Unit 1 Solution of Non-linear equations 5hrs = 16 marks

→ Numerical error (short note)

Error is the difference between actual value and calculated value.
i.e. error = $x_a - x_c$

Types:

i) Roundoff errors
→ arises from the process of rounding off the numbers during the computation.
eg: $\frac{1}{3} = 0.333---$

soln: writing more number after decimal
like, $\frac{1}{3} = 0.33333-$

ii) Truncation errors
→ arises replacing an infinite process by a finite one.
Eg: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + - - \infty$

if, $x = 2$ $= 1 + 2 + \frac{2}{2!} + \frac{2^3}{3!} + ----\infty$

$= 1 + 2 = 3$

$1 + 2 + \frac{2}{2!} = 5$. soln:

soln: writing more terms.

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iii) Inherent errors
→ error present in the input data or model before any calculations are performed. Arises from unavoidable limitations in measurements, assumptions or simplifications.
Eg: measuring a length as 5.2cm when the true value is 5.23cm.

→ 1. Root bracketing method

i) Bisection method

Algorithm:
1. Start
2. define function $f(x)$
3. write the predefined tolerable error (e).
4. Let the initial guess be x_0 and x_1, such that $f(x_0)\cdot f(x_1) < 0$
5. Compute the new approximate value,
$$x_2 = \frac{x_0+x_1}{2}$$
6. calculate $f(x_2)$
a) if $f(x_0)\cdot f(x_2) < 0$ then
Set, $x_0 = x_0$, $x_1 = x_2$
b) if $f(x_0)\cdot f(x_2) > 0$ then
Set $x_0 = x_2$, $x_1 = x_1$
c) if $f(x_0)\cdot f(x_2) = 0$ $(f(x_0) \neq 0)$
then write root $= x_2$ and goto step 8

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7. if $|f(x_2)| < e$, then write root $= x_2$ and goto step 8, otherwise goto step 5.
8. Stop

Q. Find a real root of the equation $x^3 - x - 1 = 0$, using bisection method connect to 3 decimal places.

Soln: $f(x) = x^3 - x - 1$

x	$f(x)$
0	-1
1	-1
2	5

let, $x_0 = 1$, $f(x_0) = -1 < 0$
$x_1 = 2$, $f(x_1) = 5 > 0$
$\because f(x_0)\cdot f(x_1) = -5 < 0$

now, using bisectional method,

$$x_2 = \frac{x_0+x_1}{2} = \frac{1+2}{2} = 1.5$$

$$f(x_2) = (1.5)^3 - 1.5 - 1 = 0.875 > 0$$

iterative table

itno.	$x_0(<0)$	$x_1(>0)$	$x_2 = \frac{x_0+x_1}{2}$	sign of $f(x_2)$
1.	1	2	1.5	>0
2.	1	1.5	1.25	<0
3.	1.25	1.5	1.375	>0
4.	1.25	1.375	1.3125	<0
5.	1.3125	1.375	1.3438	>0
6.	1.6125	1.3438	1.32814	>0
7.	1.3125	1.32814	1.3203	<0
8.	1.3203	1.3282	1.3243	<0
9.	1.3243	1.3282	1.3282	>0
10.	1.3243	1.3262	1.3252	>0
11.	1.3243	1.3252	1.3247	<0
12.	1.3241	1.3252	1.3248	>0

∴ The real root of the given eqn correct the 3 decimal places, using bisection method is 1.324.

Q2. Find a real root of the equation $x - \cos x = 0$ using bisection method correct to it's 3 decimal place.

Soln: $f(x) = x - \cos x$

x	$f(x)$
0	-1
1	1

$x_0 = 0$, $f(x_0) = -1$
$x_1 = 1$, $f(x_1) = 0.4596$

since, $f(x_0)\cdot f(x_1) = -1 < 0$
now, using bisection method
$x_2 = \frac{x_0+x_1}{2} = 0.5$ // x_2 की value है

iterative table

itno	x_0 (-ve)	x_1 (+ve)	x_2	sign of $f(x_2)$
1	0	1	0.5	<0
2	0.5	1	0.75	>0
3	0.5	0.75	0.625	<0
4	0.625	0.75	0.687	<0
5	0.687	0.75	0.718	<0
6	0.718	0.75	0.734	<0
7	0.734	0.75	0.742	>0
8	0.734	0.742	0.738	<0
9	0.738	0.742	0.740	>0
10	0.738	0.740	0.739	<0
11	0.739	0.740	0.739	

The real root of given eqn correct to 3 decimal place using bisection method ≈ 0.739

(Preboard, 2024)

ii) False Position Method

Algorithm:

$f_0 = f(x_0)$
$f_1 = f(x_1)$

1. start
2. Define a function $f(x)$
3. decide tolerable error (E)
4. choose x_0 and x_1, such that $f(x_0) \cdot f(x_1) < 0$
5. Calculate $x_2 = \dfrac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$
6. calculate $f(x_2)$
7. a) if $f(x_0) \cdot f(x_2) < 0$, then set $x_0 = x_0$, $x_1 = x_2$, goto step 8
   b) if $f(x_0) \cdot f(x_2) > 0$, then set $x_0 = x_2$, $x_1 = x_1$, goto step 8.
   c) if $f(x_0) \cdot f(x_2) = 0$, then x_2 is root.
8. if $|f(x_2)| < E$, then root $= x_2$, otherwise goto step 5.
9. Stop.

Q. Using false position method, find a root of the eqn ~~$x^3 - 2x - 5 = 0$~~, correct to four decimal places. $3x + \sin x - e^x = 0$

Soln: $f(x) = 3x + \sin x - e^x$

x	$f(x)$
0	-1
1	1.12318

$x_0 = 0$, $f(0) = -1$ // $f(x_1)$ को value
$x_1 = 1$, $f(1) = 1.12318$

using false position method,

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{0 \times (1.12318) - 1 \times (-1)}{1.12318 + 1}$$

$$= 0.47099, \quad f(x_2) = 0.26515$$

iterative table

itno	x_0	x_1	f_0	f_1	x_2	$f(x_2)$
1	0	1	-1	1.12318	0.47099	0.26515
2	0	0.47099	-1	0.26515	0.37227	0.02951
3	0	0.37227	-1	0.02951	0.36159	0.00292
4	0	0.36159	-1	0.00292	0.31498	-0.11566
5	0.31498	0.36159	-0.11566	0.00292	0.36044	0.00045
6	0.31498	0.36044	-0.11566	0.00045	0.36026	-0.00040
7	0.36026	0.36044	-0.00040	0.00045	0.36034	-0.00020
8	0.36034	0.36044	-0.00020	0.00045	0.34961	-0.02715
9	0.34961	0.36044	-0.02715	0.00045	0.36026	-0.00040
10	0.36026	0.36044	-0.00040	0.00045	0.36034	-0.00020
11	0.36034	0.36044	-0.00020	0.00045	0.36037	-0.00012
12	0.36037	0.36044	-0.00012	0.00046	0.36038	-0.00010

∴ The real root of the given eqn correct to 4 decimal point using false position method = -0.0001

root = 0.3603

2. Non-bracketing (open end) methods:

1) Secant method // 29, spring

Algorithm:

1. Start
2. Define $f(x)$.
3. Let x_0 & x_1 be the initial guesses and e be the tolerable error.
4. Compute $f_0 = f(x_0)$ & $f_1 = f(x_1)$.
5. Compute $x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$
6. Compute $f(x_2)$.
   if $f(x_2) < e$, then write root $= x_2$, and goto stop.
   otherwise set,
   $x_0 = x_1$, $x_1 = x_2$,
   $f_0 = f_1$, $f_1 = f_2$
7. Repeat step 5 & 6.
8. Stop.

(Contd. Part)

example: Find a root of the equation $x^3 + x^2 - 3x - 3 = 0$ by secant method such that error is less than 10^{-4}.

Soln: $f(x) = x^3 + x^2 - 3x - 3$

x	$f(x)$
0	-3
1	-4
2	3

Take, $x_0 = 1$, $x_1 = 2$
$f_0 = -4$, $f_1 = 3$

using secant method:

$$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0} = \frac{1 \times 3 - 2 \times (-4)}{3 + 4}$$

$$= 1.57142$$

Iterative table:

Itno	x_0	f_0	x_1	f_1	x_2	$f(x_2)$
1	1	-4	2	3	1.57142	-1.36323
2	2	3	1.57142	-1.36323	1.79182	0.52201
3	1.57142	-1.36323	1.79182	0.52201	1.72542	-0.06296
4.	1.79182	0.52201	1.72542	-0.06296	1.73179	-0.00291
5.	1.72542	-0.06296	1.73179	-0.00291	1.73204	-0.00005
				∴ root	= 1.7320	

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ii) Newton- Raphson Method: (2024, prebaord) reverse

24, spring

Algorithm:

1. start
2. define function $f(x)$ & $f'(x)$.
3. choose initial guess (x_0) such that $f(x_0) \neq 0$.
4. set the tolerable error (e).
5. compute, $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

6. compute $f(x_1)$
if $f(x_1) = 0$, then root $= x_1$ and goto stop.

if $f(x_1) \neq 0$,
check $|f(x_1)| < e$, then write root $= x_2$ &
goto stop.

otherwise set $x_0 = x_1$ and repeat steps.

7. Stop.

Q. find a root of $3x = \cos x + 1$, correct to 4 decimal place.

Soln: $f(x) = 3x - \cos x - 1$
$f'(x) = 3 + \sin x$

x	$f(x)$
0	-2
1	1.45969

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choose, $x_0 = 0.5$
$f(0.5) = -0.37758$
$f'(0.5) = 3.47942$

$x_1 = 0.5 + \frac{0.37758}{3.47942}$

$= 0.60851$

$f(x_1) = 0.00050$

Drawbacks:

- if initial guess is poorly choosen, it slow to converge or may fail to converge.
- may fail to converge for multiple roots

iterative table:

itno	~~current state~~ x_0	$f'(x_0)$	~~input~~ $f'(x_0)$	x_1	$f(x_1)$
1	0.5	-0.37758	3.47942	0.60851	0.000502
2	0.60851	0.000502	3.57164	0.60837	0.00452
3	0.60837	0.00452	3.57153	0.60710	-0.000005

Since, $|f(x_1)| = 0.000005 < e$

root ≈ 0.60710.

$x =$

$\log x = \ln x$

iii) Fixed Point Iterative Method

Algorithm :

1. start
2. define $f(x)$
3. rearrange $f(x) = 0$, in the form $x = g(x)$
4. choose initial guess x_0 such that $|g'(x_0)| < 1$ and define tolerable error (e).
5. compute, $x_1 = g(x_0)$
6. if $|f(x_1)| < e|$, then root $= x_1$ and goto stop. otherwise, set $x_0 = x_1$ and goto step 5.
7. Stop

example :

$f(x) = 2x - \log_{10} x - 7$

x	$f(x)$
1	-5
2	-3.30103
3	-1.47712
4	0.39794

So, root lies between 3 & 4.

Choose, $x_0 = 3.5$

$\log_a x = \frac{1}{x \ln a}$ $\quad$ $\log_{10} x$ $\quad$ $\frac{1}{x}\log_{10} e$

$\frac{d}{dx}\log_{10} x = \frac{1}{x}\log_{10} e$

now, from the eqn

$$x = \frac{7 + \log_{10} x}{2} = g(x)$$

$$g'(x) = \frac{1}{2}\frac{1}{x}\log_{10} e = 0.06204 < 1$$

so, root converges from the $g(x)$.

0.43429

$\log_a x = \frac{1}{x \ln a}$

$$x_1 = \frac{7 + \log_{10} 3.5}{2} = 3.77203$$

$$x_2 = \frac{7 + \log_{10} 3.77203}{2} = 3.78829$$

$$x_3 = \frac{7 + \log_{10} 3.78829}{2} = 3.78922$$

$$x_4 = \frac{7 + \log_{10} 3.78922}{2} = 3.78927$$

$\therefore$ root ≈ 3.7892

2A. Fall Difference between

→ Convergence and divergence of a nonlinear equation

Convergence	Divergence
Sequence approaches a finite solution.	Sequence doesnot approach a solution
error decreases over iterations	error grows indefinitely.

Unit 4

System of Linear Equation (6hrs)

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1. → Direct Method:

i) Gauss elimination method: (24, spring) same Cosmos UNIVESE

Q. $10x - 7y + 3z + 5u = 6$; // 2024, preboard

$-6x + 8y - z - 4u = 5$

$3x + y + 4z + 11u = 2$

$5x - 9y - 2z + 4u = 7$

COSMOS

→ limitations of:
- potential division by zero during elimination process.
- sensitivity to round-off errors.
- limitations with ill-conditioned matrices.

Sol^n: augmented matrix form:

$$C = \left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ -6 & 8 & -1 & -4 & 5 \\ 3 & 1 & 4 & 11 & 2 \\ 5 & -9 & -2 & 4 & 7 \end{array}\right]$$

$R_2 \to 5R_2 + 3R_1$

$R_3 \to 10R_3 - 3R_1$

$R_4 \to 2R_4 - R_1$

$$\therefore C = \left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ 0 & 19 & 4 & -5 & 43 \\ 0 & -11 & 31 & 95 & 2 \\ 0 & 11 & 3 & -3 & 8 \end{array}\right]$$

$R_3 \to 19R_3 + 11R_2$, $R_4 \to 19R_4 + 11R_2$

$$\left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ 0 & 19 & 4 & -5 & 43 \\ 0 & 0 & 545 & 1860 & 71 \\ 0 & 0 & 101 & 988 & 185 \end{array}\right]$$

note: diagonal element must not be zero // exchange row

$R_4 \to \frac{545}{101} R_4 - R_3$

$$C = \left[\begin{array}{cccc:c} 10 & -7 & 3 & 5 & 6 \\ 0 & 19 & 4 & -5 & 43 \\ 0 & 0 & 545 & 1860 & 71 \\ 0 & 0 & 0 & 350600/101 & 93654/101 \end{array}\right]$$

now, $10x - 7y + 3z + 5u = 6$ — (I)

$19y + 4z - 5u = 43$ — (II)

$545z + 1860u = 71$ — (III)

$350600u = 93654$ — (IV)

from (IV)

$u = \frac{93654}{350600} = 0.26712$

put u in (III)

$545z + 1860 \times 0.26712 = 71$

or, $z = -0.78136$

(II) becomes:

$19y + 4 \times (-0.78136) - 5 \times 0.26712 = 43$

or, $y = 2.49795$

(I) becomes:

$10x - 7 \times 2.49795 + 3 \times (-0.78136) + 5 \times 0.26712 = 6$

or, $x = 2.44941$

$$\therefore \boxed{\begin{aligned} x &= 2.44941 \\ y &= 2.49795 \\ z &= -0.78136 \\ u &= 0.26712 \end{aligned}}$$

ii) Gauss elimination with Partial pivoting
(diagonal element be largest.) ✓

COSMOS

Q. $x + 2y + 3z - u = 10$
$2x + 3y - 3z - u = 1$
$2x - y + 2z + 3u = 7$
$3x + 2y - 4z + 3u = 2$

Augumented Cofficient matrix

$$A = \left[\begin{array}{cccc:c} 1 & 2 & 3 & -1 & 10 \\ 2 & 3 & -3 & -1 & 1 \\ 2 & -1 & 2 & 3 & 7 \\ 3 & 2 & -4 & 3 & 2 \end{array}\right]$$

$R_1 \leftrightarrow R_4$

$$= \left[\begin{array}{cccc:c} 3 & 2 & -4 & 3 & 2 \\ 2 & 3 & -3 & -1 & 1 \\ 2 & -1 & 2 & 3 & 7 \\ 1 & 2 & 3 & -1 & 10 \end{array}\right]$$

$R_2 \to 3R_2 - 2R_1$
$R_3 \to 3R_3 - 2R_1$
$R_4 \to 3R_4 - R_1$

$$= \left[\begin{array}{cccc:c} 3 & 2 & -4 & 3 & 2 \\ 0 & 5 & -1 & -9 & -1 \\ 0 & -7 & 14 & 3 & 17 \\ 0 & 4 & 13 & -6 & 28 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{cccc:c} 3 & 2 & -4 & 3 & 2 \\ 0 & -7 & 14 & 3 & 17 \\ 0 & 5 & -1 & -9 & -1 \\ 0 & 4 & 13 & -6 & 28 \end{array}\right]$$

$R_3 \to 7R_3 + 5R_2$
$R_4 \to 7R_4 + 4R_2$

$$= \left[\begin{array}{cccc:c} 3 & 2 & -4 & 3 & 2 \\ 0 & -7 & 14 & 3 & 17 \\ 0 & 0 & 63 & -48 & 78 \\ 0 & 0 & 147 & -30 & 264 \end{array}\right]$$

$R_3 \leftrightarrow R_4$

$$\left[\begin{array}{cccc:c} 3 & 2 & -4 & 3 & 2 \\ 0 & -7 & 14 & 3 & 17 \\ 0 & 0 & 147 & -30 & 264 \\ 0 & 0 & 63 & -48 & 78 \end{array}\right]$$

$R_4 \to 7R_4 - 3R_3$

$$\left[\begin{array}{cccc:c} 3 & 2 & -4 & 3 & 2 \\ 0 & -7 & 14 & 3 & 17 \\ 0 & 0 & 147 & -30 & 264 \\ 0 & 0 & 0 & -246 & -264 \end{array}\right]$$

$-246u = -246$

$\underline{u = 1}$

$147z - 30 \times u = 264$

$\underline{z = 2}$

$-7y + 14z + 3u = 17$

$\underline{y = 2}$

$3x + 2y - 4z + 3u = 2$

$\underline{x = 1}$

iii) Gauss Jordan Method 24, Fall

$3x + 2y + z = 10$
$2x + 3y + 2z = 14$
$x + 2y + 3z = 14$

augmented cofficient matrix:

$$A = \left[\begin{array}{ccc:c} 3 & 2 & 1 & 10 \\ 2 & 3 & 2 & 14 \\ 1 & 2 & 3 & 14 \end{array}\right]$$

$R_2 \to 3R_2 - 2R_1, \quad R_3 \to 3R_3 - R_1$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 10 \\ 0 & 5 & 4 & 22 \\ 0 & 4 & 8 & 32 \end{array}\right]$$

$R_1 \to 5R_1 - 2R_2, \quad R_3 \to 5R_3 - 4R_2$

$$\left[\begin{array}{ccc:c} 15 & 0 & -3 & 6 \\ 0 & 5 & 4 & 22 \\ 0 & 0 & 24 & 72 \end{array}\right]$$

$R_1 \to 8R_1 + R_3, \quad R_2 = 6R_2 - R_3$

$$\left[\begin{array}{ccc:c} 120 & 0 & 0 & 120 \\ 0 & 30 & 0 & 60 \\ 0 & 0 & 24 & 72 \end{array}\right]$$

$R_1 \to R_1/120$
$R_2 \to R_2/30$
$R_3 \to R_3/24$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$\therefore \boxed{x = 1, \; y = 2, \; z = 3}$

$x^3 = 9$

2. → Factorization Method:

Let, the equations be

$a_1x_1 + a_2x_2 + a_3x_3 = d_1$
$b_1x_1 + b_2x_2 + b_3x_3 = d_2$
$c_1x_1 + c_2x_2 + c_3x_3 = d_3$

or,
$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$A \cdot X = B \quad -(I)$$

Coefficient matrix A is decomposed into lower triangular and upper triangular matrices.
i.e. $A = L \cdot U$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Dolittle factorization
$l_{11} = l_{22} = l_{33} = 1$

Crout's factorization
$u_{11} = u_{22} = u_{33} = 1$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}+u_{21}u_{11} & u_{12} & \\ & & \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11}+u_{21} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11}+l_{32}u_{21}+u_{31} & l_{31}u_{12}+l_{32}u_{22}+u_{32} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

comparing both sides, we get L and U matrices.

now, from (I), put $A = L \cdot U$

$$L(UX) = B \quad -(II)$$

put, $UX = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad -(III)$

∵ From (II)
$L \cdot Z = B$
We get the value of z.
Then, from eqn (III)
we get the solution.

