Errors

An error is defined as the difference between a true value and estimate or approximation of that value. i.e. Error = TV − EV

Types of errors:

(i) Conversion error
(ii) Human error
(iii) Machine error
(iv) Numerical error

Numerical Error

a) Round off error

The round off error is used because representing every number as a real number is not possible. A round off error is closest real number value.

For eg:

Rounding to nearest whole number

Result : 6.22

After rounding: 6

Rounding to highest whole number.

Result : 2

After rounding : 7

b) Truncation error

A truncation error occur when approximation is involved in numerical analysis.

For eg:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

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c) Absolute Error

It is the magnitude of the difference between the value 'x' and the approximate value 'x_a'.

$\therefore$ Absolute error $(E_{abs}) = |x - x_a|$

When, 'x' denote the exact value and 'x_a' denotes as the approximate value.

d) Relative Error

It is the magnitude of the difference between the true value 'x' and the approximate value 'x_a' divided by exact value.

$$(E_{rel}) = \frac{|x - x_a|}{x}$$

e) Percentage error

$$\% \text{ Error} = \frac{|x - x_a|}{|x|} \times 100$$

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(1) Define error with an example. Find the truncation error if the number 752.6835 is truncated to four significant digits.

Sol^n:-

True value (TV) = 752.6835

Truncate value (Tr.V) = 752.6 [∵ 4-digit]

∴ Truncation Error = TV − Tr.V

= 752.6835 − 752.6

= 0.0835

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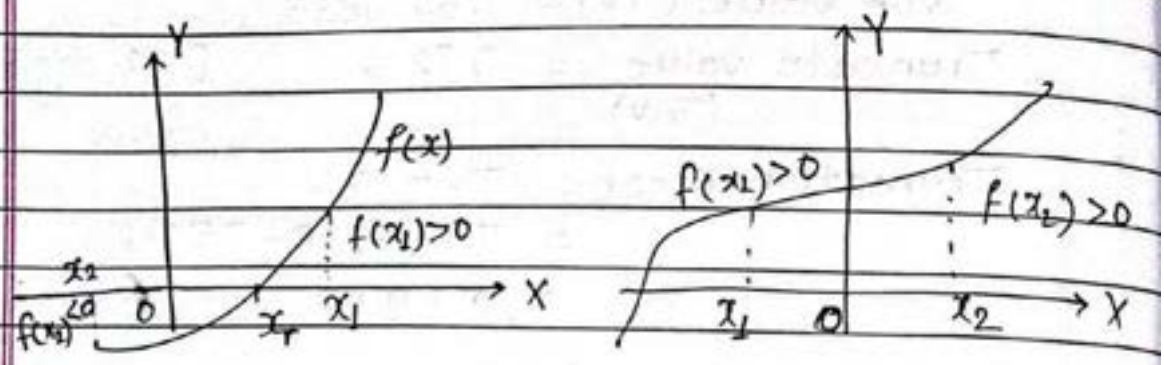
Solution of non linear equations

1) Root Bracketing Methods

Let $f(x)=0$ be the nonlinear equation



a. Bisection Method

Algorithm:-

(i) Let $f(x)=0$ be the non-linear equation, & E be the stopping criteria

(ii) Choose x_1 & x_2 such that $f(x_1)\cdot f(x_2)<0$

(iii) Calculate the new approximate value

$$x_0 = \frac{x_1+x_2}{2}$$

(iv) Compute $f(x_0)$, and if $f(x_0)=0$ go to step (vii)

v) If $f(x_0)\cdot f(x_1)<0$, set $x_1 = x_1$ & $x_2 = x_0$

Otherwise,

Set $x_1 = x_0$ & $x_2 = x_2$

vi) Repeat step 3, 4 & 5 until the stopping criteria meets.

vii) Stop

Ex. 1) Using bisection method, find a root of the equation $x^3 + x^2 - 1 = 0$ correct to two decimal places.

Sol^n:-

$f(x) = x^3 + x^2 - 1$

x	$f(x)$	
0	-1	<0
1	1	>0

Take $x_1 = 0$, $f(x_1) = -1 < 0$

$x_2 = 1$ $f(x_2) = 1 > 0$

$\therefore f(x_1) \cdot f(x_2) < -1 < 0$

Using bisection method

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$$

Calculation Table

Iteration No.	x_1(-ve)	x_2(+ve)	$x_0 = \frac{x_1 + x_2}{2}$	$f(x_0)$
1	0	1	0.5	-0.625
2	0.5	1	0.75	-0.015625
3	0.75	1	0.875	0.0355
4	0.75	0.875	0.8125	0.19653
5.	0.75	0.8125	0.7813	0.0873
6	0.75	0.7813	0.7657	0.035
7	0.75	0.7657	0.7579	0.009
8.	0.75	0.7579	0.754	-0.00282

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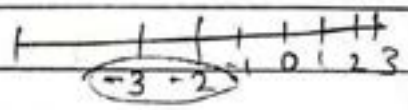
$\therefore$ Root of the given eq^n correct two decimal places $\simeq 0.754$

Ex- 2) Using bisection method, find a root of the equation $x^3 - 2x + 5 = 0$ correct four decimal places.

Sol^n:

$$f(x) = x^3 - 2x + 5$$

x	$f(x)$
0	5
1	4
-1	6
2	9
-2	1
3	26
-3	-16



Take $x_1 = -3$, $f(x_1) = -16$
$x_2 = -2$, $f(x_2) = 1$

$\therefore f(x_1) \cdot f(x_2) < -16 < 0$

Using bisection method

$$x_0 = \frac{x_1 + x_2}{2} = \frac{-3 + (-2)}{2} = \frac{-3-2}{2} = -2.5$$

Calculation Table

It. No.	x_1(-ve)	x_2(+ve)	$x_0 = \frac{x_1+x_2}{2}$	$f(x_0)$
1.	-3	-2	-2.5	-5.625
2.	-2.5	-2	-2.25	-1.8906
3.	-2.25	-2	-2.125	-0.3457
4.	-2.125	-2	-2.0625	0.351318
5.	-2.125	-2.0625	-2.09375	0.008416
6.	-2.125	-2.09375	-2.109375	-0.16683
7.	-2.109375	-2.09375	-2.10156	-0.078
8.	-2.10156	-2.09375	-2.09765	-0.0346
9	-2.09765	-2.09375	-2.0957	-0.0128
10	-2.0957	-2.09375	-2.094725	-0.0019369
11	-2.094725	-2.09375	-2.0942375	0.0035038
12	-2.094725	-2.0942375	-2.09448125	0.000783
13	-2.094725	-2.09448125	-2.09460312 5	-5.78×10^{-4}
14	-2.0946031	-2.09448125	-2.09454	1.03×10^{-4}
15	-2.0946031	-2.09454	-2.09457	

∴ Root of the given eqn correct four decimal places is -2.0945.

Trigonometric ⇒ Radian Mode

Date
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Ex. 1) Using bisection method, find a root of the equation $\cos x = 3x - 1$ correct to three decimal places.

Sol:- $\cos x = 3x - 1 \Rightarrow \cos x - 3x + 1 = 0$

$\therefore f(x) = \cos x - 3x + 1$

x	$f(x)$
0	$2 > 0$
1	$-1.45 < 0$

Take $x_1 = 0$; $f(x_1) = 2 > 0$

$x_2 = 1$ $f(x_2) = -1.45 < 0$

$\therefore f(x_1) \cdot f(x_2) = 2 \times -1.45 = -2.9 < 0$

$\therefore$ a root lies between '0' & '1'

Using bisection method

$$x_0 = \frac{x_1 + x_2}{2} = \frac{0+1}{2} = 0.5$$

$$f(x_0) = f(0.5) = 0.38 > 0$$

Iteration Table

It. No	x_1 (+ve)	x_2 (-ve)	x_0	$f(x_0)$ -
1.	0	1	0.5	0.38
2.	0.5	1	0.75	-0.52
3.	0.5	0.75	0.625	-0.064
4.	0.5	0.625	0.5625	0.186
5.	0.5625	0.625	0.59375	0.076
6.	0.59375	0.625	0.609375	-8.11×10^{-3}
7.	0.59375	0.609375	0.6015625	0.01976

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Note: $\left\{ \begin{array}{l} \log x - 3x + 5 = 0 \\ \ln x - 3x + 5 = \end{array} \right\} \rightarrow [\ln]$

$\log_{10} x - 3x + 5 \Rightarrow [\log_{\square} \square]$

8.	0.6015625	0.609375	0.60547	5.82×10^{-3}
9.	0.60547	0.609375	0.607423	-1.14×10^{-3}
10.	0.60547	0.607423	0.60643	2.33×10^{-3}
11.	0.60643	0.607423	0.6069	

$\therefore$ Root of the given eq^n correct 3 decimal places $\simeq 0.606$

Ex. 2) Using bisection method, find a root of the eq^n $2x - \log_{10} x - 7 = 0$. Correct to three decimal places.

Sol^n:-

$2x - \log_{10} x - 7 = 0$.

$\therefore f(x) = 2x - \log_{10} x - 7$

x	$f(x)$
1	-5
2	-3.30
3	-1.4
4.	0.39

Take $x_1 = 3$; $f(x_1) = -1.4 < 0$

$x_2 = 4$; $f(x_2) = 0.39 > 0$

$\therefore f(x_1) \cdot f(x_2) = -1.4 \times 0.39 = \cancel{-1.98970}\ -0.546$

$\therefore$ Root lies between '3' & '4'

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Using bisection method

$$x_0 = \frac{x_1 + x_2}{2} = \frac{3+4}{2} = 3.5$$

$$f(x_0) = f(3.5) = -0.544 < 0$$

Calculation Table:-

Iter.No.	x_1 (-ve)	x_2 (+ve)	x_0	$f(x_0)$
1	3	4	3.5	-0.544
2.	3.5	4	3.75	-0.07
3.	3.75	4	3.875	0.16
4.	3.75	3.875	3.813	0.043
5.	3.75	3.813	3.782	-0.0146
6.	3.782	3.813	3.796	0.0155
7.	3.782	3.796	3.789	-5.24×10^{-3}
8.	3.789	3.796	3.7925	6.07×10^{-3}
9	3.789	3.7925	3.79075	2.77×10^{-3}
10.	3.789	3.79075	3.789	1.125×10^{-3}
11.	3.789	3.79075	3.789	1.125×10^{-3}

$\therefore$ Root of the given eq^n correct three decimal places ≈ 3.789

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2019 Fall

Ex. Find where the graph of $y = x - 3$ and $y = \ln x$ intersect using bisection method. Obtain the result correct to four decimal places.

Soln:

Given,

$$y = x - 3 \quad \text{---(i)}$$
$$y = \ln x \quad \text{---(ii)}$$

Point of intersection of the given curve is obtained by

$$x - 3 = \ln x$$

i.e. $x - 3 - \ln x = 0$

$\therefore f(x) = x - 3 - \ln x$

x	$f(x)$
1	-2
2	-1.69
3	-1.098
4	-0.38
5	0.39

Take $x_1 = 4$; $f(x_1) = -0.38$

$x_2 = 5$; $f(x_2) = 0.39$

$\therefore f(x_1) \cdot f(x_2) = -0.1482 < 0$

$\therefore$ Root lies betn 4 and 5

Using bisection method

$$x_0 = \frac{x_1 + x_2}{2} = \frac{4+5}{2} = 4.5$$

$$f(x_0) = f(4.5) = -4.0774 \times 10^{-3} < 0$$

Calculation Table

It. No.	x_1 (-ve)	x_2 (+ve)	x_0	$f(x_0)$
1.	4	5	4.5	-4.0774×10^{-3}
2.	4.5	5	4.75	0.192
3.	4.5	4.75	4.625	1.544
4.	4.5	4.625	4.5625	0.045
5.	4.5	4.5625	4.5313	0.0202
6.	4.5	4.5313	4.51565	8.1×10^{-3}
7.	4.5	4.51565	4.50783	2.01×10^{-3}
8.	4.5	4.50783	4.50392	-1.03×10^{-3}
9.	4.50392	4.50783	4.50586	4.98×10^{-4}
10	4.50392	4.50586	4.50489	-2.73×10^{-4}
11.	4.50489	4.50586	4.505375	1.038×10^{-4}
12.	4.50489	4.505375	4.505133	-8.48×10^{-5}
13.	4.505133	4.505375	4.505254	9.72×10^{-6}
14.	4.505133	4.505254	4.505194	-3.73×10^{-5}
15.	4.505194	4.505254	4.505224	-1.36×10
16.	4.505224	4.505254	4.505239	$-1.94 \times 10^{-}$

$\therefore$ Root of the given eq^n correct ~~three~~ four decimal places $\simeq 4.5052$.

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b. False Position (or Regula-Falsi) Method

Algorithm

1) Let $f(x)=0$ be the eq^n and stopping criteria is given.

2) Choose 'x_1' & 'x_2' such that $f(x_1)\cdot f(x_2)<0$

3) Calculate $x_0 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

4) Calculate $f(x_0)$. If $f(x_0)=0$, then the root of eq^n is x_0.

5) If $f(x_0)\cdot f(x_1)<0$, then

set $x_1 = x_1$, $x_2 = x_0$

otherwise, set $x_1 = x_0$, $x_2 = x_2$

6) Repeat steps 4 & 5 until the stopping criteria meets.

7) Stop.

Eg: 1 Find a root of equation $x^3 - 2x + 5 = 0$ using false position method correct to four decimal places.

Sol^n:-

Given,

$x^3 - 2x + 5 = 0$

$f(x) = x^3 - 2x + 5$

x	$f(x)$	x	$f(x)$
0	5	-2	1
1	4	3	26
-1	6	-3	-16
2	9		

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Take $x_1 = -2$; $f(x_1) = 1 > 0$
$x_2 = -3$; $f(x_2) = -16 < 0$

$\therefore f(x_1) \cdot f(x_2) = -16 < 0$
Root lies in between -2 & -3.

Using False Position method:-

$$x_0 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(-2)(-16) - (-3)\cdot 1}{(-16) - 1}$$

$= -2.05882$

$\therefore f(x_0) = 0.3911$

Calculation table

It. No	x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
1	-2	-3	1	-16	-2.0588	0.3911
2.	-2.0588	-3	0.3911	-16	-2.0813	0.1468
3.	-2.0813	-3	0.1468	-16	-2.0896	0.0545
4.	-2.0896	-3	0.0545	-16	-2.0926	0.02074
5.	-2.0926	-3	0.02074	-16	-2.0937	8.66×10^{-3}
6.	-2.0937	-3	8.66×10^{-3}	-16	-2.0941	4.026×10^{-3}
7.	-2.0941	-3	4.026×10^{-3}	-16	-2.0943	2.49×10^{-3}
8	-2.0943	-3	2.49×10^{-3}	-16	-2.0944	1.23×10^{-3}
9.	-2.0944	-3	1.23×10^{-3}	-16	-2.0944	9.13×10^{-4}

$\therefore$ Root of the given eqn correct four decimal places $\simeq -2.0944$.

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Take $x_1 = -2$; $f(x_1) = 1 > 0$
$x_2 = -3$; $f(x_2) = -16 < 0$

$\therefore f(x_1) f(x_2) = -16 < 0$
Root lies in between -2 & -3.

Using False Position method:-

$$x_0 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{(-2)(-16) - (-3)\cdot 1}{(-16) - 1}$$

$= -2.05882$
$\therefore f(x_0) = 0.3911$
Calculation table

It. No	x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
1	-2	-3	1	-16	-2.0588	0.3911
2.	-2.0588	-3	0.3911	-16	-2.0813	0.1468
3.	-2.0813	-3	0.1468	-16	-2.0896	0.0545
4.	-2.0896	-3	0.0545	-16	-2.0926	0.02074
5.	-2.0926	-3	0.02074	-16	-2.0937	8.66×10^{-3}
6.	-2.0937	-3	8.66×10^{-3}	-16	-2.0941	4.026×10^{-3}
7.	-2.0941	-3	4.026×10^{-3}	-16	-2.0943	2.49×10^{-3}
8	-2.0943	-3	2.49×10^{-3}	-16	-2.0944	1.23×10^{-3}
9.	-2.0944	-3	1.23×10^{-3}	-16	-2.0944	9.13×10^{-4}

$\therefore$ Root of the given eqn correct four decimal places $\simeq -2.0944$.

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2. Find a root of the eq^n $\log x = \cos x$ using false position method correct to four decimal places.

Sol^n. Given,

$\log x = \cos x$

$\log x - \cos x = 0$

$\therefore f(x) = \log x - \cos x$

x	$f(x)$
1	-0.5403
2	1.1092

Take $x_1 = 1$; $f(x_1) = -0.5403$

$x_2 = 2$; $f(x_2) = 1.1092$

$\therefore f(x_1) \cdot f(x_2) = -0.5403 \times 1.1092 = -0.5993 < 0$

Root lies between 1 & 2

Using False-Position Method

$$x_0 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{1 \times 1.1092 - 2 \times (-0.5403)}{1.1092 - (-0.5403)}$$

$$= 1.3275$$

$\therefore f(x_0) = 0.0428$

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It No.	x_1	x_2	$f(x_1)$	$f(x_2)$	x_0	$f(x_0)$
1.	1	2	-0.5403	1.1092	1.3275	0.0428
2.	1	1.3275	-0.5403	0.0428	1.3034	8.6109×10^{-4}
3.	1	1.3034	-0.5403	8.6109×10^{-4}	1.3029	-8.0999×10^{-5}
4.	1.3029	1.3034	-8.0999×10^{-5}	8.6109×10^{-4}	1.3031	3.0887×10^{-4}
5.	1.3029	1.3031	-8.0999×10^{-5}	3.0887×10^{-4}	1.3029	-3.8879×10^{-5}
6.	1.3029	1.3031	-3.8879×10^{-5}	3.0887×10^{-4}	1.3029	-1.1083×10^{-4}

$\therefore$ Root of the given eqn correct four decimal places $\simeq$ 1.3029

Non- Bracketing Methods

1) Secant Method

Let,

$f(x)=0$ be the given equation.

Let, x_1 & x_2 be the initial guesses.

Then, the Secant method is given by

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

Algorithm

i) Define the eq^n $f(x)=0$ and stopping criteria.

ii) Let x_1 & x_2 be the initial ~~ease~~ guesses.

iii) Calculate $x_3 = \frac{x_1 . f_2 - x_2 f_1}{f_2 - f_1}$

iv) Calculate $f(x_3) = f_3$

v) Set $x_1 = x_2$, $x_2 = x_3$

vi) Repeat step 3, 4 & 5 until the stopping criteria meets.

vii) Stop.

Q.(i) Using Secant method, find a root of the eq^n $3x + \sin x - e^x = 0$ correct to ~~4~~ four decimal places.

Sol^n:-

$f(x) = 3x + \sin x - e^x$

x	$f(x)$
0	-1
1	1.1232

Take $x_1 = 0$, $x_2 = 1$

$f_1 = f(x_1) = -1$ $\quad f_2 = f(x_2) = 1.1232$

$\therefore$ Using Secant Method

$$x_3 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1}$$

$$= \frac{0 \cdot 1.1232 - 1 \cdot (-1)}{1.1232 - (-1)}$$

$$= 0.47098$$

$$= 0.4710$$

$f_3 = f(x_3) = 0.2652$

Iteration table

It.No	x_1	x_2	f_1	f_2	x_3	f_3
1	0	1	-1	1.1232	0.471	0.2652
2.	1	0.471	1.1232	0.2652	0.3075	-0.1348
3.	0.471	0.3075	0.2652	-0.1347	0.3626	0.0054
4.	0.3075	0.3626	-0.1347	0.0054	0.3605	0.0002
5.	0.3626	0.3605	0.0054	0.0002	0.3604	-0.00001
6.	0.3605	0.3604	0.0002	-0.00001	0.3604	-0.00001

$\therefore$ Root of the given eqn correct four decimal places $\approx$ 0.3604.

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Q.2. Using Secant method, find a root of the eqn $x\log_{10}x = 1.2$ correct to four decimal places.

Sol^n: $x\log_{10}x = 1.2 \Rightarrow x\log_{10}x - 1.2$

$f(x) = x\log_{10}x - 1.2$

x	$f(x)$
1	-1.2
2	-0.5979
3	0.2314

Take $x_1 = 2 \Rightarrow f_1 = -0.5979$

$x_2 = 3 \Rightarrow f_2 = 0.2314$

$\therefore$ Using Secant method

$$x_3 = \frac{x_1 f_2 - x_2 f_1}{f_2 - f_1} = 2.7210$$

$\therefore f_3 = f(x_3) = -0.0171$

Iteration Table

It No	x_1	x_2	f_1	f_2	x_3	f_3
1.	2	3	-0.5979	0.2314	2.7210	-0.0171
2.	3	2.7210	0.2314	-0.0171	2.7402	-0.0004
3.	2.7210	2.7402	-0.0171	-0.0004	2.7406	0.0000
4.	2.7402	2.7406	-0.0004	0.0000	2.7406	0.0000

$\therefore$ Root of the given eqn correct four decimal places $\approx$ 2.7406

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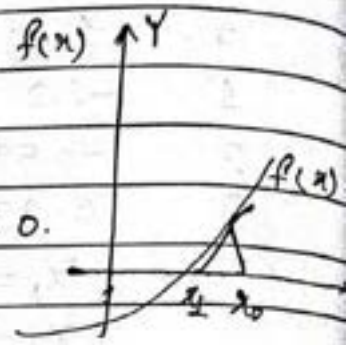
(Fastest method)

Newton-Raphson Method

Let, x_0 be the initial guess for the solution of $f(x)=0$.

