Design of Digital Filters

Introduction:

Filters are a particularly important class of linear time-invariant systems. In the design of frequency selective filters, the desired filter characteristics are specified in the frequency domain in terms of the desired magnitude and phase response of the filter. In the filter design process, we determine the coefficients of a causal FIR or IIR filter that closely approximate the desired frequency response specifications.

In practice, FIR filters are employed in filtering problems where there is a requirement for a linear-phase characteristic within the passband of the filters. As a general rule, an IIR filter has lower sidelobes. In the stopband than an FIR filter having the same number of parameters. For this reason, if some phase distortion is either tolerable or unimportant, an IIR filter is preferable, primarily because its implementation involves fewer parameter, require less memory & has lower computational complexity.

Digital Filter Design:

IIR filter and FIR filter are the two kinds of digital filters. The former one is commonly referred to as Recursive and the later as non-recursive.

Digital filters or digital signal processors are small special purpose digital computers designed to implement an algorithm that converts an input sequence x[n] into a desired output sequence y[n]. Such filters employ as hardwares, the devices such as adders, multipliers, shift registers, and delay elements. On the other hand, an analog filter employs resistors, capacitors and op-amps. As a result of this, digital processors are unaffected by factors such as component accuracy, temperature stability, long term drift, etc that affect the analog filters.

However, digital filter designs have to take into account such things as finite word size, round-off errors, aliasing, and other factors.

The difference equations that represent the IIR and FIR filters, in general, can be described as under:

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Where a_k and b_k are suitable constants, x[n] and y[n] represent the input and output sequences.

In short, the procedure of implementation of a digital filter has the following steps in order:

1. Selection of filters
2. Specification of the frequency response characteristics of the filter
3. Phase response specifications
4. Filter Design
5. Filter realization
6. Filter implementation

Comparison between analog and digital filters:

	Analog Filters		Digital Filters
1.	Both inputs and outputs are continuous-time signals.	1.	Both inputs and outputs are discrete-time signals.
2.	Implementation of such filters is carried out using passive components such as resistor, capacitor, inductors and active components such as transistors, op-amps.	2.	Digital filters are implemented on a microcontroller using DSP integrated circuits. Three basic elements such as adder, multiplier and delay elements are utilized.
3.	Analog filters operate in infinite frequency range theoretically but limited in practice by the finite maximum operating frequency of the semiconductor devices used. Eg. Op-amp functions upto 100 MHz and higher frequency are handled by microwave devices.	3.	Frequency range is restricted to half of the sampling rate. It is also restricted by maximum computational speed available in particular application. In fact, is drawback of a digital filter.
4.	Main disadvantages of analog filters are its higher noise sensitivity, non-linearities, dynamic range limitations, lack of flexibility in designing and reproductivity, errors generated due to drift and variations in the value of active and passive components used in circuits.	4.	Main advantages of digital filters are that these are insensitive to noise, higher linearities, unlimited dynamic range, flexibility in software design, high accuracy, highly reliable.
5.	Analog filters have higher frequency range as well as they can interact directly with the real analog world.	5.	Digital filters require additional ADC and DAC converters section for connecting to the physical analog world.

Comparison between IIR and FIR digital filters:

	IIR Digital Filter		FIR Digital Filter
1.	IIR digital filters are characterized by rational system function: $H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$	1.	FIR digital filters are characterized by system function which are not rational as: $H(z) = \sum_{k=0}^{M} b_k z^{-k}$
2.	Impulse response of these digital filters are computed for infinite number of samples i.e. $h[n] \neq 0$ for $0 \leq n \leq \infty$	2.	Impulse response of these filters is computed for finite no. of samples i.e. $h[n] \neq 0$ for $0 \leq n \leq M-1$ & 0 elsewhere.
3.	These filters do not have linear phase & these are used where some phase distortion is tolerable.	3.	These filters have linear phase characteristics. These filters are used in speech processing, it eliminates the adverse effects of frequency dispersion due to non-linearity of phase.
4.	Theoretically, these filters are stable. After truncation their coefficients, becomes unstable.	4.	These filters are realized by direct convolution, that is why these are stable.
5.	These filters has less flexibility for	5.	These filters have greater flexibility to control

	obtaining non –standard frequency response or for which analog filter design techniques are not available.		the shape of their magnitude response and realization efficiently.
6.	These filters are usually realized by recursive method. The present output of these filters also depends on previous outputs as well as past and present inputs. It is a feedback system.	6.	These filters are generally realize non-recursively or by direct convolution. These are not feedback systems. They are not dependent on previous outputs.
7.	They are more susceptible to round-off noise associated with finite precision arithmetic, quantization error and coefficient inaccuracies.	7.	These effects are less severe in FIR digital filters.
8.	Short-time delay.	8.	Time delay increase with increase in order.
9.	These require lesser number of arithmetic operations and these have lower computational complexity and smaller memory requirements.	9.	For sharp amplitude response, we require higher order FIR digital filter. This is a main drawback of an FIR filters.
10	IIR filters have resemblance with analog filters. The common method for IIR filter design is to design an IIR analog filter followed by analog to digital transformation methods.	10	FIR filters are unique to discrete-time domain. These cannot be derived from analog filters.
11	It requires less approximation parameter to design so, it is simple to design.	11	It requires more approximation parameter so, design procedure is complex.

Filters:

The term filter is commonly used to describe a device that discriminate, according to some attribute of the objects applied at its input, what passes through it.
Filtering in electrical world is a process by which the frequency spectrum of a signal can be modified, reshaped, or manipulated to achieve some desired objectives.
Filters allow some frequencies to pass while completely blocks other frequencies.

Why filters:

- Noise reduction
- For demodulation
- To separate distinct singals
- To limit the bandwidth of signals, etc

Types:

Filters are usually classified according to their frequency-domain characteristics as **Low-pass, High-pass, Band-pass, Band-stop or Band-elimination and Notch filters**.