2021, Fall

1. Solve the following system by using LU Crout's method. ✓ NCIT

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 14$$
$$x + 2y + 3z = 14$$

Soln: given system in matrix form:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

or, $AX = B$ ---(1)

where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

$$B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

now, using Crout's factorization method:

$$A = L \cdot U$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & u_{12}l_{21} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

comparing both sides:

$l_{11} = 3$, $l_{21} = 2$, $l_{31} = 1$

$l_{11}u_{12} = 2$, $l_{11}u_{13} = 1$

$\Rightarrow u_{12} = 2/3$, $\Rightarrow u_{13} = 1/3$

$LUX = B$
$UX = Z$
$LZ = B$

$u_{12}l_{21} + l_{22} = 3$

$\frac{2}{3} \times 2 + l_{22} = 3$

$l_{22} = 3 - \frac{4}{3} = \frac{5}{3}$

$l_{21}u_{13} + l_{22}u_{23} = 2$

$2 \times \frac{1}{3} + \frac{5}{3} \times u_{23} = 2$

or, $\frac{5}{3} u_{23} = 2 - \frac{2}{3}$

$u_{23} = \frac{4}{3} \times \frac{3}{5} = \frac{4}{5}$

$l_{31}u_{12} + l_{32} = 2$

$1 \times \frac{2}{3} + l_{32} = 2$

$l_{32} = 2 - \frac{2}{3} = \frac{4}{3}$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 3$

$1 \times \frac{1}{3} + \frac{4}{3} \times \frac{4}{5} + l_{33} = 3$

$l_{33} = 3 - \frac{1}{3} - \frac{16}{15}$

$\frac{4}{9} + \frac{20}{15}$ $\frac{41}{45}$

$l_{33} = \cancel{\frac{135 - 20 - 60}{45} = \frac{55}{45} = \frac{11}{9}}$

$= \frac{8}{5}$

$$\therefore L = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix}, U = \begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix}$$

from, eq^n (1) put, $A = LU$

$$L(UX) = B \quad -(2)$$

put, $UX = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad -(3)$

from eq^n (2)

$$L \cdot Z = B$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 8/5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$3z_1 = 10$

$z_1 = 10/3$

$2z_1 + \frac{5}{3} z_2 = 14$

or, $2 \times \frac{10}{3} + \frac{5}{3} z_2 = 14 \Rightarrow z_2 = \frac{22}{5}$

or, $z_1 + \frac{4}{3} z_2 + \frac{8}{5} z_3 = 14$

$\frac{10}{3} + \frac{4}{3} \times \frac{22}{5} + \frac{8}{5} z_3 = 14$

$z_3 = 3$

from eq^n (3) $UX = Z$

$$\begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10/3 \\ 22/5 \\ 3 \end{bmatrix}$$

$\Rightarrow z = 3$

$y + \frac{4}{5} z = \frac{22}{5}$

$y = \frac{22}{5} - \frac{4}{5} \times 3 = \frac{10}{5} = 2$

$x + \frac{2}{3} y + \frac{1}{3} z = \frac{10}{3}$

$\cancel{x + \frac{2}{3} \times \frac{18}{5} + \frac{1}{3} \times 3} = \frac{10}{3}$

$x + \frac{2}{3} \times 2 + \frac{1}{3} \times 3 = \frac{10}{3}$

$x = 1$

$\therefore$ $\boxed{\begin{matrix} x = 1 \\ y = 2 \\ z = 3 \end{matrix}}$

I matrix symmetric की अर्थ] ...

Choleskey's Factorization Method

Let the given system of equation in matrix form be $AX = B$ —(1)

if $A = A^T$. Then the matrix is called Symmetric matrix.

For the Symmetric matrix

$$A = L \cdot L^T$$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{31}l_{11} \\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{31}l_{21}+l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2 \end{bmatrix}$$

12
21
13
31

now, from (1)

$L(L^T \cdot X) = B$

$LZ = B$ —(2)

where, $Z = L^T \cdot X = Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix}$

Q.

$$3x + 2y + z = 10$$
$$2x + 3y + 2z = 14$$
$$x + 2y + 3z = 14$$

पहिला ... solve. then $L^T X$

given system in matrix form:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

$AX = B$ —(1)

where $A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,

$B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$

Here, $A = A^T$.

now, using choleskey's factorization method:

$$A = L \cdot L^T$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{31}l_{11} \\ l_{21}l_{11} & l_{21}^2+l_{22}^2 & l_{31}l_{21}+l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21}+l_{32}l_{22} & l_{31}^2+l_{32}^2+l_{33}^2 \end{bmatrix}$$

comparing: $l_{11}^2 = 3$

$l_{11} = \sqrt{3}$

$l_{11} l_{21} = 2$

$l_{21} = \frac{2}{\sqrt{3}}$

$l_{31} l_{11} = 1$

$l_{31} = \frac{1}{\sqrt{3}}$

$l_{21}^2 + l_{22}^2 = 3$

$l_{22}^2 = \cancel{3 - 4/3}$

$L_{22} = \cancel{\frac{3}{2}}$ $\quad L_{22} = \sqrt{\frac{5}{3}}$

$l_{31} l_{21} + l_{22} l_{32} = 2$

$\frac{1}{\sqrt{3}} \times \frac{2}{\sqrt{3}} + \frac{\sqrt{5}}{\sqrt{3}} \times l_{32} = 2$

$\frac{2}{3} + \frac{\sqrt{5}}{\sqrt{3}} \times l_{32} = 2$

$L_{32} = \cancel{\frac{4}{3} \times \frac{2}{3}} = \cancel{\frac{8}{9}}$ $\quad \frac{4}{\sqrt{15}}$

$l_{31}^2 + l_{32}^2 + l_{33}^2 = 3$

$\frac{1}{3} + \frac{8^2}{9^2} + l_{33}^2 = 3$

$L_{33}^2 = \cancel{152/81}$

$L_{33} = \cancel{\frac{2\sqrt{38}}{9}}$ $\quad \sqrt{\frac{8}{5}}$

$\therefore L = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & 3/2 & 0 \\ 1/\sqrt{3} & 8/9 & 2\sqrt{38}/9 \end{bmatrix}$

$L^T = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 3/2 & 8/9 \\ 0 & 0 & 2\sqrt{38}/9 \end{bmatrix}$

$2 - \frac{2}{3}$

now, from ①

$L(L^T \cdot X) = B$

$\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & 3/2 & 0 \\ 1/\sqrt{3} & 8/9 & 2\sqrt{38}/9 \end{bmatrix} [$

$L = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 2/\sqrt{3} & \sqrt{5/3} & 0 \\ 1/\sqrt{3} & 4/\sqrt{15} & \sqrt{8/5} \end{bmatrix}, \quad L^T = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{5/3} & 4/\sqrt{15} \\ 0 & 0 & \sqrt{8/5} \end{bmatrix}$

from eqⁿ ① : $AX = B$

$L\,(L^T \cdot X) = B$

$L Z = B \quad$ — ②

where, $L^T \cdot X = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad$ — ③

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from eqn (ii): $LZ = B$

$$\begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 2/\sqrt{3} & \sqrt{5/3} & 4/\sqrt{15} \\ 1/\sqrt{3} & 4/\sqrt{15} & \sqrt{8/5} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

or, $\cancel{\sqrt{3} z_1 + \frac{2}{\sqrt{3}} z_2 + \frac{1}{\sqrt{3}} z_3}$

$\cancel{\sqrt{5/3}\, z_2 +}$

$\sqrt{3} z_1 = 10$

$z_1 = 10/\sqrt{3}$

$\frac{2}{\sqrt{3}} z_1 + \sqrt{\frac{5}{3}} z_2 = 14$

$\frac{2}{\sqrt{3}} \times \frac{10}{\sqrt{3}} + \sqrt{\frac{5}{3}}\; z_2 = 14$

$z_2 \sqrt{\frac{5}{3}} = \frac{22}{3}$

$z_2 = \frac{22}{3} \times \sqrt{\frac{3}{5}}$

$= \frac{22}{\sqrt{15}}$

$\frac{1}{\sqrt{3}} z_1 + \frac{4}{\sqrt{15}} z_2 + \sqrt{8/5}\, z_3 = 14$

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or, $\frac{1}{\sqrt{3}} \times \frac{10}{\sqrt{3}} + \frac{4}{\sqrt{15}} \times \frac{22}{\sqrt{15}} + \sqrt{\frac{8}{5}} z_3 = 14$

$z_3 = \frac{24}{5} \times \sqrt{\frac{5}{8}}$

$= \frac{24}{\sqrt{40}} = \frac{48}{\sqrt{10}}$

$LZ = B$

from (B): $L^T X = Z$

$$\begin{bmatrix} \sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \sqrt{5/3} & 4/\sqrt{15} \\ 0 & 0 & \sqrt{8/5} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10/\sqrt{3} \\ 22/\sqrt{15} \\ 24/\sqrt{40} \end{bmatrix}$$

now, $\sqrt{\frac{8}{5}}\; z = \frac{24}{\sqrt{40}}$

$z = \frac{24}{\sqrt{40}} \times \sqrt{\frac{5}{8}}$

$= \frac{24}{8} = 3$

$\sqrt{5/3}\, y + \frac{4}{\sqrt{15}} z = \frac{22}{\sqrt{15}}$

$\sqrt{\frac{5}{3}} \times y = \frac{22}{\sqrt{15}} - \frac{4}{\sqrt{15}} \times \frac{24}{8}$

$\sqrt{3}\, x + \frac{2}{\sqrt{3}} \times \overset{y = 2}{2} + \frac{1}{\sqrt{3}} \times \frac{24}{8} = \cancel{24/\sqrt{40}}\; 10/\sqrt{3}$

$x = 1$

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$$\therefore \boxed{\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}}$$

3. Iterative Methods of solving System of Linear equations

(i) Jacobi Method
(ii) Gauss-Seidel Method (most fast)

To apply Iterative methods, the system of equations should be in diagonally dominant form.
→ ~~diagonate dominant form~~

Let, the given system of linear equation be,

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

i.e. $|a_1| \geq |a_2| + |a_3|$

$|b_2| \geq |b_1| + |b_3|$

$|c_3| \geq |c_1| + |c_2|$

solving these equations for x, y and z respectively.

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$$x = \frac{1}{a_1}[d_1 - a_2y - a_3z] \quad \text{—(i)}$$

$$y = \frac{1}{b_2}[d_2 - b_1x - b_3z] \quad \text{—(ii)}$$

$$z = \frac{1}{c_3}[d_3 - c_1x - c_2y] \quad \text{—(iii)}$$

Let, $x = y = z = 0$ be Initial guess.

COSMOS

a) Jacobi Method:

iteration 1: Putting the initial guesses on RHS of eq^n (i), (ii) & (iii).

$$x^{(1)} = \frac{d_1}{a_1}, \quad y^{(1)} = \frac{d_2}{b_2}, \quad z^{(1)} = \frac{d_3}{c_3}$$

iteration 2:

$$x^{(2)} = \frac{1}{a_1}\left[d_1 - a_2 \cdot \overset{y^{(1)}}{\frac{d_2}{b_2}} - a_3 \cdot \overset{z^{(1)}}{\frac{d_3}{c_3}}\right]$$

$$y^{(2)} = \frac{1}{b_2}\left[d_2 - b_1 \cdot \overset{x^{(1)}}{\frac{d_1}{a_1}} - b_3 \cdot \overset{z^{(1)}}{\frac{d_3}{c_3}}\right]$$

$$z^{(2)} = \frac{1}{c_3}\left[d_3 - c_1 \cdot \overset{x^{(1)}}{\frac{d_1}{a_1}} - c_2 \cdot \overset{y^{(1)}}{\frac{d_2}{b_2}}\right]$$

and so on.

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b) Gauss - Seidel Method / COSMOS

iteration(i):

$x^{(1)} = \frac{1}{a_1} [d_1]$

$y^{(1)} = \frac{1}{b_2} [\cancel{x^{(1)}}\, d_2 - b_1 x^{(1)} - b_3 \cdot 0]$

$y^{(1)} = \frac{1}{b_2} [d_2 - b_1 x^{(1)}]$

$z^{(1)} = \frac{1}{c_3} [d_3 - c_1 x^{(1)} - c_2 y^{(1)}]$

iteration(ii):

$x = \frac{1}{a_1} [d_1 - a_2 y^{(1)} - a_3 z^{(1)}]$

$y = \frac{1}{b_2} [d_2 - b_1 x^{(2)} - b_3 z^{(1)}]$

$z = \frac{1}{c_3} [d_3 - b_1 x^{(2)} - b_3 \cancel{y}^{(2)}]$

and, so on..

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2019, Fall (same as Q4, spring)

1. Using Gauss-seidel method, solve:

$3x + 20y - z = 18$
$2x - 3y + 20z = 25$
$20x + y - 2z = 17$

Soln: rearrange the given equation such that the system is in diagonally dominant form.

$20x + y - 2z = 17$ — (i)
$3x + 20y - z = 18$ — (ii)
$2x - 3y + 20z = 25$ — (iii)

solving these equations for x, y, z respectively,

$x = \frac{1}{20} (17 - y + 2z)$ — (1)

$y = \frac{1}{20} (18 - 3x + z)$ — (2)

$z = \frac{1}{20} (25 - 2x + 3y)$ — (3)

let, $x = 0, y = 0, z = 0$ be the initial guess.
Using Gauss-Seidel Method:

∴ iteration i:

$x^{(1)} = \frac{17}{20} = 0.85$

x_2 1

$$y = \frac{1}{20}(18 - 3 \times 0.85 + 0)$$
$$= 0.7725$$

$$z = \frac{1}{20}(25 - 2 \times 0.85 + 3 \times 0.7725)$$
$$= 1.28$$

iterative table :

it no	x from eqn (i)	y from eqn (ii)	z from eqn (iii)
1	0.85	0.77	1.28
2	0.94	0.82	1.28
3.	0.94	0.82	1.28

$$\therefore \boxed{\begin{array}{l} x = 0.94 \\ y = 0.82 \\ z = 1.28 \end{array}}$$

2. Solve the following system using any iteration method.

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

Soln: rearranging the system:

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

solving these equations for x, y, z respectively:

$$\cancel{x} = \cancel{\frac{85}{27}} \quad \cancel{\frac{85}{2}}$$

$$x = \frac{1}{27}[85 - 6y + z] \quad -(i)$$

$$y = \frac{1}{15}[72 - 6x - 2z] \quad -(ii)$$

$$z = \frac{1}{54}[110 - x - y] \quad -(iii)$$

let, $x = y = z = 0$ be the initial guess.
Using Gauss - Seidel method:

iterations :

$x = \frac{85}{27} = 3.14$

$y = \frac{1}{15}(72 - 6\times3.14 - 0) = 3.57$

$z = \frac{1}{54}(110 - 3.14 - 3.54)$

$= 1.91$

iterative table

it no	x from eqn (i)	y from eqn (ii)	z from eqn (iii)
1.	3.14	3.54	1.91
2.	2.43	3.57	1.93
3.	2.43	3.57	1.93

$\therefore x = 2.43,\ y = 3.57,\ z = 1.93$

21, spring

1 a) Solve $x\log_{10}x = 1.2$ by Newton-Raphson method correct to four decimal point

Soln: $f(x) = x\log_{10}x - 1.2$

$f'(x) = \log_{10}e + \log_{10}x$

x	f(x)
2	-0.59794
3	0.23136

$x_0 = 2.5$

$f(x_0) = -0.20515$

$f'(x_0) = 0.83223$

$x_1 = 2.5 + \frac{0.20515}{0.83223}$

iterative table

it no	x_0	$f(x_0)$	$f'(x_0)$	x_1	$f(x_1)$
1	2.5	-0.20515	0.83223	2.74650	0.005113
2	2.74650	0.005113	0.873075	2.74065	0.00000

$\therefore$ root $= 2.74065$.

24/2079

$\left[\log_{10} x = \frac{\ln x}{\ln 10}\right]$

1b) Using secant method, find root of function $2x - \log_{10} x - 7 = 0$, correct upto three decimal point.

Sol^n: $f(x) = 2x - \log_{10} x - 7$

x	$f(x)$
1	-5
2	-3.6931
3	-1.9771
4	0.3979

$x_0 = 3,\ f_0 = -1.9771$
$x_1 = 4,\ f_1 = 0.3979$

$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$
$= 3.7277$

$f(x_2) = -0.00298$

Iterative table

itno	x_0	f_0	x_1	f_1	x_2	$f(x_2)$
1	3	-1.9771	4	0.3979	3.7277	-0.00298
2	4	0.3979	3.7277	-0.00298	3.7274	-0.00359
3	3.7277	-0.00298	3.7274	-0.00359	3.7293	0.00004

root = 3.729

$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$

or

1b) Find root of equation $f(x) = x^2 - 4x - 10$ correct to three decimal place by using False Position Method.

Sol^n: $f(x) = x^2 - 4x - 10$

x	$f(x)$
0	-10
1	-13
2	-14
3	-13
4	-2
5	-5
6	2

Here,
$x_0 = 5,\ f(x_0) = -5$
$x_1 = 6,\ f(x_1) = 2$

Since,
$f(0) \cdot f(1) < 0$
root lies between 5 and 6

$x_2 = \frac{x_0 f_1 - x_1 f_0}{f_1 - f_0}$
$= \frac{5 \times 2 - 6 \times (-5)}{2 + 5}$
$= 5.7142$

$f(x_2) = 5.7142 \quad -0.2047$

Iterative table :

Itno	x_0	x_1	$f(0)$	f_1	x_2	$f(x_2)$
1.	5	6	-5	2	5.7142	-0.2047
2.	5.7142	6	-0.2047	2	5.7407	-0.0043
3.	5.7407	~~-0.0~~, 6	-0.0093	2	5.7412	-0.0034
4.	5.7412	6	-0.0034	2	5.7416	-0.0004
5.	5.7416	6	-0.0004	2	5.7416	-0.0004

∴ root = 5.7416.