Then, Newton's method is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$



Newton's method fails when $f'(x_0) = 0$.

In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots$$

Ex-1) Find a root of the eq^n $3x+\sin x - e^x = 0$ using Newton's method correct to four decimal places.

Sol^n:

Here,

$f(x) = 3x + \sin x - e^x$.

$f'(x) = 3 + \cos x - e^x$

x	$f(x)$
0	-1
1.	1.2832

Take

$$x_0 = \frac{0+1}{2} = 0.5$$

Using Newton' method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, 3 \ldots$$

$\therefore f(x_0) = f(0.5) = 0.3307$
$f'(x_0) = f'(0.5) = 2.2289$ $\neq 0$ } Note: $f(x_0)$ & $f'(x_0)$ must not $= 0$.

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.3307}{2.2289} = 0.3516$$

Iteration Table:

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	$x_0 = 0.5$	0.3307	2.2289	0.3516
1	0.3516	-0.0221	2.5175	0.3604
2	0.3604	-0.0001	2.5019	0.3604

$\therefore$ Root of the given eq^n correct to four decimal places ≈ 0.3604.

Algorithm:

1) Define the eq^n $f(x) = 0$ and stopping criteria.
2) ~~Calculate~~ Choose x_0 such that $f'(x_0) \neq 0$.
3) Calculate $f(x_0)$ & $f'(x_0)$.
4) Calculate $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
5) Calculate $f(x_1)$. If $f(x_1) = 0$, then write the root $= x_1$

   Otherwise, set $x_0 = x_1$ and repeat steps 3, 4 & 5 until the stopping criteria meets.
6) Stop.

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Q019

Find root of the eqn $\log x - \cos x = 0$ correct to three decimal places using NR method.

Soln.

$f(x) = \log x - \cos x$

$f'(x) = \frac{1}{x} + \sin x$

x	$f(x)$
1	-0.5403
2	1.1093

Take

$$x_0 = \frac{1+2}{2} = 1.5$$

Using NR-method

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}, \quad n = 0, 1, 2, 3, \ldots$$

$\therefore f(x_0) = f(1.5) = 0.3347 \neq 0$

$f'(x_0) = f'(1.5) = 1.6642$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{0.3347}{1.6642} = 1.2989$$

Iteration table

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	$x_0 = 1.5$	0.3347	1.6642	1.2989
1	$x_1 = 1.2989$	-0.0070	1.7331	1.3029
2.	1.3029	-0.0000	1.7318	1.3030
3.	1.3030	0.0001	1.7318	$1.$

* Some Deductions from Newton's method

i) To find reciprocal of a real number 'N' using Newton's method.

⇒ Let, x be the reciprocal of N.

$$\therefore \quad x = \frac{1}{N}$$

$$\text{or,} \quad N = \frac{1}{x}$$

$$\text{or} \quad \frac{1}{x} - N = 0$$

$$\therefore \quad f(x) = \frac{1}{x} - N$$

$$f'(x) = -\frac{1}{x^2}$$

$\therefore$ By Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\left[\frac{1}{x_n} - N\right]}{\left[\frac{-1}{x_n^2}\right]}$$

$$= x_n + \frac{\frac{1 - Nx_n}{x_n}}{\left(\frac{1}{x_n^2}\right)}$$

$$= x_n + (1 - Nx_n)\, x_n$$

$$= x_n + x_n - Nx_n^2$$

$$\therefore \quad x_{n+1} = 2x_n - Nx_n^2$$

guess in point (x_n)

Ex. 1) Find the reciprocal of '7' using Newton's method correct to four decimal places.

Sol^n:

The reciprocal of N using Newton's method is

$$x_{n+1} = 2x_n - Nx_n^2$$

$$n = 0, 1, 2, 3, \ldots$$

Let

$x_0 = 0.1$ be the initial guess

$N = 7$

$$\therefore x_1 = 2x_0 - 7x_0^2$$

$$= 2 \cdot 0.1 - 7 \times 0.1^2$$

$$= 0.13$$

Iteration Table

n	x_n	x_{n+1}
0	0.1	0.13
1	0.13	0.1417
2.	0.1417	0.1428
3.	0.1428	0.1428

$$\therefore \frac{1}{7} \approx 0.1428$$

Ex. 2) Find a formula to find the cube root of a real number 'N' using Newton's method. Hence obtain the cube root of 13 correct to four decimal places.

$f(x) = x^0 \rightarrow$ linear

$\neq 1$ non linear

Sol^n:-

Let x be the cube root of N

$\therefore x = N^{1/3}$

or, $x^3 = N$ [$\because$ Cubing on both side

$\therefore x^3 - N = 0$

$f(x) = x^3 - N$

$f'(x) = 3x^2$

$\therefore$ Using Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{x_n^3 - N}{3x_n^2}\right)$$

$$= \frac{3x_n^3 - (x_n^3 - N)}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + N}{3x_n^2}$$

$$\therefore x_{n+1} = \frac{2x_n^3 + N}{3x_n^2}$$

$n = 0, 1, 2, \ldots$

Obtain the cube root of '13'.

Let,

$x_0 = 2$ be the initial guess

$N = 13$

$$\therefore x_1 = \frac{2(2)^3 + 13}{3(2)^2} = 2.417$$

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Iteration table

n	x_n	x_{n+1}
0	2	2.4167
1.	2.4167	2.3531
2.	2.3531	2.3513
3	2.3513	2.3713

$\therefore \sqrt[3]{13} \approx 2.3513$

Ex. 3) Find a formula to find $\frac{1}{\sqrt{N}}$ using Newton's method. Hence obtain the value of $\frac{1}{\sqrt{11}}$ correct to four decimal places.

Sol:

Let, 'x' be the $\frac{1}{\sqrt{N}}$ of N

$\therefore x = \frac{1}{\sqrt{N}}$

or, $x = (N)^{-1/2}$

Squaring on both side

$x^2 = N^{-1}$

$x^2 = \frac{1}{N}$

$\therefore x^2 - \frac{1}{N} = 0$

$\therefore f(x) = x^2 - \frac{1}{N}$

$f'(x) = 2x$

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$\therefore$ Using Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(x_n^2 - 1/N)}{(2x_n)} = x_n - \frac{\left(\frac{Nx_n^2 - 1}{N}\right)}{2x_n}$$

$$\cancel{= \frac{2x_n^2 - \left(\frac{Nx_n^2 - 1}{N}\right)}{2x_n}} = x_n - \frac{Nx_n^2 + 1}{2Nx_n}$$

$$= \frac{2Nx_n^2 - Nx_n^2 + 1}{2Nx_n}$$

$$x_{n+1} = \frac{Nx_n^2 + 1}{2Nx_n}$$

Now,

$x_0 = 0.1$ be the initial guess

$N = 11$

$$\therefore x_1 = \frac{11 \cdot (0.1)^2 + 1}{2 \cdot 11 \cdot 0.1} = 0.5045$$

Iteration table

n	x_n	x_{n+1}
0	0.1	0.5045
1	0.5045	0.3423
2	0.3423	0.3039
3.	0.3039	0.3015
4.	0.3015	0.3015

$$\therefore \frac{1}{\sqrt{11}} \approx 0.3015$$

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Fixed Point Iteration Method

Algorithm

1) Define a eq^n $f(x)=0$ and the stopping criteria.
2) Rewrite the given eq^n in the form
$$x = g(x)$$
Such that $|g'(x)| < 1$ for any initial value x_0.
3) Write the formula
$$x_{n+1} = g(x_n)$$
$$n = 0, 1, 2, \ldots$$
4) If $|x_{i+1} - g(x_i)| \approx 0$
then stop.

Ex. 1) Find a root of the eq^n
$$\cos x = 3x - 1$$
Using fixed point method correct to four decimal places.

Sol^n:

We have
$$\cos x = 3x - 1$$
$$3x = \cos x + 1$$
$$\therefore \quad x = \frac{\cos x + 1}{3}$$

Let $g(x) = \dfrac{\cos x + 1}{3} = \dfrac{1}{3}(\cos x + 1)$

$$g'(x) = \frac{1}{3}(-\sin x) = -\frac{1}{3}\sin x$$

Now,

x	$f(x)$
0	$2 > 0$
1	$-1.4597 < 0$

Fixed Point Iteration Method

Algorithm

1) Define a eq^n $f(x)=0$ and the stopping criteria.
2) Rewrite the given eq^n in the form

$$x = g(x)$$

Such that $|g'(x)| < 1$ for any initial value x_0

3) Write the formula

$$x_{n+1} = g(x_n)$$

$$n = 0, 1, 2, \ldots$$

4) If $|x_{i+1} - g(x_i)| \approx 0$

then stop.

Ex. 1) Find a root of the eq^n

$$\cos x = 3x - 1$$

using fixed point method correct to four decim places.

Sol^n:

We have

$$\cos x = 3x - 1$$

$$3x = \cos x + 1$$

$$\therefore \quad x = \frac{\cos x + 1}{3}$$

Let $g(x) = \dfrac{\cos x + 1}{3} = \dfrac{1}{3}(\cos x + 1)$

$$g'(x) = \frac{1}{3}(-\sin x) = -\frac{1}{3}\sin x$$

Now,

x	$f(x)$
0	$2 > 0$
1	$-1.4597 < 0$

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$|g'(x_0)| = |g'(0.5)| = |-\frac{1}{3}\sin 0.5|$

$= 0.1598 < 1$

$\therefore \quad x_{n+1} = g(x_n)$

$= \frac{\cos x_n + 1}{3}$

When, $n=0$

$x_1 = \frac{1}{3}(\cos x_0 + 1)$

$= \frac{1}{3}(\cos 0.5 + 1)$

$= 0.6259$

$x_2 = \frac{1}{3}(\cos x_1 + 1)$

$= 0.6035$

$x_3 = \frac{1}{3}(\cos x_2 + 1)$

$= 0.6078$

$x_4 = 0.6070 \quad [\because \frac{1}{3}\cos x_3 + 1]$

$x_5 = 0.6071$

$x_6 = 0.6071$

$\therefore$ Root ≈ 0.6071

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Ex. 2) Find a root of the eqn

$$x^3+x^2-1=0$$

Using FP method correct to four decimal places.

Soln.

Given,

$$x^3+x^2-1=0$$

x	$f(x)$
0	-1
1	1

$\Rightarrow x_0 = \frac{0+1}{2} = 0.5$

Now,

$$x^3+x^2-1=0$$

$$x^3 = 1-x^2$$

Eg. $\therefore \quad x = (1-x^2)^{1/3}$

let $g'(x) = \frac{1}{3}(1-x^2)^{1/3-1} \cdot -2x$

$\therefore g'(x) = -\frac{2}{3}(1-x^2)^{-2/3}\, x$

$g'(x_0) = |g'(0.5)| = \left|-\frac{2}{3}(1-0.5)^2\right)^{-2/3} \cdot 0.5$

$= |-0.40381|$

$= 0.4038$

$\therefore \quad x_{n+1} = \cancel{-\frac{2}{3}(1-x_n^2)^{-2/3}}\ (1-x_n^2)^{1/3}$

Now, $n = 0, 1, 2, 3, \ldots$

$x_1 = \frac{-2}{3}(1-x_0^2)^{-2/3} \quad x_0 = 0.3031$

$x_1 = (1-x_0^2)^{1/3} = (1-0.4038^2)^{1/3} = 0.9424$

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$x_2 = 0.4819$
$x_3 = 0.9157$
$x_4 = 0.5446$
$x_5 = 0.8893$
$x_6 = 0.5935$

$\therefore x_{n+1} = (1 - x_n^2)^{1/3}$

$x_1 = (1 - 0.5^2)^{1/3} = 0.9086$

$x_2 = 0.5588$	0.7648
$x_3 = 0.8827$	0.7460
$x_4 = 0.6044$	0.7626
$x_5 = 0.8594$	~~0.7479~~
$x_6 = 0.6395$	~~0.7479~~
$x_7 = 0.8392$	
$x_8 = 0.6662$	
$x_9 = 0.8224$	
$x_{10} = 0.6866$	
$x_{11} = 0.8085$	
0.7022	
0.7973	
0.7142	
0.7883	
0.7234	
0.7812	
0.7305	
0.7755	
0.7359	
0.7711	
0.7401	
0.7675	
0.7434	

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System of Linear Equations

Methods :

1) Gauss - elimination
2) Gauss - Jordan method
3) Factorization (3)

} Direct method

4) Gauss - Jacobi
5) Gauss - Seidel
6) Relaxation

} Iterative method

* Gauss-elimination

Let,

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

The augmented coefficient matrix of the given system is:

$$A = \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Change the matrix 'A' into upper triangular form using elementary row operations.

Let, the upper triangular form of the matrix 'A' be

$$A \sim \left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a'_{22} & a'_{23} & b'_2 \\ 0 & 0 & a''_{33} & b''_3 \end{array}\right]$$

Corresponding system of eq^n

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a'_{22}y + a'_{23}z = b'_2$$
$$a''_{33}z = b''_3$$

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Solve these equations to get the solution.

Ex. 1) Using Gauss-elimination method, solve:

$$10x - 7y + 3z + 5u = 6$$
$$-6x + 8y - z - 4u = 5$$
$$3x + y + 4z + 11u = 2$$
$$5x - 9y - 2z + 4u = 7$$

The augmented coeff. matrix of the given system

$$A = \left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ -6 & 8 & -1 & -4 & 5 \\ 3 & 1 & 4 & 11 & 2 \\ 5 & -9 & -2 & 4 & 7 \end{array}\right]$$

$R_2 \to 5R_2 + 3R_1$

$R_3 \to 10R_3 - 3R_1$

$R_4 \to 2R_4 - R_1$

$$A \sim \left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ 0 & 19 & 4 & -5 & 43 \\ 0 & 31 & 31 & 95 & 2 \\ 0 & -11 & -7 & 3 & 8 \end{array}\right]$$

$R_3 \to 19R_3 - 31R_2$

$R_4 \to 19R_4 + 11R_2$

$$A \sim \left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ 0 & 19 & 4 & -5 & 43 \\ 0 & 0 & 465 & 1960 & -1295 \\ 0 & 0 & -89 & 2 & 625 \end{array}\right]$$

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$R_4 \to 465R_4 + 89R_3$

$$A \sim \left[\begin{array}{cccc|c} 10 & -7 & 3 & 5 & 6 \\ 0 & 19 & 4 & -5 & 43 \\ 0 & 0 & 465 & 1960 & -1295 \\ 0 & 0 & 0 & 175370 & 175370 \end{array}\right]$$

Corresponding system of eqn

$$10x - 7y + 3z + 5u = 6$$
$$19y + 4z - 5u = 43$$
$$465z + 1960u = -1295$$
$$175370u = 175370$$

$\therefore u = 1$

$465z + 1960 = -1295$

$465z = -1295 - 1960$

$z = -7$

$19y + 4(-7) - 5 = 43$

$y = \frac{43 + 5 + 28}{19}$

$y = 4$

$10x - 7 \times 4 + 3 \times (-7) + 5 \times 1 = 6$

$x = 5$

$\therefore x$

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Gauss elimination with partial pivoting

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

$$\left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

Partial	Complete
Step 1: $\lvert a_1\rvert > \lvert b_1\rvert, \lvert c_1\rvert$	Step 1: $\lvert a_1\rvert > \lvert b_1\rvert, \lvert c_1\rvert, \lvert a_2\rvert, \lvert a_3\rvert$
Step 2: $\lvert b_2\rvert > \lvert c_2\rvert$	Step 2: $\lvert b_2\rvert > \lvert c_2\rvert, \lvert b_3\rvert$

Ex. 1) Using Gauss elimination with partial pivoting (if necessary), solve

$$x + 4y - z = -5,$$
$$x + y - 6z = -12$$
$$3x - y - z = 4$$

Sol:

The augmented coefficient matrix

$$A = \left[\begin{array}{ccc|c} 1 & 4 & -1 & -5 \\ 1 & 1 & -6 & -12 \\ 3 & -1 & -1 & 4 \end{array}\right]$$

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$R_1 \leftrightarrow R_3$

$$A \sim \left[\begin{array}{ccc|c} 3 & -1 & -1 & 4 \\ 1 & 1 & -6 & -12 \\ 1 & 4 & -1 & -5 \end{array}\right]$$

$R_2 \to 3R_2 - R_1$

$R_3 \to 3R_3 - R_1$

$$A \sim \left[\begin{array}{ccc|c} 3 & -1 & -1 & 4 \\ 0 & 4 & -17 & -40 \\ 0 & 13 & -2 & -19 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$A \sim \left[\begin{array}{ccc|c} 3 & -1 & -1 & 4 \\ 0 & 13 & -2 & -19 \\ 0 & 4 & -17 & -40 \end{array}\right]$$

$R_3 \to 13R_3 - 4R_2$

$$A \sim \left[\begin{array}{ccc|c} 3 & -1 & -1 & 4 \\ 0 & 13 & -2 & -19 \\ 0 & 0 & -213 & -444 \end{array}\right]$$

Corresponding equations using back substitution method,

$-213z = -444$

$z = 2.0845$

$13y - 2z = -19$

$13y - 2 \times 2.0845 = -19$

$y = \frac{-19 + 4.016}{13}$

$y = -1.1408$

$3x - y - z = 4$

$3x + 1.1408 - 2.0845$

$x = 1.648$

$x = 1.648$

$y = -1.1408$

$z = 2.0845$

Ans

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Ex 2) Using Gauss elimination with partial pivoting (if necessary), solve

$$x_1 + x_2 - 2x_3 = 3$$
$$4x_1 - 2x_2 + x_3 = 5$$
$$3x_1 - x_2 + 3x_3 = 8$$

Sol^n:

The augmented coeff. of matrix:

$$A = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 4 & -2 & 1 & 5 \\ 3 & -1 & 3 & 8 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$A \sim \left[\begin{array}{ccc|c} 4 & -2 & 1 & 5 \\ 1 & 1 & -2 & 3 \\ 3 & -1 & 3 & 8 \end{array}\right]$$

$R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 - 3R_1$

$$A \sim \left[\begin{array}{ccc|c} 4 & -2 & 1 & 5 \\ 0 & 6 & -9 & 7 \\ 0 & 2 & 9 & 17 \end{array}\right]$$

$R_3 \to 6R_3 - 2R_2 \Rightarrow 3R_3 - R_2$

$$A \sim \left[\begin{array}{ccc|c} 4 & -2 & 1 & 5 \\ 0 & 6 & -9 & 7 \\ 0 & 0 & 36 & 44 \end{array}\right]$$

Corresponding eq^n using back substitution meth

$$36z = 44$$
$$z = 11/9$$

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$6y - 9z = 7$

$6y - 9 \times \frac{11}{9} = 7$

$y = -2/3$

$4x - 2y + z = 5$

$4x - 2(-2/3) + \frac{11}{9} = 5$

$4x + \frac{4}{3} + \frac{11}{9} = 5$

$4x = 5 - \frac{4}{3} - \frac{11}{9}$

$x = \frac{11}{18}$

$\therefore x = 11/8$

$y = -2/3$

$z = 11/18$

* Gauss-Jordan Method

Let, the augmented coefficient matrix be

$$A = \left[\begin{array}{ccc|c} a_1 & a_2 & a_3 & d_1 \\ b_1 & b_2 & b_3 & d_2 \\ c_1 & c_2 & c_3 & d_3 \end{array}\right]$$

Using elementary row operations, change the matrix 'A' into the form

$$A \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & d_1' \\ 0 & 1 & 0 & d_2' \\ 0 & 0 & 1 & d_3' \end{array}\right]$$

$x_1 = d_1'$

$x_2 = d_2'$

$x_3 = d_3'$

Ex-1) Using Gauss Jordan method, solve:

$x_1 + x_2 - 2x_3 = 3$

$4x_1 - 2x_2 + x_3 = 5$

$3x_1 - x_2 + 3x_3 = 8$

Soln.