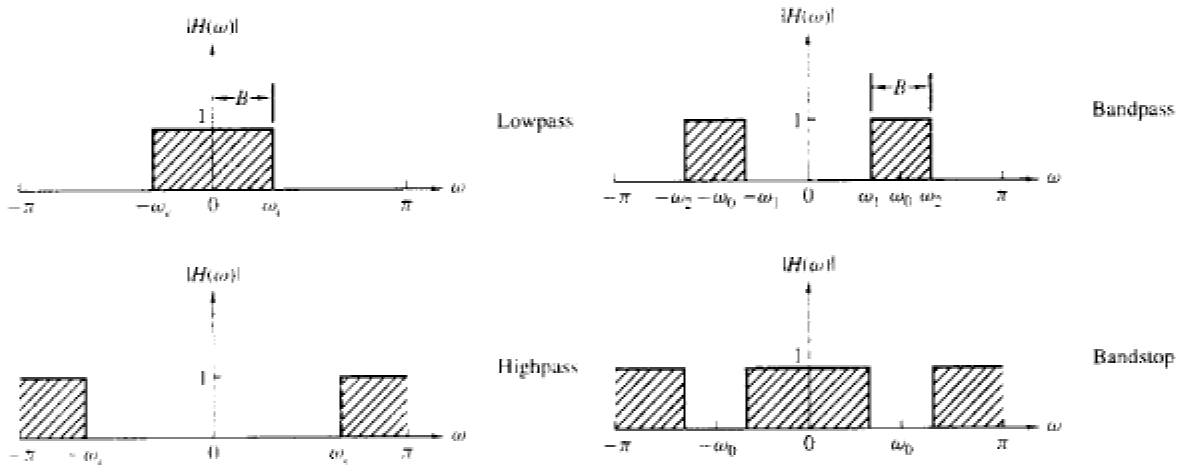


Figure: Magnitude responses for some ideal frequency-selective discrete-time filters.

Notch Filter

A notch filter is a filter that contains one or more deep notches or, ideally, perfect nulls in its frequency response characteristic. Figure below illustrates the frequency response characteristic of a notch filter with nulls at frequencies ω_0 and ω_1.

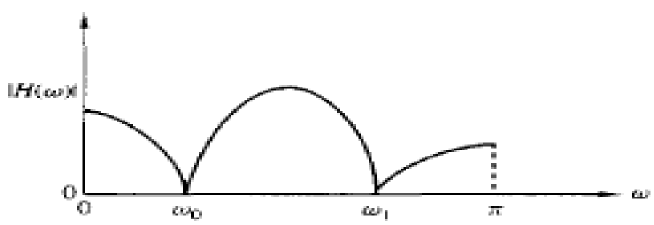


Figure: Frequency response characteristics of Notch filter

To create a null in the frequency response of a filter at frequency ω_0, we simply introduce a pair of complex conjugate zeros on the unit circle at an angle ω_0.

$$i.e.\, z_{1,2} = e^{\pm j\omega_0}$$

Thus system function for an FIR notch filter is

$$H(z) = b_0\left(1 - e^{j\omega_0}z^{-1}\right)\left(1 - e^{-j\omega_0}z^{-1}\right)$$
$$= b_0(1 - 2cos\omega_0 z^{-1} + z^{-2})$$

The problem with the FIR notch filter is that the notch has relatively larger bandwidth, which means that other frequency components around the desired null are severely attenuated.

Suppose we place a pair of complex conjugate poles at ,

$$p_{1,2} = re^{\pm j\omega_0}$$

The effect of the poles is to introduce a resonance in the vicinity of the null and thus reduce the bandwidth of the notch. The system function for the resulting filter is

$$H(z) = \frac{b_0(1 - 2cos\omega_0 z^{-1} + z^{-2})}{1 - 2rcos\omega_0 z^{-1} + r^2 z^{-2}}$$

Notch filters are useful in many applications where specific frequency components must be eliminated. For example, instrumentation and recording systems require that the power-line frequency of 60 Hz and its harmonics be eliminated.

Advantages of using digital filters:

1. A digital filter is programmable, i.e. its operation is determined by a program stored in the processor's memory. This means the digital filter can easily be changed without affecting the circuitry (hardware). An analog filter can be changed by redesigning the filter circuit.
2. Digital filters are easily designed, tested and implemented on a general-purpose computer or workstation.
3. The characteristics of analog filter circuits (particularly those containing active components) are subject to drift and are dependent on temperature. Digital filters do not suffer from these problems, and so are extremely stable with respect both to time and temperature.
4. Unlike their analog counterparts, digital filters can be handle low frequency signals accurately. As the speed of DSP technology continues to increase, digital filters are being applied to high frequency signals in the RF (radio frequency) domain, which in the past was the exclusive preserve of analog technology.
5. Digital filters are very much more versatile in their ability to process signals in a variety of ways: this includes the ability of some types of digital filter to adapt to changes in the characteristics of the signal.
6. Fast DSP processors can handle complex combinations of filters in parallel or cascade (series), making the hardware requirements relatively simple and compact in comparison with the equivalent analog circuitry.

Characteristics of Practical Frequency-Selective Filters:

Ideal Low-pass filter have sharp cut-off from passband to stopband. There is no gap between passband and stopband in ideal lowpass filter. In practice there are some ripples tolerable in passband and stopband and some gap between passband and stopband is also observed. The practical lowpass filter is shown in figure below:

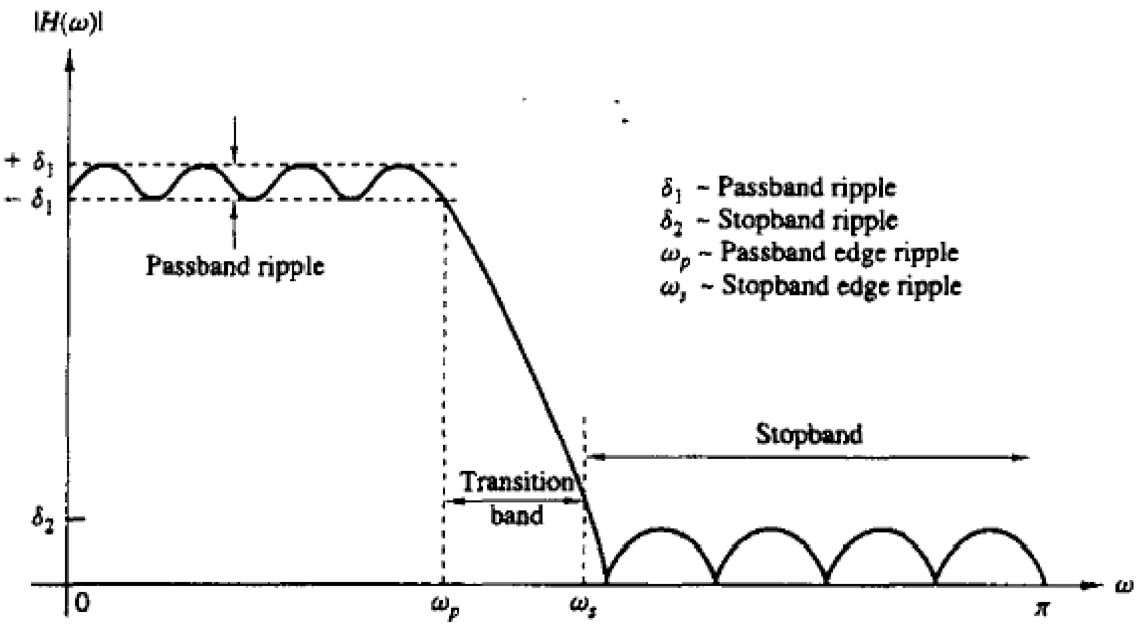


Figure: Magnitude Characteristics of Physically realizable lowpass filters.