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4(a) Solve: $3x+2y+z=10$
$2x+3y+2z=14$
$x+2y+3z=14$
By Gauss elimination method.

Sol^n: Augmented matrix:

$$A=\left[\begin{array}{ccc:c} 3 & 2 & 1 & 10 \\ 2 & 3 & 2 & 14 \\ 1 & 2 & 3 & 14 \end{array}\right]$$

$R_2 \to 3R_2-2R_1,\quad R_3 \to 3R_3-R_1$

$$A=\left[\begin{array}{ccc:c} 3 & 2 & 1 & 10 \\ 0 & 5 & 4 & 22 \\ 0 & 4 & 8 & 32 \end{array}\right]$$

$R_3 \to 5R_3-4R_2$

$$=\left[\begin{array}{ccc:c} 3 & 2 & 1 & 10 \\ 0 & 5 & 4 & 20 \\ 0 & 0 & 24 & 72 \end{array}\right]$$

$24z=72$ —(i)
$z=3$

$5y+4z=20$ —(ii)
$y=\dfrac{20-12}{5}$
$=8/5$

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$3x+2y+z=10$ —(iii)
$3x+2\times 8/5+3=10$

$x=\dfrac{19}{15}$

$\therefore$ $x=19/15$
$y=8/5$
$z=3$

Q) Solve using Crout method:

$x+y+z = 4$
$x+4y+3z = 8$
$x+6y+2z = 6$

Sol^n: given System in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$$

$AX = B$ — (1)

where, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 2 \end{bmatrix}$.

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$B = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$

$$A = L \cdot U = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}\begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 6 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

$l_{11} = 1, \quad l_{21} = 1, \quad l_{31} = 1$

$l_{11} \times u_{12} = 1$
$u_{12} = 0$
$l_{11} + u_{13} = 1$
$u_{13} = 0$

$l_{21}u_{12} + l_{22} = 4$
$1 \times u_{12} + l_{22} = 4$
$l_{22} = 4$

$l_{21}u_{13} + l_{22}u_{23} = 3$
$1 \times 0 + 4 \times u_{23} = 3$
$u_{23} = 3/4$

$l_{31}u_{12} + l_{32} = 6$
$1 \times 0 + l_{32} = 6$
$l_{32} = 6$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2$
$1 \times 0 + \cancel{4} \times 3/\cancel{4} + l_{33} = 2$
$l_{33} = -1$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 6 & -1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix}$

from eq^n (1)

$A = LU$
$\therefore L(UX) = B$ — (11)

$$UX = Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \quad \text{--- (iii)}$$

from eqn (i) $\quad LZ = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 6 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_1 + 4z_2 \\ z_1 + 6z_2 - z_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \Rightarrow \begin{array}{l} z_1 = 4 \\ 4 + 4 \times z_2 = 8 \\ z_2 = 1 \\ 4 + 6 \times 1 - z_3 = 6 \\ z_3 = 4 \end{array}$$

$\therefore$ from eqn (iii) $\quad UX = Z$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y + 3/4\, z \\ 1z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

$x = 4$

$z = 3$

$y + 3/4 \times 3 = 1$

$y = 1 - 9/4 = -5/4$

23 Fall

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

1a) Write down the drawbacks of N-R method. Find real root of $x^2 - 3x + 2 = 0$, in the neighbourhood of $x = 0$, correct to 5 decimal points.

Soln: Drawbacks of N-R method:

→ sensitive in initial guess i.e. $f'(x_0) \neq 0$

→ a poorly choosen initial guess can lead to divergence, convergence to a different root or even failure to converge.

II part

$f(x) = x^2 - 3x + 2$

$f'(x) = 2x - 3$

x	$f(x)$
0	2
1	0
2	3
~~3~~	~~2~~

$x_0 = 0.5 \quad f(x_0) = -0.250000$ $\quad$ 0.750000

iterative table, $\quad f'(x_0) =$

itno	x_0	$f(x_0)$	$f'(x_0)$	x_1	$f(x_1)$
1	1.5	+0.750000	-2	0.875000	0.140625
2	0.875000	0.140625	-1.250000	0.987500	0.012656
3	0.987500	0.012656	-1.025000	0.999847	0.000153
4	0.999847	0.000153	-1.000306	1.000000	0.000000

$\therefore$ root = 1.00000

$$z = \frac{1}{54}(110 - 3.14 - 3.54)$$
$$= 1.91$$

iterative table

itno	x from eqn (i)	y from eqn (ii)	z from eqn (iii)
1	3.14	3.54	1.91
2	2.43	3.57	1.92
3	2.43	3.57	1.92

$$\boxed{\begin{aligned} x &= 2.43 \\ y &= 3.57 \\ z &= 1.92 \end{aligned}}$$

Unittest SetA

2b) Solve, using iterative method:

$$27x + 6y - z = 85$$
$$x + y + 54z = 110$$
$$6x + 15y + 2z = 72$$

rearranging the system:

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Solving these equations for x, y, z respectively:

$$x = \frac{1}{27}[85 - 6y + z] \quad \text{— (i)}$$

$$y = \frac{1}{15}[72 - 6x - 2z] \quad \text{— (ii)}$$

$$z = \frac{1}{54}[110 - x - y] \quad \text{— (iii)}$$

put, $x = y = z = 0$ for initial guess.
using gauss seidel law,

$$x = \frac{1}{27}[85] = \frac{85}{27} = 3.14$$

$$y = \frac{1}{15}\left[72 - \frac{6 \times 85}{27}\right] = \frac{478}{135} = 3.54$$

Ill-condition System (short notes)

An ill condition system is a mathematical problem where a small change in input leads to a large change in output, making the solution highly sensitive to errors.

<u>formal Definition</u>: for a matrix $A \in R^{n \times n}$, the condition number $K(A) = \|A\| \cdot \|A^{-1}\|$ $\gg 1$, then it is called ill-condition system.

if a system matrix A has $|A|$ is nearly close to 0, then the system is ill-condition system.

<u>example</u>: Consider the following system of linear equations:

$$x_1 + 2x_2 = 3$$
$$2x_1 + 4.0001x_2 = 6.0001$$

in matrix form: $AX = B$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4.0001 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 6.0001 \end{bmatrix}$$

$|A| = 4.0001 - 4 = 0.0001$ (near to 0)

so, the system is ill-conditioned.

20) Using Gauss Jordan method, solve

$$3x_1 + 2x_2 + 7x_3 = 4$$
$$2x_1 + 3x_2 + x_3 = 5$$
$$3x_1 + 4x_2 + x_3 = 7$$

Soln: Augmented matrix:

$$A = \left[\begin{array}{ccc:c} 3 & 2 & 7 & 4 \\ 2 & 3 & 1 & 5 \\ 3 & 4 & 1 & 7 \end{array}\right]$$

$R_2 \to 3R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 3 & 2 & 7 & 4 \\ 0 & 5 & -11 & 7 \\ 0 & 2 & -6 & 3 \end{array}\right]$$

$R_1 \to 5R_1 - 2R_2$, $R_3 \to 5R_3 - 2R_2$

$$\left[\begin{array}{ccc:c} 15 & 0 & 57 & 6 \\ 0 & 5 & -11 & 7 \\ 0 & 0 & 8 & 1 \end{array}\right]$$

$R_1 \to 8R_1 - 57R_3$, $R_2 \to 8R_2 + 11R_3$

$$\left[\begin{array}{ccc:c} 120 & 0 & 0 & -9 \\ 0 & 40 & 0 & 67 \\ 0 & 0 & 8 & 1 \end{array}\right] \therefore A = \left[\begin{array}{ccc:c} 1 & 0 & 0 & -3/40 \\ 0 & 1 & 0 & 67/40 \\ 0 & 0 & 1 & 1/8 \end{array}\right]$$

$R_1 \to R_1/120$
$R_2 \to R_2/40$
$R_3 \to R_3/8$

$x_1 = 3/40$
$x_2 = 67/40$
$x_3 = 1/8$
x_4

1. b) Derive a formula to find the reciprocal of a number $(N \neq 0)$ using Newton's method. Hence, find reciprocal of 17 correct to 4 decimal places.

Soln:
$$x = \frac{1}{N}$$
$$f(x) = \frac{1}{x} - N, \quad f'(x) = -\frac{1}{x^2}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n + \frac{1/x_n - N}{1/x_n^2}$$
$$x_{n+1} = x_n + \frac{(1 - Nx_n)x_n}{Nx_n^2}$$
$$= 2x_n - Nx_n^2$$

let, initial guess, $x_0 = 0.05$, $N = 17$

itn no	x_n	x_{n+1}
1	0.05	0.05750
2	0.05750	0.05879
3	0.05879	0.05883
4	0.05883	0.05882

$\therefore$ root $= 0.0588$
reciprocal of 17 $= 0.0588$

$x = N^{1/3}$
$x^3 - N = 0$

$N =$

$x = N^{1/3}$
$x^3 = N$

$$x = \frac{1}{N}$$

x_0, x_1

$$f(x) = \frac{1}{x} - N$$
$$f'(x) = \frac{-1}{x^2}$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{1/x_n - N}{-1/x_n^2}$$
$$x_{n+1} = x_n + x_n(1 - Nx_n)$$
$$= 2x_n - Nx_n^2$$

24, Fall

4 a) Find the inverse of matrix using Gauss Jordan Method.

$$\begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$A: \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 3 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 3R_1$
$R_3 \rightarrow R_3 + 2R_1$

$$= \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 1 \\ 0 & 0 & -6 & -3 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 & 0 & 1 \\ 0 & 0 & -6 & -3 & 1 & 0 \end{array}\right]$$

$R_1 \rightarrow 2R_1 + R_2$

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 8 & 4 & 0 & 3 \\ 0 & -2 & 2 & 2 & 0 & 1 \\ 0 & 0 & -6 & -3 & 1 & 0 \end{array}\right]$$

$R_1 \rightarrow 3R_1 + 4R_3$
$R_2 \rightarrow 3R_2 + R_3$

$$\left[\begin{array}{ccc:ccc} 6 & 0 & 0 & 0 & 4 & 9 \\ 0 & -6 & 0 & 3 & 1 & 3 \\ 0 & 0 & -6 & -3 & 1 & 0 \end{array}\right]$$

$R_1 \rightarrow R_1/6$
$R_2 \rightarrow R_2/-6$
$R_3 \rightarrow R_3/-6$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 4/6 & 9/6 \\ 0 & 1 & 0 & 3/-6 & 1/-6 & 3/-6 \\ 0 & 0 & 1 & -3/-6 & 1/-6 & 0 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 2/3 & 3/2 \\ -1/2 & -1/6 & -1/2 \\ 1/6 & -1/6 & 0 \end{bmatrix}$$

Unit) Power method

→ to find the largest eigen value of a matrix.

let, A be the given square matrix.

$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

let, $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ = initial eigen vector of the matrix.

$AX_0 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ b_1/a_1 \end{bmatrix}$ if $|a_1| > |b_1|$

a_1 ठूलो छ।

$= \lambda_1 \cdot X_1$

$A \cdot X_1 = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} 1 \\ b/a_1 \end{bmatrix} = \lambda_2 . X_2$

(लगातार, λ_1, λ_2 मिल्दैन।)

example1:

Find the largest (dominant) eigen value and the corresponding eigen vector of the matrix

$A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

using power method correct to 3 decimal places.

(imp) 1 fix

Soln: Here, $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

let, $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be the initial guess.

$\therefore AX_0 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$= 3 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$

$= 3 \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix} = \lambda_1 X_1$

$\lambda_1 = 3, \ X_1 = \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix}$

$A.X_1 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0.3333 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 4.3333 \\ 2.9999 \end{bmatrix} \cancel{= \lambda_2 X_2} \quad 4.3333 \begin{bmatrix} 1 \\ 0.6922 \end{bmatrix} = \lambda_2 X_2$

$A \cdot X_2 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} \cancel{4.3333} \\ \cancel{1.4444} \end{bmatrix}$

$= \begin{bmatrix} \cancel{16.3329} & \cancel{5.4444} \\ \cancel{18} & \cancel{5.8888} \end{bmatrix}$

$= 5.8888 \begin{bmatrix} 0.9245 \\ 1 \end{bmatrix} = \lambda_3 X_3$

Date:
Page:

$A \cdot X_2 = 4.3846 \begin{bmatrix} 0.8596 \\ 1 \end{bmatrix} = \lambda_3 X_3$

$A \cdot X_3 = 4.8596 \begin{bmatrix} 1 \\ 0.9422 \end{bmatrix} = \lambda_4 X_4$

$A \cdot X_4 = 4.8844 \begin{bmatrix} 0.9763 \\ 1 \end{bmatrix} = \lambda_5 X_5$

$A \cdot X_5 = 4.9829 \begin{bmatrix} 0.9987 \\ 1 \end{bmatrix} = \lambda_6 X_6$

$A \cdot X_6 = 4.9987 \begin{bmatrix} 1 \\ 0.9995 \end{bmatrix} = \lambda_7 X_7$

$A \cdot X_7 = 4.9999 \begin{bmatrix} 0.9999 \\ 1 \end{bmatrix} = \lambda_8 X_8$

$= 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ The largest eigen value $\simeq 5$
and the corresponing vector $\approx \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2023 Spring (Same as 2?, Fall)

Q. Find the largest eigen value and corresponding eigen vector of the vector matrix.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$ using power method

Soln: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

let, $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial guess.

$$\therefore AX_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.80 \\ 0.20 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.80 \\ 0.20 \end{bmatrix}$$

$$= \begin{bmatrix} 2.8 \\ -2.80 \\ 1.20 \end{bmatrix}$$

$$= 2.80 \begin{bmatrix} 1 \\ -1 \\ 0.4285 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0.4285 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -3.4285 \\ 1.8570 \end{bmatrix} = 3.4285 \begin{bmatrix} 0.8750 \\ -1 \\ 0.5416 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.8750 \\ -1 \\ 0.5416 \end{bmatrix}$$

$$= \begin{bmatrix} 2.7500 \\ -3.4166 \\ 2.0832 \end{bmatrix} = \lambda_5 X_5 = 3.4166 \begin{bmatrix} 0.8048 \\ -1 \\ 0.6097 \end{bmatrix}$$

14

$$AX_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.8098 \\ -1 \\ 0.6097 \end{bmatrix}$$

$$= \begin{bmatrix} 2.6096 \\ -3.4145 \\ 2.2194 \end{bmatrix} = 3.4145 \begin{bmatrix} 0.7642 \\ -1 \\ 0.6499 \end{bmatrix}$$

$$= \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.7642 \\ -1 \\ 0.6499 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5284 \\ -3.4141 \\ 2.2998 \end{bmatrix} = 3.4141 \begin{bmatrix} 0.7405 \\ -1 \\ 0.6736 \end{bmatrix}$$

$\because$ largest eigen value $(\lambda) \approx 3.414$

eigen vector $\approx \begin{bmatrix} 0.7405 \\ -1 \\ 0.6736 \end{bmatrix}$

2021, Fall

$$A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

let, initial guess,

$$X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AX_0 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ 0.0400 \\ 0.0800 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0400 \\ 0.0800 \end{bmatrix}$$

$$= \begin{bmatrix} 25.2000 \\ 1.1200 \\ 2.3200 \end{bmatrix} = 25.2000 \begin{bmatrix} 1 \\ 0.0444 \\ 0.9206 \end{bmatrix}$$

$$= \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0444 \\ 0.9206 \end{bmatrix}$$

$$= \begin{bmatrix} 26.8856 \\ 1.1332 \\ 5.6824 \end{bmatrix} = 26.8856 \begin{bmatrix} 1 \\ 0.0421 \\ 0.2113 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0421 \\ 0.2113 \end{bmatrix}$$

$$= \begin{bmatrix} 25.4647 \\ 1.1263 \\ 2.8452 \end{bmatrix} = 25.4647 \begin{bmatrix} 1 \\ 0.0442 \\ 0.1117 \end{bmatrix}$$

$$= \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0442 \\ 0.1117 \end{bmatrix}$$

$$= \begin{bmatrix} 25.2676 \\ 1.1326 \\ 2.4468 \end{bmatrix} = 25.2676 \begin{bmatrix} 1 \\ 0.0448 \\ 0.0968 \end{bmatrix}$$

$$= \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0448 \\ 0.0968 \end{bmatrix} = \begin{bmatrix} 25.2384 \\ 1.1344 \\ 2.3872 \end{bmatrix}$$

$$= 25.2384 \begin{bmatrix} 1 \\ 0.0449 \\ 0.0945 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0449 \\ 0.0945 \end{bmatrix}$$

$$= \begin{bmatrix} 25.2339 \\ 1.1347 \\ 2.3780 \end{bmatrix} = 25.2339 \begin{bmatrix} 1 \\ 0.0449 \\ 0.0942 \end{bmatrix}$$

$$= \lambda_7 X_7$$

$$AX_7 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.0449 \\ 0.0942 \end{bmatrix}$$

$$= \begin{bmatrix} 25.2333 \\ 1.1347 \\ 2.3768 \end{bmatrix} = 25.2333 \begin{bmatrix} 1 \\ 0.0449 \\ 0.0942 \end{bmatrix}$$

$\therefore$ largest eigen value ≈ 25.233

eigen vector $\approx \begin{bmatrix} 1 \\ 0.0449 \\ 0.0942 \end{bmatrix}$ #

Unit 2 Interpolation and Approximation — 16 marks

(both equally & unequally spaced)

1. Lagrange's interpolation formula (simple & switch)

Let, the given data be

x:	x_0	x_1	x_2	x_3
y:	y_0	y_1	y_2	y_3

→ data may be equally spaced or unequally spaced.

The Lagrange's interpolation formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \cdot y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \cdot y_3$$

x = The point at ~~least~~ which the value is required.

same formula for ...

