dfrac{y-\beta}{-2} = \dfrac{z-\gamma}{3}$

If this meets the plane $z = 0$, then

$\dfrac{x-\alpha}{1} = \dfrac{y-\beta}{-2} = \dfrac{-\gamma}{3}$

$\therefore$ $x = \alpha - \dfrac{\gamma}{3},\ y = \beta + \dfrac{2\gamma}{3}$

If this point $(x, y, 0)$ lies on the ellipse $x^2 + 2y^2 = 1$, then

$$\left(\alpha - \frac{\gamma}{3}\right)^2 + 2\left(\beta + \frac{2\gamma}{3}\right)^2 = 1$$

or, $(3\alpha - \gamma)^2 + 2(3\beta + 2\gamma)^2 = 9$

or, $9\alpha^2 + \gamma^2 - 6\alpha\gamma + 18\beta^2 + 8\gamma^2 + 24\beta\gamma - 9 = 0$

or, $9\alpha^2 + 18\beta^2 + 9\gamma^2 - 6\alpha\gamma + 24\beta\gamma - 9 = 0$

or, $3\alpha^2 + 6\beta^2 + 3\gamma^2 - 2\alpha\gamma + 8\beta\gamma - 3 = 0$

Thus, the locus of $P(\alpha, \beta, \gamma)$ is $3x^2 + 6y^2 + 3z^2 - 2xz + 8yz - 3 = 0$

which is the required equation of the cylinder.

3. **Find the equation of cylinder whose generators are parallel to $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and guiding curve is the ellipse $x^2+2y^2=1$, $z=3$.**

Solution:

The given line is

$$\frac{x}{1}=\frac{y}{-2}=\frac{z}{3} \quad \ldots\ldots (i)$$

and the guiding curve is the ellipse

$$x^2+2y^2=1,\ z=3 \quad \ldots\ldots (ii)$$

Let $P(\alpha,\beta,\gamma)$ be any point on the cylinder. Then the equations of the generator through $P(\alpha,\beta,\gamma)$ and parallel to (i) are $\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{z-\gamma}{3}$

If this meets the plane $z=3$, then

$$\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{3-\gamma}{3}$$

$\therefore$ $x=\alpha-\frac{\gamma}{3}+1,\ y=\beta+\frac{2\gamma}{3}-2$

If this point $(x, y, 0)$ lies on the ellipse $x^2+2y^2=1$, then

$$\left(\alpha-\frac{\gamma}{3}+1\right)^2+2\left(\beta+\frac{2\gamma}{3}-2\right)^2=1$$

or, $(3\alpha-\gamma+3)^2+2(3\beta+2\gamma-6)^2=9$

or, $9\alpha^2+\gamma^2+9+18\alpha-6\gamma-6\alpha\gamma+18\beta^2+8\gamma^2+72+24\beta\gamma-48\gamma-72\beta-9=0$

or, $3\alpha^2+6\beta^2+3\gamma^2-2\alpha\gamma+8\beta\gamma+6\alpha-24\beta-18\gamma+24=0$

Thus, the locus of $P(\alpha,\beta,\gamma)$ is $3x^2+6y^2+3z^2-2xz+8yz+6x-24y-18z+24=0$

which is the required equation of the cylinder.

4. **Find the equation of cylinder whose generators are parallel to $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and guiding curve is the ellipse $x^2+y^2=16$, $z=0$.**

Solution:

The given line is

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \quad \ldots\ldots (i)$$

and the guiding curve is the ellipse

$$x^2+y^2=16,\ z=0 \quad \ldots\ldots (ii)$$

Let $P(\alpha,\beta,\gamma)$ be any point on the cylinder. Then the equations of the generator through $P(\alpha,\beta,\gamma)$ and parallel to (i) are

$$\frac{x-\alpha}{1}=\frac{y-\beta}{2}=\frac{z-\gamma}{3}$$

If this meets the plane $z=0$, then

$$\frac{x-\alpha}{1}=\frac{y-\beta}{2}=\frac{-\gamma}{3}$$

$\therefore$ $x=\alpha-\frac{\gamma}{3},\ y=\beta+\frac{2\gamma}{3}$

If this point $(x, y, 0)$ lies on the ellipse $x^2+y^2=16$, then

$$\left(\alpha-\frac{\gamma}{3}\right)^2+\left(\beta+\frac{2\gamma}{3}\right)^2=16$$

or. $(3\alpha-\gamma)^2+2(3\beta+2\gamma)^2=144$

or. $9\alpha^2+\gamma^2-6\alpha\gamma+18\beta^2+8\gamma^2+24\beta\gamma-144=0$

or. $9\alpha^2+18\beta^2+9\gamma^2-6\alpha\gamma+24\beta\gamma-144=0$

or. $3\alpha^2+6\beta^2+3\gamma^2-2\alpha\gamma+8\beta\gamma-48=0$

Thus, the locus of $P(\alpha,\beta,\gamma)$ is $3x^2+6y^2+3z^2-2xz+8yz-48=0$

which is the required equation of the cylinder.