Pass-band:

The frequency range over which a filter passes signal energy. At passband, the filters' frequency response is equal to or greater than -3dB.

Stop-band:

The frequency range over which a filter eliminates or reject signal energy. It is the band of frequency attenuated by digital filters. At stopband, the filters' frequency response is less than -3dB.

Transition band:

The frequency range between the passband and stopband.

Ripple:

It refers to fluctuations in passband or stopband of a filters' frequency versus magnitude curve. A small amount of ripple in the passband and stopband is usually tolerable.

Specifications of Analog Filters:

Gain, $G = 20\log_{10}|H(j\omega)|$ dB

Attenuation, $\alpha = -20\log_{10}|H(j\omega)|$ dB

- The width of the transition band is ω_s-ω_p
- The width of the passband is usually called the bandwidth of the filter
- The ripple in the passband is $20\log_{10}\delta_1$ decibels (dB)
- The ripple in the stopband is $20\log_{10}\delta_2$ decibels (dB)

In any filter design problem we can specify

1. The Maxmimum tolerable passband ripple
2. The maximum tolerable stopband ripple
3. The passband edge frequency, ω_p
4. The stopband edge frequency, ω_s.

Based on these specification, we can select the parameters $\{a_k\}$ and $\{b_k\}$ in the frequency response characteristics, $H(j\omega)$ which best approximates the desired specifications.

Design Procedure:

-Firslty lowpass filter is designed
-Then we change LPF to any other types
LPF ---(frequency transformation)→ Other types (BP, BS, HP)

Digital filter types:

1) **FIR (Non-Recursive)**
2) **IIR (Recursive) (We will study IIR filter in chapter 7)**

Design of Digital FIR filters:

- If a linear phase characteristics is required then FIR design, else FIR or IIR filter design.
- As a general rule, an IIR filter has lower sidelobes in the stopband than an FIR filter having the same number of parameters.
- IIR implementation involves a fewer parameters, require less memory and has lower computational complexity.

Why FIR filters?

In many digital signal processing applications, FIR filters are preferred over their IIR counterparts. The main advantages of the FIR filter designs over their IIR equivalents are the following:

i. They can have an exact linear phase.
ii. There exist computationally efficient realizations for implementing FIR filters. These include both nonrecursive and recursive realizations.
iii. FIR filters realized nonrecursively are inherently stable and free of limit cycle oscillations when implemented on a finite-wordlength digital system.
iv. The design methods are generally linear.
v. They can be realized efficiently in hardware.
vi. The filter start up transients have finite duration.
vii. The output noise due to multiplication roundoff errors in an FIR filter is usually very low and the sensitivity to variations in the filter coefficients is also low.

The main disadvantage of conventional FIR filter designs is that they require, especially in applications demanding narrow transition bands, considerably more arithmetic operations and hardware components, such as multiplier, adders, and delay elements than do comparable IIR filters.

Causality and its implications:

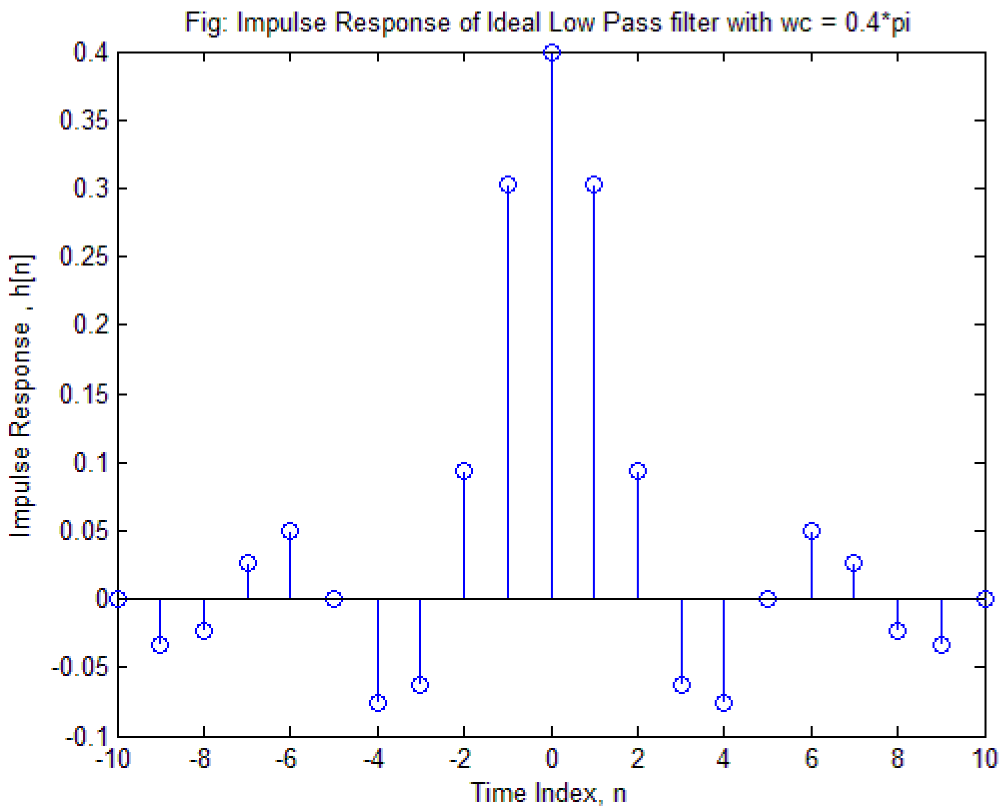
Let us examine the impulse response h[n] of an ideal lowpass filter with frequency response characteristics:

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < \omega \le \pi \end{cases}$$

The impulse response of this filter is

$$h[n] = \begin{cases} \omega_c/\pi, & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin\omega_c n}{\omega_c n}, & n \neq 0 \end{cases}$$

Which is a Sinc function and it is clear that the ideal lowpass filter is noncausal and hence it cannot be realized in practice.



One possible solution is to introduce a large delay n_0 in h[n] and arbitrarily to set h[n] = 0 for $n < n_0$. However, the resulting system no longer has an ideal frequency response characteristics. If we set h[n] = 0 for $n < n_0$, the Fourier series expansion of H(ω) results in the Gibbs Phenomenon.