2021, spring.

i) From the following data evaluate $f(2.5)$ by using Lagrange's method

x:	1	2	4	5	7
$f(x)$:	1	1.414	1.782	2.002	2.6

Soln: using Lagrange's interpolation formula:

~~$f(2.5) = (x-x_1)(x-x_2)(x-x_3)(x-x_4)$~~

$$f(2.5) = \frac{(2.5-2)(2.5-4)(2.5-5)(2.5-7)}{(1-2)(1-4)(1-5)(1-7)} \cdot 1 + \frac{(2.5-1)(2.5-4)(2.5-5)(2.5-7)}{(2-1)(2-4)(2-5)(2-7)} \cdot 1.414 + \frac{(2.5-1)(2.5-2)(2.5-5)(2.5-7)}{(4-1)(4-2)(4-5)(4-7)} \cdot 1.782 + \frac{(2.5-1)(2.5-2)(2.5-4)(2.5-7)}{(5-1)(5-2)(5-4)(5-7)} \cdot 2.002 + \frac{(2.5-1)(2.5-2)(2.5-4)(2.5-5)}{(7-1)(7-2)(7-4)(7-5)} \cdot 2.6$$

$= -0.1171 + (\cancel{-0.8437}) + \cancel{1.8750} - \cancel{1.7578}\ 0.4218$

$(\cancel{+1.6875})$ $\quad 0.8118 \quad + 0.0406$

1.1930

$= 1.5065$

Q(2) The following table gives the viscosity of an oil as a function of Temperature. (x)

Temp (°F) :	110	130	160	190
viscosity :	10.8	8.1	5.5	4.8

using Lagrange's interpolation formula to find the viscosity of oil at Temp. 140° F.

Soln: Let, Temp °F = x
viscosity = y

$$y(140) = \frac{(140-130)(140-160)(140-190)}{(110-130)(110-160)(110-190)} \times 10.8$$

$$+ \frac{(140-110)(140-160)(140-190)}{(130-110)(140-110)(140-190)} \times 8.1$$

$$+ \frac{(140-110)(140-130)(140-190)}{(160-110)(160-130)(160-190)} \times 5.5$$

$$+ \frac{(140-110)(140-130)(140-160)}{(190-110)(190-130)(190-160)} \times 4.8$$

$= -1.35 + 6.75 + 1.833 - 0.2$
$= 7.033$

preboard, 2022.

2(a) Estimate the number of workers getting wages between Rs.100 and Rs 150 on the basis of following data:

Wages Rs.	0-100	100-200	200-300	300-400
No of workers	9	30	35	42

Soln:

wages less than (x)	no of workers (y)	1st difference	2nd difference	3rd difference
100	9	30	5	2
200	39	35	7	
300	74	42		
400	116			

using newton's forward interpolation: $P = \frac{150-100}{100} = 0.5$

$$y(150) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$$= 9 + 0.5 \times 30 + \frac{0.5(0.5-1)}{2!} \times 5 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times 2$$

$= 23.50$

no. of worker whose wages is less than 150 = 23.5
" " " " " " 100 = 9
" " " " " between 100 & 150 =
23.5 - 9
= 19 #

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Interpolation for equally spaced data:

1. Newton's Interpolation formula:

Let, the given data be

x:	x_0	x_0+h	x_0+2h	x_0+3h
y:	y_0	y_1	y_2	y_3

(2024, Fall)

a) Newton's forward Interpolation formula:

$$y(x) = y_0 + p.\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots$$

जुन point मा value चाहियो

$$\text{where, } p = \left(\frac{x - x_0}{h}\right)$$

$\Delta y_0 = y_1 - y_0$ (first forward difference)

$\Delta y_1 = y_2 - y_1$ (")

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - 2y_1 + y_0$ (second forward difference)

$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$ (third forward ")

and so on.

> Note: this formula is used for interpolating the values of y near the begining of a set of tabulated values.

Page:

b) Newton's backward Interpolation Formula:

$$y(x) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \dots$$

$$\text{Where, } q = \frac{x - x_n}{h}$$

$\nabla y_n = y_n - y_{n-1}$ (1st backward difference)

$\nabla^2 y_n = \nabla y_n - 2\nabla y_{n-1} + y_n$ (2nd backward formula)

$\nabla^3 y_n = \nabla^2 y_{n-1} - \nabla^2 y_n$ (third formula)

Finite difference (Δy_0) table:

x	y	1st difference	2nd difference	3rd difference
x_0	y_0	$y_1 - y_0 = a$		
x_0+h	y_1	$y_2 - y_1 = b$	$b - a = d$ $(\Delta^2 y_0)$	$e - d = f$ $(\Delta^3 y_0)$
x_0+2h	y_2	$y_3 - y_2 = c$	$c - b = e$ $(\nabla^2 y_n)$	$(\nabla^3 y_n)$
x_0+3h	y_3	∇y_n		

> Note: this formula is used for interpolating the value of y near the begining of a set of tabulated values.

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example: Given:

x:	40	50	60	70	80
y:	184	204	226	250	276

estimate the value of y at $x = 44$ & $x = 78$

Soln:

Finite difference table:

x	y	1st difference	2nd difference	3rd difference	4th difference
40	184	20	2	0	0
50	204	22	2	0	
60	226	24	2		
70	250	26			
80	276				

(i) To estimate y at $x = 44$, we use Newton's forward interpolation formula:

$$y(44) = y_0 + P\Delta y_0 + \frac{P(P-1)\Delta^2 y_0}{2!} + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)\Delta^4 y_0}{4!}$$

$$P = \frac{x - x_0}{h} = \frac{44-40}{10} = 0.4$$

$$\therefore y(44) = 184 + 0.4\times 20 + \frac{0.4(0.4-1)\cdot 2}{2!} + \frac{0.4(0.4-2)\times 0}{3!} + 0$$

$= 191.76$ #

(ii) To estimate y at $x = 78$, we use Newton's backward formula:

$$y(78) = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\times\nabla^2 y_n + \frac{(q+1)(q+2)}{3!}\nabla^3 y_n + \frac{(q+1)(q+2)(q+3)}{4!}\nabla^4 y_n$$

where,

$$q = \frac{78-80}{10} = -0.2$$

$$\therefore y(78) = 276 + (-0.2)\times 26 + \frac{(-0.2)\times(0.2+1)\times 2}{2!} + 0$$

$= ~~201.44~~$ # 270.56 #

$\frac{x - x_n}{}$

$44 - 86$

-3.6

87

2024, Spring (v.v.imp)

1. From the following table, estimate the no. of students who obtained marks between
i) 50 and 55
ii) 65 and 70

marks:	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
no. of students:	31	42	51	35	31

Soln: to apply Newton's interpolation formula change the given data into less than cumulative frequency table:

Marks less than (x)	no. of Student (y)	1st difference	2nd difference	3rd difference	4th difference
40	31	42	9	-25	37
50	73	51	-16	12	
60	124	35	-4		
70	159	31			
80	190				

(i) To estimate the no. of students who obtained marks between 50 & 55.

we apply newton's forward formula with $x = 55$. $p = \frac{x - x_0}{h} = \frac{55-40}{10} = 1.5$

$$Y(55) = Y_0 + p\Delta Y_0 + \frac{p(p-1)}{2!}\Delta^2 Y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 Y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 Y_0$$

$$= 31 + 1.5 \times 42 + \frac{1.5(1.5-1)}{2!} \times 9 + \frac{1.5(1.5-1)(1.5-2)}{3!} \times (-25) + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!} \times 12$$

$= 99.80$

≈ 100

no. of students whose marks is less than 55 = 100

" " " " " 50 = 73

$\therefore$ no. of students whose marks lies between 50 and 55 $= 100 - 73$

$= 27$ #

ii) we apply newton's backward with $x = 65$

$q = \frac{65-80}{10} = -1.5$

$$Y(65) = Y_n + q\nabla Y_n + \frac{q(q+1)}{2!}\nabla^2 Y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 Y_n + \frac{q(q+1)(q+2)(q+3)}{4!}\nabla^4 Y_n$$

$$= 190 + (-1.5) \times 31 + \frac{-1.5 \times (-0.5)}{2!} \times (-4) + \frac{(-1.5)(-1.5+1)(-1.5+2)}{3!} \times 12 + \frac{-1.5(-1.5+1)(-1.5+2)(-1.5+3)}{4!} \times 37$$

$y(65) = 149.61 \approx 149$

No. of students whose marks is less than 65 = 144

" " " " " " 70 = 159

" " " " between 60 & 65 = 15

Q. 2019, fall

Find the missing term in the table using suitable interpolation formula.

not actual

x:	0	1	2	3	4
y:	1	3	9	?	81

Soln: using lagrange's formula.

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times y_1 +$$

$$y(3) = \frac{(3-1)(3-2)\cancel{(3-3)}(3-4)}{(0-1)(0-2)\cancel{(0-3)}(0-4)} \times 1 + \frac{(3-0)(3-2)\cancel{(3-3)}(3-4)}{(1-0)(1-2)\cancel{(1-3)}(1-4)} \times 3 + \frac{(3-0)(3-1)\cancel{(3-3)}(3-4)}{(2-0)(2-1)\cancel{(2-3)}(2-4)} \times 9 +$$

$$\frac{(3-0)(3-1)(3-2)}{(4-0)(4-1)(4-2)} \times 81 = 30.50\#$$

2019, Fall — || 2021 spring (Lagrange's)

1) Given the value of a function $y = f(x)$ as (7, 3), (8, 1), (9, 1) and (10, 9). Find the value of y for $x = 9.5$ using lagrange's interpolation formula.

Soln:

x:	7	8	9	10
y:	3	1	1	9

$$y(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)} \times 3 + \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)} \times 1 + \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)} \times 1 + \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)} \times 9$$

$$= 3.625\#$$

(Sarees 2022, Fall) using NP [unequal space = Lagrange's]

Q. given that

θ : 0 30 45 60 90

$\sin\theta$: 0 $1/2$ $1/\sqrt{2}$ $\sqrt{3}/2$ 1

find $\sin 50°$ using interpolation formula.

Sol^n: using Lagrange's formula

$$\sin 50 = \frac{(50-30)(50-45)(50-60)(50-90)}{(0-30)(0-45)(0-60)(0-90)} \times 0 +$$

$$\frac{(50-0)(50-45)(50-60)(50-90)}{(30-0)(30-45)(30-60)(30-90)} \times \frac{1}{2} +$$

$$\frac{(50-0)(50-30)(50-60)(50-90)}{(45-0)(45-30)(45-60)(45-90)} \times \frac{1}{\sqrt{2}} +$$

$$\frac{(50-0)(50-30)(50-45)(50-90)}{(60-0)(60-30)(60-45)(60-90)} \times \frac{\sqrt{3}}{2} +$$

$$\frac{(50-0)(50-30)(50-45)(50-60)}{(90-0)(90-30)(90-45)(90-60)} \times 1$$

= 0.79376 # = 0.79 #

x गरेको y (dependent)

[estimate गरेको हो।]

Curve Fitting

Approximation

i) fitting a linear curve using Least Square method

Let $y = a + bx$ —(1) be the required linear curve.

the values of a & b are obtained by solving the normal equations

$$\Sigma y = na + b\Sigma x$$

and $$\Sigma xy = a\Sigma x + b\Sigma x^2$$

Q. The result of measurement of electric resistance 'R' of a copper bar at various temperature (t) are listed below.

t(°C) :	19	25	30	36	40	45	50
R (ohm) :	76	77	79	80	82	83	85

fit a linear curve of the form $R = a + bt$ and estimate the Resistance (R) at temperature $t = 35°C$

n = 7

	t(x)	R(y)	t_iR_i (xy)	t^2 (x^2)
	19	76	1444	361
	25	77	1925	625
	30	99	1170	900
	36	80	2880	1296
	40	82	3280	1600
	45	83	3735	2025
	50	85	4250	2500
Total	245	562	18884	9307

The linear cuve is:

$$R = a + bt \quad -(1)$$

To find a & b, solve the normal equations:

$$\Sigma R = na + b\Sigma t$$

and $\Sigma tR = a\Sigma t + b\Sigma t^2$

$$562 = 7a + 245b$$

and

$$19884 = 245a + 9307b$$

solving these two:

$$a = 70.05$$
$$b = 0.29$$

from eq^n (1): $R = 70.05 + 0.29t$

at $t = 35°C$:

$$R(35°C) = 70.05 + 0.29 \times 35$$
$$= 80.2 \ \Omega hm \ \#$$

Q. Given that

x:	2	4	6	8	10
y:	6.07	12.85	31.47	57.38	91.29

using least square method, fit a curve $y = a + bx + cx^2$ to the given data. Hence, estimate y at $x = 5$.

Sigma lagaune
Σx ra multiply

$\Sigma, \Sigma x, \Sigma x^2$

Sol^n:

Here, $n = 5$

we have,

$$y = a + bx + cx^2 - (1)$$

normal equations are:

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad -(i)$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad -(ii)$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad -(iii)$$

x	y	xy	x^2	x^2y	x^3	x^4
2	6.07	12.14	4	24.28	8	16
4	12.85	51.40	16	205.60	64	256
6	31.47	188.82	36	1132.92	216	1296
8	57.38	459.04	64	3672.32	512	4096
10	91.29	912.9	100	9129	1000	10000
total, 30	199.06	1624.30	220	14164.12	1800	15664

(1) becomes:

$$199.06 = 5a + 30b + 220c$$
$$1624.30 = 30a + 220b + 1800c$$
$$14164.12 = 220a + 1800b + 15664c$$

solving those:

$$a = 6.096$$
$$b = -2.440$$
$$c = 1.099$$

at, $x = 5$

$$y = 6.096 - 2.440 \times 5 + 1.099 \times 5^2$$
$$= 21.371 \ \#$$

(v.v.imp)

$N = ab^t$

ii) Function of curve $y = ae^{bx}$ — (i)

Sol: Taking ln on both sides:

$\ln y = \ln(ae^{bx})$ $[\ln(e) = 1]$

$= \ln(a) + \ln e^{bx}$

$= \ln(a) + bx \ln(e)$

$= \ln(a) + bx \cdot 1$

$\ln y = \ln a + bx$

$Y = A + BX$ — (ii)

where $\boxed{\begin{aligned} Y &= \ln y \\ A &= \ln a \\ B &= b \\ X &= x \end{aligned}}$, $e^A = a$

$e^{\ln a} = a$

A & B in eq^n (ii) are obtained by solving the normal equations.

$\Sigma Y = nA + B\Sigma X$ and

$\Sigma XY = A \cdot \Sigma X + B\Sigma X^2$

$X = x$	y	$Y = \ln y$	XY	X^2

imp. ($\log$)

$N = ab$

iii) Form $y = ax^b$ — (i)

Taking ln on both sides:

$\ln y = \ln ax^b$

$= \ln a + b\ln x$

$Y = A + BX$

where $\boxed{\begin{aligned} Y &= \ln y \\ A &= \ln a, \quad a = e^A \\ X &= \ln x \\ B &= b \end{aligned}}$

normal equation.

$\Sigma Y = nA + B\Sigma X$, $\Sigma XY = A\Sigma X + B\Sigma X^2$

$X = \ln x$	y	$Y = \ln y$	XY	X^2

iv) $xy^a = b$ $(PV^\alpha = \beta)$

$\Rightarrow \ln x + a\ln y = \ln b$

$\ln y = \dfrac{(\ln b - \ln x)}{a}$

or, $\ln y = \left(\dfrac{\ln b}{a}\right) - \dfrac{1}{a}\ln x$

$Y = A + BX$

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Imp

$$\boxed{\begin{aligned} \text{where, } Y &= \ln y \\ X &= \ln x \\ A &= \frac{\ln b}{a} \\ B &= -\frac{1}{a} \end{aligned}}$$

2019, Spring

2a) The voltage V accross the capacitor at time t is given below:

Time (t in s) :	0	2	4	6	8
Voltage (V in V) :	150	63	28	12	5.6

using least square method approximation, fit a curve of the form $V = ae^{kt}$. Also, estimate the voltage at $t = 2.6$ seconds.

Sol^n: we have,

$V = ae^{kt}$ — (i)

taking $\ln$ on both sides:

$\ln V = \ln a + kt$

$Y = A + BX$ — (ii)

where, $Y = \ln V$, $A = \ln a$, $B = k$, $X = t$

to determine A and B at eq^n (ii) solve:

$\Sigma Y = nA + B\Sigma X$ — (i) and

$\Sigma XY = \Sigma XA + B\Sigma X^2$ — (ii)

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	$X = t$	V	$Y = \ln V$	XY	X^2	XY
	0	150	5.010	~~751.500~~	0	0
	2	63	4.143	~~261.009~~	4	8.286
	4	28	3.332	~~93.296~~	16	13.32
	6	12	2.484	~~29.808~~	36	14.904
	8	5.6	1.722	~~9.647~~	64	13.77
Total:	20		16.691	~~1145.260~~	120	

from eq^n (i) & (ii):

$16.691 = 5A + 20B$

50.28 ~~1145.260~~ $= 20A + 120B$

$A = 4.94$

$B = -0.41 = k$

$\because A = \ln a = 4.94$

$a = e^{4.94} = 146.05$

$\therefore$ The equation is

$\boxed{V = 146.05e^{-0.41t}}$ #

estimate, at $t = 2.6$

$V = 146.05 \times e^{-0.41 \times 2.6}$

$= 50.1$ volt #

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2019 fall

V:	38.4	20	8.51	4.44	3.03
P:	10	20	50	100	150

Fit a curve of the form $PV^\alpha = \beta$. (dependent)

$PV^\alpha = \beta$ — ①

Soln: taking ln:

$\ln P + \alpha \ln V = \ln\beta$

$\ln V = \frac{\ln\beta - \ln P}{\alpha}$

$= \frac{\ln\beta}{\alpha} - \frac{1}{\alpha}\ln P$

$Y = AX + B$ — ②

where, $Y = \ln V$

$A = \frac{\ln\beta}{\alpha}$

$B = -\frac{1}{\alpha}$, $X = \ln P$

to determine A and B at eqn ② solve:

$\Sigma Y = nA + B\Sigma X$ — (i)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ — (ii)

P	X = ln P	V	Y = ln V	XY	X^2
10	2.302	38.4	3.648	8.397	5.299
20	2.995	20	2.995	8.970	8.970
50	3.912	8.51	2.141	8.375	~~15.188~~ 303
100	4.605	4.44	1.490	6.861	21.206
150	5.010	303	1.108	5.551	25.100
total	18.824		11.382	38.254	~~70.763~~ 75.878

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$11.382 = 5A + 18.824B$

$38.154 = 18.824A + \cancel{70.763B}\ 75.878B$

~~α A = -165.181~~

$A = 5.806$

$B = -0.937$

$B = -\frac{1}{\alpha}$

$-0.937 = -\frac{1}{\alpha}$

$\Rightarrow \alpha = 1.067$

$A = \frac{\ln\beta}{\alpha}$

$5.806 \times 1.067 = \ln\beta$

$\beta = e^{6.196}$

$= 490.781$

$\therefore$ the equation is:

$PV^{1.067} = 490.781$ #

$x: x_0 \; x_1 \; x_2 \; \dots \; x_n$
$y: y_1 \; y_1 \; y_2 \; \dots \; y_n$

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Cubic Spline Interpolation (NCT7)