The augmented coeff. matrix.

$$A = \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 4 & -2 & 1 & 5 \\ 3 & -1 & 3 & 8 \end{array}\right]$$

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$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$$A \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & -6 & 9 & -7 \\ 0 & -4 & 9 & -1 \end{array}\right]$$

$R_1 \to 6R_1 + R_2$

$R_3 \to 3R_3 - 2R_2$

$$A \sim \left[\begin{array}{ccc|c} 6 & 0 & -3 & 11 \\ 0 & -6 & 9 & -7 \\ 0 & 0 & 9 & 11 \end{array}\right]$$

$R_1 \to 3R_1 + R_3, \quad R_2 \to R_2 - R_3$

$$A \sim \left[\begin{array}{ccc|c} 18 & 0 & 0 & 44 \\ 0 & -6 & 0 & -18 \\ 0 & 0 & 9 & 11 \end{array}\right]$$

$R_1 \to R_1/18, \quad R_2 \to R_2/-6, \quad R_3 \to R_3/9$

$$A \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 44/18 \\ 0 & 1 & 0 & -18/-6 \\ 0 & 0 & 1 & 11/9 \end{array}\right]$$

$$A \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 22/9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 11/9 \end{array}\right]$$

$\therefore x_1 = 22/9$

$x_2 = 3$

$x_3 = 11/9$ Ans

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* Factorization Method

let $a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

or,

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & b_2 & b_3 \\ a_3 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$A.\ X = B$ ---- (i)

Let $A = L.U$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

i) Dolittle method

$l_{11} = l_{22} = l_{33} = 1$.

ii) Crout's Method

$u_{11} = u_{22} = u_{33} = 1$

From eqn (i)

$L.UX = B$ ---- (ii)

Put $UX = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ ---- (iii)

From eqn (ii)

$L.Z = B$

Obtain the value of $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Then solve eq^n (iii) to get $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Ex. 1) Apply Dolittle factorization method to solve

$$3x + 2y + 7z = 4$$
$$2x + 3y + z = 5$$
$$3x + 4y + z = 7$$

Sol^n:

Matrix form of the given system is:

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

or,

$$A \cdot X = B \quad ---- \quad (i)$$

Let $A = LU$

For the Dolittle method

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore \quad A = L \cdot U$

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21} \cdot u_{11} & l_{21} u_{12} + u_{22} & l_{21} u_{13} + u_{23} \\ l_{31} \cdot u_{11} & l_{31} u_{12} + u_{22} l_{32} & l_{31} u_{13} + l_{32} u_{23} \end{bmatrix}$$

Comparing on both sides,

$u_{11} = 3, \quad u_{12} = 2, \quad u_{13} = 7$

$l_{21} \cdot u_{11} = 2 \Rightarrow l_{21} = 2/3$

$l_{21} \cdot u_{12} + u_{22} = 3$

$\frac{2}{3} \cdot 2 + u_{22} = 3$

$\therefore \quad u_{22} = 3 - \frac{4}{3} = \frac{9-4}{3}$

$\therefore \quad u_{22} = \frac{5}{3}$

$l_{21} u_{13} + u_{23} = 1 \Rightarrow \frac{2}{3} \cdot 7 + u_{23} = 1$

$= u_{23} = -11/3$

$l_{31} u_{11} = 3$

$l_{31} u_{11} = 3$

$l_{31} = 3/3$

$\therefore l_{31} = 1$

$l_{31} u_{12} + u_{22} l_{32} = 4$

$1 \cdot 2 + \frac{5}{3} l_{32} = 4$

$l_{32} = \frac{4-2}{5/3}$

$l_{32} = \frac{2 \times 3}{5} = \frac{6}{5}$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 1$$

$$1 \cdot 7 + \frac{6}{5}\left(-\frac{11}{3}\right) + u_{33} = 1$$

$$7 - \frac{66}{15} + u_{33} = 1$$

$$u_{33} = 1 + \frac{66}{15} - 7$$

$$u_{33} = -\frac{8}{5}$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix}$$

Put $A = L \cdot U$ in eq^n (i)

$$L \cdot UX = B \quad --- \text{(ii)}$$

Put $UX = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

From eq^n (ii)

$$L \cdot Z = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ \frac{2}{3}z_1 + z_2 \\ z_1 + 6/5\, z_2 + z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

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Comparing the corresponding elements on both side.

$$z_1 = 4$$

$$\frac{2}{3} z_1 + z_2 = 5$$

$$\frac{2}{3} \times 4 + z_2 = 5$$

$$z_2 = 5 - \frac{8}{3}$$

$$\therefore z_2 = \frac{15-8}{3} = \frac{7}{3}$$

$$z_1 + \frac{6}{5} z_2 + z_3 = 7$$

$$4 + \frac{6}{5} \times \frac{7}{3} + z_3 = 7$$

$$4 + \frac{42}{15} + z_3 = 7$$

$$\Rightarrow z_3 = 7 - 4 - \frac{42}{15}$$

$$\Rightarrow z_3 = \frac{105 - 60 - 42}{15}$$

$$\therefore z_3 = \frac{3}{15} = \frac{1}{5}$$

$$\therefore Z = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

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From eq^n (1)

$Ux = z$

$$\begin{bmatrix} 3 & 2 & 7 \\ 0 & 5/3 & -11/3 \\ 0 & 0 & -8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

$$\begin{bmatrix} 3x + 2y + 7z \\ 5/3\, y - 11/3\, z \\ -8/5\, z \end{bmatrix} = \begin{bmatrix} 4 \\ 7/3 \\ 1/5 \end{bmatrix}$$

$\therefore -\frac{8}{5} z = 1/5$

$z = -1/8$

$\frac{5}{3} y - \frac{11}{3} z = 7/3$

$\frac{5}{3} y - \frac{11}{3}(-1/8) = 7/3$

$\frac{5}{3} y + \frac{88}{3} = \frac{7}{3}$

$\frac{5}{3} y = \frac{7}{3} - \frac{88}{3}$

$\frac{5}{3} y = -\frac{81}{3}$

$y = -\frac{81}{5}$

$y = 9/8$

$3x + 2y + 7z = 4$

$3x + 2 \times \frac{9}{8} + 7 \times (-1/8) = 4$

$3x + \frac{9}{4} - \frac{7}{8} = 4$

$x = 7/8$

$\therefore x = 7/8$
$y = 9/8$
$z = -1/8$ } Ans

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Ex 2) Apply crout's factorization method to solve

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_3 + 3x_3 = 14$$

Sol^n:

Matrix form of the given eq^n

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

or, $AX = B$ --- (i)

let $A = L \cdot U$

For the crout's method,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A = LU$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}\cdot u_{12} & l_{11}\cdot u_{13} \\ l_{21} & l_{21}\cdot u_{12}+l_{22} & l_{21}\cdot u_{13}+l_{22}\cdot u_{23} \\ l_{31} & l_{31}\cdot u_{12}+l_{32} & l_{31}\cdot u_{13}+l_{32}\cdot u_{23}+l_{33} \end{bmatrix}$$

Comparing on both sides -

$\therefore l_{11} = 3$

$l_{21} = 2$

$l_{31} = 1$

$l_{11}\cdot u_{12} = 2$ $u_{12} = 2/l_{11} = 2/3$	$l_{11}\cdot u_{13} = 1$ $u_{13} = 1/3$	$l_{21}\cdot u_{12} + l_{22} = 3$ $l_{22} = 5/3$

$l_{21}\cdot u_{13} + l_{22} u_{23} = 2$ $u_{23} = 4/5$	$l_{31}\cdot u_{12} + l_{32} = 2$ $\therefore l_{32} = 4/3$

$l_{31}\cdot u_{13} + l_{32}\cdot u_{23} + l_{33} = 3$

$l_{33} = 8/5$

$$\therefore L = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix}$$

Put, $A = L\cdot U$ in eq^n (i),

$LUX = B$ --- (ii)

Put $UX = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ --- (iii)

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from (ii)

$L \cdot Z = B$

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 3z_1 \\ 2z_1 + 5/3\, z_2 \\ z_1 + 4/3\, z_2 + 8/5\, z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Comparing the corresponding elements on both side

$\therefore z_1 = 10/3$

$z_2 = 22/5$

$z_3 = 3$

$$\therefore Z = \begin{bmatrix} 10/3 \\ 22/5 \\ 3 \end{bmatrix}$$

from eqn (iii)

$UX = Z$

$$\begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10/3 \\ 22/5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x + \frac{2}{3}y + 1/3\, z \\ y + 4/5\, z \\ z \end{bmatrix} = \begin{bmatrix} 10/3 \\ 22/5 \\ 3 \end{bmatrix}$$

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Comparing the corresponding elements.

$z = 3$
$y = 2$
$x = 1$

* Choleskey's Factorization Method

If the coeff. matrix A is a symmetric matrix.
i.e. $A^T = A$.

Then,

$$A = L \cdot L^T.$$

Where,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$\therefore L^T = \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Ex 1) Decompose the matrix 'A' using Choleskey's method:

$$A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

Sol:

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Here,

$$A^T = A = \begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix}$$

$\therefore$ A is a symmetric matrix.

$\therefore$ Using Choleskey's method

$$A = L.L^T$$

$$\begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 4 & 12 & -16 \\ 12 & 37 & -43 \\ -16 & -43 & 98 \end{bmatrix} = \begin{bmatrix} l_{11}l_{11} & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{21}l_{31}+l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2 \end{bmatrix}$$

Comparing the corresponding elements on bothsid

$l_{11}^2 = 4 \Rightarrow l_{11} = 2$

$l_{21}l_{11} = 12 \Rightarrow l_{21} = 6$

$l_{31}l_{11} = -16 \Rightarrow -8$

$\therefore l_{31} =$

$l_{21}^2 + l_{22}^2 = 37$

$6^2 + l_{22}^2 = 37$

$l_{22}^2 = 1$

$\therefore l_{22} = 1$

$l_{31} \cdot l_{21} + l_{32} l_{22} = -43$

$-8 \times 6 + l_{32} \cdot 1 = -43$

$l_{32} = -43 + 48$

$l_{32} = 5$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 98$

$(-8)^2 + (5)^2 + l_{33}^2 = 98$

$l_{33}^2 = 98 - 64 - 25$

$l_{33}^2 = 9$

$\therefore l_{33} = 3$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 6 & 1 & 0 \\ -8 & 5 & 3 \end{bmatrix} \quad \& \quad L^T = \begin{bmatrix} 2 & 6 & -8 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Ex 2) Using factorization method, solve:

$$4x_1 + 10x_2 + 8x_3 = 44$$
$$10x_1 + 26x_2 + 26x_3 = 128$$
$$8x_1 + 26x_2 + 61x_3 = 214$$

Sol^n.

Matrix form of the given system

$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

or, $A \cdot X = B$ - - - - (i)

Here, $A = A^T = \begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & 26 \\ 8 & 26 & 61 \end{bmatrix}$

$\therefore$ A is a symmetric matrix.

$\therefore$ Using cholesky's method,

$A = LL^T$

$$A = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

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$$\begin{bmatrix} 4 & 10 & 8 \\ 10 & 26 & \\ 8 & 26 & 61 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

$\therefore l_{11}^2 = 4 \Rightarrow l_{11} = 2$

$l_{21}l_{11} = 10 \Rightarrow l_{21} = 5$

$l_{31}l_{11} = 8 \Rightarrow l_{31} = 4.$

$l_{21}^2 + l_{22}^2 = 26$

$5^2 + l_{22}^2 = 26$

$\cancel{l_{22}^2 = 26 - \frac{25}{4} = \frac{79}{4}}$

$\cancel{l_{22} = \frac{\sqrt{79}}{2}}$

$l_{22} = 1$

$l_{32} = 6$

$l_{33} = 3$

$$\therefore L = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix}, \quad L^T = \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

From (i), put $A = LL^T$

$\therefore L.L^T X = B$ --- (ii)

Put

$$L^T X = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \text{ --- (iii)}$$

$\therefore$ From eq^n (iii)

$L.Z = B$

$$\begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$$\begin{bmatrix} 2z_1 \\ 5z_1 + z_2 \\ 4z_1 + 6z_2 + 3z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

$\therefore\ 2z_1 = 44 \Rightarrow z_1 = 22$

$5z_1 + z_2 = 128 \Rightarrow 5 \times 22 + z_2 = 128 \Rightarrow z_2 = 128 - 110$

$= 18$

$4z_1 + 6z_2 + 3z_3 = 214$

$4 \times 22 + 6 \times 18 + 3z_3 = 214$

$z_3 = 6$

From eqn (iii)

$$\begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 5x_2 + 4x_3 \\ x_2 + 6x_3 \\ 3x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

$\therefore\ 3x_3 = 6$

$x_3 = 2$

$x_2 + 6x_3 = 18$

$x_2 + 6x_3 = 18$

$x_2 = 18 - 6 \times 2$

$x_2 = 6$

$2x_1 + 5x_2 + 4x_3 = 22$

$2x_1 = 22 - 5 \times 6 - 4 \times 2$

$2x_1 = 22 - 30 - 8$

$2x_1 = -8 - 8$

$x_1 = \frac{-16}{2}$

$\therefore\ x_1 = -8$

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* Iterative Methods for Solving System of linear Eqn.

1) Jacobi Method

Let, the given system of linear equations be

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

The system should be in diagonally dominant form i.e.

$$|a_1| > |b_1| + |c_1|$$
$$|b_2| > |a_2| + |c_2|$$
$$|c_3| > |a_3| + |b_3|$$

Then, solving for x, y, & z respectively.

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z)$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

Let, $x = y = z = 0$ be the initial guess.

Iteration 1

$$x^{(1)} = \frac{d_1}{a_1}, \quad y^{(1)} = \frac{d_2}{b_2}, \quad z^{(1)} = \frac{d_3}{c_3}$$

Iteration 2

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

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$$y^{(2)} = \frac{1}{b_1}\left(d_2 - a_2 x^{(1)} - c_2 z^{(1)}\right)$$

$$z^{(2)} = \frac{1}{c_3}\left(d_3 - a_3 x^{(1)} - b_3 y^{(1)}\right)$$

and so on.

Ex. 1) Using Jacobi method, solve

$$x + 5y - 2z = 18$$
$$-2x + 2y + 7z = 19$$
$$9x - 2y + z = 50$$

Solⁿ:-

Rearrange the given system so that it is in diagonally dominant form

$$9x - 2y - z = 50$$
$$x + 5y - 2z = 18$$
$$-2x + 2y + 7z = 19$$

Solving for x, y, z respectively

$$x = \frac{1}{9}(50 + 2y - z) \quad \text{----- (i)}$$

$$y = \frac{1}{5}(18 - x + 2z) \quad \text{---- (ii)}$$

$$z = \frac{1}{7}(19 + 2x - 2y) \quad \text{--- (iii)}$$

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Let, $x = y = z = 0$ be the initial guess.

Then,

$$x = \frac{50}{9} = 5.56$$

$$y = \frac{18}{5} = 3.6$$

$$z = \frac{19}{7} = 2.72$$

Iteration Table:

It. No	x from (i)	y from (ii)	z from (iii)
1	5.56	3.6	2.72
2.	6.05	3.58	3.27
3.	5.99	3.7	3.42
4.	~~6.01~~ 6.00	3.~~87~~	3.3~~7~~
5.	~~6.03~~ 6.02	~~3.74~~ 3.75	~~3.47~~ 3.35
6.	6.02	3.74	3.36
7.	6.01	3.74	3.37

$\therefore x \simeq 6.02$

$y \simeq 3.74$

$z \simeq 3.37$

Ex. 2) Using Jacobi method solve

$$10x - 2y - 3z = 205$$
$$-2x - y + 10z = 120$$
$$-2x + 10y - 2z = 154$$

Sol^n.

Rearrange the given system. So that it is in diagonally dominant ~~from~~ form

$$10x - 2y - 3z = 205$$
$$-2x + 10y - 2z = 154$$
$$-2x - y + 10z = 120$$

Solving for x, y, z respectively

$$x = \frac{1}{10}(205 + 2y + 3z) \quad \text{--- (i)}$$

$$y = \frac{1}{10}(154 + 2x + 2z) \quad \text{--- (ii)}$$

$$z = \frac{1}{10}(120 + 2x + y) \quad \text{--- (iii)}$$

Let, $x = y = z = 0$ be the initial guess.

$$x = \frac{205}{10} = 20.5$$

$$y = \frac{154}{10} = 15.4$$

$$z = \frac{120}{10} = 12$$

Iteration Table

It no.	x from (i)	y from (ii)	z from (iii)
1)	20.5	15.4	12
2.	27.18	21.9	17.64
3.	30.17	24.36	19.63
4.	31.26	25.36	20.47
5.	31.71	25.75	20.79
6.	31.89	25.9	20.92
7.	31.96	25.96	20.97
8.	31.98	25.99	20.99

$x \simeq 31.9$

$y \simeq 25.9$

$z \simeq 20.9$

Gauss Seidel Method

In a diagonal dominant system

let,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \quad \text{---(i)}$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \quad \text{---(ii)}$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \quad \text{---(iii)}$$

Let, $y = z = 0$ be the initial guess

Iteration 1:

$$x^{(1)} = \frac{d_1}{a_1}$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)})$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

Iteration 2

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1 y^{(1)} - c_1 z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2 x^{(2)} - c_2 z^{(1)})$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3 x^{(2)} - b_3^{(2)})$$

and so on..

Q. Using Gauss-Seidel method

$$2x - 3y + 20z = 25$$
$$3x + 20y - z = 18$$
$$20x + y - 2z = 17$$

Sol^n:- Rearrange the given system such that it is in diagonal dominant.

$$20x + y - 2z = 17$$
$$3x + 20y - z = 18$$
$$2x - 3y + 20z = 25$$

Solving for x, y & z respectively

$$x = \frac{1}{20}(17 - y + 2z) \quad --- \text{(i)}$$

$$y = \frac{1}{20}(18 - 3x + z) \quad ---- \text{(ii)}$$

$$z = \frac{1}{25}(25 - 2x + 3y) \quad --- \text{(iii)}$$

Let $y = z = 0$ be the initial guess;

$$\therefore x = \frac{17}{20} = 0.85$$

$$y = \frac{1}{20}(18 - 3 \times 0.85) = 0.77$$

$$z = \frac{1}{20}(25 - 2 \times 0.85 + 3 \times 0.77)$$
$$= 1.28$$

Iteration table

It No.	x	y	z
1	0.85	0.77	1.28
2	0.94	0.82	1.28
3	0.94	0.82	1.28

$x \approx 0.94$

$y \approx 0.82$

$z \approx 1.28$

Power Method

→ Power method is used to determine the numerical largest eigen value & corresponding eigen vector using iterative method.

Let, A be the square matrix of order 3.

Let $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ be the initial guess for the eigen vector.

$$A X_0 = X_1' = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 X_1$$

$A \cdot X_1 = \lambda_2 X_2$

and so on.

Ex.1) Using power method, find the largest eigen value and corresponding eigen vector of the matrix.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Sol:-

Let $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial guess for the eigen vector.

$$\therefore A \cdot X_0 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0.333 \\ 0.333 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.333 \\ 0.333 \end{bmatrix} = \begin{bmatrix} 3.666 \\ 2.998 \\ 3.331 \end{bmatrix}$$

$$= 3.666 \begin{bmatrix} 1 \\ 0.817 \\ 0.908 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.817 \\ 0.908 \end{bmatrix} = \begin{bmatrix} 4.727 \\ 6.09 \\ 7.181 \end{bmatrix}$$

$$= 7.181 \begin{bmatrix} 0.658 \\ 0.848 \\ 1 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0.658 \\ 0.848 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.822 \\ 6.050 \\ 7.354 \end{bmatrix} = 7.354 \begin{bmatrix} 0.520 \\ 0.823 \\ 1 \end{bmatrix} = \lambda_4 X_4$$

$$\therefore AX_4 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0.520 \\ 0.823 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.383 \\ 5.812 \\ 7.166 \end{bmatrix} = 7.166 \begin{bmatrix} 0.472 \\ 0.811 \\ 1 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0.472 \\ 0.811 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.227 \\ 5.716 \\ 7.094 \end{bmatrix}$$

$$= 7.094 \begin{bmatrix} 0.455 \\ 0.806 \\ 1 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0.455 \\ 0.806 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.171 \\ 5.679 \\ 7.067 \end{bmatrix}$$

$$= 7.067 \begin{bmatrix} 0.449 \\ 0.804 \\ 1 \end{bmatrix} = \lambda_7 X_7$$

$$AX_7 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0.449 \\ 0.804 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.151 \\ 5.665 \\ 7.057 \end{bmatrix}$$

$$= 7.057 \begin{bmatrix} 0.447 \\ 0.803 \\ 1 \end{bmatrix} = \lambda_8 X_8$$

$$AX_8 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0.447 \\ 0.803 \\ 1 \end{bmatrix} = \begin{bmatrix} 3.144 \\ 5.659 \\ 7.053 \end{bmatrix}$$

$$= 7.053 \begin{bmatrix} 0.446 \\ 0.802 \\ 1 \end{bmatrix} = \lambda_9 X_9$$

$$AX_9 = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 0.446 \\ 0.802 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.14 \\ 5.654 \\ 7.050 \end{bmatrix} = 7.05 \begin{bmatrix} 0.445 \\ 0.802 \\ 1 \end{bmatrix}$$

$\therefore$ The largest eigen value ≈ 7.05 & corresponding eigen vector

$$\approx \begin{bmatrix} 0.445 \\ 0.802 \\ 1 \end{bmatrix}$$

* Ill-Conditioned System of Linear Equations.