Magnitude Response and Phase Response of Digital Filters:

The discrete-time Fourier transform of a finite sequence impulse response h[n] is given by

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h[n]e^{-j\omega n} = |H(e^{j\omega})|e^{j\varphi(\omega)}$$

The magnitude and phase responses are given by:

$$M(\omega) = |H(e^{j\omega})| = \sqrt{(H_R(e^{j\omega})^2 + H_I(e^{j\omega})^2)}$$

$$\varphi(\omega) = \tan^{-1} {H_I(e^{j\omega})}\Big/{H_R(e^{j\omega})}$$

Where $H_R(e^{j\omega}) = Re\{H(e^{j\omega})\}$ & $H_I(e^{j\omega}) = Im\{ H(e^{j\omega})\}$

Filters can have a linear or non-linear phase, depending upon the delay function, namely the phase delay and group delay. The phase and group delays of the filter are given by

$$\tau_p = -\frac{\varphi(\omega)}{\omega} \qquad \& \qquad \tau_g = -\frac{d\varphi(\omega)}{d\omega}$$

Linea phase filters are those filters in which the phase delay and group delay are constants i.e. independent of frequency. Linear phase filters are also called constant time delay filters. For the phase response to be linear

$$\frac{\varphi(\omega)}{\omega} = -\tau, \qquad -\pi \leq \omega \leq \pi$$
$$\therefore\ \varphi(\omega) = -\omega\tau$$

Where τ is a constant phase delay expressed in number of samples.

$$\varphi(\omega) = \tan^{-1}\frac{H_I(e^{j\omega})}{H_R(e^{j\omega})} = -\omega\tau$$
$$or, -\omega\tau = \tan^{-1}\frac{\sum_{n=0}^{M-1} h[n]\sin\omega n}{\sum_{n=0}^{M-1} h[n]\cos\omega n}$$
$$or, \tan(\omega\tau) = \frac{\sum_{n=0}^{M-1} h[n]\sin\omega n}{\sum_{n=0}^{M-1} h[n]\cos\omega n}$$

Simplifying, we get

$$\sum_{n=0}^{M-1} h[n]\sin(\omega\tau - \omega n) = 0$$

And the solution is given by,

$$\tau = \frac{M-1}{2} \quad \& \; h[n] = h[M-1-n] \; for\; 0 < n < M-1$$

If these conditions are satisfied, then the FIR filter will have constant phase and group delays and thus the phase of the filter will be linear. The phase and group delays of the linear phase FIR filter are equal and constant over the frequency band. Whenever a constant group delay alone is preferred, the impulse response will be of the form,

$$h[n] = -h[M-1-n] -\!\!\rightarrow Antisymmetric\ impulse\ response\ sequence.$$

FIR filter design based on windowed Fourier series:

FIR filters are described by a transfer function that is a polynomial in z^{-1} and require different approaches for their design.

A direct and straightforward method is based on truncating the Fourier series representation of the prescribed frequency response.

The second method is based on the observation that for a length of N FIR digital filter, N distinct equally spaced frequency samples of its frequency response constitute the N-point DFT of its impulse response and hence, the impulse response sequence can be readily computed by applying an inverse DFT to these frequency samples.

Impulse Response of Ideal Filter:

Ideal lowpass filter has zero-phase frequency response

$$H_{LP}\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The corresponding impulse response,

$$h_{LP}(n) = \begin{cases} \omega_c/\pi, & n = 0 \\ \dfrac{\omega_c}{\pi}\dfrac{\sin \omega_c n}{\omega_c n}, & n \ne 0 \end{cases} = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty \le n \le \infty$$

The impulse response of an ideal lowpass filter is doubly infinite, not absolutely summable, and therefore unrealizable. By setting all impulse response coefficients outside the range $-M \le n \le M$ equal to zero, we arrive at a finite length non-causal approximation of length $N = 2M + 1$, which is shifted to right yield the coefficients of a causal FIR lowpass filter:

$$\hat{h}_{LP}(n) = \begin{cases} \dfrac{\sin \omega_c (n-m)}{\pi (n-m)}, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

FIR filter design by windowing:

If $H_d(\omega)$ is desired frequency response and $h_d[n]$ be its corresponding unit sample response. We begin with $H_d(\omega)$ & determine the $h_d[n]$. The relations are given by Fourier transform.

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d[n] e^{-j\omega n} - - - (i)$$

Where,

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\,d\omega - - - (ii)$$

Here, the unit sample response from (i) is infinite duration and it is also non-causal and unrealizable. In order to obtain a realizable filter it must be truncated at some point. Let n = M-1 to yield the FIR filter of length M. i.e.

$$h[n] = \begin{cases} h_d[n], & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

Now, frequency response corresponding to the finite duration sequence is,

$$H(\omega) = \sum_{n=0}^{M-1} h_d[n]e^{-j\omega n} - - - (iii)$$

The process of obtaining (iii) from (i) is **windowing**. That is, truncating $h_d[n]$ to length M is equivalent to multiplying $h_d[n]$ by a Rectangular window, defined as

$$w[n] = \begin{cases} 1, & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

So, the unit sample response of FIR filter becomes,

$$h[n] = h_d[n]w[n] = \begin{cases} h_d[n], & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

note that the multiplication of $h_d[n]$ and w[n] results that the convolution of $H_d(\omega)$ and $W(\omega)$ will be the frequency domain representation of designed FIR filter. i.e.

$$H(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(v)W(\omega - v)dv$$

Gibbs Phenomenon:

The causal FIR filters, obtained by simply truncating the impulse response coefficients of the ideal filters given in the previous section exhibit an *oscillating behavior*, in their magnitude response is known as Gibb's Phenomenon.

The oscillation behavior of the magnitude response on both sides of the cut-off frequency is clearly visible in both cases. Moreover, as the length of the filter increased, the number of ripples in both passband and stopband increases with a corresponding decrease in the width of the ripples. However, the heights of the largest ripples, which occurs in both sides of the cut-off frequency remains same independent of the filter length and are approximately 11 percent of the difference between the passband and stopband magnitude of the ideal filters.