// short notes + LQ

Defn: A function $f(x)$ is said to be cubic spline if

i) $f(x)$ is linear outside the range of $[x_0, x_n]$

ii) $f(x)$ is cubic in each of the subintervals $(x_0,x_1), (x_1,x_2), \dots (x_{n-1},x_n)$

iii) $f'(x)$ and $f''(x)$ are continuous in each of the subintervals. ans

Cubic Spline polynomial is given by:

$$f_i(x) = \frac{1}{6} a_{i-1} \cdot u_i (h_i^2 - u_i^2) - \frac{1}{6} a_i \cdot u_{i-1} (h_i^2 - u_{i-1}^2)$$

$$+ \frac{1}{h_i} (y_i u_{i-1} - y_{i-1} u_i) \quad \text{--- (1)}$$

where, a_{i-1}, a_i are constants and determine by solving the equations

$$h_i a_{i-1} + 2(h_i + h_{i+1}). a_i + h_{i+1} a_{i+1}$$

$$= 6 \left[\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i} \right] \quad \text{--- (2)}$$

$i = 1, 2, 3, \dots, n-1$

$a_0 = a_n = 0$

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$h_i = x_i - x_{i-1}$
$h_{i+1} = x_{i+1} - x_i$
$u_i = x - x_i$
$u_{i-1} = x - x_{i-1}$

a technique of constructing lower degree polynomials in each sub-interval of data is called spline interval. A 3 degree polynomials is constructed in each sub-interval cubic:

2021, Spring

2b) Find the cubic spline interpolations for the following data.

	x_0	x_1	x_2	x_3	x_4
x:	1	2	3	4	5
y:	1	0	1	0	1

Soln: To find cubic spline interpolations: put $i = 1, 2, 3, 4$ in

$$f_i(x) = \frac{1}{6} a_{i-1} \cdot u_i (h_i^2 - u_i^2) - \frac{1}{6} a_i u_{i-1} (h_i^2 - u_{i-1}^2)$$

$$+ \frac{1}{h_i} (y_i u_{i-1} - y_{i-1} u_i) \quad \text{--- (I)}$$

$$f_1(x) = \frac{1}{6} a_0 u_1 (h_1^2 - u_1^2) - \frac{1}{6} a_1 u_0 (h_1^2 - u_0^2) + \frac{1}{h_1} (y_1 u_0 - y_0 u_1) \quad \text{--- (I)}$$

$$f_2(x) = \frac{1}{6} a_1 u_2 (h_2^2 - u_2^2) - \frac{1}{6} a_2 u_1 (h_2^2 - u_1^2) + \frac{1}{h_2} (y_2 u_1 - y_1 u_2) \quad \text{--- (II)}$$

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$f_3(x) = \frac{1}{6} a_2 u_3 (h_3^2 - u_3^2) - \frac{1}{6} a_3 u_2 (h_3^2 - u_2^2) +$
$\frac{1}{h_3} (y_3 u_2 - y_2 u_3)$ — (iii)

$f_4(x) = \frac{1}{6} a_3 u_4 (h_4^2 - u_4^2) - \frac{1}{6} a_4 u_3 (h_4^2 - u_3^2) +$
$\frac{1}{h_4} (y_4 u_3 - y_3 u_4)$ — (iv)

Here, $a_0 = a_4 = 0$

$h_i = x_i - x_{i-1}$
$h_1 = 2 - 1 = 1$
$h_2 = 3 - 2 = 1$
$h_3 = 4 - 3 = 1$
$h_4 = 5 - 4 = 1$

$u_i = x - x_i$
$u_0 = x - 1$
$u_1 = x - 2$
$u_2 = x - 3$
$u_3 = x - 4$
$u_4 = x - 5$

putting these on the above equations:

(i) becomes:

$f_1(x) = \frac{1}{6} \times 0 - \frac{1}{6} a_1 (x-1) \{1 - (x-1)^2\} +$
$\{0 \ (x-1) - 1 \ (x-2)\}$

$1 - (x^2 - 4x + 4)$ $\quad 1 - x^2 + 6x$
$1 - x^2 + 4x - 4$

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$= -\frac{a_1}{6} (x-1) \ (-x^2 + 2x) - x + 2$ — (v)

$f_2(x) = \frac{1}{6} a_1 \cdot (x-3) \ [1^2 - (x-3)^2] -$
$\frac{1}{6} a_2 (x-2) (1^2 - (x-2)^2) +$
$\frac{1}{1} [1 \times (x-2)) - 0 \times (x-3)]$

$= \frac{1}{6} a_1 (x-3) \ (-x^2 + 6x - 8) -$
$\frac{1}{6} a_2 (x-2) \ (-x^2 + 4x - 3) + (x-2)$
— (vi)

$f_3(x) = \frac{1}{6} a_2 (x-4) [1 - (x-4)^2] -$
$\frac{1}{6} a_3 (x-3) \ [1^2 - (x-3)^2] +$
$\frac{1}{1} \ [1 \times (x-4) - 0]$

$= \frac{1}{6} a_2 (x-4) \ (-x^2 + 8x - 15)$
$- \frac{1}{6} a_3 (x-3) \ (-x^2 + 6x - 8) +$
$- 1 \times (x-4)$ — (vii)

$f_4(x) = \frac{1}{6} a_3 (x-5)(-x^2 + 10x - 24) -$
$\frac{1}{6} a_4 (x-4) (-x^2 + 8x - 15) - \frac{1}{1} \{1 (x-4)$ — (viii)

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$h_{i} a_{i-1} + 2(h_i + h_{i+1}) \cdot a_i + h_{i+1} a_{i+1} = 6\left[\frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i}\right]$

now, from eqn (2), put $i = 1, 2, 3, 4$

$h_1 a_0 + 2(h_1 + h_2) a_1 + h_2 a_2 = 6\left(\frac{y_2 - y_1}{h_2} - \frac{y_1 - y_0}{h_1}\right)$

$2(1+1) a_1 + a_2 = 6\left(\frac{1-0}{1} - \frac{0-1}{1}\right)$ — (ix)

$4a_1 + a_2 = 12$

$1 \times a_1 + 2(1+1) \times a_2 + 1 \times a_3 = 6\left(\frac{0-1}{1} - \frac{1-0}{1}\right)$ — (x)

$a_1 + 4a_2 + a_3 = -12$

$a_2 + 2(1+1) \cdot a_3 + 1 \cdot a_4 = 6\left(\frac{1-0}{1} - \frac{0-1}{1}\right)$ — (xi)

or, $a_2 + 4a_3 + a_4 = 12$

$a_3 + 4a_4 + a_5 = 6\left[\frac{1-0}{1} - \frac{0-1}{1}\right] = 12$

$a_3 + 0$

$4a_1 + a_2 = 12$ — (ix)

$a_1 + 4a_2 + a_3 = -12$ (x)

$a_2 + 4a_3 = 12$ — (xi)

$a_1 = 4.13$

$a_2 = -4.52$

$a_3 = 1.96$

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putting these values in (v), (vi), (vii) & (viii) we get require polynomials.

(v) becomes:

$= \frac{4.13}{6}(x-1)(-x^2+2x) - x + 2$

$= -0.69(-x^3 + 3x^2 - 2x) - x + 2$

$= 0.69x^3 - 2.07x^2 + 0.38x + 2$ #

(vi) becomes:

$= \frac{1}{6} \times 4.13 (x-3)(-x^2+6x-8) - \frac{1}{6} \times (-4.52)(x-2)(-x^2+4x-3) + (x-2)$

$= 0.69(-x^3 + 9x^2 - 6x + 24) + 0.76(-x^3 + 6x^2 - 11x + 6) + (x-2)$

$= -1.45x^3 + 10.77x^2 - 11.50x + 19.12$ #

(vii) becomes:

$= \frac{1}{6}(-4.52)(x-4)(-x^2+8x-15) - \frac{1}{6} \times 1.96 (x-3)(-x^2+6x-8) - (x-4)$

$= -0.76(-x^3 + 12x^2 - 47x + 60) - 0.32(-x^3 + 9x^2 - 26x + 24) - (x-4)$

$= 1.08x^3 - 12x^2 + 43.09x - 49.28$ #

(viii) becomes:

$= \frac{1}{6} \times 1.96 (x-5)(-x^2+10x-24) - \frac{1}{4}(x-4)$

$= 0.32(-x^3 + 15x^2 - 64x + 120) - x + 4$

$= -0.32x^3 + 4.80x^2 - 21.48x + 42.40$ #

Chapter 3 Numerical Differentiation and Integration

1. Numerical Differentiation :

Let, the given data be

$x: x_0 \quad x_1 \quad x_2 \quad x_3$
$y: y_0 \quad y_1 \quad y_2 \quad y_3$

Find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$. (most asked)

case i: equally spaced data.

By Newton's forward interpolation formula,

$$y = y_0 + p.\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}.\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}.\Delta^4 y_0$$

where, $p = \frac{x-x_0}{h}$

$$\frac{dy}{dx} = \frac{dy}{dp}\left(\frac{dp}{dx}\right) = \frac{dy}{dp}.\frac{1}{h}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \Delta^4 y_0.\frac{4p^3-18p^2+22p-6}{4!}\right]}$$ #

$(2p^2-p)(p-2) \quad 2p^3-4p^2-p^2+2p$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6p-6}{3!}\Delta^3 y_0 + \dots\right]$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0 \dots\right]}$$ #

Similarly,

By Newton's Backward interpolation formula,

$$y = y_n + q\nabla y_n + \frac{q(q+1)}{2!}\nabla^2 y_n + \frac{q(q+1)(q+2)}{3!}\nabla^3 y_n + \dots$$

where, $q = \frac{x-x_n}{h}$

now, $$\frac{dy}{dx} = \frac{dy}{dq}\left(\frac{dq}{dx}\right) = \frac{1}{h}\frac{dy}{dq}$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2q+1}{2!}\nabla^2 y_n + \frac{3q^2+6q+2}{3!}\nabla^3 y_n + \dots\right]$$ #

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (q+1)\nabla^3 y_n + \dots\right]$$ #

Case ii: unequally spaced data:

- use lagrange's interpolation formula.
- obtain a polynomial
- find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$.

2019, spring

3. a) Velocity of a vehicle is given in the table below. // unequal data lagrange's

	t_0	t_1	t_2	t_3
Time, t (second):	1	2	4	5
velocity, v (m/s):	0.25	1	2.2	4

Find the acceleration at $t = 1.1$ sec & $t = 2.5$ sec.

Soln: acceleration $= \frac{dv}{dt}$.

using lagrange's interpolation formula,

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} . v_0 +$$

$(t-t_0)\ (t.$

(t^2-6t+8)
$t^3 - 3t^2 - 6t^2 + 30t$
$+18t - 40$

$$v = \frac{(t-2)(t-4)(t-5)}{(1-2)(1-4)(1-5)} \times 0.25 + \frac{(t-1)(t-4)(t-5)}{(2-1)(2-4)(2-5)} . 1 + \frac{(t-1)(t-2)(t-5)}{(4-1)(4-2)(4-5)} . 2.2 + \frac{(t-1)(t-2)(t-4)}{(5-1)(5-2)(5-4)} \times 4$$

$$= -0.02\,(t^3 - 11t^2 + 38t - 40) + 0.17\,(t^3 - 10t^2 + 29t - 20) + -0.37\,(t^3 - 8t^2 + 17t - 10) + 0.33\,(t^3 - 7t^2 + 14t - 8)$$

$$a = \frac{dv}{dt} = -0.02\,(3t^2 - 22t + 38) - 0.17\,(3t^2 - 20t + 29) - 0.37\,(3t^2 - 16t + 17) + 0.33\,(3t^2 - 14t + 14)$$

$$\boxed{a = \frac{dv}{dt}}$$

at, $t = 1.1$,

$a = -0.34 - 1.80 - 1.12 + 0.73$

$= -2.52\ m/s^2$

at $t = 2.5$

$a(1.1)$ —

$a(2.5) = -0.03 - 0.38 - 0.83 - 0.74$

$= -1.98\ m/s^2$

Exercise: Given:

x:	1.0	1.1	1.2	1.3	1.4
y:	7.99	8.40	8.78	9.13	9.45

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.0$ and $x = 1.3$

Soln: Finite difference table:

x	y	1st difference	2nd difference	3rd difference	4th difference
1.0	7.99	0.41	-0.03	0	0
1.1	8.40	0.38	-0.03	0	
1.2	8.78	0.35	-0.03		
1.3	9.13	0.32			
1.4	9.45				

at, $x = 1.0$, using newton's forward interpolation,

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2p-1)}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0\right]$$

$p = \frac{x - x_0}{h}$

$p = \frac{1-1}{0.1} = 0$

$$= \frac{1}{0.1}\left[0.41 + \frac{2\times 0-1}{2!}\times(-0.03)\right]$$

$= 4.40$ #

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left(\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{12p^2-36p+22}{4!}\Delta^4 y_0\right)$$

$$= \frac{1}{0.1^2}\left(-0.03 + 0\right)$$

$= -3.0$ #

at, $x = 1.3$

using newton's backward interpolation:

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2q+1}{2!}\nabla^2 y_n\right]$$

$q = \frac{1.3 - 1.4}{0.1}$

$= -1$

$$\frac{dy}{dx} = \frac{1}{0.1}\left[0.32 + \frac{2\times(-1)+1\times(-0.03)}{2}\right]$$

$= 3.35$ #

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n\right]$$

$$= \frac{1}{0.1^2}\times(-0.03) = -3.0\ \#$$

CamScanner

2. Numerical Integration

$$I = \int_a^b f(x)\,dx$$

$h = \frac{b-a}{n}$, n = no. of subintervals.

x : a $a+h$ $a+2h$ ... b
$f(x) = y$: y_0 y_1 y_2 ... y_n

a) Trapezoidal rule

$$I = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})\right]$$

no of subintervals (n) can take any value.

b) Simpson's $\frac{1}{3}$ Rule

$$I = \frac{h}{3}\left[y_0 + y_n + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 \dots)\right]$$

$\boxed{\text{n should be even number}}$ $\boxed{n = 6}$ we take

c) Simpson's $\frac{3}{8}$ Rule

$$I = \frac{3}{8}h\left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \dots) + 2(y_3 + y_6 + y_9 + \dots)\right]$$

$\boxed{\text{n should be multiple of 3}}$

2019, fall
3 a) Evaluate $\int_{-2}^{2} \frac{x}{x+2e^x}\,dx$ using trapezoidal, simpson's $\frac{1}{3}$ and simpson's $\frac{3}{8}$ rules.

$a = -2$, $b = 2$, $f(x) = \frac{x}{x+2e^x}$

Soln. $h = \frac{b-a}{n} = \frac{2-(-2)}{6} = \frac{4}{6} = \frac{2}{3}$

x	$f(x) = y$	
-2	1.156	y_0
$-2+\frac{2}{3} = -\frac{4}{3}$	1.653	y_1
$-\frac{2}{3}$	-1.850	y_2
0	0	y_3
$\frac{2}{3}$	0.146	y_4
$\frac{4}{3}$	0.149	y_5
$\frac{6}{3} = 2$	0.119	$y_n = y_6$

i) Trapezoidal rule

$$\int_{-2}^{2} \frac{x}{x+2e^x} = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{2}{3 \times 2}\left[1.156 + 0.119 + 2(1.653 - 1.850 + 0 + 0.146 + 0.149)\right]$$

$$= 0.490$$

ii) Simpson's $\frac{1}{3}$ Rule:

$$= \frac{2}{3 \times 3}\left[1.156 + 0.119 + 4(1.653 + 0 + 0.149) + 2(-1.850 + 0.146)\right]$$

$= \cancel{1.572}$ 1.13 #

iii) Simpson's 3/8 rule

$= \frac{3}{8} \times \frac{2}{3} \left[1.156 + 0.119 + 3(1.653 + -1.850 + 0.196 + 0.199) + 2(0) \right]$

$= 0.392$ #

2024, spring:

1) $\int_0^1 e^{-x^2} dx$

Soln:

$a = 0, b = 1, \quad f(x) = e^{-x^2}$

$h = \frac{1-0}{6} = \frac{1}{6}$

x	$f(x)$	
0	1	y_0
1/6	0.972 ~~1.028~~	y_1
2/6 = 1/3	0.894 ~~1.117~~	y_2
3/6 = 1/2	0.778 ~~1.284~~	y_3
4/6 = 2/3	0.641 ~~1.559~~	y_4
5/6 = 5/6	0.499 ~~2~~	y_5
6/6 = 1	0.367 ~~2.718~~	$y_n = y_6$

$\int_0^1 e^{-x^2} dx =$

(i) Trapezoidal rule:

$= \frac{h}{2} \left[y_0 + y_n + 2(y_1 + y_2 + y_3 + y_4 + y_5) \right]$

$= \frac{1}{6 \times 2} \left[\cancel{1 + 2.718 + 2(1.028 + 1.117 + 1.284 + 1.559 + 2)} \right]$

~~30.756~~ # ~~1.474~~ # 0.893 #

(ii) Simpson's $\frac{1}{3}$ rule $\int_0^1 e^{-x^2} dx$:

$= \frac{h}{3} \left[y_0 + y_n + 4[y_1 + y_3 + y_5] + 2(y_2 + y_4) \right]$

$= \frac{1}{6 \times 3} \times \cancel{26.30}$ 0.746 #

~~= 1.461~~

(iii) Simposon's $\frac{3}{8}$ rule

$= \frac{3h}{8} \left[y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5) + 2y_3 \right]$

$= 0.746$ #

Romberg Integration

$\int_a^b f(x)\,dx,\ h = \frac{b-a}{n}$

(R_2 स्याम मिलेर गर्ने)

$$\begin{array}{ll}
n=1 & R_{0,0} \to R_{1,1} \to R_{2,2} \to R_{3,3} \\
n=2 & R_{1,0} \\
n=4 & R_{2,0} \to R_{2,1} \to R_{3,2} \\
n=8 & R_{3,0} \to R_{3,1} \\
n=16 & R_{4,0}
\end{array}$$

↓ Trapezoidal rule

improved estimates

$$R_{ij} = \frac{4^j R_{i,j-1} - R_{i-1,j-1}}{4^j - 1},$$

where, $\boxed{i,j \geq 1}$

e.g.

$$R_{11} = \frac{4^1 R_{1,0} - R_{0,0}}{4^1 - 1}$$

$$R_{32} = \frac{4^2 R_{3,1} - R_{2,3}}{4^2 - 1}$$

Example 1) Evaluate $\int_0^1 \frac{1}{1+x}\,dx$ using Romberg integration correct to 3 decimal places.