The System of linear equations are said to be ill condition if the determinant of coefficient matrix is almost singular. These equations are very sensitive to round off errors. A slight change in the coefficient or constants, produces a large change in the solution.

For eg: (1)

$$x + 2y = 4$$
$$2x + 3.999y = 7.999$$

It's solution is $(x, y) = (2, 1)$

and the solution of

$$x + 2y = 4.001$$
$$2x + 3.999y = 7.98$$
$$(x, y) = (-3.999, 4.00)$$

(2)

$$1.01x + 2y = 2.01$$
$$x + 2y = 2$$
$$(x, y) = (1, 0.5)$$

and the sol^n of

$$1.01x + 2y = 2.01$$
$$x + 2.01y = 2$$
$$(x, y) = (1.33, 0.33)$$

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Ex.1) Using factorization method, solve :

$$3x+2y+z=10$$
$$2x+3y+2z=14$$
$$x+2y+3z=14$$

Sol^n.

Using Dolittle factorization method,

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$AX = B$ ---(i)

Let, $A = LU$

For the Dolittle method,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$\therefore A = L.U$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & & \\ & & \\ & & \end{bmatrix}$$

Using Choleskey's method:

$$A^T = A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$\therefore$ A is a symmetric matrix

$\therefore$ Using Choleskey's method:

$$A = L \cdot L^T$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11} l_{21} & l_{11} l_{31} \\ l_{21} l_{11} & l_{21}^2 + l_{22}^2 & l_{21} l_{31} + l_{22} l_{32} \\ l_{31} l_{11} & l_{31} l_{21} + l_{33} l_{23} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Comparing corresponding elements on both side

$l_{11}^2 = 3$

$l_{11} = \sqrt{3} = 1.73$

$l_{11} l_{21} = 2$

$\therefore l_{21} = 2/l_{11} = \frac{2}{\sqrt{3}} = 1.15$

$l_{11} l_{31} = 1$

$l_{31} = 1/2 = 0.5$

$l_{21}^2 + l_{22}^2 = 3$

$l_{22} = \sqrt{3 - l_{21}^2}$

$= \sqrt{3 - \frac{2}{\sqrt{3}}}$

$= 1.35$

$l_{31} l_{21} + l_{33} l_{22} = 2$

$0.5 \times 1.15 + l_{33} \times 1.35 = 2$

$l_{33} = 1.055$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 3$

$(0.5)^2 + l_{32}^2 + (1.055)^2 = 3$

$l_{32}^2 = 3 - 0.5^2 - 1.055^2$

$= 1.63$

$l_{32} = 1.27$

$$\therefore L = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & \sqrt{3-\frac{2}{\sqrt{3}}} & 0 \\ 1/2 & 1.27 & 1.055 \end{bmatrix} \quad \& \quad L^T = \begin{bmatrix} \sqrt{3} & 2/3 & 1/2 \\ 0 & \sqrt{3-\frac{2}{\sqrt{3}}} & 1.27 \\ 0 & 0 & 1.055 \end{bmatrix}$$

From (i), put $A = LL^T$

$\therefore L.L^T X = B$ ---(ii)

Put,

$$L^T X = Z := \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \text{ ---(iii)}$$

$\therefore$ From eq^n (iii)

$L.Z = B$

$\sqrt{3} \quad 2/3$

0

0

1, 2, 3

$$\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & \sqrt{3-\frac{2}{\sqrt{3}}} & 0 \\ 1/2 & 1.27 & 1.055 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$$\sqrt{3}\, z_1 = 10$$

$$\frac{2}{\sqrt{3}} z_1 + \sqrt{3-\frac{2}{\sqrt{3}}}\, z_2 = 14$$

$$\frac{1}{2} z_1 + 1.27 z_2 + 1.055 z_3 = 14$$

$$z_1 = \frac{10}{\sqrt{3}}, \quad z_2 = \left(14 - \frac{2}{\sqrt{3}} \times \frac{10}{\sqrt{3}}\right) \Big/ \sqrt{3-\frac{2}{\sqrt{3}}}$$

$$= 5.39$$

&

$$\frac{1}{\cancel{2}} \times \frac{\cancel{10}\,5}{\sqrt{3}} + 1.27 \times 5.39 + 1.055 z_3 = 14$$

$$z_3 = \left(14 - \frac{5}{\sqrt{3}} - 1.27 \times 5.39\right) / 1.055$$

$$= 4.045$$

From (iii)

$$L^T X = Z$$

$$\begin{bmatrix} \sqrt{3} & 2/3 & 1/2 \\ 0 & \sqrt{3-2/\sqrt{3}} & 1.27 \\ 0 & 0 & 1.05 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10/\sqrt{3} \\ 5.39 \\ 4.045 \end{bmatrix}$$

Interpolation & Approximation

1) Lagrange's Interpolation Method

Let, the given set of data be

x:	x_0	x_1	x_2	x_3
y:	y_0	y_1	y_2	y_3

Estimate the $y(x)$

Lagrange's interpolation formula is given by

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3.$$

Note: Lagrange's method is used for equally spaced or unequally spaced data.

Ex. 1) Given that

	x_0	x_1	x_2	x_3
x:	0	1	2	4
y:	1 (y_0)	3 (y_1)	9 (y_2)	81 (y_3)

Estimate $y(3)$ using Lagrange's formula.

Sol^n:

Using Lagrange's interpolation formula

$$y(3) = \frac{(3-1)(3-2)(3-4)}{(0-1)(0-2)(0-4)} \cdot 1 + \frac{(3-0)(3-2)(3-4)}{(1-0)(1-2)(1-4)} \cdot 3 + \frac{(3-0)(3-1)(3-4)}{(2-0)(2-1)(2-4)} \cdot 9$$

$$+ \frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)} 81$$

$= \frac{1}{4}\,0 - 3 + 13.5 + 20.25$

$= 31$

Ex. 2) Find a polynomial $f(x)$ using Lagrange's formula and hence estimate $f(3)$ from the data

x:	x_0 0	x_1 1	x_2 2	x_3 5
$f(x)$ or y:	2 y_0	3 y_1	12 y_2	147 y_3

Sol^n:- Using Lagrange's interpolation formula

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} \cdot 2 + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} \cdot 3$$

$$+ \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} \cdot 147 + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} \cdot 12$$

$$= \frac{(x^2-2x-x+2)(x-5)}{(-1)\times(-2)\times(-5)} \times 2 + \frac{x(x^2-5x-2x+10)}{1\times(-1)\times(-4)} \times 3$$

$$+ \frac{x(x^2-5x-x+5)}{2\times 1\times(-3)} \times 12 + \frac{x(x^2-2x-x+2)}{5\times 4\times 3} \times 147$$

$$= \frac{(x^3-3x^2+2x-5x^2+10x+5x-10)}{-10} \times 2 + \frac{x^3-7x^2+10x}{4} \times 3$$

$$+ \frac{x^3-6x^2+5x}{-6} \times 12 + \frac{x^3-3x^2+2x}{60} \times 147$$

$$= \frac{x^3-8x^2+17x-10}{-5} \times 2 + \frac{x^3-7x^2+10x}{4} \times 3 + \frac{x^3-6x^2+5x}{(-1/2)} + 0$$

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$$+ \frac{x^3 + 6x^2 + 5x}{60} \times 147$$

$$= -\frac{1}{5}(x^3 - 8x^2 + 17x - 10) + \frac{3}{4}(x^3 - 7x^2 + 10x) - 2(x^3 - 6x^2 + 5x)$$
$$+ \frac{147}{60}(x^3 + 6x^2 + 5x)$$

$$= \frac{-12[x^3 - 8x^2 + 17x - 10] + 3 \times 15[x^3 - 7x^2 + 10x] - 120[x^3 - 6x^2 + 5x) + 147[x^3 + 6x^2 + 5x]}{60}$$

$$= \frac{-12x^3 + 96x^2 - 204x + 120 + 45x^3 - 315x^2 + 450x - 120x^3 + 720x^2 - 600x + 147x^3 + 882x^2 + 735x}{60}$$

$$= \frac{60x^3 + 589x^2 + 381x + 120}{60}$$

$$\therefore f(3) = \frac{60 \times (3)^3 + 589(3)^2 + 381 \times 3 + 120}{60}$$

$$= 136.35$$

Finite Differences

Let the given data be

x:	x_0	x_0+h	x_0+2h	x_0+3h
$f(x)$:	$f(x_0)$	$f(x_0+h)$	$f(x_0+2h)$	$f(x_0+3h)$
y:	y_0	y_1	y_2	y_3

Forward difference :

$$\left.\begin{aligned} f(x_0+h) - f(x_0) &= y_1 - y_0 = \Delta y_0 \\ f(x_0+2h) - f(x+h) &= y_2 - y_1 = \Delta y_1 \\ y_3 - y_2 &= \Delta y_2 \end{aligned}\right] \text{First forward difference.}$$

Second Forward difference

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1 = (y_3 - y_2) - (y_2 - y_1) = y_3 - 2y_2 + y_1$$

$$\Delta^2 y_2 =$$

So on...

Third Forward difference

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

In Short

Forward difference table

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
		Δy_0		
x_0+h	y_1		$\Delta^2 y_0$	
		Δy_1		$\Delta^3 y_0$
x_0+2h	y_2		$\Delta^2 y_1$	
		Δy_2		
x_0+3h	y_3			

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Backward Difference

$$f(x_0+h) - f(x_0) = y_1 - y_0 = \Delta y_1$$
$$f(x_0+2h) - f(x_0+h) = y_2 - y_1 = \Delta y_2$$
$$y_3 - y_2 = \Delta y_3$$

First Backward Difference

Backward difference Table:

x	y	∇y	$\Delta^2 y$	$\Delta^2 y$
x_0	y_0	$y_1 - y_0 = \Delta y_1$	$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$
x_0+h	y_1	Δy_2	$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$	
x_0+2h	y_2	Δy_3		
x_0+3h	y_3			

Ex. 1) Find the different forward differences ~~form~~ from the data :

x :	10	20	30	40	50
$f(x)$:	15	17	25	30	37

Sol^n:

Forward difference table

x -	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	15	2			
20	17		6		
30	25	8		-9	
40	30	5	-3		14
50	37	7	2	5	

Ex. 2) Find the first forward difference of $f(x) = \tan^{-1} x$

or find $\Delta(\tan^{-1} x)$

$$\Delta f(x) = \Delta \tan^{-1} x$$
$$= f(x+h) - f(x)$$
$$= \tan^{-1}(x+h) - \tan^{-1} x$$
$$= \tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$$
$$= \tan^{-1}\left(\frac{1}{1+x^2+xh}\right)$$

Divided differences

Let, the given data be

x:	x_0	x_1	x_2	x_3
y:	y_0	y_1	y_2	y_3

First divided difference

$$\Delta y_0 = \frac{y_1 - y_0}{x_1 - x_0}$$

$$\Delta y_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Delta y_2 = \frac{y_3 - y_2}{x_3 - x_1}$$

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Second divided difference

$$\Delta^2 y_0 = \frac{\Delta y_1 - \Delta y_0}{x_2 - x_0}$$

$$\Delta^2 y_1 = \frac{\Delta y_2 - \Delta y_1}{x_3 - x_1}$$

~~$\Delta^3 y_2$~~ Third divided diff.

$$\Delta^3 y_0 = \frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0}$$

Divided difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0	$\frac{y_1 - y_0}{x_1 - x_0} = \Delta y_0$		
x_1	y_1		$\frac{\Delta y_1 - \Delta y_0}{x_2 - x_0} = \Delta^2 y_0$	$\frac{\Delta^2 y_1 - \Delta^2 y_0}{x_3 - x_0} = \Delta^3 y_0$
x_2	y_2	Δy_1	$\frac{\Delta y_2 - \Delta y_1}{x_3 - x_1}$	
x_3	y_3	Δy_2		
~~x_4~~	~~y_4~~			

Newton's Divided Difference Formula

Let the given data be

x:	x_0	x_1	x_2	x_3
$y = f(x)$:	y_0	y_1	y_2	y_3

Then, Newton's divided difference formula is

$$y(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0$$

EX1) Given that

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

Evaluate $f(9)$ using Newton's divided difference formula.

Sol^n:

Newton's divided difference formula

$$f(x) = f_0 + (x - x_0)\Delta f_0 + (x - x_0)(x - x_1)\Delta^2 f_0 + (x - x_0)(x - x_1)(x - 2)\Delta^3 f_0 + (x - x_0)(x - x_1)(x - 2)(x - x_3)\Delta^4 \quad \text{--- (i)}$$

Divided difference table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
5	$\boxed{150}$	$\frac{392-150}{7-5} = \boxed{121}$			
7	392		$\boxed{24}$		
11	1452	265		$\boxed{1}$	$\boxed{0}$
13	2366	457	32		
17	5202	709	42		

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$\therefore$ from (i)

$f(9) = 150 + (9-5) \cdot 121 + (9-5)(9-7) \cdot 24 + (9-5)(9-7)(9-1)$
$\cdot 1 + 0$
$= 810$

Ex.2) Given:

$f(0) = -18, \quad f(1) = 0, \quad f(3) = 0, \quad f(5) = -248$
$f(6) = 0$

Find $f(x)$ using Newton's divided difference formula.

Hence estimate $f(4)$.

Sol^n:-

~~Divided d~~

x	$f(x)$
0	-18
1	0
3	0
5	-248
6	0

Newton's divided difference formula:

~~x $f(x)$ Δ~~

$$f(x) = f_0 + (x - x_0)\Delta f + (x - x_0)(x - x_1)\Delta^2 f + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\Delta^4 f$$

Divided difference formula

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
0	$\boxed{-18}$	$\boxed{18}$	$\boxed{-6}$		
1	0	0		$\boxed{-5}$	
3	0	-124	-31		$\boxed{6}$
5	-248		124	31	
6	0	248			

$f(x) =$

$$-18 + (x-0)18 + (x-0)(x-1)(-6) + (x-0)(x-1)(x-3)-5 + (x-0)(x-1)(x-3)(x-5)6$$

$$= -18 + x \cdot 18 - 6x(x-1) + x(x-1)(x-3)-5 + (x(x-1)(x-3)(x-5)6$$

$$= -18 + 18x - 6x(x-1) - 5x(x-1)(x-3) + 6x(x-1)(x-3)(x-5)$$

$$= -18 + 18x - 6x^2 + 6x - (5x^2+5x)(x-3) + (6x^2-6x)(x-3)(x-5)$$

$$= -18 + 18x - 6x^2 + 6x - [5x^3 - 15x^2 + 5x^3 - 15x] + (6x^2-6x)[x^2 - 5x - 3x + 15]$$

$$= -18 + 18x - 6x^2 + 6x - 5x^3 + 15x^2 - 5x^3 + 15x + \{6x^2 - 6x)(x^2 - 8x + 5)\}$$

$$= -18 + 18x - 6x^2 + 6x - 5x^3 + 15x^2 - 5x^3 + 15x + [6x^4 - 48x^3 + 30x^2 - 6x^3 + 48x^2 - 30x]$$

$$= -18 + 24x - 9x^2 - 10x^3 + 15x + 6x^4 - 48x^3 + 30x^2 - 6x^3 + 48x^2 - 30x$$

$$\cancel{= 6x^4} \; 6x^4 - 64x^3 + 87x^2 + 69x - 18$$

$$= -150$$

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Q3. Given,

x : 0 2 5 7

$f(x) = e^x + 5$: 6 12.39 153.41 1101.63

Limitation

Estimate $(e^4 + 5)$ using Newton's divided difference formula

Sol: ∴ Newton's difference formula :-

$$f(x) = f_0 + (x - x_0)\Delta f_0 + (x - x_0)(x - x_1)\Delta^2 f_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 f_0$$

divided difference table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	$\boxed{6}$	$\frac{12.39-6}{2-0} = \boxed{3.16}$	$\frac{47.01-3.16}{5-0} = \boxed{8.77}$	$\frac{85.42 - 8.77}{7-0} = \boxed{10.95}$
2	12.39	$\frac{153.41-12.39}{5-2} = 47.01$	$\frac{474.11-47.01}{7-2} = 85.42$	
5	153.41	$\frac{1101.63-153.41}{7-5} = 474.11$		
7	1101.63			

$$f(4) = (e^4 + 5) = 6 + (4-0)\times 3.16 + (4-0)(4-2)\times 8.77 + (4-0)(4-2)(4-5)\times 10.95$$

$$= \cancel{\ldots} \; 1.2$$

Part 1

Newton's forward interpolation formula

↳ equally space

let, the given data be

x : x_0 x_0+h x_0+2h x_0+3h

$f(x)$: $\boxed{f_0}$ f_1 f_2 $\boxed{f_3}$ ⟸ f_n

forward backward

Then, newtons forward interpolation formula is

$$f(p) = f_0 + p.\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \cdots$$

where, $p = \frac{x - x_0}{h}$

NOTE : This formula is used to interpolate or extrapolate near the beginning of the given data.

Newton's Backward interpolation Formula.

$$f(q) = f_n + q.\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 f_n + \cdots$$

NOTE :- This formula is used to interpolate or extrapolate near the end of given data.

where, $q = \frac{x - x_n}{h}$

common difference

e. Using suitable interpolation formula, estimate f(43) & f(85) from following data:

x :	40	50	60	70	80
$f(x)$:	184	204	226	250	276

Sol: The difference table

x	$f(x)$	1st diff.	2nd diff	3rd diff	4th diff
40	184	20			
50	204	22	2		
60	226	24	2	0	
70	250	26	2	0	0
80	276				

To estimate f(43) we use newton's forward interpolation formula.

$n = 43,\ n_0 = 40,\ h = 10$

$p = \frac{n - n_0}{h} =$

$\therefore p = \frac{43-40}{10} = 0.3$

newton's forward interpolation formula is given by

$$f(p) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \cdots$$

$$f(43) = f(0.3) = 184 + 0.3 \times 20 + \frac{0.3 \times (0.3-1)}{2} \times 2 + 0 \cdots$$

$$\therefore \boxed{f(43) = 189.79}$$

To estimate f(85) we use newton's backward extrapolation formula

$q = \frac{x - x_n}{h} =$ $\quad n = 85,\ x_n = 80,\ h = 10$

$q = \frac{85-80}{10} = 0.5$

$$f(q) = f_n + q\nabla f_n + \frac{q(q+1)}{2!}\nabla^2 f_n + \cdots$$

$$f(0.5) = f(25) = 276 + 0.5 \times 70 + \frac{(0.5)(0.5+1)}{2!} \times 2$$

$$= 283.75$$

#

Q Given;

$\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$

$\sin 60 = 0.8660$

using Newton's interpolation formula, find the $(\sin 38°)$ & $(\sin 57°)$.

Sol

x	$f(x)$	1st diff	2nd diff	3rd diff
45	0.7071	0.0589	-0.0037	-0.0007
50	0.7660	0.0572	-0.0044	
55	0.8192	0.0468		
60	0.8660			

at $\sin(38)$; $p = \frac{x - x_0}{h} = \frac{38 - 45}{5}$

$= -1.4$

$$f(p) = f_0 + p\Delta f_0 + \frac{p(p-1)}{2!}\Delta^2 f_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 f_0 + \cdots$$

$$f(38) = 0.7071 + (-1.4) \times (0.0589) + \frac{(-1.4)(-1.4-1)}{2} \times (-0.0037) + \frac{(-1.4)(-1.4-1)(-1.4-2)}{6} \times (0.0007) + 0$$

$\therefore f(38) = 0.616$

at 91.1 $\quad q = \frac{x - x_n}{h} = \frac{57 - 60}{5}$

$q = -0.6$

$$f(q) = f_n + q\nabla f_n + \frac{q(q+1)}{2}\nabla^2 f_n + \frac{q(q+1)(q+2)}{6}\nabla^3 f_n + \cdots$$

$$f(-0.6) = 0.8660 + (-0.6)\times(0.0468) + \frac{(-0.6)(-0.6+1)}{2}\times(-0.0064) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}\times(-0.0007)$$

$f(57) = 0.839$ #

Q3. from the following table estimate the number of student who obtain marks,

(i) betn 40 and 45.

(ii) betn 70 and 75.