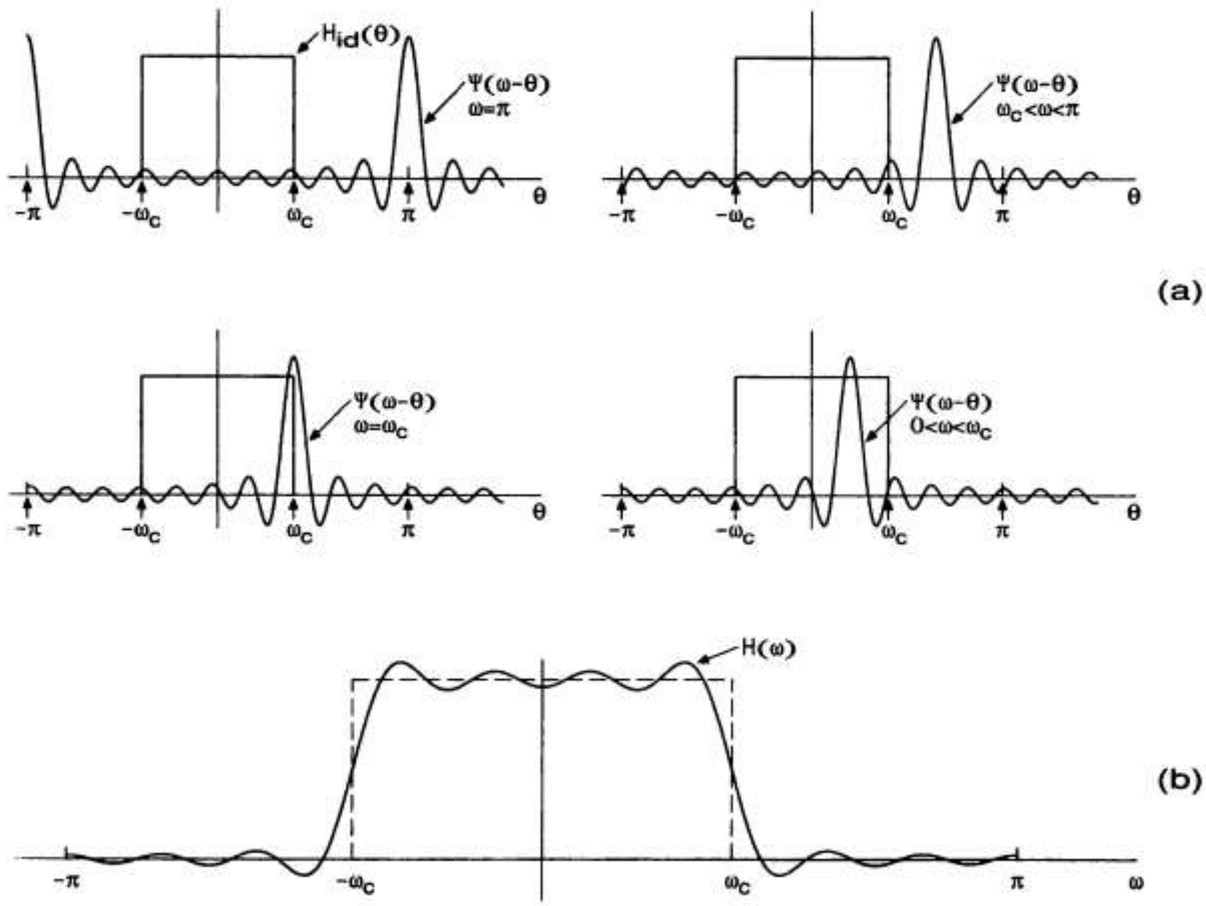


Figure: Illustration of effect of windowing in the frequency domain & Gibbs phenomenon.

The reason behind the Gibb's phenomenon can be explained by considering the truncation operations as multiplication by a finite-length window sequence ω(n) and by examining the windowing process in the frequency domain.

Rectangular window:

The rectangular window is defined as,

$$w[n] = \begin{cases} 1, & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

Then its Fourier transform is,

$$W(\omega) = \sum_{n=0}^{M-1} w[n]\, e^{-j\omega n}$$

On expansion we get,

$$W(\omega) = e^{-j\omega(M-1)/2} \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$

This window function has a magnitude response

$$|W(\omega)| = \left| \frac{\sin(\omega M/2)}{\sin(\omega/2)} \right|, \qquad -\pi \le \omega \le \pi$$

Here, the width of the main lobe is $4\pi/M$. As the value of M increases the main lobe becomes narrower. However the sidelobes of $|W(\omega)|$ remain unaffected by value of M.

Note: (i) when M is increased, the width of main lobe is decreased and the transition band is reduced. (ii) Attenuation in side lobes is independent of M but it depends upon types of windows. (iii) A window function with minimum stop band, attenuation has maximum main lobe width. So proper window should be chosen in order to achieve a desire stop band attenuation.

Types of Windows:

1. Fixed Window Function:

Type	**w[n] for $0 \le n \le M$**	**Transition width of main lobes**	**Minimum stopband attenuation**
Rectangular	1	$\frac{4\pi}{M+1}$	-21 dB
Bartlett	$1 - \frac{\lvert n \rvert}{M+1}$	$\frac{8\pi}{M}$	-25 dB
Hanning	$0.5\left(1 - \cos\frac{2\pi n}{M-1}\right)$	$\frac{8\pi}{M}$	-44 dB
Hamming	$0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$	$\frac{8\pi}{M}$	-53 dB
Blackman	$0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$	$\frac{12\pi}{M}$	-74 dB
Blackman	$0.42 - 0.5\cos\frac{2\pi n}{M-1} + 0.08\cos\frac{4\pi n}{M-1}$	$\frac{12\pi}{M}$	-74 dB

Note:

a. The $H_d(\omega)$ of an ideal LP filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)}, & for\ |\omega| \le \omega_c \\ 0, & elsewhere \end{cases}$$

b. The order of the filter is

$$N = k\left[\frac{2\pi}{\omega_s - \omega_p}\right]$$

The value of k can be obtained from the width of the main lobes.

c. The width of the main lobe $= k\left(\frac{2\pi}{M}\right)$

d. The phase delay $\tau = \frac{M-1}{2}$

2. Adaptive Window

a. Kaiser window:

$$w[n] = \frac{I_0\left[\beta\left(\sqrt{1-\left({}^{n}/_{M}\right)^2}\right)\right]}{I_0(\beta)}\ for - M \le n \le M$$

β = adjustable parameter

$I_0(x)$ = zeroth order Bessel function

$$I_0(x) = 1 + \sum_{r=1}^{\infty}\left[\frac{\left({}^{x}/_{2}\right)^r}{r!}\right]^2$$

β controls the minimum attenuation $\alpha_s = -20\log_{10}\delta_s$, in the stopband of the windowed filter response.