Soln: $f(x) = \frac{1}{1+x}$

$a = 0,\ b = 1$

(i) Take, $n = 1$, $h = \frac{b-a}{n} = \frac{1-0}{1} = 1$

x	$f(x)$
0	1 — y_0
1	$1/2 = 0.5$ — y_n

∴ Using Trapezoidal Rule:

$$R_{0,0} = \frac{h}{2}[y_0 + y_n]$$
$$= \frac{1}{2}[1 + 0.5] = 0.75$$

(ii) take $n = 2$, $h = \frac{1}{2} = 0.5$

x	$f(x)$
0	1
0.5	0.6667
1	0.5

$$R_{1,0} = \frac{0.5}{2}[1 + 0.5 + 2 \times 0.6667]$$
$$= 0.7084$$

(iii) take $n = 4$, $h = \frac{1}{4} = \cancel{0.3333}\ 0.25$

x		$f(x)$
0		1
0.25	~~0.333~~	~~0.750~~ 0.8
0.50	~~0.666~~	~~0.6002~~ 0.6667
0.75	~~0.999~~	~~0.5~~ 0.5714
1		0.5

$\Rightarrow R_{2,0} = \frac{0.333}{2}\left(1 + 0.5 + 2(\cancel{0.750 + 0.6002})\right)$ $0.8 + 0.6667 + 0.5714$

$= \cancel{0.6994}$ 0.6968

now, calculate improved estimate using the formula,

$$R_{i,j} = \frac{4^j R_{i,j-1} - R_{i-1,j-1}}{4^j - 1}$$

$= \frac{dx}{dt}$

$x =$

$$\therefore R_{1,1} = \frac{4R_{1,0} - R_{0,0}}{4-1}$$

$$= \frac{4 \times 0.7084 - 0.75}{3} = 0.6945$$

$$R_{2,1} = \frac{4R_{2,0} - R_{1,0}}{4^2 - 1} = 4\times$$

$= 0.6982$

$R_{2,2} = \; = 0.6982$

using Romberg Integration)

$= \int_0^1 \frac{1}{1+x} dx = 0.6932$ #

$\frac{1}{1+x^2} dx$ (2024) $\frac{h}{2}[y_0 + y_n + 2(y_1+y_2 \ldots)]$

2023, Spring // same ; Spring

3a) Using Romberg Integration, evaluate $\int_0^2 \frac{e^x + \sin x}{1+x^2} dx$ correct to two decimal places.

Soln: Here, $f(x) = \frac{e^x + \sin x}{1+x^2}$

$h = \frac{b-a}{n}$

$a = 0, b = 2$

① take $n = 1$

$h = \frac{2}{1} = 2$

x	$f(x)$	
0	1	y_0
1	~~1.7799~~	~~y_1~~
2	1.6597	y_n

$R_{0,0} = \frac{2}{2}[1 + 1.6597 + 2\times1.7799]$

$= \cancel{6.2195}$ 2.6597

$R_{1,0}$,

(ii) take, $n = 2$

$h = \frac{2}{2} = 1$

x	$f(x)$
0	1
1	1.7799
2	1.6597

$R_{1,0} = \cancel{6.2195}$ 3.1098

(iii) take, $n = 4$, $h = \frac{2}{4} = 0.\cancel{2}5$

x	$f(x)$
0	1
~~0.25~~	
0.50	1.7025
1 ~~0.75~~	1.7799
1.5	1.6859
2	1.6597

$R_{2,0} = \frac{0.5}{2}\left[1 + 1.6597 + 2\left(\begin{matrix}1.7025 + 1.7799 + \\ 1.6859\end{matrix}\right)\right]$

$= 3.2491$

now, calculate improve estimate using the formula,

$R_{i,j} = \frac{4^j R_{i,j-1} - R_{i-1,j-1}}{4^j - 1}$

3.29

$R(1,1) = \frac{4^1 R_{1,0} - R_{0,0}}{3} = \frac{4 \times 6.2195 - 2.6597}{3}$ 3.1098

$= \cancel{7.4061}$ 3.2598

$R(2,1) = \frac{4^2 R_{2,0} - R_{1,0}}{4 - 1} = \cancel{15.2554}$ 3.2955

$R(2,2) = \frac{4^2 R_{2,1} - R_{1,1}}{4^2 - 1}$

$= 3.2979$ #

$\therefore$ using Romberg integration

$\int_0^2 \frac{e^x + \sin x}{1+x^2}\,dx = 3.29$ #

2024 spring evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$

Here,

$f(x) = \frac{1}{1+x^2}$, $a = 0, b = 1$

$h = \frac{b-a}{n}$

(i) take, $n = 1$

$h = \frac{1}{1} = 1$

x	$f(x)$
0	1
1	0.5

Page:

$R_{0,0} = \frac{h}{2}[y_0 + y_n]$

$= \frac{1}{2}[1+0.5] = 0.75$

(ii) take $n = 2$

$h = \frac{1}{2} = 0.5$

x	$f(x)$
0	1
0.5	0.80
1	0.5

$R(1,0) = \frac{0.5}{2}[1 + 0.5 + 2 \times 0.80]$

$= 0.7750$

(iii) take $n = 4$, $h = \frac{1}{4} = 0.25$

x	$f(x)$
0	1
0.25	0.9412
0.50	0.80
0.75	0.64
1.0	0.5

$R(2,0) = \frac{0.25}{2}\left[1 + 0.5 + 2(0.9412 + 0.80 + 0.64)\right]$

$= 0.7828$

$\frac{4^j R_{i,j-1} - R_{i-1,j-1}}{4^j - 1}$

Using formula, Calculate improved estimate:

$R_{1,1} = \frac{4 R_{1,0} - R_{0,0}}{4-1}$

$= \frac{4 \times 0.7750 - 0.75}{3}$

$= 0.7833$

$R(2,1) = \frac{4^1 R_{2,0} - R_{1,0}}{3}$

$= \frac{4 \times 0.7828 - 0.7750}{3}$

$= 0.7854$

$R_{2,2} = \frac{4^2 R_{2,1} - R_{1,1}}{4^2 - 1} = 0.7855$ #

$\therefore$ Using Romberg integration:

$\int_0^1 \frac{1}{1+x^2}\,dx = 0.785$ #

Gaussian Integration (24,1...)

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3) + \ldots$$

where, $w_1, w_2, w_3, \ldots$ are called weights.

$x_1, x_2, x_3, \ldots$ " " sampling points.

(i) for, $n = 2$

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{2} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2)$$

where, $w_1 = w_2 = 1$, $x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$

(ii) for, $n = 3$

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

where, $w_1 = w_3 = \frac{5}{9}$, $w_2 = \frac{8}{9}$

$x_1 = -x_3 = -\sqrt{3/5} \Rightarrow x_1 = -\sqrt{3/5}$, $x_3 = \sqrt{\frac{3}{5}}$

$x_2 = 0$

$$\int_{-1}^{1} f(x) = \frac{5}{9} f(-\sqrt{3/5}) + \frac{5}{9} f(\sqrt{3/5}) + \frac{8}{9} f(0)$$

$$= \frac{5}{9} \left[f(-\sqrt{3/5}) + f(\sqrt{3/5}) \right] + \frac{8}{9} f(0) \quad \#$$

(i) for $n = 2$

$$\int_{-1}^{1} f(x)\,dx = f(-1/\sqrt{3}) + f(1/\sqrt{3}) \quad \#$$

To evaluate $\int_a^b f(x)dx$ using gauss elimination integration.

Then, put $x = \frac{b-a}{2} \cdot t + \frac{b+a}{2}$

$$dx = \frac{b-a}{2}\, dt$$

$$\therefore I = \int_a^b f(x)\,dx = \int_{-1}^{1} G(t) \cdot \left(\frac{b-a}{2}\right) dt$$

$$\int_a^b f(x)\,dx = \frac{b-a}{2} \int_{-1}^{1} G(t)\cdot dt \quad \#$$

(i) for, $n = 2$

$$I = \frac{b-a}{2} \left[G(-1/\sqrt{3}) + G\left(\frac{1}{\sqrt{3}}\right) \right] \#$$

(ii) for, $n = 3$

$$I = \frac{b-a}{2} \left[\frac{5}{9} \left\{ G(-\sqrt{3/5}) + G(\sqrt{3/5}) \right\} + 8/9\, G(0) \right]$$

#

2021, Spring

3(b) Evaluate:

$$\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$$, using Gauss Integration for $n=2$ & $n=3$.

Sol^n, Here, $a=0, b=2$, $f(x)=\frac{x^2+2x+1}{1+(x+1)^4}$

put, $x=\frac{b-a}{2}t+\frac{b+a}{2}$

$= t+1$

$dx = dt$

$f(x)=\frac{x^2+2x+1}{1+(x+1)^4}$

$G(t) = \frac{(t+1)^2+2(t+1)+1}{1+(t+1+1)^4}$

$= \frac{t^2+2t+1+2t+2+1}{1+(t+2)^4}$

$= \frac{t^2+4t+4}{1+(t+2)^4}$

for $n=2$,

$I = \frac{b-a}{2}\left[G\left(-\frac{1}{\sqrt3}\right)+G\left(\frac{1}{\sqrt3}\right)\right]$

(t^2+4t+4)

$= 1\left[\frac{\left(-1/\sqrt3+1\right)^2+2\left(-1/\sqrt3+1\right)+1}{1+\left(-1/\sqrt3+2\right)^4}\right]+\frac{(1/\sqrt3+1)^2+2(1/\sqrt3+1)+1}{1+(1/\sqrt3+2)^4}$

$= 0.5443$ #

for, $n=3$,

$I = \frac{b-a}{2}\left[\frac{5}{9}\left\{G\left(-\sqrt{3/5}\right)+G\sqrt{3/5}\right)+\frac{8}{9}G(0)\right]$

$= 1\cdot\left[\frac{5}{9}\left\{\frac{\left(-\sqrt{3/5}+1\right)^2+2\left(-\sqrt{3/5}\right)+1}{1+\left(-\sqrt{3/5}+2\right)^4}+\frac{\left(\sqrt{3/5}+1\right)^2+2\left(\sqrt{3/5}+1\right)}{1+\left(\sqrt{3/5}+2\right)^4}\right\}+\frac{8}{9}\cdot\frac{1+2+1}{1+2^4}\right]$

$= 0.536$ #

2019, Fall

evaluate $\int_0^1 \frac{1}{1+x}\,dx = I$ using gaussian integration with $n=3$

$a=0,\ b=1$

$f(x) = \frac{1}{1+x}$

put, $x = \frac{b-a}{2}t + \frac{b+a}{2}$

$= \frac{1}{2}t + \frac{1}{2}$

// $dx = \frac{1}{2}dt$ (no need)

$G(t) = \frac{1}{1+\frac{1}{2}(t+1)}$

$= \frac{2}{3+t}$

for $n=2$

$I = \frac{b-a}{2}\left[G(-1/\sqrt{3}) + G(1/\sqrt{3})\right]$

$= \frac{1}{2}\left[\frac{1}{1+\frac{1}{2}(-1/\sqrt{3}+1)} + \frac{1}{1+1/2(1/\sqrt{3}+1)}\right]$

$= 0.6923$ #

for, $n=3$

$I = \frac{b-a}{2}\left[\frac{5}{9}\left\{G(-\sqrt{3/5}) + G(\sqrt{3/5})\right\} + \frac{8}{9}G(0)\right]$

$= \frac{1}{2}\left[\frac{5}{9}\left(\frac{2}{3-\sqrt{3/5}} + \frac{2}{3+\sqrt{3/5}}\right) + \frac{8}{9}\cdot\frac{2}{3}\right]$

$= 0.6931$ #

2021, Fall
2019, Fall

$I = \int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin x}}\,dx$

Soln: $a=0,\ b=\pi/2,\ f(x) = \frac{\cos x}{\sqrt{1+\sin x}}$

put, $x = \frac{b-a}{2}t + \frac{b+a}{2}$

$= \frac{\pi}{4}t + \frac{\pi}{4} = \frac{\pi}{4}(t+1)$

for $n=2$

~~$I = \frac{\pi}{4}[$~~ $G(t) = \frac{\cos \pi/4(t+1)}{\sqrt{1+\cos\pi/4(t+1)}}$

for, $n=2$

$I = \frac{\pi}{4}\left[\frac{\cos \pi/4(-1/\sqrt{3}+1)}{\sqrt{1+\cos\pi/4(-1/\sqrt{3}+1)}} + \frac{\cos\pi/4(1/\sqrt{3}+1)}{\sqrt{1+\cos\pi/4(1/\sqrt{3}+1)}}\right]$

Date
Page

$$= \frac{\pi}{4}\left[\frac{\frac{1}{\sqrt{2}}\times\frac{-1}{\sqrt{3}}+\frac{1}{\sqrt{2}}}{\sqrt{1+\frac{1}{\sqrt{2}}\times\left(\frac{-1}{\sqrt{3}}\right)+\frac{1}{\sqrt{2}}}} + \frac{\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}}{\sqrt{1+\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{2}}}}\right]$$

$$= \frac{\pi}{4}\left(0.2622 + 0.7669\right)$$

$$= 0.8082 \#$$

at, $n = 3$

$$I = \frac{\pi}{4}\left[\frac{5}{9}\left\{\frac{\cos\pi/4\left(-\sqrt{3/5}+1\right)}{1+\cos\pi/4\left(-\sqrt{3/5}+1\right)} + \frac{\cos\pi/4\left(\sqrt{3/5}+1\right)}{1+\cos\pi/4\left(\sqrt{3/5}+1\right)}\right\} + \frac{8}{9}\cos\pi/4\,(0+1)\right]$$

$$= \frac{\pi}{4}\left[\frac{5}{9}\times\left\{\frac{\frac{1}{\sqrt{2}}\times-\sqrt{\frac{3}{5}}+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}\times\sqrt{3/5}+\frac{1}{\sqrt{2}}} + \frac{\frac{1}{\sqrt{2}}\times\sqrt{\frac{3}{5}}+\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}\times\sqrt{3/5}+\frac{1}{\sqrt{2}}}\right\} + \frac{8}{9}\times\frac{1}{\sqrt{2}}\right]$$

$h = \frac{b-a}{n}$

$$= \frac{\pi}{4}\left[\frac{5}{9}\times\left(0.1375 + 0.5565\right) + 0.6283\right]$$

$$= 0.7964 \#$$

Unit 5 Numerical solution of Ordinary Differential Equation (ODE) (6 hr = 20 marks)

$y' = f(x,y),\ y(x_0) = y_0$ find y at $x = x_n$

Q) Runge-Kutta Methods of order one, two and four. // maximum 10 step

a) RK method of order one (Euler's method)

Let, the given differential equation be

$$y' = f(x,y),\quad y(x_0) = y_0$$

Euler's method is given by

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$n = 0, 1, 2, 3, \ldots$$

example:

$y' = x^2 + y^2$, $y(0) = 1$, using euler's method, estimate y at $x = 1$.

soln: we have,

$$y' = f(x,y) = x^2 + y^2,$$

$$x_0 = 0,\ y_0 = 1$$

Take, $h = 0.1$

By Euler's method,

$$y_{n+1} = y_n + hf(x_n, y_n)$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.1(0^2 + 1^2)$$

$$= 1.1$$

$y = x^2 + y^2$

$y_1 = y(x_1) = y(x_0+h) = y(0+0.1) = y(0.1) = 1.1$

$h = 0.1$

n	x_n	y_n	$f(x_n, y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$
0	0	1	1	1.1
1	0.1	1.1	1.22	~~1.24~~ 1.222
2	0.2	1.222	1.533	1.3759
3	0.3	1.3759	1.9815	1.5734
4	0.4	1.573	2.634	1.836
5	0.5	1.836	3.622	2.198
6	0.6	2.198	5.192	2.7172
7	0.7	2.717	7.873	3.504
8	0.8	3.504	12.92	4.796
9	0.9	4.796	23.81	7.177
10	1.0			~~7.177~~

एक step अगाडि

$\therefore y(1) = 7.177$

Ex: $y' = \frac{y-x}{y+x}$, $y(0) = 1$

using Euler's Method, estimate y at $x = 1$, with $h = 0.2$

Soln: we have,

$y_{n+1}' = f(x, y) = \frac{y-x}{y+x}$

$x_0 = 0$, $y_0 = 1$

$h = 0.2$

By euler's method:

$y_{n+1} = y_n + hf(x_n, y_n)$

$y_1 = y_0 + 0.2 f(x_0, y_0)$

$= 1 + 0.2 \times \frac{1-0}{1+0}$

$= 1.2$

$y_1 = y(x_1) = y(x_0+h) = y(0.2) = 1.2$

n	x_n	y_n	$f(x_n, y_n)$	$y_{n+1} = y_n + hf(x_n, y_n)$
0	0	1	1	1.2
1	0.2	1.2	0 ~~.0909~~ 0.71	~~1.218~~ 1.342
2	0.4	1.218	0.545	1.~~319~~ 45
3	0.6	1.319	0.474	1.~~393~~ 532
4	0.8	1.393	0.378	1.~~447~~ 594
5	1.0			

$\therefore y(1) = 1.594$ #

$\frac{1-0}{1+0} = 1$

$1 + 0.2 \times 1 =$

2021, spring.

5 b) Apply euler's method to approximate the value of $y(0.3)$ for the diff. eqn.

$$y' = x + y, \quad y(0) = 1$$

Take, $h = 0.05$

Solⁿ: Here,

$y' = f(x, y) = x + y$

$x_0 = 0, \; y_0 = 1 \qquad h = 0.05$

using euler's method ~~estimate y at x = 0.3~~

$y_{n+1} = y_n + h f(x_n, y_n)$

$y_1 = y_0 + h f(0, 1)$

$= 1 + 0.05 \times 1$

$= 1.05$

n	x_n	y_n	$f(x_n, y_n)$	$y_{n+1} = y_n + h f(x_n, y_n)$
0	0.00	1	1	1.05
1	0.05	1.05	~~1.05~~ 1.10	1.~~1525~~ 105
2	0.10	1.1025	1.205	1.165
3	0.15	1.165	1.3153	1.2308
4	0.20	1.2308	1.4308	1.3023
5	0.25	1.3023	1.5523	1.3799
6	0.30	1.3799		

$\therefore \; y(0.3) = 1.3799$ #

6) Runge-Kutta Second Order Method (Heun's Method)

9/5 step

Let, the given diff. eqn be

$y' = f(x, y), \; y(x_0) = y_0$

By Heun's method:

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

where, $k_1 = f(x_n, y_n)$

$k_2 = f(x_n + h, y_n + k_1 h)$

$n = 0, 1, 2, 3, \ldots$

for, $n = 0$,

$y_1 = y_0 + \frac{h}{2}(k_1 + k_2)$

$k_1 = f(x_0, y_0)$,

$k_2 = f(x_0 + h, y_0 + k_1 h)$

2021, spring

5 b) Find the solution of

$$\frac{dy}{dx} + 0.4y = 3e^{-x}, \quad y(0) = 5$$

in the interval $0 \le x \le 0.5$ with $h = 0.25$ using Heun's method.