Marks:	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
no. of std:	31	42	51	35	31

Sol^n change the given data into less than cumulative frequency table

marks less than	cumulative freq
40	31
50	73
60	124
70	159
80	190

x	y	1st diff	2nd diff	3rd diff	4th diff
40	31	42	9	-25	17
50	73	51	-16	42	
60	124	35	-4		
70	159	31			
80	190				

Q) To estimate the number of students who obtain marks less than 45; we use forward interpolation formula

$$p = \frac{x - x_0}{h} = \frac{45 - 40}{10} = \frac{1}{2} = 0.5$$

$$y(p) = y_0 + p.\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0.$$

$$y(0.5) = 31 + 0.5 \times 42 + \frac{0.5 \times (0.5-1)}{2} \times 9 + \frac{(0.5).(0.5-1)(0.5-2)}{3!} \times (-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!} \times 17$$

$= 47.86 \simeq 48$ (no of students)

$\therefore$ no of student ~~been~~ ~~to~~ who obtain marks less than of 45 $= 48 - 31$
$= 17$

$$\therefore y(0.5) = 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{3 \times 2} \times -25$$

(ii) To estimate no of student who obtain more than 75 we use backward interpolation formula.

$$q = \frac{x - x_n}{h} = \frac{75 - 80}{10} = (-0.5)$$

$$y(-0.5) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n$$

$$+ \frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 y_0$$

$$= 190 + (-0.5) \times 31 + \frac{(-0.5) \times (-0.5-1) \times (-4)}{2}$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6} \times 12$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24} \times 17$$

$$= 190 - 15.5 - \frac{3}{2} - \frac{3}{4} - \frac{85}{128}$$

$$= 172.80$$

$$\simeq 173$$

no of student who obtain below 70 & 75

$$= 173 - 159$$

$$= 14$$

3. From the following data estimate the number of persons having income betn 2000 & 2500

income:	below 500	500-1000	1000-2000	2000-3000
no of persons:	6000	10250	13850	15350

(income in Rs) x	(persons) y	∇y	$\nabla^2 y$	$\nabla^3 y$
500	6000	8.5	$\frac{3.6-8.5}{2000-500} = -0.0033 \approx 0$	
1000	10250	3.6		0
2000	13850	1.5	$\frac{1.5-3.6}{3000-1000} = -0.0011 \approx 0$	
3000	15350			

∴ To estimate the no of person whose income is less than 2500, we use newton's divided diff formula (unequally spaced)

$x = 2500$

$f(2500) = f_0 + (x - x_0)\nabla f_0 + \ldots$

$= 6000 + (2500-500) \times 8.5 + \ldots$

$= 23000$

Ans (2000 and) $23000 - 13850$

$= 9150$ Ans

EX. 1) From the following table, estimate the number of students who obtain marks

i) between 40 and 45

ii) between 70 and 75

Marks:	30-40	40-50	50-60	60-70	70-80
No. of students:	31	42	51	35	31

Sol^n:-

Change the given data into less than cumulative frequency table.

Marks Less than (x)	Cumulative frequency (y)	First diff	2nd diff	3rd diff	4th diff
40	31	42			
50	73	51	9	-25	-45 37
60	124	35	-16	-12	
70	159	31	-4		
80	190				

(i) between 40 & 45

To estimate the number of students who obtain marks less than 45,

We use forward interpolation formula:

$$y(p) = y_0 + p.\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$p = \frac{x - x_0}{h}, \quad x_0 = 40, \quad x = 45, \quad h = 10$$

$$\therefore p = \frac{45-40}{10} = 0.5$$

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$$\therefore y(0.5) = 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{3 \times 2} \times -25$$

$$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4 \times 3 \times 2} \times 37$$

$$= \frac{6127}{128} = 47.86$$

$$\approx 48$$

$\therefore$ No. of students who obtain marks between 40 & 45 $= 48 - 31 = 17$

(ii) between 70 & 75

To estimate the no. of students who obtain marks less than 75, we use backward interpolation formula.

$$y(q) = y_n + q\,\Delta y_n + \frac{q(q+1)}{2!}\Delta^2 y_n + \frac{q(q+1)(q+2)}{3!}\Delta^3 y_n$$

$$+ \frac{q(q+1)(q+2)(q+3)}{4!}\Delta^4 y_n$$

$q = \frac{x - x_n}{h}$, $x = 75$, $x_n = 80$, $h = 10$

$$\therefore q = \frac{75 - 80}{10} = \frac{-5}{10} = -0.5$$

$$\therefore y(-0.5) = 190 + (-0.5)\,31 + \frac{(-0.5)(-0.5+1)}{2!}(-4) +$$

$$\frac{-0.5 \times (-0.5+1)(-0.5+1)(-0.5+2)}{3!}(-12) + \frac{(-0.5)(-0.5+1)}{}$$

$$\frac{(-0.5+2)(-0.5+3)}{4!} \times 37$$

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Spline Interpolation

Def^n:

The technique of constructing lower degree polynomials in each sub-interval of the given data is called spline interpolation. If a third degree polynomial is constructed in each sub-interval then it is called cubic spline interpolation. A cubic spline interpolation polynomial satisfies the following conditions:

i) $S_i(x)$, $S_i'(x)$ & $S_i''(x)$ are continuous at each of connecting points $x_1, x_2, x_3 \ldots x_{n-1}$.

ii) $S_i(x)$ is linear at the begining & end point.

Ex 1)

	$S_1(x)$	$S_2(x)$	$S_3(x)$	
x :	0	2	5	6
y :	3	4	4.5	5

$S_1(x) = a_0x^3 + b_0x^2 + c_2x + d_1 \quad 0 \le x \le 2$

$S_2(x) = a_1x^3 + b_1x^2 + c_1x + d_1 \quad 2 \le x \le 5$

$S_3(x) = a_2x^3 + a_2x^2 + c_2x + d_2 \quad 5 \le x \le 6$

Cubic Spline Interpolation Formula

$$S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 \cdot u_i - u_i^3) + \frac{a_i}{6h_i}(u_{i-1}^3 - h_i^2 \cdot u_{i-1})$$

$$+ \frac{1}{h_i}(y_i \cdot u_i - y_{i-1} u_i)$$

Where, $h_i = x_i - x_{i-1}$

$u_i = x - x_i$

$i = 1, 2, 3, \ldots, n-1$

To determine the ~~theons~~ constants we use $h_i a_{i-1} + 2(h_i + h_{i+1}) a_i + h_{i+1} a_{i+1} = 6\left(\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right)$

$a_0 = a_n = 0$

$i = 1, 2, 3, \ldots, n-1$

Ex1) Given:

x:	1	2	5
y:	-8	-1	18
	y_0	y_1	y_2

Compute $y(1.5)$ using cubic spline interpolation formula.

Sol^n:

Cubic Spline interpolation formula

$$S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 \cdot u_i - u_i^3) + \frac{a_i}{6h_i}(u_{i-1}^3 - h_i^2 \cdot u_{i-1}) + \frac{1}{h_i}(y_i \cdot u_i - y_{i-1} \cdot u_i)$$

The target point $x = 1.5$ lies in the first interval.

$\therefore$ put $i = 1$

$$S_1(x) = \frac{a_0}{6h_1}(h_1^2 \cdot u_1 - u_1^3) + \frac{a_1}{6h_1}(u_0^3 - h_1^2 \cdot u_0) + \frac{1}{h_1}(y_1 \cdot u_1 - y_0 \cdot u_1)$$

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$\boxed{f = y}$

$h_i = x_i - x_{i-1}$

$\therefore h_1 = x_1 - x_0 = 2 - 1 = 1$

$h_2 = x_2 - x_1 = 5 - 2 = 3$

$u_i = x - x_i$

$u_1 = x - x_1 = 1.5 - 2 = -0.5$

$u_0 = x - x_0 = 1.5 - 1 = 0.5$

Here, $a_0 = a_n \Rightarrow a_0 = a_2 = 0$

$\therefore S_1(1.5) = 0 + \frac{a_1}{6\times1}\left(0.5^3 - 1\times0.5\right) + \frac{1}{1}\left(-1\times0.5 - (-8)(-0.5)\right)$

$= \frac{a_1}{6}\left(0.125 - 0.5\right) + 4.5$

$\therefore S_1(1.5) = -0.06259a_1 - 4.5 \quad --(i)$

To determine the value of a_1, we use

$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1}a_{i+1} =$

$$6\left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right)$$

Put $i = 1$

$$h_1 a_0 + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left(\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right)$$

$\therefore 0 + 2a_1(1+3) + 0 = 6\left[\frac{18+1}{3} - \frac{-1+8}{1}\right]$

$2a_1 \times 4 = 6\left[\frac{19}{3} - 7\right]$

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$$8a_1 = \frac{19-21}{3} \times 6 \Rightarrow a_1 = -0.5$$

$$a_1 = \frac{19-21}{3 \times 8} \times 6 = \frac{2}{3 \times 8_4}$$

From (i)

$$S_1(1.5) = -0.0625 \cdot (-0.5) - 4.5$$

$$S(1.5) = -4.46875$$

$$\therefore y(1.5) = -4.46875$$

Ex2) Given,

x:	-1	0	1	2
y:	-1	1	3	35
	f_0	f_1	f_2	f_3

Estimate $y(0.8)$ using cubic spline interpolation

Sol^n:-

Cubic Spline Interpolation formula

$$S_i(x) = \frac{a_{i-1}}{6h_i}(h_i^2 u_i - u_i^3) + \frac{a_i}{6h_i}(u_{i-1}^3 - h_i^2 \cdot u_{i-1})$$

$$+ \frac{1}{h_i}(f_i \cdot u_i - f_{i-1} \cdot u_i)$$

The target point 0.8 lies in the 2nd interval

$\therefore$ Put $i = 2$

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$$S_2(x) = \frac{a_1}{6h_2}\left[h_2^2 u_2 - u_2^3\right] + \frac{a_2}{6h_2}\left[u_1^3 - h_2^2 \cdot u_1\right]$$

$$+ \frac{1}{h_2}\left[f_2 \cdot u_2 - f_1 u_2\right]$$

$h_i = x_i - x_{i-1} \Rightarrow h_1 = x_1 - x_0 \Rightarrow 0-(-1) = 1$

$\Rightarrow h_2 = x_2 - x_1 \Rightarrow 1 - 0 = 1$

$\Rightarrow h_3 = x_3 - x_2 \Rightarrow 2 - 1 = 1$

$u_i = x - x_i \Rightarrow u_1 = x - x_1 \Rightarrow 0.8 - 0 = 0.8$

$u_2 = x - x_2 \Rightarrow 0.8 - 1 = -0.2$

$f_1 = 1,\ f_2 = 3$ & $a_0 = a_3 = 0$

$\therefore$

$$S_2(0.8) = \frac{a_1}{6 \times 1}\left[1^2(-0.2) - (-0.2)^3\right] + \frac{a_2}{6 \times 1}\left[0.8^3 - 1^2 \times 0.8\right] +$$

$$\frac{1}{1}\left[3 \cdot (0.8) - 1 \cdot (-0.2)\right]$$

$$= \frac{a_1}{6}\left[\frac{-24}{125}\right] + \frac{a_2}{6}\left[\frac{-36}{125}\right] + \left[-\frac{2}{5}\right]$$

$$= -0.0329a_1 - 0.0489a_2 + 2.6$$

$\therefore S_2(0.8) = -0.0329a_1 - 0.0489a_2 + 2.6$ --- (i)

To determine a_1 & a_2, we use,

$$h_i a_{i-1} + 2a_i(h_i + h_{i+1}) + h_{i+1} a_{i+1} = 6\left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right) \quad \text{--- (iii)}$$

Put $i=1$ in eq^n (iii)

$$h_1 a_0 + 2a_1(h_1 + h_2) + h_2 a_2 = 6\left(\frac{f_2 - f_1}{h_2} - \frac{f_1 - f_0}{h_1}\right)$$

$$2a_1(1+1) + a_2 = 6\left(\frac{3-1}{1} - \frac{1-1}{1}\right)$$

$$\therefore 4a_1 + a_2 = 0 \quad \text{--- (a)}$$

Put $i = 2$ in eq^n (iii)

$$h_2 a_1 + 2a_2(h_2 + h_3) + h_3 a_3 = 6\left(\frac{f_3 - f_2}{h_3} - \frac{f_2 - f_1}{h_2}\right)$$

$$a_1 + 2a_2(1+1) + 1\,a_3 = 6\left(\frac{35-3}{1} - \frac{3-1}{1}\right)$$

$$a_1 + 4a_2 + a_3 = 6(32 - 2)$$

$$\boxed{\therefore a_3 = 0}$$

$$a_1 + 4a_2 = 180 \quad \text{--- (b)}$$

Solving (a) & (b)

$a_1 = -12$

$a_2 = 48$

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$\therefore$ from (ii)

$$S_2(0.8) = -0.032 \times (-12) - 0.048 \times 48 + 2.6$$
$$= 0.68.$$

* Curve Fitting

i) Fit a linear curve : $y = a + bx$ ----- (1) using least square approximation method.

→ To determine the values of 'a' & 'b', solve

$$\Sigma y = na + b\Sigma x \text{ ----- (i)}$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 \text{ ---- (ii)}$$

OR

$$b = \frac{n\Sigma x \cdot y - \Sigma x \cdot \Sigma y}{n\Sigma x^2 - (\Sigma x)^2}$$

$$a = \frac{1}{n}(\Sigma y - b\Sigma x)$$

Ex. 1) If 'p' is the pull required to lift a load W by means of a pully block. Find a linear curve of the form

$p = aW + b$

from the data:

P:	12	15	21	25
W:	50	70	100	120

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Compute 'P' when W = 150 unit.
Calculate the error at W = 70 unit.

Sol^n:

We have,

$P = aW + b$ ..... (i)

a and b are obtained by solving the equations

$\Sigma P = nb + a\Sigma W$ ---- (i)

& $\Sigma PW = b\Sigma W + a\Sigma W^2$ -- (ii)

W	P	W·P	W^2
50	12	600	2500
70	15	1050	4900
100	21	2100	10000
120	25	3000	14400
$\Sigma W =$ 340	$\Sigma P = 73$	$\Sigma W \cdot P =$ 6750	$\Sigma W^2 =$ 31800

$n = 4$

From (i)	From (ii)
$\Sigma P = nb + a\Sigma W$	$\Sigma PW = b\Sigma W + a\Sigma W^2$
$73 = 4 \times b + a \cdot 340$	$6750 = b \times 340 + a \times 31800$
$73 = 340a + 4b$	$6750 = 31800 + 340b$

Solving these eq^n

$a = 0.1879$

$b = 2.2758$

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$\therefore$ From (i)

$P = aw + b$

$\therefore P = 0.1879w + 2.275$

At $w = 150$

$P = 0.1879 \times 150 + 2.275 = 30.46$

To find the error

Put $w = 70$

$P = 0.1879 \times 70 + 2.275$

$= 15.428$

$\therefore$ Error at $w = 70$

$e = |15 - 15.428|$

$= 0.428$ Lt

Q.) Given,

x:	0	5	10	15
y:	1.80	1.45	1.18	1

Fit a linear curve to determine the value of y for $x = 12$

Sol^n:

Let, the linear curve be

$y = a + bx$ --- (i)

To determine values of 'a' & 'b'

$\Sigma y = na + b\Sigma x$ ---(i)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ ---(ii)

x	y	xy	x^2
0	1.80	0	0
5	1.45	7.25	25
10	1.18	11.8	100
15	1	15	225
$\Sigma x = 30$	5.43	34.05	350

$n = 4$

From (i)

$\Sigma y = na + b\Sigma x$

$5.43 = 4a + 30b$

From (ii)

$34.05 = 30a + 350b$

Q Given,

x :	0	5	10	15
y :	1.80	1.45	1.18	1

Fit a linear curve to determine the value of y for $x=12$.

Sol: we have.

~~y=a~~

let linear curve be

$$y = a + bx \quad \text{--- (I)}$$

a & b are obtained by solving the eqn

$$\Sigma y = na + b\Sigma x \quad \text{--- (II)}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \text{--- (III)}$$

x	y	xy	x^2
0	1.80	0	0
5	1.45	7.25	25
10	1.18	11.8	100
15	1	15	225
$\Sigma x = 30$	$\Sigma y = 5.43$	$\Sigma xy = 34.05$	$\Sigma x^2 = 350$

$\therefore n = 4$

from (I) & (II)

$30(5.43 = 4a + b.30)$

$4(34.05 = a \times 30 + b \times 350)$

$\Rightarrow 162.9 = 120a + 900b$

$136.2 = 120a + 1400b$

$26.7 = -500b$

$\therefore b = -0.0534$

$5.43 = 4a + 30 \times (-0.0534)$

$5.43 = 4a - 1.602$

$a = 1.758$

from (I)

$y = a + bx = 1.758 + (-0.0534 \times x)$

at $x = 12$

$y = 1.758 + (-0.0534 \times 12)$

$\therefore y = 1.1172$

Non-Linear Curves

i) Second degree polynomial.

let $y = a + bx + cx^2$ --- (1) be the second degree polynomial.

The values of a, b, & c are determined by solving the equations

$\Sigma y = na + b\Sigma x + c\Sigma x^2$ --- (i)

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$ --- (ii)

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$ --- (iii)

Solving these equations (i), (ii) & (iii) we get, a, b & c

ii) Fitting curve of the form

a. $y = a + bx^2$ --- (i)

Put, $x^2 = X$

$\therefore y = a + bX$ ---- (ii)

To find a & b in eqn (ii)

Solve,

$\Sigma y = na + b \cdot \Sigma X$ --- (a)

$\Sigma Xy = a\Sigma X + b\Sigma X^2$ -- (b)

x	y	$X = x^2$	$X \cdot y$	X^2

b. $y = ax + bx^2$ --- ①

$\frac{y}{x} = a + bx$

$Y = a + bx$ --- ②

Where, $Y = \frac{y}{x}$

x	y	$Y = y/x$	$x \cdot Y$	x^2

c. $y = ax + b/x$

$xy = ax^2 + b$

Put, $x \cdot y = Y$

$x^2 = X$

$\therefore Y = b + a \cdot X$

x	y	$X = x^2$	$Y = x \cdot y$	$X \cdot Y$	X^2

d. $y = ax^2 + \frac{b}{x}$

$xy = ax^3 + b$

Put $xy = Y$, $x^3 = X$

$\therefore Y = aX + b$

x	y	X	Y	XY	X^2

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2023, Spring

Ex 1) Using the method of least squares, fit a curve of the form $y = ax^2 + \frac{b}{x}$ to the data.

x:	1	2	3	4
y:	-1.52	0.96	8.88	17.66

Estimate y at $x = 2.5$.

Sol^n:-

We have.

$$y = ax^2 + \frac{b}{x} \quad --- (1)$$

$$xy = ax^3 + b \qquad [\text{put, } xy = Y,\ x^3 = X]$$

$$Y = aX + b \quad --- (2)$$

From The values of a & b in (2) are obtained from the equations.

$$\Sigma Y = a\,\Sigma X + nb \quad --- (i)$$

$$\Sigma XY = a\,\Sigma X^2 + b\,\Sigma X \quad --- (ii)$$

x	y	$X = x^3$	$Y = xy$	XY	X^2
1	-1.52	1	-1.52	-1.52	1
2	0.96	8	1.92	15.36	64
3	8.88	27	26.64	719.28	729
4	17.66	64	70.64	4520.96	4096
Total		100	97.68	5254.08	4890

$n = 4$

Now, eq^n (i) & (ii) becomes.

$$97.68 = 100a + 4b \quad --- (a)$$

$$5254.08 = 4890a + 100b \quad --- (b)$$

2023, Spring

Ex 1) Using the method of least squares, fit a curve of the form $y = ax^2 + \frac{b}{x}$ to the data.

x:	1	2	3	4
y:	-1.52	0.96	8.88	17.66

Estimate y at $x = 2.5$.