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7) & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) & \text{for } 21 < \alpha_s \le 50 \\ 0 & for\ \alpha_s < 21 \end{cases}$$

The order of filter N is estimated by using the formula

$$N = \begin{cases} \dfrac{\alpha_s - 7.95}{14.36\Delta f} + 1 & \text{for } \alpha_s > 21 \\ \dfrac{0.9222}{\Delta f} + 1 & \text{for } \alpha_s \le 21 \end{cases}$$

$$\Delta f = f_s - f_p = \frac{\omega_s - \omega_p}{2\pi}\ (transition\ width)$$

FIR filters design by Frequency Sampling method:

In this method

- A set of samples is found from the desired frequency response, which is considered as DFT coefficients.
- Then IDFT of these samples are calculated which gives filter coefficients.

$$H_d(\omega) \xrightarrow{sample} H(k) \xrightarrow{IDFT} h(n)$$

The $H_d(\omega)$ is sampled at

$$\omega_k = \frac{2\pi}{M}(k+\alpha), \qquad k = 0,1,2,---,K$$

$$K = \begin{cases} \frac{M-1}{2} & if\ M\ odd \\ \frac{M}{2} - 1 & if\ M\ even \end{cases}$$

If α = 0 , Type I design, else if α = ½ , Type II design.

Type I Design of frequency sampling method:

Consider the design of the FIR filter whose desired frequency response is denoted by $H_d(\omega)$. This frequency response is sampled uniformly at M points. Such frequency samples are given at,

$$\omega_k = \frac{2\pi}{M}k, \ \ k = 0,1,2,3,\dots,M-1$$

Such sampled desired frequency response is a Discrete Fourier Transform, it can be denoted by

$$H(k) = H_d(\omega)|_{\omega=\omega_k}, \ \ k = 0,1,2,\dots,M-1$$

$$or\ H(k) = H_d\left(\frac{2\pi k}{M}\right), \ \ k = 0,1,2,\dots,M-1$$

Hence H(k) is M-point DFT. By taking inverse DFT of H(k), we get h[n]. This h[n] is unit sampled response of FIR filter.

$$h[n] = \frac{1}{M}\sum_{k=0}^{M-1} H(k)e^{\frac{j2\pi kn}{M}}, \quad n = 0,1,2,3,\dots,M-1$$

hence, the unit sample response of FIR filter of length M is obtained using frequency sampling method. The above impulse response can be simplified for linear phase condition as,

$$h[n] = \frac{1}{M}\left[H(0) + 2\sum_{k=1}^{K} Re\left\{H(k)e^{\frac{j2\pi kn}{M}}\right\}\right]$$

$$K = \begin{cases} \frac{M-1}{2} & if\ M\ odd \\ \frac{M}{2} - 1 & if\ M\ even \end{cases}$$

Derive an expression for system function if the unit sample response h[n] is obtained using frequency sampling method.

Solution: ➔

The system function H(z) is given as z-transform of h[n] i.e.

$$H(z) = \sum_{n=0}^{M-1} h[n]z^{-n}$$

Substituting for h[n]

$$H(z) = \sum_{n=0}^{M-1}\left[\frac{1}{M}\sum_{k=0}^{M-1} H(k)e^{\frac{j2\pi kn}{M}}\right] z^{-n}, n = 0, 1, 2, 3, \dots, M-1$$

Interchanging the order of summation in above equation, we get

$$H(z) = \frac{1}{M}\sum_{k=0}^{M-1}\left[H(k)\sum_{n=0}^{M-1} e^{\frac{j2\pi kn}{M}} z^{-n}\right]$$

The inner summation is in the form of,

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

$$e^{\frac{j2\pi kn}{M}} z^{-n} = \left(e^{\frac{j2\pi k}{M}} z^{-1}\right)^n$$

Now applying above result to the summation,

$$\sum_{n=0}^{M-1}\left(e^{\frac{j2\pi k}{M}} z^{-1}\right)^n = \frac{1-\left(e^{\frac{j2\pi k}{M}} z^{-1}\right)^M}{1-e^{\frac{j2\pi k}{M}} z^{-1}} = \frac{1-e^{j2\pi k}z^{-M}}{1-e^{\frac{j2\pi k}{M}} z^{-1}} = \frac{1-z^{-M}}{1-e^{\frac{j2\pi k}{M}} z^{-1}}$$

$$\therefore H(z) = \frac{1-z^{-M}}{M}\sum_{k=0}^{M-1}\left[\frac{H(k)}{1-e^{\frac{j2\pi k}{M}} z^{-1}}\right]$$

This equation gives the system function H(z).

Design a LP FIR filter using frequency sampling method having cutoff frequency of π/2 rad/samples. The filter should have linear phase & length of 17.

Solution: →

i) To determine the desired frequency response:

The desired frequency response is $H_d(\omega)$. For the linear phase FIR lowpass filter, it is given as under:

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)}, & for\ 0 \le \omega \le \omega_c \\ 0\ , & \omega_c \le \omega \le \pi \end{cases}$$

The length of the filter is M = 17 and the cutoff frequency is $\omega_c = \pi/2$ rad/samples.

$$H_d(\omega) = \begin{cases} e^{-j\omega 8}, & for\ 0 \le \omega \le \dfrac{\pi}{2} \\ 0\ , & for\ \dfrac{\pi}{2} \le \omega \le \pi \end{cases}$$

This is the desired frequency response of required lowpass filter.

ii) To obtain $H_d(\omega)$

$$H(k) = H_d(\omega)|_{\omega=\omega_k}$$

$$\omega_k = \frac{2\pi}{M}k,\ \ k = 0,1,2,3,\dots,M-1$$

$$H(k) = \begin{cases} e^{-j\frac{2\pi}{17}k8}, & for\ 0 \le \dfrac{2\pi}{17}k \le \dfrac{\pi}{2} \\ 0\ , & for\ \dfrac{\pi}{2} \le \dfrac{2\pi}{17}k \le \pi \end{cases}$$

Now for $\frac{2\pi}{17}k = 0, k = 0$ & $\frac{2\pi}{17}k = \frac{\pi}{2}, k = 17/4$

$$H(k) = \begin{cases} e^{-j\frac{16\pi k}{17}}, & for\ 0 \le k \le \dfrac{17}{4} == 0 \le k \le 4.25 \approx 4 \\ 0\ , & for\ \dfrac{17}{4} \le k \le \dfrac{17}{2} == 4.25 \approx 5 \le k \le 8 \end{cases}$$

The above equation gives sampled version of $H_d(\omega)$ i.e. H(k)

iii) To obtain h[n]

For odd value of M, h[n] is given as under

$$h[n] = \frac{1}{M}\left[H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left\{H(k)e^{\frac{j2\pi kn}{M}}\right\}\right]$$

$$= \frac{1}{17}\left[1 + 2\sum_{k=1}^{8} Re\left\{H(k)e^{\frac{j2\pi kn}{17}}\right\}\right] = \frac{1}{17}\left[1 + 2\sum_{k=1}^{4} Re\left\{e^{-j\frac{16\pi k}{17}}e^{\frac{j2\pi kn}{17}}\right\}\right]$$

Solving,

$$h[n] = \frac{1}{17}\left[1 + 2\sum_{k=1}^{4}\cos\left(\frac{2\pi k(8-n)}{17}\right)\right],\ \ n = 0,1,2,3,\dots,16$$

This is the unit sample response of the FIR filter obtained using frequency sampling method.