Solⁿ: we have,

$$\frac{dy}{dx} = 3e^{-x} - 0.4y = f(x,y)$$

$$x_0 = 0, \quad y_0 = 5$$
$$h = 0.25$$

using Heun's method:

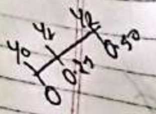
$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$$

$$y_1 = y_0 + \frac{h}{2}(k_1 + k_2) \quad \text{— (1)}$$
$$= 5 + \frac{0.25}{2}(k_1 + k_2)$$

where,

$$k_1 = f(x_0, y_0) = f(0,5) = 3e^{-0} - 0.4 \times 5 = 1$$

$$k_2 = f(x_0 + h, y_0 + k_1 h)$$
$$= f(0 + 0.25, 5 + 1 \times 0.25)$$
$$= f(0.25, 5.25)$$
$$= 0.236$$



$$\therefore y_1 = 5 + \frac{0.25}{2}(1 + 0.236)$$
$$= 5.1545$$

$$\therefore y(0.25) = 5.1545 \ \#$$

also,

$$y_2 = \cancel{5+1} \; 5.1545 + \frac{0.25}{2}(k_1 + k_2) \quad \text{— (2)}$$

$$k_1 = f(x_1, y_1)$$
$$= f(0.25, 5.1545)$$
$$= 0.274$$

$$k_2 = f(0.25 + 0.25, 5.1545 + 0.274 \times 0.25)$$
$$= f(0.50, 5.223)$$
$$= -0.2696$$

$$\therefore y_2 = 5.1550$$

$$\therefore y(0.50) = 5.1550 \ \#$$

Example: Solve:

$$y' = \frac{2x+e^x}{x^2+xe^x}$$

$y(1) = 0$
in the interval $1 \le x \le 1.04$, using Heun's method.
with, $h = 0.01$

Sol^n: $y' = f(x,y) = \frac{2x+e^x}{x^2+xe^x}$

$x_0 = 1$, $y_0 = 0$
$h = 0.01$

using Heun's method:

$$y_1 = y_0 + \frac{h}{2}(k_1+k_2)$$

$$= 0 + \frac{0.01}{2}(k_1+k_2) \quad \text{— (I)}$$

where, $k_1 = f(x_0, y_0)$
$= f(1, 0)$
$= \frac{2+e}{1+e} = 1.2689$

$k_2 = f(x_0+h, y_0+k_1h)$
$= f(1+0.01, 0+1.2689\times0.01)$
$= 1.2564$

$\therefore y_1 = 0.0126$

$\therefore y(1.01) = 0.0126$ #

$$y_2 = y_1 + \frac{0.01}{2}(k_1+k_2)$$

$$= 0.0126 + \frac{0.01}{2}(k_1+k_2) \quad \text{— (II)}$$

where, $k_1 = f(x_1, y_1)$
$= f(1.01, 0.0126)$
$= 1.3245$

$k_2 = f(1.02, 0.0258)$
$= 1.2440$

$\therefore y_2 = 0.0254$

$\therefore y(1.02) = 0.0254$ #

$$y_3 = 0.0254 + \frac{0.01}{2}(k_1+k_2) \quad \text{— (III)}$$

where, $k_1 = f(1.02, 0.0254)$
$= 1.2440$

$k_2 = f(1.03, 0.0378)$
$= 1.2319$

$\therefore y_3 = y(1.03) = 0.0378$ #

$y_4 = 0.0975 + \frac{0.01}{2}(K_1+K_2)$ — (iv)

where,

$K_1 = f(1.03, 0.0975)$
$= 1.2919$

$K_2 = f(1.04, 0.0501)$
$= 1.22$

$= 0.0501$ #

$\therefore y_4 = y(1.04) = 0.0501$ #

Q. Solve:

$\frac{dy}{dx} - y^2 - xy = 0, \quad y(1) = 1$

estimate $y(2)$ using RK-2 method with $h = 0.25$.

Sol^n:

$\frac{dy}{dx} = f(x,y) = y^2 + xy$

$x_0 = 1, y_0 = 1, \quad h = 0.25$

using Heun's method:

$y_1 = y_0 + \frac{h}{2}(K_1+K_2)$

$= 1 + \frac{0.25}{2}(K_1+K_2)$ — (i)

where, $K_1 = f(1,1) = 2$

$K_2 = f(x_0+h, y_0+K_1h)$
$= f(1.25, 1.50)$
$= 4.1250$

$\therefore y_1 = 1.7656$
$y(1.25) = 1.7656$

$y_2 = 1.7656 + \frac{0.25}{2}(K_1+K_2)$ — (ii)

where, $K_1 = f(1.25, 1.7656)$
$= 5.3243$

$K_2 = f(1.50, 3.0967)$
$= 14.2346$

$y_2(1.50) = 4.2105$ #

$y_3 = 4.2105 + \frac{0.25}{2}(K_1+K_2)$ — (iii)

where, $K_1 = f(1.50, 4.2105)$
$= 24.0441$

$K_2 = f(1.75, 10.2215)$
$= 122.3667$

$y_3 = 22.5118$

$y(1.75) = 22.5118$ #

CamScanner

$y_4 = 22.5118 + \frac{0.25}{2}(24.0971 + K_1 + K_2)$ — (14)

Where $K_1 = f(1.75, 22.5118)$
$= 546.179$

$K_2 = f(2.0, 159.05)$
$= 25617.0861$

$y_4 = 3292.9199$ #

1 imp
c) Runge-Kutta Fourth Order Method (Classical RK method) $\Rightarrow$ 2 steps (1 fix)

Let, the given diff. eq^n be

$\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$

Then, By R-K 4th order method:

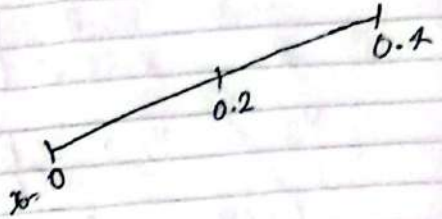
$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where, $k_1 = f(x_0, y_0)$

$k_2 = f(x_0 + h/2, y_0 + k_1 h/2)$

$k_3 = f(x_0 + h/2, y_0 + k_2 h/2)$

$k_4 = f(x_0 + h, y_0 + k_3 h)$



2019, Spring

5b) Solve, $y' = y + e^x$,
$y(0) = 0$
for $y(0.2)$ and $y(0.4)$ by RK 4th order method.

Sol^n: $y' = f(x, y) = y + e^x$
$x_0 = 0$, $y_0 = 0$
let, $h = 0.2$

$\because$ Using RK 4th order method:

$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ — (1)

where, $k_1 = f(x_0, y_0)$
$= f(0, 0)$
$= 0 + e^0 = 1$

$k_2 = f(0 + 0.2/2, 0 + 1 \times 0.2/2)$
$= f(0.1, 0.1)$
$= 1.205$

$\frac{0.1}{\times 2}$

$k_3 = f(0.1, 0.1205)$
$= 1.2257$

$k_4 = f(0.1, 0.2451)$
$= 1.3503$

$\because y_1 = 0.2404$
$\because y(0.2) = 0.2404$

$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ — (ii)

where, $k_1 = f(0.2, 0.2404)$
$= 1.4618$

$k_2 = f(0.3, 0.3868)$
$= 1.7367$

$k_3 = f(0.3, 0.4141)$
$= 1.7639$

$k_4 = f(0.4, 0.5932)$
$= 1.9430$

$\because y_2 = y(0.4) = 0.5873$ #

2023, spring — // same as 24 preboard
24, spring

6(a) Use RK 4th order method to solve
$\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}$, $y_0 = 0$, $x_0 = 1$

for $y(1.2)$ and $y(1.4)$

Soln: $f(x,y) = \frac{2xy + e^x}{x^2 + xe^x}$, $y_0 = 0$, $x_0 = 1$

let, $h = 0.2$

$\because$ Using RK 4th order method:

$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ — (i)

where, $k_1 = f(1, 0)$
$= \cancel{0.2689}\ 0.7310$

$k_2 = \cancel{f(1.1, 0.026)}\ f(1.1, 0.073)$
$= 0.701$

$k_3 = f(1.1, 0.070)$
$= 0.700$

$k_4 = f(1.2, 0.140)$
$= \cancel{0.700}\ \cancel{0.734}\ 0.674$

$\because y_1 = y(1.2) = 0.140$ #

$\frac{2xy+e^x}{x^2+xe^x}$

y_0 ... y_2

$y_2 = 0.140 + \frac{0.2}{6}(k_1+2k_2+2k_3+k_4)$ — (11)

$k_1 = f(1.2, 0.140)$
$= 0.674$

$k_2 = f(1.3, 0.207)$
$= \cancel{0.704}\ 0.691$

$k_3 = f(1.3, \cancel{0.210}\ 0.209)$
$= \cancel{0.705}\ 0.652$

$k_4 = f(1.4, 0.270)$
$= 0.630$

$\therefore y_2 = 0.273$

$\therefore y(1.4) = 0.273$ #

Second Order Differential equation

Let, the second order differential equation be

$$\frac{d^2y}{dx^2} = y'' = f(x, y, y')$$

The initial conditions are

$y(x_0) = y_0$
$y'(x_0) = y_0'$

put, $y' = z = f_1(x, y, z)$

$\because y'' = z' = f_2(x, y, z)$

initial condition:

$y(x_0) = y_0$
$z(x_0) = z_0$

1) Euler's method

$y_{n+1} = y_n + h f_1(x_n, y_n, z_n)$
$z_{n+1} = z_n + h f_2(x_n, y_n, z_n)$
$n = 0, 1, 2, \ldots$

$y_1 = y_0 + h f_1(x_0, y_0, z_0)$
$z_1 = z_0 + h f_2(x_0, y_0, z_0)$

Example: Using Euler's method solve

$y'' + y^2 = x(y')^2$

to estimate $y(1)$ and $y'(1)$ with $h = 0.25$.

initial Conditions are:

$x=0,\ y=1,\ y'=0$

Soln: x_0 0.25 0.50 0.75 x_1

put $y' = z = f_1(x,y,z)$

from given differential eqn:

$y'' = x(y')^2 - y^2$

$z' = xz^2 - y^2 = f_2(x,y,z)$

initial condition:

$x_0 = 0,\ y_0 = 1,\ y_0' = z_0 = 0,\ h = 0.25$

∴ By euler's method:

$y_1 = y_0 + h f_1(x_0,y_0,z_0)$

$= 1 + 0.25 f_1(0,1,0)$

$= 1 + 0.25 \times 0 = 1$

∴ $y(0.25) = 1$

$z_1 = z_0 + h f_2(0,1,0)$

$= 0 + 0.25(0 \times 0^2 - 1^2)$

$z(0.25) = -0.25$

$y_2 = 1 + 0.25 \times f(0.25, 1, -0.25)$

$= 1 + 0.25 \times (-0.25)$

$y(0.50) = 0.9376$

$z_2 = -0.25 + 0.25 f_2(0.25, 1, -0.25)$

$= -0.5039$

$y_3 = 0.9376 + 0.25 f_1(0.50, 0.9376, -0.5039)$

$y(0.75) = 0.812$

$z_3 = -0.5039 + 0.25 f_2(0.50, 0.9376, -0.5039)$

$= -0.692$

$y_4 = 0.812 + 0.25 f_1(1.0, 0.812, -0.692)$

$y(1) = 0.639$

$z_4 = -0.692 + 0.25 f_2(1, 0.812, -0.692)$

$= -0.737$

∴ $y(1) = 0.639$ #

$y'(1) = \cancel{0.639}\ -0.737$ #

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2) RK 2^{nd} order method for 2^{nd} order diff. equation (Heun's method) // 2 step

let the given diff. eq^n be,

$y'' = f(x, y, y')$ ∮ $y(x_0) = y_0$ & $y'(x_0) = y_0'$ | $y(0.2)$ $y'(0.2)$.

put,

$y' = z = f_1(x, y, z)$

$y'' = z' = f_2(x, y, z)$

By Heun's method.

$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2)$

$z_{n+1} = z_n + \frac{h}{2}(m_1 + m_2)$

where,

$k_1 = f_1(x_n, y_n, z_n)$

$k_2 = f_1(x_n + h, y_n + k_1 h, z_n + m_1 h)$

$m_1 = f_2(x_n, y_n, z_n)$

$m_2 = f_2(x_n + h, y_n + k_1 h, z_n + m_1 h)$

Example: $\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ // same as Q4, preboard

$\Rightarrow y'' = -xy' + y$, $y_0 = 1, y_0' = 0$, $x_0 = 0$

$z_0 = 0$

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put,

$y' = z = f_1(x, y, z)$

$z' = -xz + y$, take $h = 0.1$

$z' = f_2(x, y, z) = y - xz$

By Heun's method:

$y_1 = y_0 + \frac{0.1}{2}(k_1 + k_2)$

$z_1 = z_0 + \frac{0.1}{2}(m_1 + m_2)$

where,

$k_1 = f_1(x, y, z)$
$= f_1(0, 1, 0)$
$= 0$

~~$k_2 = f_1(0 + 0.1, 0,$~~

$m_1 = f_2(0, 1, 0)$
$= 1 - 0 = 1$

$k_2 = f_1(0.1, 1, 0.1)$
$= 0.1$

$1 + 0 + 0.1$ $m_2 = f_2(0.1, 1, 0.1)$
$= 0.99$

$\therefore y_1 = y(0.1) = 1.005$

$z_1 = 0.10$

$f_2 = y + 2z$

$$\begin{array}{r} 0.1 \\ \times 2 \\ \hline 0.2 \end{array}$$

$y_2 = 1.005 + \frac{0.1}{2}(k_1 + k_2)$

$z_2 = 0.10 + \frac{0.1}{2}(m_1 + m_2)$

where, $k_1 = f_1(0.1, 1.005, 0.10)$
$= 0.10$

$m_1 = f_2(0.1, 1.005, 0.10)$
$= \cancel{1.005}\ 0.995$

$k_2 = f_1(0.2, 1.015, 0.200)$
$= 0.200$

$m_2 = f_2(0.2, 1.015, 0.200)$
$= 0.975$

$y(0.2) = 1.020$ #

$y'(0.2) = 0.199$ #

b) Solve:
$y'' - y' - 2y = 3e^{2x}$, find $y(0.2)$
given that,
$y(0) = 0,\ y'(0) = -2$

Soln: put $y' = z = f_1(x, y, z)$

now, $z' = z + 2y + 3e^{2x} = f_2(x, y, z)$

given that, $y(0) = 0 = y_0$
$y'(0) = -2 = z_0$
$x_0 = 0$, now take $h = 0.1$

$y_1 = y_0 + \frac{h}{2}(k_1 + k_2) = 0 + \frac{0.1}{2}(k_1 + k_2)$

$z_1 = z_0 + \frac{h}{2}(m_1 + m_2) = -2 + \frac{0.1}{2}(m_1 + m_2)$

where,
$k_1 = f_1(x, y, z) = f_1(0, 0, -2)$
$= -2$

$m_1 = f_2(0, 0, -2)$
$= -2 + 3e^0 = 1$

$k_2 = f_1(0.1, -0.20, -1.90)$
$= -1.90$

$m_2 = f_2(0.1, -0.20, -1.90)$
$= 8.293$

$\therefore y_1 = y(0.1) = -0.195$
$z_1 = -1.535$

$y_2 = -0.195 + \frac{0.1}{2}(k_1 + k_2)$

$z_2 = -1.535 + \frac{0.1}{2}(m_1 + m_2)$

$k_1 = f_1(0.1, -0.195, -1.535)$
$= -1.535$

$m_1 = f_2(0.1, -0.195, -1.535)$
$= 1.739$

$k_2 = f_1(0.2, -0.349, -1.495)$
$= -1.495$

$Ma = f_2(0.2, 0.949, -1.495)$

$= 9.738$

$y_2 = y(0.2) = -0.344$ #

$\frac{dy}{dx} = x+z$

Q1) fall.

(b)(ii) Solve: $\frac{dy}{dx} = x+z$, $\frac{dz}{dx} = x-y$ for $x = 1.5$ given that $y = z = 1$ at $x = 1$ by using Euler's method (take h = 0.1)

Soln:

$\frac{dz}{dx} = x-y$

$y' = x+z = f_1(x,y,z)$

$x_0 = 1$

$y(x_0) = z(x_0) = 1$

$h = 0.1$

$z' = f_2(x,y,z) = x-y$

By euler's method:

$y_1 = y_0 + hf_1(1, 1, 1) =$

$y(1.1) = 1 + 0.1 \times (1+1) = 2.1$

$z_1 = 1 + 0.1 \times f_2(1, 1, 1)$

$= 1 + 0.1 \times (1-1) = 1$

$y(1.2) = 3.1 + 0.1 \times f_1(1.1, 3.1, 1) = 2.1$

$z_2 = 1 + 0.1 \times f_2(1.1, 3.1, 1) = 0.80$

$y(1.3)\ y_3 = 2.1 + 0.1 \times f_1(1.2, 2.1, 0.80) = 2.30$

$z_3 = 0.80 + 0.1 \times f_2(1.2, 2.1, 0.80) = 0.120$

$y(1.4) = 2.492$

$z(1.4) = 2.20$

$\therefore y(1.5) = 2.802$

$z(1.5) = 2.07$

$y(1.5) = 2.802$

$z(1.5) = 2.07$

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3) RK 4^{th} order method (1 step)

Let, the given diff. eq^n be

$$y'' = f(x, y, y')$$
$$y(x_0) = y_0$$
$$y'(x_0) = y'_0$$

put, $y' = z = f_1(x, y, z)$, $y(x_0) = y_0$

$y'' = z' = f_2(x, y, z)$, $z(x_0) = z_0$

by RK - 4^{th} order method,

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{n+1} = z_n + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$$

where, $k_1 = f_1(x_n, y_n, z_n)$

$m_1 = f_2(x_n, y_n, z_n)$

$k_2 = f_1(x_n + h/2, y_n + k_1h/2, z_n + m_1h/2)$

$m_2 = f_2(x_n + h/2, y_n + k_1h/2, z_n + m_1h/2)$

$k_3 = f_1(x_n + h/2, y_n + k_2h/2, z_n + m_2h/2)$

$m_3 = f_2(x_n + h/2, y_n + k_2h/2, z_n + m_2h/2)$

$k_4 = f_1(x_n + h, y_n + k_3h, z_n + m_3h)$

$m_4 = f_2(x_n + h, y_n + k_3h, z_n + m_3h)$

CS CamScanner

4th

example: using classical RK-method, solve $y'' = x(y')^2 - y^2$, $y = 1$, $y' = 0$ at $x = 0$ for $y(0.2)$ & $y'(0.2)$.