5. **Find the equation of right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$.**

Solution:

The axis is the line $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$

The direction cosines of the axis are proportional to 2, 1, 2.

Also, we have the radius of the cylinder is 2.

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2}+R^2$$

or, $$(x-1)^2+(y-2)^2+(z-3)^2=\frac{[2(x-1)+1(y-2)+2(z-3)]^2}{2^2+1^2+2^2}+2^2$$

which is the required equation of the cylinder.

6. **Find the equation of right circular cylinder of radius 3 and having its axis the line $\frac{x-1}{2}=\frac{y-0}{2}=\frac{z-3}{1}$.**

Solution:

The axis is the line $\frac{x-1}{2}=\frac{y-0}{2}=\frac{z-3}{1}$

The direction cosines of the axis are proportional to 2, 2, 1.

Also, we have the radius of the cylinder is 3.

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=\frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]^2}{l^2+m^2+n^2}+R^2$$

or, $$(x-1)^2+(y-0)^2+(z-3)^2=\frac{[2(x-1)+1(y-0)+2(z-3)]^2}{2^2+1^2+2^2}+3^2$$

which is the required equation of the cylinder.

7. **Find the equation of a right circular cylinder whose guiding curve is $x^2+y^2+z^2=9$, $x-2y+2z=3$.**

Solution:

The given circle is

$$x^2+y^2+z^2=9,\ x-2y+2z=3 \quad \ldots\ldots (i)$$

Centre of sphere of circle (i) is (0, 0, 0)

and radius of sphere (i) is $r=3$

Length of perpendicular from centre of the sphere to the plane

$x-2y+2z-3=0$ is

$$p = \left| \frac{0 - 0 + 0 - 3}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} \right| = \frac{3}{3} = 1$$

Thus radius of the circle $(R) = \sqrt{r^2 - p^2}$

$= \sqrt{9 - 1}$

$= \sqrt{8}$

The axis of the cylinder passing through origin $O(0, 0, 0)$ and perpendicular to the plane $x - 2y + 2z = 3$ is

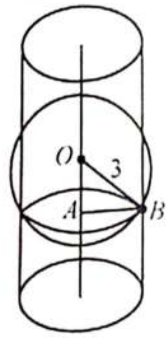
$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{2} \quad \cdots\cdots (ii)$$

where direction ratios of line (ii) are to $1, -2, 2$.

Thus, the direction cosines of line (ii) are $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

Let $P(x, y, z)$ be any point on the cylinder. Join OP and draw $PM \perp$ to the axis, then

PM = radius of the cylinder (or circle) $= \sqrt{8}$



Also, $OP^2 = x^2 + y^2 + z^2$

Again,

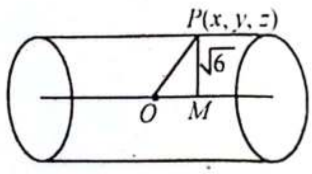
OM = projection of OP on the axis of cylinder.

$$= (x - 0).\frac{1}{3} + (y - 0).\left(-\frac{2}{3}\right) + (z - 0).\frac{2}{3}$$

$$= \frac{x - 2y + 2z}{3}$$

From right angled triangle OPM, we have

$OP^2 = OM^2 + PM^2$



or, $x^2 + y^2 + z^2 = \left(\frac{x - 2y + 2z}{3}\right)^2 + (\sqrt{8})^2$

or, $9(x^2 + y^2 + z^2) = (x - 2y + 2z)^2 + 72$

or, $9x^2 + 9y^2 + 9z^2 = x^2 + y^2 + z^2 - 4xy - 8yz + 4xz + 72$

or, $9x^2 + 9y^2 + 9z^2 = x^2 + y^2 + z^2 - 4xy - 8yz + 4xz + 72$

or, $8x^2 + 8y^2 + 8z^2 + 4xy + 8yz - 4xz - 72 = 0$

which is the required equation of the right circular cylinder.