Design of Optimum Equiripple Linear-Phase FIR filters:

The windowing method and frequency sampling method of FIR design are simple however they lack precise control in ω_p and ω_s.

This method is viewed as an optimum design criterion in the sense that the weighted approximation error between the desired frequency response and the actual frequency response is spread evenly across the passband and stopband of the filter minimizing the maximum error. The resulting filter designs have ripples in both the passband and the stopband.

To describe the design procedure, let us consider the design of lowpass filter with passband edge frequency ω_p and stopband edge frequency ω_s, then the filter frequency response satisfies the condition

$$1-\delta_1 \le H_r(\omega) \le 1+\delta_1 \qquad |\omega| \le \omega_p$$

Similarly, in the stopband, the filter frequency response is specified to fall between the limits $\pm\delta_2$

$$-\delta_2 \le H_r(\omega) \le \delta_2 \qquad |\omega| > \omega_s$$

Thus δ_1 represents the ripple in the passband and δ_2 represents the attenuation or ripple in the stopband. The remaining filter parameter is M, the filter length or the number of filter coefficients.

Let us focus on the four different cases that result in a linear phase FIR:

Case 1: Symmetric unit sample response & M odd

$$h[n] = h[M-1-n]$$

$$H_r(\omega) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\omega\left(\frac{M-1}{2}-n\right)$$

If we let k = (M-1)/2 – n & define a new set of filter parameters {a(k)}

$$a(k) = \begin{cases} h\left(\frac{M-1}{2}\right) & k=0 \\ 2h\left(\frac{M-1}{2}-k\right) & k = 1,2,3,---,\frac{M-1}{2} \end{cases}$$

Then

$$H_r(\omega) = \sum_{k=0}^{(M-1)/2} a(k)\cos\omega k$$

Case 2: Symmetric unit sample response & M even

$$h[n] = h[M-1-n]$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\omega\left(\frac{M-1}{2}-n\right)$$

Again, we change the summation index from n to k = M/2 – n & define a new set of filter parameters {b(k)} as

$$b(k) = 2h\left(\frac{M}{2}-k\right), \qquad k = 1,2,3,--,\frac{M}{2}$$

Then

$$H_r(\omega) = \sum_{k=1}^{M/2} b(k)\cos\omega\left(k-\frac{1}{2}\right)$$
$$= \cos\frac{\omega}{2}\sum_{k=0}^{\frac{M}{2}-1} \hat{b}(k)\cos\omega k$$

Where,

$$\hat{b}(0) = \frac{1}{2}b(1)$$

$$\hat{b}(k) = 2b(k) - \hat{b}(k-1) \;\; k = 1,2,---,\frac{M}{2}-2$$
$$\hat{b}\left(\frac{M}{2}-1\right) = 2b\left(\frac{M}{2}\right)$$

Case 3: Antisymmetric unit sample response and M odd:

$$h[n] = -h[M-1-n]$$

$$H_r(\omega) = 2\sum_{n=0}^{(M-3)/2} h(n)\sin\omega\left(\frac{M-1}{2}-n\right)$$

Suppose k = (M-1)/2 - n

$$c(k) = 2h\left(\frac{M-1}{2}-k\right) \quad k = 1,2,3,---,\frac{M-1}{2}$$

Then

$$H_r(\omega) = \sum_{k=1}^{(M-1)/2} c(k)\sin\omega k$$
$$= \sin\omega \sum_{k=0}^{(M-3)/2} \hat{c}(k)\cos\omega k$$

Where,

$$\hat{c}\left(\frac{M-3}{2}\right) = c\left(\frac{M-1}{2}\right)$$
$$\hat{c}\left(\frac{M-5}{2}\right) = 2c\left(\frac{M-3}{2}\right)$$

$$\hat{c}(k-1) + \hat{c}(k+1) = 2c(k)\ 2 \le k \le \frac{M-5}{2}$$
$$\hat{c}(0) + \frac{1}{2}\hat{c}(2) = c(1)$$

Case 4: Antisymmetric unit sample response & M even:

$$h[n] = -h[M-1-n]$$

$$H_r(\omega) = 2\sum_{n=0}^{(M/2)-1} h(n)\sin\omega\left(\frac{M-1}{2} - n\right)$$

Suppose k = (M/2) - n

$$d(k) = 2h\left(\frac{M}{2} - k\right)\quad k = 1,2,3,---,\frac{M}{2}$$

Then

$$H_r(\omega) = \sum_{k=1}^{M/2} d(k)\sin\omega\left(k - \frac{1}{2}\right)$$
$$= \sin\frac{\omega}{2}\sum_{k=0}^{(M/2)-1} \hat{d}(k)\cos\omega k$$

Where,

$$\hat{d}\left(\frac{M}{2} - 1\right) = 2d\left(\frac{M}{2}\right)$$

$$\hat{d}(k-1) - \hat{d}(k) = 2d(k) \; 2 \le k \le \frac{M}{2} - 1$$

$$\hat{d}(0) - \frac{1}{2}\hat{d}(1) = d(1)$$

Now, to summarize, we know

$$H_r(\omega) = Q(\omega)P(\omega)$$

Where

$$Q(\omega) = \begin{cases} 1 & case\ 1 \\ \cos\frac{\omega}{2} & case\ 2 \\ \sin\omega & case\ 3 \\ \sin\frac{\omega}{2} & case\ 4 \end{cases}$$

And the P(ω) has the common form

$$P(\omega) = \sum_{k=0}^{L} \alpha(k)\cos\omega k$$

With {α(k)} representing the parameters of the filter which are linearly related to the unit response h(n) of the FIR filter. Suppose $H_{dr}(\omega)$ be the real valued desired frequency response and weighting function be W(ω). The weighting function on the approximation error allows us to choose the relative size of the errors in the different frequency bands.