Sol^n:-

We have.

$$y = ax^2 + \frac{b}{x} \quad \text{--- (1)}$$

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Total		100	97.68	5254.08	4890

$n = 4$

Now, eq^n (i) & (ii) becomes.

$$97.68 = 100a + 4b \quad \text{--- (a)}$$

$$5254.08 = 4890a + 100b \quad \text{--- (b)}$$

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On solving

$a = 1.1766 \qquad b = -4.9950$

From eq^n ①

~~$y = ax + b$~~

$$y = ax^2 + \frac{b}{x}$$

$$y = 1.1766x^2 - \frac{4.9950}{x}$$

At $x = 2.5$

$$y = 1.1766 \times (2.5)^2 - \frac{4.9950}{2.5}$$

$$y = 5.35$$

Ex. 1) Using the method of least squares, fit a curve of the form $y = ax + \frac{b}{x}$ to the data:

x:	-1	1	3	4
y:	2	5	3	9

Also, estimate $y(5)$ and calculate the error at $x = 3$

$\log x^n = n\log x$

IMP $\log_{10} e =$

Exponential Type of Curves

1) Form $y = ae^{bx}$ (Exponential model)
2) Form $y = ax^b$ (power Function model)
3) Form $xy^a = b$ (or $PV^{\alpha} = \beta$)

Taking 'ln' on both sides
From (1)

$$\ln y = \ln(ae^{bx})$$
$$= \ln a + \ln(e^{bx})$$
$$= \ln a + bx \cdot \underbrace{\ln e}_{1}$$

$$\ln y = \ln a + bx$$

or $Y = A + bx$ ..... (i)
Where $Y = \ln y$
$A = \ln a$
$a = e^A$ ---- #

The constant A anb 'b' in eq^n (i) are obtained by solving the equations

$$\Sigma Y = nA + b\Sigma x \quad \text{---(ii)}$$
$$\Sigma xY = A\Sigma x + b\Sigma x^2 \quad \text{---(iii)}$$

Solving these equations, we get A and b.

Form 2) $y = ax^b$

Taking ln on both sides

$\ln y = \ln a + b \cdot \ln x$

$Y = A + b \cdot X$

Where, $Y = \ln y$

$X = \ln x$

$A = \ln a \Rightarrow a = e^A$

Form 3) $xy^a = b$

Taking 'ln' on both sides

$\ln(xy^a) = \ln b$

$\ln x + \ln y^a = \ln b$

$\ln x + a \ln y = \ln b$

$a \ln y = \ln b - \ln x$

$\ln y = \frac{\ln b}{a} - \frac{\ln x}{a}$

$\ln y = \frac{1}{a}(\ln b - \ln x)$

$\ln y = \frac{1}{a} \ln b - \frac{1}{a} \ln x$

$Y = A + B \cdot X$

Where,

$Y = \ln y$

$X = \ln x$

$A = \frac{1}{a} \ln b$

$B = -\frac{1}{a}$

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2019, Spring

The voltage 'V' across a capacitor at time 't' in seconds is given in the table

voltage (V) :	150	63	28	12	5.6	1.24
Time(t)(seconds) :	0	2	4	6	8	10

If the relationship between voltage 'V' & time 't' is of the form $v = ae^{bt}$. Using least square method estimate the voltage at $t = 2.6$ seconds.

Sol:-

We have,

$$V = ae^{bt} \quad \text{--- (i)}$$

Taking 'ln' on both sides

$$\ln v = \ln a + bt$$

$$Y = A + B \cdot X \quad \text{---- (ii)}$$

where,

$$Y = \ln v$$

$$A = \ln a$$

$$X = t$$

Using least square approximation method,

We have

$$\Sigma Y = nA + b\Sigma X \quad \text{--- (i)}$$

$$\Sigma X \cdot Y = A\Sigma X + b\Sigma X^2 \quad \text{--- (ii)}$$

$$n = 6$$

t	V	$Y = \ln V$	XY	X^2
0	150	5.011	0	0
2	63	4.143	8.286	4
4	28	3.332	13.328	16
6	12	2.485	14.91	36
8	5.6	1.723	13.784	64
10	1.24	0.215	2.150	100
Σ		16.909	52.458	220

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From (i) & (ii)

$16.9 = 6A + 30b$

& $52.458 = 30A + \cancel{16.9b}\ 220b$

Solving these eqn

$\cancel{A = 1.613, \quad b = 0.241}$

$A = 5.11$

$a = e^{A} = e^{5.11} = 165.67$

$b = -0.458$

~~From~~

$\cancel{a = e^{A}}$

$a = e^{1.613} =$

∴ From (i), the required curve is

$$V = 165.67\, e^{-0.458t}$$

At $t = 2.6$ seconds

$$V = 165.67\, e^{-0.458 \times 2.6}$$

$$= 50.36 \text{ unit.}$$

Ans

2019, Fall

The pressure & volume of a gas are given ~~the~~ in the table

V:	384	20	8.51	4.44	3.03
P:	10	20	50	100	150

Fit a curve of the form $PV^{\alpha} = \beta$ using least square method. Also estimate the volume at $P = 30$ unit

Sol^n: We have,

$$PV^{\alpha} = \beta$$

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at $t = 2.6$

$$v = 144.027 \cdot e^{-0.43 \times 2.6}$$

$$v = 47.09 \text{ unit}$$

Q

V:	38.4	20	8.51	4.44	3.03
P:	10	20	50	100	150

pressure & volm of a gas are given in the table. fit a curve of a form $pV^{\alpha} = \beta$ using least sq. method. Also estimate the volume at $p = 30$ unit.

$my^q = \beta$

Sol^n

given $pV^{\alpha} = \beta$

taking (ln) on b.s

$$\ln(pV^{\alpha}) = \ln(\beta)$$

$$\ln(p) + \alpha \ln(V) = \ln(\beta)$$

$$\ln(V) = \frac{1}{\alpha}\ln(\beta) - \frac{1}{\alpha}\ln(p)$$

$$Y = A + BX$$

where $Y = \ln(V)$, $A = \frac{1}{\alpha}\ln(\beta)$

$B = -\frac{1}{\alpha}$, $X = \ln(p)$

using least sq.

$\cancel{\Sigma y = nA + BX}$

$\cancel{\Sigma xy = AZ}$

$\Sigma y = nA + B\Sigma x$ —(i)

$\Sigma xy = A\Sigma y + B\Sigma x^2$ —(ii)

V	p	$x = \ln(p)$	$y = \ln(V)$	xy	x^2
38.4	10	2.3	3.65	8.40	5.29
20	20	3.0	3.0	9	9
8.52	50	3.91	2.14	8.37	15.29
4.44	100	4.61	1.49	6.87	21.25
3.03	150	5.01	1.11	5.56	25.10
		$\Sigma x = 18.83$	$\Sigma y = 11.39$	$\Sigma xy = 38.2$	$\Sigma x^2 = 75.93$

$\therefore n = 5$

$11.39 = 5A + B\,18.83$

$38.2 = A \cdot 11.39 + B\,75.93$

$A = 0.88$

$B = 0.37$

$\therefore B = \frac{-1}{a}$

$0.37 = \frac{-1}{a}$

$\boxed{\therefore a = \frac{-1}{0.37} = -2.70}$

$\therefore A = \frac{1}{a}\ln(\beta)$

$0.88 = -2.70 \cdot \ln(\beta)$

$-0.33 = \ln(\beta)$

$e^{-0.33} = \beta$

$\boxed{\therefore \beta = 0.72}$

$\therefore p(V)^{-2.70} = 0.72$

at $p = 30$

$30\,(V)^{-2.70} = 0.72$

$(V)^{-2.70} = 0.024$

$V =$

Numerical Differentiation & Integration

Derivative of continuous functions.

Let, $y=f(x)$ be the given function

Then,

$$\frac{dy}{dx} = f'(x) = \frac{f(x+h)-f(x)}{h} =$$ forward difference formula for first derivative

$$= \frac{f(x)-f(x-h)}{h} =$$ backward diff. formula for first derivative.

$$= \frac{f(x+h)-f(x-h)}{2h} =$$ central diff. formula for first derivative

Again,

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Q. 2023, Spring

1) Estimate approximate derivative of $f(x) = x^2$ at $x=1$ for $h = 0.1, 0.2, 0.05, 0.01$, Also, find the corresponding error.

Sol^n:-

Here,

$$f(x) = x^2$$
$$f(x+h) = (x+h)^2$$
$$f(x-h) = (x-h)^2$$

Date
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Using central difference formula

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$\therefore f'(1) = \frac{f(1+h) - f(1-h)}{2h}$$

For $h = 0.1$

$$f'(1) = \frac{f(1+0.1) - f(1-0.1)}{2 \times 0.1} = \frac{f(1.1) - f(0.9)}{0.2}$$

$$= \frac{(1.1)^2 - (0.9)^2}{0.2} = \frac{1.21 - 0.81}{0.2} = 2$$

$\therefore$ Error = |Exact − calculated|

$= |2 - 2| = 0$

For $h = 0.2$

$$f'(0.2) = \frac{f(1+0.2) - f(1-0.2)}{2 \times 0.2} = \frac{(1.2)^2 - (0.8)^2}{0.4}$$

$= 2$

$\therefore$ Error = 0.

For $h = 0.05$

$$f'(0.05) = \frac{f(1+0.05) - f(1-0.05)}{2 \times 0.05}$$

$$= \frac{f(1.05) - f(0.95)}{0.1} = 2$$

⋮

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$$f'(1) = \frac{f(1+0.1) - f(1)}{0.1} = 2.1$$

$$\therefore \text{error} = |2 - 2.1| = 0.1$$

Ex find first & second order derivative of $f(x)$
$f(x) = e^x \sin x$ at $x = 1$ for $h = 0.1$ & $h = 0$
Also, find corresponding error.

Soln

$$f(x) = e^x \sin x$$

$$f'(x) = \underline{e^x \cos x} + \underline{e^x \sin x} = e^x(\cos x + \sin x)$$

$$f''(x) = \underline{e^x(\sin x) + e^x \cos x} + \underline{e^x \cos x + e^x \sin x}$$

$$= e^x(-\cancel{\sin x} + \cos x + \cos x + \cancel{\sin x})$$

$$= e^x(2\cos x)$$

$$= 2e^x \cos x$$

at $x = 1$

$$f'(x) = e^1(\cos(1) + \sin(1))$$

$$= 3.756$$

$$f''(x) = 2 \cdot e^1 \cos(1)$$

$$= 2.937$$

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$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}$$

at $x = 1$ & $h = 0.1$

$$f'(1) = \frac{(f(1+0.1) - f(1-0.1))}{2 \times 0.1}$$

$$= \frac{f(1.1) - f(0.9)}{0.2}$$

$$= \frac{e^{1.1} \cdot \sin(1.1) - e^{0.9} \cdot \sin(0.9)}{0.2}$$

$$= 3.753$$

Error = |Exact − calculated|

$= (3.756 - 3.753)$

$= 0.003$

at $x = 1$ & $h = 0.01$

$$f'(1) = \frac{f(1+0.01) - f(1-0.01)}{2 \times 0.01}$$

$$= \frac{f(1.01) - f(0.99)}{2 \times 0.01} = 3.756$$

$\therefore$ Error $= |3.756 - 3.756|$

$= 0$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

at $x=1$, $h=0.1$

$$f''(1) = \frac{f(1+0.1) - 2f(1) + f(1-0.1)}{(0.1)^2}$$

$$= \frac{f(1.1) - 2f(1) + f(0.9)}{0.01}$$

$$= 2.93$$

$$\therefore \text{error} = |2.937 - 2.93| = 0.007$$

at $x=1$, $h=0.01$

$$f''(1) = \frac{f(1+0.01) - 2f(1) + f(1-0.01)}{(0.001)^2}$$

$$= \frac{f(1.01) - 2f(1) - f(0.99)}{(0.001)^2}$$

$$= 2.937$$

$$\therefore \text{error} = |2.937 - 2.937| = 0$$

#

Numerical Differentiation

Let, the given set of data points be.

x :	x_0	x_1	x_2	x_3
y :	y_0	y_1	y_2	y_3

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ at any point x.

for Equally spaced data:

↳ Newton's forward difference formula

~~$y = y_0 + (x - x_0)\Delta y_0 + x - x_0$~~

$$\left[y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right]$$

$$\left[\therefore p = \frac{x - x_0}{h} \right]$$

$$\frac{dp}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dp} \cdot \left(\frac{dp}{dx}\right) \quad \text{// Chain rule}$$

$$= \frac{1}{h}\left[\frac{dy}{dp}\right]$$

$$= \frac{1}{h}\left[0 + 1 \cdot \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!}\Delta^3 y_0 + \cdots\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \cdots\right]$$

Let the given set of data points be

x:	x_0	x_1	x_2	x_3
y:	y_0	y_1	y_2	y_3

Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ at any point 'x'.

For equally spaced data;
Newton's Forward difference formula

$$y = y_0 + p \cdot \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0$$

$$p = \frac{x - x_0}{h} \Rightarrow \frac{dp}{dx} = \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dp}\left(\frac{dp}{dx}\right)$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0 + \dots\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dp}\left(\frac{dy}{dx}\right)\frac{dp}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0 + \dots\right]$$

Ex.1) Given,

x:	1.0	1.1	1.2	1.3	1.4
y:	8.0	8.4	8.8	9.1	9.5

Find $\frac{dy}{dx}$ & $\frac{dy^2}{dx^2}$ at $x = 1.0$ & $x = 1.3$

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Sol^n:-

Here,

The difference table

x	y	1st diff.	2nd diff.	3rd diff.	4th diff.
1.0	8.0	0.4	0		
1.1	8.4	0.4		-0.1	
1.2	8.8	0.3	-0.1		0.3
1.3	9.1	0.4	0.1	0.2	
1.4	9.5				

To find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x=1$, we use

Newton's Forward difference formula

$$y = y_0 + p.\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

$$p = \frac{x-x_0}{h} \Rightarrow \frac{dp}{dx} = \frac{1}{x},$$

Here, $h = 0.1$,

$$\therefore \frac{dy}{dx} = \frac{dy}{dp}\left(\frac{dp}{dx}\right)$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0\right]$$

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x	1.0	1.1	1.2	1.3	1.4
y	8.0	8.4	8.8	9.1	9.5

Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 1.0$ & $x = 1.3$

Sol^n The difference table

x	y	1st diff	2nd diff	3rd diff	4th diff
1.0	8.0	0.4	0	-0.1	0.3
1.1	8.4	0.4	-0.1	0.2	
1.2	8.8	0.3	0.1		
1.3	9.1	0.4			
1.4	9.5				

To find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 1$, we use newton's forward difference formula.

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$\therefore p = \frac{x - x_0}{h} \Rightarrow \frac{dp}{dx} = \frac{1}{h}$ here, $\boxed{h = 0.1}$

$$\frac{dy}{dx} = \frac{d(y)}{dp} \cdot \left(\frac{dp}{dx}\right)$$

$$= \frac{1}{h}\left[0 + \Delta y_0 + \frac{2p-p}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0\right.$$

$\therefore p = \frac{x - x_0}{h} = \frac{\frac{x-x_0}{h}}{h} = \frac{1-1}{h} = 0$

$$\frac{dy}{dx} = \frac{1}{0.1}\left[0.4 + 0 + \frac{2}{3!} \times (-0.1) - \frac{6}{4!} \times 0.3\right]$$

$$= 2.9166$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{12-36p+22}{4!}\Delta^4 y_0\right]$$

$$\therefore p = \frac{x - x_0}{h} = \text{at } x = 1$$

$$p = \frac{1-1}{h} = 0$$

$$= \frac{1}{(0.1)^2}\left[0 + (0-1)\cdot(-0.1) + \frac{22}{24} \times 0.3\right] = 37.5$$

Again to find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 1.3$

we use newtons backward interpolation formula.

$$y = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 y_n$$

$$\therefore q = \frac{x - x_n}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dq}\left(\frac{dq}{dx}\right) = \frac{dq}{dx}\cdot\frac{dy}{dq}$$

$$= \frac{1}{h}\left[0 + \nabla y_n + \frac{(2q+1)}{2!}\nabla^2 y_n + \frac{3q^2+6q+2}{3!}\nabla^3 y_n + \frac{4q^3+18q^2+22q+6}{4!}\nabla^4 y_n\right]$$

$$\frac{dy}{dx} = \frac{1}{0.1}\left[0.4 + 0 + \frac{2}{3!} \times (-0.1) - \frac{6}{4!} \times 0.3\right]$$

$$= 2.9166$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{12 - 36p + 22}{4!}\Delta^4 y_0\right]$$

$$\therefore p = \frac{x - x_0}{h} = \text{at } x = 1$$

$$p = \frac{1-1}{h} = 0$$

$$= \frac{1}{(0.1)^2}\left[0 + (0-1)\cdot(-0.1) + \frac{22}{24} \times 0.3\right] = 37.5$$

Again to find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ at $x = 1.3$

we use newtons backward interpolation formula.

$$y = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 y_n$$

$$\therefore q = \frac{x - x_n}{h}$$

$$\frac{dy}{dx} = \frac{dy}{dq}\left(\frac{dq}{dx}\right) = \frac{dq}{dx}\cdot\frac{dy}{dq}$$

$$= \frac{1}{h}\left[0 + \nabla y_n + \frac{(2q+1)}{2!}\nabla^2 y_n + \frac{3q^2 + 6q + 2}{3!}\nabla^3 y_n + \frac{4q^3 + 18q^2 + 22q + 6}{4!}\nabla^4 y_n\right]$$

$$\frac{d^2y}{dx^2}(x_n) = \frac{q^3 + 3q^2 + 2q + 4}{(q+1)(q+2)(q+3)(q^2+5q+6)}$$

$$q^4 + 5q^3 + 6q^2 + 4q^2 + 2q$$

$$\therefore q = \frac{x - x_n}{h} = \frac{1.3 - 1.4}{0.1} = \frac{-0.1}{0.1} = -1$$

$$\therefore \frac{dy}{dx} = \frac{1}{0.1}\left[0 + 0.4 + \frac{-1 \times (0.2)}{2} + \frac{3 + -6 + 2}{6}(0.2) + \frac{-4 + 18 - 22 + 6}{24}(0.3)\right]$$

$$= \frac{1}{0.1}\left[0.4 - \frac{0.1}{2} - \frac{1 \times 0.2}{6} - \frac{2}{24} \times (0.3)\right]$$

$$= 2.916$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (q+1)\nabla^3 y_n + \frac{12q^2 + 36q + 22}{4!}\nabla^4 y_n\right]$$

$$= \frac{1}{(0.1)^2}\left[0.1 + 0 + \frac{12 - 36 + 22}{4}(0.3)\right]$$

$$= 7.5$$

(9)

Time (s) :	1	2	4	5
velocity (m/s) :	0.25	1	2.2	4

$a = \frac{dv}{dt}$

find the acc^n of vehicle at $t = 1.1$ second & $t = 2.5$ sec. using suitable formula.

Sol^n

The divided difference table is.

Time	velocity	1st diff	2nd diff	3rd diff
1	$\boxed{0.25}$	$\boxed{0.75}$	$\boxed{-0.05}$	$\boxed{0.1125}$
2	1	0.6	0.4	
4	2.2	1.8		
5	4			

Now, find acc^n of vehicle at $t = 1.1$ second & $t = 2.5$ sec we use newton's divided difference.

$$y(x) = y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0$$

$$\frac{dy}{dx} = \Delta y_0 + \{2x - (x_0 + x_1)\}\Delta^2 y_0 + \{3x^2 - 2x_1 x - 2x_0 x - 2x_2 x + x_1 x_2 + x_0 x_2\}\Delta^3 y_0$$

$$\frac{d^2y}{dx^2} = 2\Delta^2 y_0 + (6x - 2x_1 - 2x_0 - 2x_2)\Delta^3 y_0$$

at $t = 1.1$ i.e. $x = 1.1$

$$\frac{d^2y}{dx^2} = 2\times(-0.05) + (6\times 1.1 - 2\times 2 - 2\times 1 - 2\times 4)\times 0.1125$$

$$= -0.9325 \text{ m/s}^2$$

at $t = 2.5$,

$$\frac{d^2y}{dx^2} = 2\times(-0.05) + (6\times 2.5 - 2\times 2 - 2\times 1 - 2\times 4)\times 0.1125$$

$$= 0.0125 \text{ m/s}^2$$

#

Numerical Integration

$\boxed{n = 6}$

(1) Trapazoidal Rule :

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

where, $h = \frac{b-a}{n}$; n = no. of subintervals.

(2) Simpson's 1/3 rule.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\right]$$

$\therefore (n)$ should be even number.

(3) Simpson's 3/8 rule :-

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) + 2(y_3 + y_6 + y_9 + \cdots)\right]$$

$\therefore (n)$ should be multiple of three.