It is convenient to normalize W(ω) to unity in stopband. So,

$$W(\omega) = \begin{cases} \delta_2/\delta_1 & : Passband \\ 1 & : Stopband \end{cases}$$

Then weighting approximation error is given as

E(ω) $= W(\omega)[H_{dr}(\omega) - H_r(\omega)]$

$= W(\omega)\,[H_{dr}(\omega) - Q(\omega)P(\omega)]$

$= W(\omega)Q(\omega)[\,H_{dr}(\omega)/Q(\omega) - P(\omega)]$

$= \widehat{W}(\omega)[\widehat{H}_{dr}(\omega) - P(\omega)$

$\widehat{W}(\omega)$= modified weighting function

$\widehat{H}_{dr}(\omega)$ = modified desired frequency response

Given, the error function E(ω), the chebyshev approximation problem is to determine the filter parameters {α(k)} then minimize the maximum value of E(ω) over particular frequency band.

i.e.

$$\underset{over\{\alpha(k)\}}{min}\left[\underset{\omega \in s}{max}|E(\omega)|\right] = \underset{over\{\alpha(k)\}}{min}\left[\underset{\omega \in S}{max}\left|\widehat{W}(\omega)\left[\widehat{H}_{dr}(\omega) - \sum_{k=0}^{L}\alpha(k)\cos\omega k\right]\right|\right]$$

Where S is the set of frequency bands over which the optimization is to be performed.

Solution to the above problem is given by alternation theorem.

Alternation theorem:

Let S be a compact subset of the interval [0, π). A necessary and sufficient condition for

$$P(\omega) = \sum_{k=0}^{L}\alpha(k)\cos\omega k$$

To be unique, nest weighted chebyshev approximation to $\widehat{H}_{dr}(\omega)$ in S, is that the error function E(ω) exhibit at least L + 2 extremal frequencies in S. That is there must exist at least L + 2 frequencies {ω_i} in S such that $\omega_1 < \omega_2 <$ -- $< \omega_{L+2}$, $E(\omega_i) = E(\omega_{i+1})$, and

$$|E(\omega_i)| = \underset{\omega \in S}{max}|E(\omega)| \quad i = 1,2,---,L+2$$

We note that the error function E(ω) alternates in sign between two successive extremal frequencies. Hence the theorem is called the alternation theorem.

Now for the unique solution for the chebyshev optimization problem, at the desired extemal frequencies {ω_n}, we have the set of equations,

$$\widehat{W}(\omega_n)\left[\widehat{H}_{dr}(\omega_n) - P(\omega_n)\right] = (-1)^n\delta \quad n = 0,1,2,--,L+2$$

Where δ = maximum value of E(ω). (δ = δ_2)

$$P(\omega_n) + \frac{(-1)^n\delta}{\widehat{W}(\omega_n)} = \widehat{H}_{dr}(\omega_n) \quad n = 0,1,2,--,L+2$$

$$\sum_{k=0}^{L}\alpha(k)\cos\omega_n k + \frac{(-1)^n\delta}{\widehat{W}(\omega_n)} = \widehat{H}_{dr}(\omega_n) \quad n = 0,1,2,--,L+2$$

We should find α(k) & δ. So can be expressed in matrix form.

$$\begin{bmatrix} 1 & \cos\omega_0 & \cos 2\omega_0 & \cdots & \cos L\omega_0 & 1/\widehat{W}(\omega_0) \\ 1 & \cos\omega_1 & \cos 2\omega_1 & \cdots & \cos L\omega_1 & -1/\widehat{W}(\omega_1) \\ & \vdots & & \ddots & & \vdots \\ 1 & \cos\omega_{L+1} & \cos 2\omega_{L+1} & \cdots & \cos L\omega_{L+1} & (-1)^{L+1}/\widehat{W}(\omega_{L+1}) \end{bmatrix} \begin{bmatrix} \alpha(0) \\ \alpha(1) \\ \vdots \\ \vdots \\ \alpha(L) \\ \delta \end{bmatrix} = \begin{bmatrix} \widehat{H}_{dr}(\omega_0) \\ \widehat{H}_{dr}(\omega_1) \\ \vdots \\ \vdots \\ \widehat{H}_{dr}(\omega_{L+1}) \end{bmatrix}$$

Initially, we know neither the set of external frequency $\{\omega_n\}$ nor the parameter $\{\alpha(k)\}$. To solve for the parameter, we use an iterative algorithm, called REMEZ EXCHANGE ALGORITHM.

<table>
<tr><th>Remez Exchange Algorithm:</th><th>Flow chart for R Remez Exchange Algorithm</th></tr>
<tr><td>

1. Guess L+2 extremal frequencies.
2. Determine $P(\omega)$ and δ.
3. Compute the error function $E(\omega)$.
4. If $|E(\omega)| \geq \delta$, determine the another set of L+2 extremal frequencies.
5. Repeat 1 to 4 untill it converges to the optimal set of extremal frequencies.
6. Obtain the best approximation.

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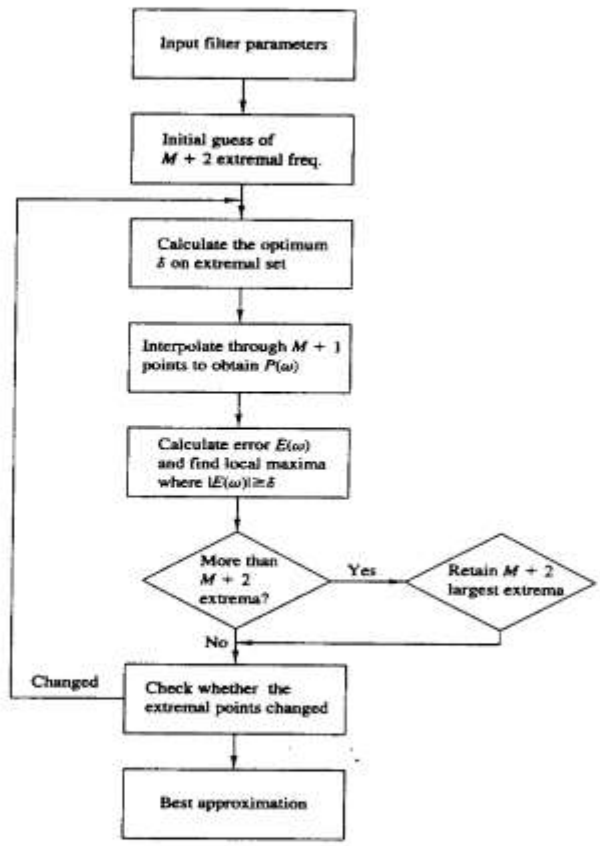

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