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Q/ evaluate $\int_{a=0}^{\pi/2} \sin x \, dx$ using trapezoidal, simpson's $1/3$ & simpson's $3/8$ rules

Sol

$f(x) = \sin x$

$a = 0,$

$b = \pi/2$

Take $n = 6$

$h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$ {Fraction में करने}

	x	$y = f(x) = \sin x$
lower limit	0	$0 = y_0$
(n+1)	$\frac{\pi}{12}$	$0.259 = y_1$
	$\frac{2\pi}{12}$	$0.5 = y_2$
	$\frac{3\pi}{12}$	$0.707 = y_3$
	$\frac{4\pi}{12}$	$0.866 = y_4$
	$\frac{5\pi}{12}$	$0.966 = y_5$
upper limit तक जाएगा	$\frac{6\pi}{12} = \frac{\pi}{2}$	$1 = y_n$

using Trapezoidal rule:

$$\int_0^{\pi/2} \sin x\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$$

$$= \frac{\pi}{24}\left[(0+1) + 2(0.259+0.5+0.707+0.866+0.966)\right]$$

$$= 0.9943$$

(कुछ term Repeat नहीं)

using Simpsons 1/3 Rule

$$\int_0^{\pi/2} \sin x\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots) + 2(y_2+y_4+\cdots)\right]$$

$$= \frac{\pi}{36}\left[(0+1) + 4(0.259+0.707+0.966) + 2(0.5+0.866)\right]$$

$$= 1.00007$$

using simpsons 3/8 rule:

$$\int_0^{\pi/2} \sin x\,dx = \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots) + 2(y_3+y_6+\cdots)\right]$$

$$= 3\times\frac{\pi}{12\times 8}\left[(0+1) + 3(0.259+0.5+0.866+0.966) + 2(0.707)\right]$$

$$= 1.0001$$

To find error

also, exact value :-

$$\int_0^{\pi/2} \sin x \, dx = \left[-\cos x\right]_0^{\pi/2} = 1$$

E_T = Error in Trapezoidal rule

$= |1 - 0.9943| = 0.0057$

$E_{S1/3}$ = error in Simpsons 1/3 rule

$= |1 - 0.00007| = 0.00007$ (least error).

$E_{S3/8}$ = error in simpson's 3/8 rule

$= |1 - 1.0001| = 0.0001$

2017 F.Y

(a) Evaluate $\int_{-2}^{2} \frac{x}{x+2e^x}\,dx$.

$\therefore f(x) = \frac{x}{x+2e^x}$.

$a = -2$ $\quad n = 6$

$b = 2$

$h = \frac{b-a}{n} = \frac{2+2}{6} = \frac{4}{6} = \frac{2}{3}$.

x	$y = f(x) = \frac{x}{x+2e^x}$
-2	1.157 (y_0)
$-2+\frac{2}{3} = -\frac{4}{3}$	1.654 (y_1)
$-\frac{4}{3}+\frac{2}{3} = -\frac{2}{3}$	-1.851 (y_2)
$-\frac{2}{3}+\frac{2}{3} = 0$	0 (y_3)
$0+\frac{2}{3} = \frac{2}{3}$	0.146 (y_4)
$\frac{2}{3}+\frac{2}{3} = \frac{4}{3}$	0.149 (y_5)
$\frac{4}{3}+\frac{2}{3} = \frac{6}{3} = 2$	0.119 (y_n)

(1) using Trapozoidal rule:

$$\int_{-2}^{2} \frac{x}{x+2e^x}\,dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+y_3 - - + y_{n-1})\right]$$

$$= \frac{2}{6}\left[(1.157+0.119) + 2(1.654 + -1.851 + 0 + 0.146 + 0.149)\right]$$

$$= 0.491$$

using simpsons 1/3 rule:

$$\int_{-2}^{2} \frac{x}{x+2e^x}\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\cdots)+2(y_2+y_4+\cdots)\right]$$

$$= \frac{2}{3\times 3}\left[(1.157+0.119)+4(1.654+0+0.149)+2(-1.852+0.146)\right]$$

$$= 1.128$$

using simpsons 3/8 rule:-

$$\int_{-2}^{2} \frac{x}{x+2e^x}\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots)+2(y_3+y_6+\cdots)\right]$$

$$= \frac{2\times 3}{3\times 8}\left[(1.157+0.119)+3(1.654-1.852+0.146+0.149)+2(0)\right]$$

$$= 0.393$$

also exact value :-

$$\int_{-2}^{2} \frac{x}{x+2e^{x}}\,dx =$$

Romberg Integration

$$\begin{array}{lccccc}
n=1 & R_{00} & & & & \\
 & \downarrow & \searrow & & & \\
n=2 & R_{10} & \longrightarrow & R_{11} & & \\
 & \downarrow & \searrow & & \searrow & \\
n=3 & R_{20} & \longrightarrow & R_{21} & \longrightarrow & R_{22} \\
 & \downarrow & \searrow & & \searrow & \\
n=4 & R_{30} & \longrightarrow & R_{31} & \longrightarrow & R_{32} \longrightarrow R_{33}
\end{array}$$

Improved estimates

Trapezoidal with double n in each successive steps.

$$R_{ij} = \frac{4^{j} R_{i,j-1} - R_{i-1,j-1}}{4^{j}-1}$$

eg $R_{\substack{21\\ij}}$ $\quad i=2,\ j=1$

$$R_{21} = \frac{4^{1} \cdot R_{20} - R_{10}}{4^{1}-1}$$

Romberg Integration

$$
\begin{array}{lllll}
n=1 & R_{00} & & & \\
 & \downarrow & & & \\
n=2 & R_{10} & R_{11} & & \\
 & \downarrow & & & \\
n=4 & R_{20} & R_{21} & R_{22} & \\
 & \downarrow & & & \\
n=8 & R_{30} & R_{31} & R_{32} & R_{33}
\end{array}
$$

Trapezoidal with double 'n' in each successive steps

Improved estimates

$$R_{ij} = \frac{4^j R_{i\,j-1} - R_{i-1\,j-1}}{4^j - 1}$$

eg: $i = 2,\ j = 1$

$$R_{21} = \frac{4R_{20} - R_{10}}{4-1}$$

2023, Spring

Ex.1) Using Romberg Integration, evaluate

$$\int_0^2 \frac{e^x + \sin x}{1+x^2}$$ correct upto two decimal places.

Sol^n:-

Here,

$$f(x) = \frac{e^x + \sin x}{1+x^2}$$

$a = 0,\ b = 2$

$$\therefore h = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

Take $n=1$

$\therefore h = \frac{2}{1} = 2$

x	$f(x)$	
0	1	$\rightarrow y_0$
2	~~1.08~~ 1.659	$\rightarrow y_n$

$\therefore$ Using Trapezoidal rule

$$R_{00} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots)\right]$$

$$= \frac{2}{2}\left[(1+1.659)\right]$$

$$= 2.659$$

Take $n=2$

$h = \frac{2}{2} = 1$

x	$f(x)$	
0	1	$\rightarrow y_0$
1	1.779	$\rightarrow y_1$
2	1.659	$\rightarrow y_n$

$$R_{10} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2)\right]$$

$$= \frac{1}{2}\left[(1+1.659) + 2\times 1.779\right]$$

$$= 3.108$$

$\therefore$ Improved estimate

$$R_{11} = 4R_{10} - R_{00}$$

Take $n=1$

$\therefore h = \frac{2}{1} = 2$

x	$f(x)$	
0	1	$\rightarrow y_1$
2	~~1.48~~ 1.659	$\rightarrow y_n$

$\therefore$ Using Trapezoidal rule

$$R_{00} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots)\right]$$

$$= \frac{2}{2}\left[(1+1.659)\right]$$

$$= 2.659$$

Take $n=2$

$h = \frac{2}{2} = 1$

x	$f(x)$	
0	1	$\rightarrow y_0$
1	1.779	$\rightarrow y_1$
2	1.659	$\rightarrow y_n$

$$R_{10} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2)\right]$$

$$= \frac{1}{2}\left[(1+1.659) + 2\times 1.779\right]$$

$$= 3.108$$

$\therefore$ Improved estimate

$$R_{11} = \frac{4R_{10} - R_{00}}{4-1}$$

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$$= \frac{4 \times 3.108 - 2.669}{3} = 3.257$$

Take $n = 4$

$$h = \frac{2}{4} = \frac{1}{2} = 0.5$$

x	$f(x)$
0	1
0.5	1.7038
1	1.779
1.5	1.685
2	1.659

$\therefore$ Using Trapezoidal Rule,

$$R_{20} = \frac{0.5}{2}\left[(1 + 1.659) + 2(1.703 + 1.779 + 1.685)\right]$$

$$= 3.249$$

Improved estimate

$$R_{21} = \frac{4R_{20} - R_{10}}{4 - 1} = \frac{4 \times 3.249 - 3.108}{3} = 3.296$$

$$R_{22} = \frac{4^2(R_{21} - R_{11})}{4^2 - 1} = \frac{16 \times 3.296 - 3.257}{15} = 3.29$$

$$\therefore \int_0^2 \frac{e^x + \sin x}{1 + x^2} = 3.299$$

Ex. 2) Using Romberg Integration

evaluate $\int_1^2 (e^{x^2}-1)\,dx$ correct to two decimal places.

Sol^n:

Here,

$$f(x) = (e^{x^2}-1)$$

$a = 1, \quad b = 2$

$$h = \frac{b-a}{n} = \frac{1}{n}$$

Take, $n = 1$

$\therefore h = 1/1 = 1$

x	$f(x)$	
1	1.72	$\rightarrow y_0$
2	53.6	$\rightarrow y_n$

$\therefore$ Using Trapezoidal

$$R_{00} = \frac{h}{2}[y_0 + y_n] = \frac{1}{2}(1.72 + 53.6) = 27.66$$

Take, $n = 2$

$h = 1/2 = 0.5$

~~x~~	~~$f(x)$~~	x	$f(x)$
~~1~~	~~1.28~~	1	1.72
~~1.5~~	~~1.72~~	1.5	8.49
2	53.6	2	53.6

$$\cancel{R_{10} = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\dots)\right]}$$

$$\cancel{= \frac{0.5}{2}\left[(0.28+53.6)+2(1.72)\right]}$$

$$R_{10} = \frac{h}{2}\left[(y_0+y_n) + 2y_1\right]$$

$$= \frac{0.5}{2}\left[(\cancel{0.28+1.72}\ 1.72+53.6) + 2\times 8.49\right]$$

$$R_{10} = 18.08$$

Take $n = 3$

$$h = \frac{1}{3} = 0.33$$

x	$f(x)$
1.	
1.33	

Improved estimate

$$R_{11} = \frac{4R_{10} - R_{00}}{4-1}$$

$$= 14.89$$

Take $n = 4$

$$h = \frac{1}{4} = 0.25$$

x	$f(x)$
1	1.72
1.25	3.77
1.5	8.49
1.75	20.38
2	53.6

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Using Trapezoidal Rule

$$R_{20} = \frac{0.25}{2}\left[(1.72 + 53.6) + 2(3.77 + 8.49 + 20.38)\right]$$

$$= 15.08$$

Improved estimate

$$R_{21} = \frac{4R_{20} - R_{10}}{4-1} = 14.08$$

$$R_{22} = \frac{4^2 R_{21} - R_{11}}{4^2 - 1} = 14.03$$

$$\therefore \int_1^2 (e^{x^2} - 1) \simeq 14.03$$

Gaussian Integration

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + \dots w_n f(x_n)$$

w_i : are called weights

x_i are called sampling points.

1) Gaussian Integration for $n=2$

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) + \omega_2 f(x_2)$$

$$\omega_1 = \omega_2 = 1$$

$$x_1 = -x_2 = -\frac{1}{\sqrt{3}} \Rightarrow \int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

2) Gaussian Integration for $n=3$

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3)$$

$$\omega_1 = \omega_3 = \frac{5}{9}, \quad \omega_2 = 8/9$$

$$x_1 = -x_3 = -\sqrt{\frac{3}{5}}, \quad x_2 = 0$$

$$\int_{-1}^{1} f(x)\,dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

$$= \frac{5}{9}\left[f\left(-\frac{\sqrt{3}}{\sqrt{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0).$$

Gaussian Integration for the Integral

$$\int_a^b f(x)\,dx$$

Put, $x = \frac{b-a}{2} t + \frac{b+a}{2}$

or, $dx = \frac{b-a}{2}\,dt$

$$\therefore \int_a^b f(x)\,dx = \int_{-1}^{1} G(t)\,\frac{b-a}{2}\,dt$$

$$= \frac{b-a}{2} \int_{-1}^{1} G(t)\,dt$$

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Ex.1) Using Gaussian integration technique evaluate

$$\int_{-1}^{1} \frac{1}{1+x^2}\, dx \text{ with } n = 2, 3$$

Sol^n:-

Here, $f(x) = \frac{1}{1+x^2}$

Gaussian integration with $n=2$

$$\int_{-1}^{1} \frac{1}{1+x^2}\, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\left(-\frac{1}{\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{3}{4} + \frac{3}{4}$$

$$= 1.5$$

Gaussian integration with $n=3$

$$\int_{-1}^{1} \frac{1}{1+x^2}\, dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$$

$$= \frac{5}{9}\left(\frac{5}{8} + \frac{5}{8}\right) + \frac{8}{9} * 1$$

$$= 1.58$$

Exact value

$$\int_{-1}^{1} \frac{1}{1+x^2}\, dx = \left[\tan^{-1} x\right]_{-1}^{1}$$

$$= \tan^{-1} 1 - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \pi/2 = 1.57$$

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2019, Spring

Ex. 2) Using Gaussian integration formula evaluate

$$\int_{0.5}^{1.5} e^{x^2} dx \quad \text{with } n = 2, 3$$

Sol^n:-

Here,

$$f(x) = e^{x^2}$$

$$a = 0.5, \quad b = 1.5$$

Put $x = \frac{b-a}{2} t + \frac{b+a}{2}$

$$= \frac{1.5-0.5}{2} t + \frac{1.5+0.5}{2}$$

$$= \frac{1}{2} t + 1 = \frac{t}{2} + 1$$

$$dx = \frac{dt}{2}$$

$$\therefore e^{x^2} = e^{(t/2+1)^2}$$

$$\therefore \int_{0.5}^{-1.5} e^{x^2} = \int_{-1}^{1} \frac{1}{2} e^{(t/2+1)^2} dt$$

$$= \frac{1}{2} \int_{-1}^{1} e^{1/4 (t+2)^2} dt$$

$\therefore$ For $n = 2$

$$\int_{0.5}^{1.5} e^{x^2} dx = \frac{1}{2} \left[e^{1/4 \left(-\frac{1}{\sqrt{3}}+2\right)^2} + e^{1/4 \left(\frac{1}{\sqrt{3}}+2\right)^2} \right]$$

$$= 3.461$$

& For $n = 3$

$$\int_{0.5}^{1.5} e^{x^2} dx = \frac{1}{2} \left[\frac{5}{9} \left\{ e^{1/4 \left(-\sqrt{\frac{3}{5}}+2\right)^2} + e^{1/4 \left(\sqrt{\frac{3}{5}}+2\right)^2} \right\} + \frac{8}{9} e^{1/4 (0+2)^2} \right]$$

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$$= \frac{1}{2}\left[4.616 + 2.416\right] = 3.516$$ Ans

2021 Fall

Ex. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}}\,dx$ with $n = 2, 3$

Soln.

$$f(x) = \frac{\cos x}{\sqrt{1+\sin x}}$$

$a = 0, \quad b = \pi/2$

Put $x = \frac{b-a}{2}t + \frac{b+a}{2} = \frac{\pi/2 - 0}{2}t + \frac{\pi/2 + 0}{2}$

$= \frac{\pi}{4}t + \frac{\pi}{4}$

$dx = \frac{\pi}{4}\,dt$

$$\therefore \frac{\cos x}{\sqrt{1+\sin x}} = \frac{\cos\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)}{\sqrt{1+\sin\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)}}$$

$$\therefore \int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}} = \int_{-1}^{1} \frac{\cos\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)}{\sqrt{1+\sin\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)}} \cdot \frac{\pi}{4}\,dt$$

$$= \frac{\pi}{4}\int_{-1}^{1} \frac{\cos\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)}{\sqrt{1+\sin\left(\frac{\pi}{4}t + \frac{\pi}{4}\right)}}\,dt$$

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For $n=2$

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}} = \frac{\pi}{4}\left[\frac{\cos\left(\frac{\pi}{4}\left(-\frac{1}{\sqrt{3}}\right)+\frac{\pi}{4}\right)}{\sqrt{1+\sin\left(\frac{\pi}{4}\left(-\frac{1}{\sqrt{3}}\right)+\frac{\pi}{4}\right)}} + \frac{\cos\left(\frac{\pi}{4\sqrt{3}}+\frac{\pi}{4}\right)}{\sqrt{1+\sin\left(\frac{\pi}{4\sqrt{3}}+\frac{\pi}{4}\right)}}\right]$$

$$= \frac{\pi}{4}\left[0.821 + 0.234\right]$$

$$= 0.829$$

For $n=3$

$$\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}} = \frac{\pi}{4}\left[\frac{5}{9}\left\{\frac{\cos\left(-\frac{\pi}{4}\sqrt{\frac{3}{5}}+\frac{\pi}{4}\right)}{\sqrt{1+\sin\left(-\frac{\pi}{4}\sqrt{\frac{3}{5}}+\frac{\pi}{4}\right)}} + \frac{\cos\left(\frac{\pi}{4}\sqrt{\frac{3}{5}}+\frac{\pi}{4}\right)}{\sqrt{1+\sin\left(\frac{\pi}{4}\sqrt{\frac{3}{5}}+\frac{\pi}{4}\right)}}\right\} + \frac{8}{9}\,\frac{\cos\left(0+\frac{\pi}{4}\right)}{\sqrt{1+\sin\left(0+\frac{\pi}{4}\right)}}\right]$$

$$= \frac{\pi}{4}\left[\frac{5}{9}\left(0.908+0.125\right) + \frac{8}{9}\times 0.541\right]$$

$$= 0.828$$

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Numerical Solution of Ordinary Differential Equations (ODE)

1) Runge-Kutta Method of Order One (or Euler's Method)

Let, $y' = f(x, y)$, $y(x_0) = y_0$ be the given differential equation of order one.

Then, by Euler's method

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

$n = 0, 1, 2, 3, \ldots$

$h =$ step size

e.g. at $n = 0$

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

at $n = 1$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$x_1 = x_0 + h$$

and so on.

Ex.1) Using Euler's method

Solve $y' = x + y$, $y(0) = 1$ for $y(1)$.

Sol^n:-

We have,

$$y' = x + y$$

$$\therefore f(x, y) = x + y$$

$$y(0) = 1 \Rightarrow x_0 = 0, \; y_0 = 1$$

Using Euler's Method

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$$y_{n+1} = y_n + h f(x_n, y_n) \quad \text{---- (i)}$$

$$h = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

Take, $n=5$

$\therefore h = \frac{1}{5} = 0.2$

At $n=0$, from (i)

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.2\ f(0,1)$$
$$= 1 + 0.2\,(0+1)$$
$$= 1 + 0.2$$
$$= 1.2$$

$y_1 = y(x_1)$
$= y(x_0+h) \Rightarrow y(x_0+h)$
$= y(0.2) = 1.2$

$[\because f(x,y) = x+y]$

At $n=1$, from (i)

$$y_2 = y_1 + h \cdot f(x_1, y_1)$$
$$= 1.2 + 0.2\ f(0.2, 1.2)$$
$$= 1.2 + 0.2 \times 1.4$$
$$= 1.48$$

$\therefore y(0.4) = y_2 = 1.48$

$x_2 = 0.4$

At $n=2$, from eq^n (i)

$$y_3 = y_2 + h f(x_2, y_2)$$
$$= 1.48 + 0.2(0.4 + 1.48)$$
$$= 1.856$$

$y_3 = y(0.6)$
$= (1.856)$

$\therefore x_3 = 0.6$

At $n=3$, from eq^n (i)

$$y_4 = y_3 + h(f(x_3, y_3)$$
$$= 1.856 + 0.2(1.856 + 0.6)$$
$$= 2.347$$

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or

we have

$y' = x^2 - y^2$

$f(x, y) = x^2 - y^2$

$y(0) = 2$

$x_0 = 0, \; y_0 = 2$

<u>using</u> <u>eulers</u> <u>method</u>

$y_{n+1} = y_n + h \cdot f(x_n, y_n)$ —— ①

at <u>$n=0$</u> $\quad \therefore h = 0.2$

$y_1 = y_0 + h \cdot f(x_0, y_0)$

$= 2 + 0.2 \times [f(0, 2)]$

$= 2 + 0.2 \times [0^2 - 2^2]$

$= 2 + 0.2 \times 4$

$= 2.8$

$y_1 = y(x_1)$

$y_1 = y(x_0 + h) = y(0 + 0.2) = y(0.2) = 2.8$

$(y_1 = 2.8 \;\&\; x_1 = 0.2)$

at <u>$n=1$</u>

$y_2 = y_1 + h \, f(x_1, y_1)$

$= 2.8 + 0.2 \times [(0.2)^2 - (2.8)^2]$

$= 1.24$

$y_2 = y(x_2) = y(x_1 + h) = y(0.2 + 0.2) = y(0.4)$
$= 1.24$

at <u>$n=2$</u>

$y_3 = 1.24 + 0.2 \times [(0.4)^2 - (1.24)^2] = 0.96448$

$y_3 = y(x_3) = y(x_2 + h) = y(0.4 + 0.2) = y(0.6)$

$y_3 = 0.96448$

$x_3 = 0.6$

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$= 2.976$

$$\Rightarrow \boxed{y_5 = y(1) = y_5 = 2.976}$$

Q. Solve $y' = x^2 - y^2$; $y(0) = 2$ to estimate $y(1)$ with $h = 0.2$ using Euler's method.

at $n=3$

$$y_4 = y_3 + 0.2 \times [(0.6)^2 - (0.6419)^2] = 0.850$$

$$y_4 = y(x_4) = y(0.6+0.2) = y(0.8) = 0.850$$

$x_4 = 0.8$
$y_4 = 0.850$

at $n=4$

$$y_5 = y_4 + 0.2[(0.8)^2 - (0.850)^2]$$
$$= 0.850 + 0.2 \times (-0.0825)$$
$$= 0.8335$$

$$y_5 = y(x_5) = y(0.8+0.2) = y(1) = 0.8335$$ #

(2) Heun's method (R.K. 2nd order method) [$n=4$ ये h दिखा है न मानें]

let, $y' = f(x, y)$, $y(x_0) = y_0$
be the given differential eqn

Then, Heun's method is given by

$$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2) \quad \text{—— (1)}$$

where, $m_1 = f(x_n, y_n)$

$m_2 = f(x_n + h, y_n + m_1 h)$

Q given that

$$\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$$

$y = 0$ at $x = 1$
using Heun's method estimate y in the interval $1 \le x \le 2$ with $h = 0.25$.

$m_0 = 1$
$y_0 = 0$

$\Rightarrow n=4$ for Heun's Method

I.M.P.

2) Heun's Method (or RK 2nd order method)

Let $y' = f(x, y)$, $y(x_0) = y_0$ be the given differential eqn.

Then, Heun's method is given by

$$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2) \quad \text{---- (i)}$$

Where,

$$m_1 = f(x_n, y_n)$$
$$m_2 = f(x_n + h, y_n + m_1 h)$$

Ex. 1) Given that,

$$\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x} \quad \text{and } y = 0, \text{ at } x = 1$$

using Heun's method, estimate 'y' in the interval $1 \le x \le 2$ with $h = 0.25$

Sol^n:-

Here, $\frac{dy}{dx} = f(x, y) = \frac{2xy + e^x}{x^2 + xe^x}$

Initial values, $x_0 = 1$, $y_0 = 0$, $h = 0.25$

By Heun's method

$$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2) \quad \text{----- (i)}$$

For $n = 0$

$$y_1 = y_0 + \frac{h}{2}(m_1 + m_2) \quad \text{----- (ii)}$$

$$m_1 = f(x_0, y_0)$$
$$= f(1, 0) = \frac{2 \times 1 \times 0 + e^1}{1^2 + 1 \cdot e^1} = 0.731$$

$$m_2 = f(x_0 + h, y_0 + m_1 h)$$
$$= f(1 + 0.25, 0 + 0.731 \times 0.25)$$
$$= f(1.25, 0.183)$$

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$$= \frac{2 \times 1.25 \times 0.183 + e^{1.25}}{(1.25)^2 + 1.25 \times e^{1.25}} = 0.667$$

$\therefore$ From eq^n (ii)

$$y_1 = 0 + \frac{0.25}{2}(0.731 + 0.666) = 0.174$$

$\therefore y(1.25) = 0.175$

Put $n=1$, in eq^n (i)

$$y_2 = y_1 + \frac{h}{2}(m_1 + m_2) \quad \text{---- (iii)}$$

$$m_1 = f(x_1, y_1)$$

$$= f(1.25, 0.175)$$

$$= \frac{2 \times 1.25 \times 0.175 + e^{1.25}}{(1.25)^2 + 1.25 \cdot e^{1.25}} = 0.663$$

$$m_2 = f(x_1 + h, y_1 + m_1 h)$$

$$= f(1.25 + 0.25, \ 0.175 + 0.663 \times 0.25)$$

$$= f(1.5, 0.34)$$

$$= 0.613$$

From eq^n (iii)

$$y_2 = 0.175 + \frac{0.25}{2}(0.663 + 0.613)$$

$$= 0.335$$

$y(1.5) = 0.335$

Put, $n = 2$ in eq^n (i)

$$y_3 = y_2 + \frac{h}{2}(m_1 + m_2) \quad \text{--- (iv)}$$

$m_1 = f(x_2, y_2)$, $x_2 = x_2 + h = 1.5$

$= f(1.5, 0.335)$

$= 0.611$

$m_2 = f(x_2 + h, y_2 + m_1 h)$

$= f(1.5 + 0.25, 0.335 + 0.611 \times 0.25)$

$= f(1.75, 0.486)$

$= \dfrac{2 \times 1.75 \times 0.486 + e^{1.75}}{(1.75)^2 + 1.75 \times e^{1.75}}$

$= 0.568$

From eqn (iv)

$y_3 = y_2 + \frac{h}{2}(m_1 + m_2)$

$= 0.335 + \frac{0.25}{2}(0.611 + 0.568)$

$= 0.482$

at $y(1.75) = 0.482$

Put $n=3$ in eqn (i)

$y_4 = y_3 + \frac{h}{2}(m_1 + m_2)$ ---- (v)

$m_1 = f(x_3, y_3)$, $x_3 = x_2 + h = 1.75$

$= f(1.75, 0.482)$

$= \cancel{0.566}\ \cancel{0.568}\ 0.566$

$= 0.$

$m_2 = f(x_3 + h, y_3 + m_1 h)$

$= f(2, 0.482 + 0.567 \times 0.25)$

$= f(2, 0.623)$

$= 0.526$

$\therefore$ from (v)

$y_4 = 0.482 + \frac{0.25}{2}(0.567 + 0.526)$

$= 0.618$

$\therefore y(2) = 0.618$

Ex-2) $y' = y + xy$, $y(0) = 1$

Estimate $y(1)$ with $h = 0.25$ using Heun's method.

Sol^n:

Here,

$y' = f(x, y) = y + xy$

Initial values, $x_0 = 0$, $y_0 = 1$, $h = 0.25$

By Heun's method

$$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2) \quad \text{---- (1)}$$

For $n = 0$

$$y_1 = y_0 + \frac{h}{2}(m_1 + m_2) \quad \text{---- (2)}$$

$m_1 = f(x_0, y_0)$

$= f(0, 1)$

$= 1 + 0 \times 1$

$= 1$

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$= f(x_3, y_3) = f(1.75, 0.482)$

$= 0.567$

$m_2 = f(x_3 + h, y_3 + m_1 \cdot h)$

$= f(1.75 + 0.25, 0.482 + (0.567 \times 0.25))$

$= 0.526$

$y_4 = 0.482 + \frac{0.25}{2}(0.567 + 0.526)$

$= 0.618$

$\boxed{y(2) = 0.618}$

Q: $y' = y + xy$, $y(0) = 1$ estimate $y(1)$ with $h = 0.25$ using Heun's method.

Sol:

$y' = y + xy$

$\frac{dy}{dx} = f(x, y) = y + xy$ — (1)

initial values are $x_0 = 0$, $y_0 = 1$. $\therefore h = 0.25$

by Heun's method,

$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2)$ — (2)

for $n = 0$,

$y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$

$m_1 = f(x_0, y_0) = 1$

$m_2 = f(x_0 + h, y_0 + m_1 \cdot h) = f(0.25, 1.25)$

$= 1.56$

$\therefore y_1 = 1.32$ $y(0.25) = 1.32$

for $n=1$,

$$y_2 = y_1 + \frac{h}{2}(m_1 + m_2)$$

$$m_1 = f(x_1, y_1) = f(0.25, 1.32) = 1.65$$

$$m_2 = f(x_1 + 0.25, 1.32 + (m_1 \times 0.25))$$

$$= f(0.5, 1.73)$$

$$= 2.600$$

$$y_2 = 1.32 + \frac{0.25}{2}(1.65 + 2.60)$$

$$y(0.5) = 1.851$$

for $n=2$,

$$y_3 = y_2 + \frac{h}{2}(m_1 + m_2)$$

$$m_1 = f(x_2, y_2) = f(0.5, 1.851) = 2.777$$

$$m_2 = f(x_2 + h, y_2 + m_1 h)$$

$$= f(0.75, 1.851 + 2.777 \times 0.25)$$

$$= f(0.75, 2.545)$$

$$= 4.454$$

$$y_3 = 1.811 + \frac{0.25}{2}(2.777 + 4.454)$$

$$y(0.75) = 2.775$$

put $n = 3$

$$y_4 = y_3 + \frac{h}{2}(m_1 + m_2)$$

where $m_1 = f(x_3, y_3) \Rightarrow f(0.75, 2.755)$

$= 4.821$

$m_2 = f(x_3 + h, y_3 + m_1 h)$

$= f(1, 3.560)$

$= 7.920$

$\therefore y_4 = 2.755 + \frac{0.25}{2}(4.821 + 7.920)$

$$\boxed{y(1) = 4.348}$$

#

Runge Kutta Method of order Four (or Classical RK method)

Let $y' = f(x,y)$, $y(x_0) = y_0$ be the given differential eqn.

Then, 4th RK method is given by

$$y_{n+1} = y_n + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

$$n = 0, 1, 2, 3, \ldots$$

where, $m_1 = f(x_n, y_n)$

$$m_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{m_1 h}{2}\right)$$

$$m_3 = f\left(x_n + \frac{h}{2}, y_n + \frac{m_2 h}{2}\right)$$

$$m_4 = f(x_n + h, y_n + m_3 h)$$

2023, Spring

Ex.1) Using RK 4th order method, solve for 'y' at $x=1.2$ and $x = 1.4$.

Given that $\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + x \cdot e^x}$, at $x=1$, $y=0$

Soln:-

We have,

$$\frac{dy}{dx} = f(x,y) = \frac{2xy + e^x}{x^2 + x \cdot e^x}$$

$x_0 = 1$, $y_0 = 0$. Take $h = 0.2$

Using R.K. 4th order method

$$y_1 = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \quad \text{-----} (1)$$

$m_1 = f(x_0, y_0)$

$= f(1, 0)$

$= 0.731$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right)$

$= f\left(1 + \frac{0.2}{2}, 0 + \frac{0.731 \times 0.2}{2}\right)$

$= f(1.1, 0.0731)$

$= 0.701$

$m_3 = f\left(1 + \frac{0.2}{2}, 0 + \frac{0.701 \times 0.2}{2}\right)$

$= f(1.1, 0.0701)$

$= 0.7$

$m_4 = f(1 + 0.2, 0 + 0.7 \times 0.2)$

$= f(1.2, 0.14)$

$= 0.674$

From (i)

$y_1 = 0 + \frac{0.2}{6}(0.731 + 2 \times 0.701 + 2 \times 0.7 + 0.674)$

$= 0.14$

$\therefore y_1 = y(1.2) = 0.14$

$x_1 = x_0 + h = 1.2$

$\therefore y_2 = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$ --- (2)

$m_1 = f(x_1, y_1)$

$= f(1.2, 0.14)$

$= 0.674$

$m_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{m_1 h}{2}\right)$

$= f\left(1.2 + \frac{0.2}{2}, 0.14 + \frac{0.674 \times 0.2}{2}\right)$

$= f(1.3, 0.207)$

$= 0.651$

$m_3 = f\left(1.2 + \frac{0.2}{2}, 0.14 + \frac{m_2 \times h}{2}\right)$

$= f(1.3, 0.205)$

$= 0.651$

$m_4 = f(1.2 + 0.2, 0.14 + 0.651 \times 0.2)$

$= f(1.4, 0.27)$

$= 0.630$

From (2)

$y_2 = 0.14 + \frac{0.2}{6}(0.674 + 2 \times 0.651 + 2 \times 0.651 + 0.63)$

$= 0.27$

$\therefore y_2 = y(1.4) = 0.27$

2019, Spring

Ex. 1) Solve, $y' = y + e^x$, $y(0)=0$ for $y(0.2)$ and $y(0.4)$ by RK 4th order method.

Sol^n.

$y' = y + e^x$

$x_0 = 0$ Take $h = 0.2$

$y_0 = 0$

Using RK-4th order method

$$y_1 = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \quad \text{-----} \quad (1)$$

$m_1 = f(x_0, y_0) = f(0, 0) = 1$

$m_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{m_1 h}{2}\right) = f\left(0 + \frac{0.2}{2}, 0 + \frac{1 \times 0.2}{2}\right) = f(0.1, 0.1)$

$= 1.205$

$m_3 = f\left(0 + \frac{0.2}{2}, \cancel{1.205}\ 0 + \frac{1.205 \times 0.2}{2}\right)$

$= f(0.1, 0.121)$

$= 1.229$

$m_4 = f(0 + 0.2, 0 + 1.229 \times 0.2)$

$= f(0.2, 0.246)$

$= 1.479$

From (1)

$y_1 = 0 + \frac{0.2}{6}(1 + 2 \times 1.205 + 2 \times 1.229 + 1.479)$

$= 0.245$

$y(0.2) = 0.245$ & $x_1 = x_0 + h = 0 + 0.2$
$= 0.2$

$x_0 = 0, \quad y_0 = 0$

take $h = 0.2$

Using R-K 4th order method

$$y_1 = y_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \quad \text{--- (I)}$$

$$m_1 = f(x_0, y_0) = f(0,0) = 1$$

$$m_2 = f\left(x_0 + \frac{0.2}{2}, \; y_0 + \frac{0.2 \times m_1}{2}\right)$$

$$= f(0.1, 0.1)$$

$$= 1.205$$

$$m_3 = f\left(0.1, \; y_0 + \frac{0.2 \times m_2}{2}\right)$$

$$= f(0.1, 0.121)$$

$$= 1.226$$

$$m_4 = f(0 + 0.2, \; y_0 + m_3 \times 0.2)$$

$$= f(0.2, 0.245)$$

$$= 1.466$$

$$y_1 = 0 + \frac{0.2}{6} \times (1 + 2 \times 1.205 + 2 \times 1.226 + 1.466)$$

$$y_1 = 0.233 \Rightarrow \underline{y(0.2) = 0.233}$$

$$y_2 = y_1 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \quad \text{--- (II)}$$

$$m_1 = f(x_1, y_1) = f(0.2, 0.233)$$

$$= \underline{1.454}$$

$$m_2 = f\left(x_1 + \frac{0.2}{2}, \; y_1 + \frac{m_1 \times 0.2}{2}\right) = f(0.3, 0.378)$$

$$= \underline{1.728}$$

$$m_3 = f\left(0.3, \; 0.233 + \frac{1.728 \times 0.2}{2}\right) = f(0.3, 0.406)$$

$$= \underline{1.756}$$

$$m_4 = f(0.2 + 0.2, \; 0.233 + 1.756 \times 0.2)$$

$$= f(0.4, 0.584) = 2.076$$

now (11)

$$y_2 = 0.233 + \frac{0.2}{6} \times \left(1.454 + (2 \times 1.728) + (2 \times 1.756) + 2.076 \right)$$

$$y_2 = 0.583$$

$$\therefore y(0.4) = 0.583$$ #

Simultaneous first order differential eqn :-

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Simultaneous first order differential equation.

Let, $\frac{dy}{dx} = f(x, y, z)$

$\frac{dz}{dx} = g(x, y, z)$

$y(x_0) = y_0, \quad z(x_0) = z_0$

Euler's Method,

$y_{n+1} = y_n + hf(x_n, y_n, z_n)$

$z_{n+1} = z_n + hg(x_n, y_n, z_n)$

$n = 0, 1, 2, \ldots$

Heun's Method

$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2)$

$z_{n+1} = z_n + \frac{h}{2}(k_1 + k_2)$

$m_1 = f(x_n, y_n, z_n)$

$k_1 = g(x_n, y_n, z_n)$

$m_2 = f(x_n + h, y_n + m_1 h, z_n + k_1 h)$

$k_2 = g(x_n + h, y_n + m_1 h, z_n + k_1 h)$

Ex 1) Solve:

$\frac{dx}{dt} = xy + 2t,$

$\frac{dy}{dt} = 2ty + x,$

At, $t = 0$, $x = 1$, $y = -1$

to estimate $x(1)$, $y(1)$ with $h = 0.25$ using Euler's Method.

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Sol^n:

Let, $\frac{dx}{dt} = f(t, x, y) = xy + 2t$

$\frac{dy}{dt} = g(t, x, y) = 2ty + x$

$t_0 = 0, \quad x_0 = 1 \quad \& \quad y_0 = -1$

$h = 0.25$

By Euler's method

$x_{n+1} = x_n + hf(t_n, x_n, y_n)$

$y_{n+1} = y_n + hg(t_n, x_n, y_n)$

For $n = 0$,

$x_1 = x_0 + hf(t_0, x_0, y_0) = 1 + 0.25f(0, 1, -1) = 0.75$

$y_1 = y_0 + hg(t_0, x_0, y_0) = -1 + 0.25g(0, 1, -1) = -0.75$

For $n = 1$,

$x_2 = x_1 + hf(t_1, x_1, y_1)$ $\quad [t_1 = t_0 + h = 0 + 0.25]$

$= 0.75 + 0.25(0.75(-0.75) + 2 \times 0.25)$

$= 0.734$

$y_2 = -0.75 + 0.25g(2 \times 0.25(-0.75) + 0.75)$

$= -0.656$

$x_3 = x_2 + hf(t_2, x_2, y_2)$

$= 0.734 + 0.25f(0.5, 0.734, -0.656)$

$= 0.864$

$y_3 = -0.656 + 0.25g(t_2, x_2, y_2)$

$= -0.656 + 0.25(2 \times 0.5 \times -0.656 + 0.734)$

$= -0.637$

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$x_4 = x_3 + hf(t_3, x_3, y_3)$

$= 0.864 + 0.25 f(0.75, 0.864, -0.637)$

$= 1.101$

$y_4 = y_3 + hg(t_3, x_3, y_3)$

$= -0.637 + 0.25 g(0.75, 0.864, -0.637)$

$= -0.66$

$x_5 = x_4 + hf(t_4, x_4, y_4)$

$= 1.101 + 0.25(1, 1.101, -0.66)$

$=$

$x_4 = 1.101 = x(1)$

$y_4 = -0.66 = y(1)$ } Ans

Ex. 2) Solve: $\frac{dy}{dx} = x + z$, $\frac{dz}{dx} = x - y^2$

$y(0) = 2$, $z(0) = 1$

Estimate $y(0.25)$, $z(0.25)$, $y(0.5)$ & $z(0.5)$ using Heun's method.

Sol^n:-

Let $\frac{dy}{dx} = f(x, y, z) = x + z$

$\frac{dz}{dx} = g(x, y, z) = x - y^2$

$x_0 = 0,\ y_0 = 2,\ z_0 = 1$

Take, $h = 0.25$

By Heun's method,

$$y_{n+1} = y_n + \frac{h}{2}(m_1 + m_2)$$
$$z_{n+1} = z_n + \frac{h}{2}(k_1 + k_2)$$

$n = 0, 1$

$\therefore\ y_1 = y_0 + \frac{h}{2}(m_1 + m_2)$ —— (I)

$z_1 = z_0 + \frac{h}{2}(k_1 + k_2)$ —— (II)

$m_1 = f(x_0, y_0, z_0) = f(0, 2, 1) = 0.0 + 1 = 1$

$k_1 = g(0, 2, 1) = 0 - 2^2 = -4$

$m_2 = f(x_0 + h,\ y_0 + m_1 h,\ z_0 + k_1 h)$

$= f(0 + 0.25,\ 2 + 1 \times 0.25,\ 1 + (-4) \times 0.25)$

$= f(0.25,\ 2.25,\ 0)$

$= (0.25)$

$k_2 = g(0.25, 2.25, 0) = 0.25 - (2.25)^2 = -4.813$

From (I)

$y_1 = 2 + \frac{0.25}{2}(1 + 0.25) = 2.156 = y(0.25)$

$z_1 = 1 + \frac{0.25}{2}(-4 - 4.813) = -0.102$

$z(0.25)$

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$$y_2 = y_1 + \frac{h}{2}(m_1 + m_2) \quad -(3)$$

$$z_2 = z_1 + \frac{h}{2}(k_1 + k_2) \quad -(4)$$

$$m_1 = f(x_1, y_1, z_1)$$
$$= f(0.25, 2.156, -0.102)$$
$$= 0.148$$

$$k_1 = g(0.25, 2.156, -0.102)$$
$$= -4.398$$

$$m_2 = f(x_1 + h, y_1 + m_1 h, z_1 + k_1 h)$$
$$= f(0.5, 2.193, -1.202)$$
$$= -0.702$$

$$k_2 = f(x_1 + h, y_1 + m_1 h, z_1 + k_1 h)$$
$$= f(0.5, 2.193, -1.202)$$
$$= -4.309$$

$$y_2 = 2.156 + \frac{0.25}{2}(0.148 + -0.702)$$
$$= 2.087$$

$$z_2 = -0.102 + \frac{0.25}{2}(-4.398 - 4.309)$$
$$= -1.190$$