Soln: put, $y' = z = f_1(x,y,z)$

$y'' = z' = f_2(x,y,z) = xz^2 - y^2$

given that,

$x_0 = 0,\ y_0 = 1,\ y_0' = z_0 = 0$

Let, $h = 0.2$

By R-K 4th order method.

$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ — ①

$z_1 = z_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4)$ — ②

where,

$k_1 = f_1(0,1,0) = 0$

$m_1 = f_2(0,1,0) = 0\times0 - 1^2 = -1$

$k_2 = f_1(0.1, 1, -0.1)$
$= -0.1$

$m_2 = f_2(0.1, 1, -0.1) = -0.999$

$k_3 = f_1(0.1, 0.995, -0.09) = -0.09$

$m_3 = f_2(0.1, 0.995, -0.09) = -0.97$

$k_4 = f_1(0.2, 0.982, -0.197)$
$= -0.197$

$m_4 = -0.95$

from eqn ①:

$y_1 = 1 + \frac{0.1}{6}(0 - 0.1\times2 - 2\times0.09 - 0.197)$

$\boxed{y(0.2) = 0.9801}$ #

$z_1 = y'(0.2) = 0 + \frac{0.1}{6}[-1 - 2\times(-0.999) - 2\times0.97 - 0.95]$

$\boxed{y'(0.2) = 0.1970}$ #

2021, Fall – using classical RK method solve

$$y'' + xy' - y = 0, \quad y = 1, \; y' = 0 \text{ at } x = 0$$
for $y(0.2)$

Solution:

put, $y' = z = f_1(x, y, z)$

$z' = y'' = f_2(x, y, z) = y - xz$

$x_0 = 0, \; y(x_0) = 1, \; y'(x_0) = 0$

take, $h = 0.2$

by RK - 4th method:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{--- (i)}$$

$$z_{n+1} = z_0 + \frac{h}{6}(m_1 + 2m_2 + 2m_3 + m_4) \quad \text{--- (ii)}$$

where,

$k_1 = f_1(0, 1, 0) = 0$
$m_1 = f_2(0, 1, 0) = 1 - 0 = 1$

$k_2 = f_1(0.1, 1, +0.1) = 0.1$
$m_2 = f_2(0.1, 1, 0.1) = 0.990$

$k_3 = f_1(0.1, 1.01, 0.099)$
$= 0.099$
$m_3 = 1$

$k_4 = $
$m_4 = f_1(0.2, 0.020, 0.20) = 0.20$
$= 0.020$

$y_n + k_3 h \quad (x_n + h/2, \; y_n + k_1 h/2, \; z_n + m_1 h/2)$

$\therefore$ from eqn (i):

$$y_1 = 1 + \frac{0.2}{6}(0 + 2\times0.1 + 2\times0.099 + 0.20)$$

$$y(0.2) = 1.020 \#$$

$$z_1 = 0 + \frac{0.2}{6}(1 + 2\times0.99 + 2\times1 - 0.020)$$

$$y'(0.2) = 0.165 \#$$

(Short note) LQ

Boundary Value Problem (BVP)

general form: $y'' = f(x, y, y')$

$y(x_0) = y_0$
$y(x_n) = y_n$

i) Shooting Method: // at least 2 step.

It is used to solve BVP by converting it into equivalent initial value problem. Let,

$y'' = f(x, y, y')$
$y(x_0) = y_0$ // 0
$y(x_n) = y_n$

be the given BVP.

put, $y'(x_0) = p_0$ // 0
(first guess)

Apply Euler's method or Heun's method with suitable h and estimate $y(x_n)$.
let $y(x_n) = q_0$

if, $y(x_n) = y_n$, then the solution is obtained by put $y_1, y_2, \ldots$

if $y(x_n) \neq y_n$, then change the first guess.
put, $y'(x_0) = p_1$ (second guess).

repeat the same step to estimate $y(x_n)$.

Solution imposing → conditions at more than one point

Let, $y(x_n) = q_1$

if $y(x_n) = q_1 = y_n$, then the solution is achieved.

If $y(x_n) = q_1 \neq y_n$, then make new guess for $y'(x_0)$.

Using the formula,

$$p_2 = p_0 + \frac{p_1 - p_0}{q_1 - q_0}(y_n - q_0)$$

∴ with this $y'(x_0) = p_3$
(third guess)

Repeat the steps.

→ BVP is a type of differential equation problem whose the solution is determined by imposing conditions at more than one point. (typically at the boundaries of domain)

$$p_3 = p_0 + \frac{p_1 - p_0}{q_1 - q_0}(y_n - q_0)$$

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BCA, 2024, Fall

6. a) Using shooting method, solve

$y'' = 6x$,
$y(1) = 2$, $y(2) = 9$

interval $1 \le x \le 2$ with $h = 0.25$.

Soln: put,
$y' = z = f_1(x, y, z) = z$

$\therefore y'' = z' = f_2(x, y, z) = 6x$

Let,

$y(1) = 2$ & $y'(1) = \underline{2} = P_0$ (first guess)

Apply Euler's method to estimate $y(2)$;

$y_1 = y_0 + h f_1(x_0, y_0, z_0) = 2 + 0.25 f_1(1, 2, 2)$
$z_1 = z_0 + h f_2(x_0, y_0, z_0) = 2 + 0.25 f_2(1, 2, 2)$

$y(1.25) = y_1 = 2.5$
$z_1 = \cancel{2.5}\ 3.5$

$y(1.50) = y_2 = 2.5 + 0.25 f_1(1.25, 2.5, 3.5) = 3.375$
$z_2 = 3.5 + 0.25 f_2(1.25, 2.5, 3.5) = 5.375$

$y(1.75) = y_3 = 3.375 + 0.25 f_1(1.50, 3.375, 5.375)$
$= 4.718$

$z_3 = 5.375 + 0.25 f_1(1.50, 3.375, 5.375) = 7.625$

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$y(2) = y_4 = 4.718 + 0.25 f_1(1.75, 4.718, 7.625)$

$= \underline{6.624} = q_0$

Since, $y_4 = 6.624 \neq 9$

$f(x, y, y')$: Let, new guess. Change the guess for

$y'(1) = \underline{3} = P_1$ (second guess)

$x_0 = 1, y_0 = 2, z_0 = 3$

using euler's method:

$y_1 = 2 + 0.25 f_1(1, 2, 3) = 2.75$
$z_1 = 3 + 0.25 f_2(1, 2, 3) = 4.5$

$y_2 = 2.75 + 0.25 f_1(1.25, 2.75, 4.5) = 3.875$
$z_2 = 4.5 + 0.25 f_1(1.25, 2.75, 4.5) = 6.875$

$y_3 = 5.468$
$z_3 = 8.625$

$y_4 = 7.625 = q_1$

$y_4 \neq y(2)$

now, apply the formula,

$$P_3 = P_0 + \frac{P_1 - P_0}{q_1 - q_0}(y_n - q_0)$$

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$$= 2 + \frac{3-2}{7.625-6.624}(9-6.624)$$

$= 4.374$

put, $y'(1) = z(1) = 4.374$

$x_0 = 1,\ y_0 = 2,\ z_0 = 4.374$

using, Euler's method:

$y(1.25) = y_1 = 2 + 0.25 f(1, 2, 4.374)$
$= 3.094$

$z_1 = 4.374 + 0.25 (1, 2, 4.374) = 5.874$

$y_2 = 3.094 + 0.25 f(1.25, 3.094, 5.874) =$ 4.563
$z_2 = 7.749$
$y_3 = \cancel{6.149}\ 6.50$
$z_3 = 9.999$
$y_4 = \cancel{8.672}\ 9.0$ #

$\because$ Solutions are:

$y_1 = 3.094$
$y_2 = 4.563$
$y_3 = 6.50$
$y_4 = 9.0$

#

$-4 \times 1 \times 1$

Unit 6 4hr = 13 (Overview of PDEs, 24 Fall short note)

Partial Differential Equation (PDE)

General form of second order p.d.e.

$$A.u_{xx} + B u_{xy} + C.u_{yy} = f(x, y, u, u_x, u_y)$$

$\left\{ \begin{array}{l} u = \text{dependent} \\ x, y = \text{independent} \end{array} \right\}$

$\boxed{u = f(x, y)}$

NCIT, preboard (Short note)

The second order p.d.e is said to be

i) Elliptic if $B^2 - 4AC < 0$

ii) Parabolic if $B^2 - 4AC = 0$

iii) Hyperbolic if $B^2 - 4AC > 0$

e.g. $4u_{xx} - 2u_{xy} + 3u_{yy} = 0$
$A = 4,\ B = -2,\ C = 3$

$B^2 - 4AC = (-2)^2 - 4 \times 4 \times 3$
$= -44 < 0$

4-48

So elliptic.

given pde. is elliptic

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$$u_x = \frac{\partial u}{\partial x} = \frac{u(x+h,y) - u(x-h,y)}{2h}$$

$$u_y = \frac{u(x,y+k) - u(x,y-k)}{2k}$$

$$u_{xx} = \frac{u(x+h,y) - u(x-h,y) - 2u(x,y)}{h^2}$$

$$u_{yy} = \frac{u(x,y+k) - u(x,y-k) - 2u(x,y)}{k^2}$$

or, $u_{xx} = \frac{1}{h^2}\left[u(i+1,j) + u(i-1,j) - 2u(i,j)\right]$

$$u_{yy} = \frac{1}{k^2}\left[u(i,j+1) + u(i,j-1) - 2u(i,j)\right]$$

i) Laplace Equation:

$u_{xx} + u_{yy} = 0$ (elliptic eq^n)

putting the values of u_{xx} & u_{yy}.

$$\Rightarrow \frac{1}{h^2}\left[u(i+1,j) + u(i-1,j) - 2u(i,j)\right] + \frac{1}{k^2}\left[u(i,j+1) + u(i,j-1) - 2u(i,j)\right] = 0$$

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For a squaremesh, $h = k$.

$$u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j) = 0$$

$$\boxed{\therefore u(i,j) = \frac{1}{4}\left[u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1)\right]}$$

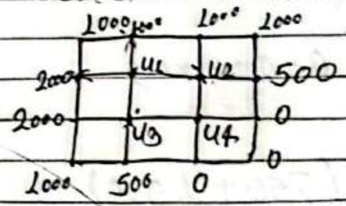
#

$$u_1 = \frac{1}{4}(u_2 + 8 + 15 + u_3)$$

$$4u_1 - u_2 - u_3 = 23 \quad \text{—(1)}$$

(like this, 24 spring)

Example: The values of $u(x,y)$ on the boundary of a square is given in the diagram. Evaluate the function $u(x,y)$ satisfying the laplace equation $\nabla^2 u = 0 \Rightarrow u_{xx} + u_{yy} = 0$ at the interior points of the figure using gauss seidel method.



Let, u_1, u_2, u_3 and u_4 be the interior points. The solution of Laplace for these points is given by

$u_1 = \frac{1}{4}(u_2 + 2000 + 1000 + u_3)$

or, $u_1 = \frac{1}{4}(u_2 + u_3 + 3000)$ — (1)

$u_2 = \frac{1}{4}(1000 + u_4 + u_1 + 500)$

$= \frac{1}{4}(u_1 + u_4 + 1500)$ — (2)

$u_3 = \frac{1}{4}(u_1 + 2500 + u_4)$ — (3)

$u_4 = \frac{1}{4}(\cancel{500} + u_2 + u_3)$ — (4)

By, Gauss Seidel method:

let, $u_2 = u_3 = u_4 = 0$, be the initial guess

from (1):

$u_1 = \frac{1}{4}(0+0+3000) = 750$

from (2):

$u_2 = \frac{1}{4}(750 + 1500) = 562.6$

from (3):

$u_3 = \frac{1}{4}(750 + 2500 + 0) = 812.50$

from (4):

$u_4 = \frac{1}{4}(562.5 + 812.50)$

$= 343.75$

(अगाडिको value मिलाए पुग्छ, point पाछे पढैना)

iterative table:

itno	u_1 from (I)	u_2 from (II)	u_3 from (III)	u_4 from (IV)
1	750	562.5	812.5	343.75
2	1093.75	734.375	984.375	429.688
3	1179.688	777.344	1027.344	451.172
4	1201.758	788.233	1038.233	456.616
5	1206.616	790.694	1040.694	457.847
6	1207.847	791.423	1041.423	458.211
7	1208.211	791.605	1041.605	458.303
8	1208.303	791.651	1041.651	458.325
9	1208.325	791.663	1041.663	458.331
10	1208.331	791.665	1041.665	458.333
11	1208.333	791.666	1041.666	458.333

$\therefore u_1 = 1208.33$

$u_2 = 791.66$

$u_3 = 1041.66$

$u_4 = 458.33$

Solution of Poisson's Equation

→ Poisson's equation is given by:

$$u_{xx} + u_{yy} = f(x,y) \quad -(1)$$

putting, $u_{xx} = \frac{1}{h^2}\left[u(i+1,j) + u(i-1,j) - 2u(i,j)\right]$

$$u_{yy} = \frac{1}{K^2}\left[u(i,j+1) + u(i,j-1) - 2u(i,j)\right]$$

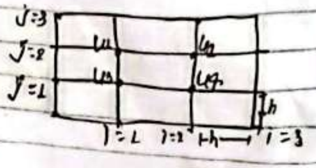
$$x = ih, \quad y = jK$$

for a square mesh, $h = k$

∵ from (1):

$$\frac{1}{h^2}\left[u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j)\right] = f(ih, jh)$$

$$u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j) = h^2 f(ih,jh) \quad -(2)$$

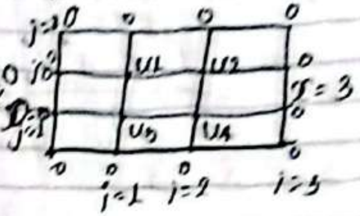


(ask many times in exam)

Example

1. Solve the equation $\nabla^2 u = -10(x^2+y^2+10)$ over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length $h = 1$.

Sol^n:



From the given dimensions, the diagram is obtained as shown in figure

The solution of Poisson equation is given by eq^n (2):

for, $u(i,j) = u_1$, $i = 1$, $j = 2$

$$f(ih,jh) = -10\left[(ih)^2 + (jh)^2 + 10\right]$$

$$= -10\left[(1\times1)^2 + (2\times1)^2 + 10\right]$$

$$= -150$$

∵ $u_2 + 0 + 0 + u_3 - 4u_1 = -1^2 \times 150$

or, $u_2 + u_3 - 4u_1 = -150$

$$u_1 = \frac{150 + u_2 + u_3}{4} \quad -(i)$$

for, $u(i,j) = u_2$, $i=2, j=2$

$f(ih, jh) = -10(2^2+2^2+10)$
$= -180$

$\because u_1 + 0\cdot 4u_2 + u_4 = -180$

$$u_2 = \frac{180 + u_1 + u_4}{4} \quad -(ii)$$

for $u(i,j) = u_3$, $i = 1, j=1$

$f(ih, jh) = -10(1^2+1^2+10)$
$= -120$

$\because 0 + 2u_1 + u_4 + 0\cdot 4u_3 = -120$

$$u_3 = \frac{120 + u_1 + u_4}{4} \quad -(iii)$$

for, $u(i,j) = u_4$, $i=2, j=1$

$$u_4 = \frac{1}{4}(150 + u_2 + u_3) \quad -(iv)$$

from eq^n (ii) & (iv)

$$u_1 = u_4 = \frac{1}{4}(150 + u_2 + u_3) - (v)$$

from (ii)

$$u_2 = \frac{1}{4}(180 + 2u_1)$$
$$= \frac{1}{2}(90 + u_1) \quad -(vi)$$

from (iii)

$$u_3 = \frac{1}{4}(120 + 2u_1)$$
$$= \frac{1}{2}(60 + u_1) \quad -(vii)$$

solving (v), (vi) & (vii) using Gauss-Seidel Method:

Let, $u_2 = u_3 = 0$ be the initial guess:

from (v), $u_1 = \frac{150}{4} = 37.5$

from (vi)
$$u_2 = \frac{1}{2}(90 + 37.5) = 63.75$$

from (vii)
$$u_3 = \frac{1}{2}(60 + 37.5) = 48.75$$

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iterative table :

itno	u_1 from (v)	u_2 from (vi)	u_3 from (vii)
1	37.6	63.75	48.75
2	65.625	77.819	62.813
3	72.656	81.328	66.328
4	74.414	82.207	67.207
5	74.853	82.426	67.426

$\therefore \; u_1 = u_4 = 74.85$

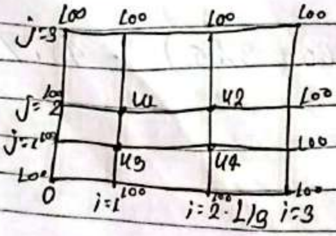
$u_2 = 82.42$

$u_3 = 67.42$

like this 24, spring

2023, fall

Solve the Poisson's equation $u_{xx} + u_{yy} = 243(x^2+y^2)$ over a square domain $0 \le x \le 1$, $0 \le y \le 1$ with step size $h = \frac{1}{3}$ and $u = 100$ on the boundary.



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from given dimensions, the diagram is obtained as shown in figure,

sol^n of Possions eq^n is given by:

$$u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j) = h^2 f(ih,jh) \quad -(1)$$

for, $u(i,j) = u_1$, $i=1, j=2$

$$f(ih,jh) = 243\,((ih)^2+(jh)^2)$$

$$= 243\,[(1/3)^2 + (2\times 1/3)^2]$$

$f(ih,jh) = 27(i^2+j^2) = 135$

$\therefore \; 100 + 100 + u_2 + u_3 - 4u_1 = 135$

$$u_1 = \frac{\cancel{135} - 65 - u_2 - u_3}{-4}$$

$$= \frac{65+u_2+u_3}{4} \quad -(i)$$

for, $u(i,j) = u_2$, $i=2, j=2$

$f(ih,jh) = 27(2^2+2^2) = 108$

$\therefore \; u_1 + 100 + 100 + u_4 - 4u_2 = 108$

$$u_2 = \frac{92 + u_1 + u_4}{4} \quad -(ii)$$

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for, $u(i,j) = u_3$, $i=1, j=1$

$f(ih, jh) = 27(1+1)$
$= 54$

$\because u_1 + 100 + 100 + u_4 - 4u_3 = 54$

$u_3 = \dfrac{146 + u_1 + u_4}{4}$ — (III)

for, $u(i,j) = u_4$, $i = 2, j = 1$

$f(ih, jh) = 27(2^2 + 1^2)$
$= 135$

$\therefore u_3 + u_2 + 100 + 100 - 4u_4 = 135$

$u_4 = \dfrac{65 + u_2 + u_3}{4}$ — (IV)

$u_1 = u_4 = \dfrac{65 + u_2 + u_3}{4}$ — (V)

from (II): $u_2 = \dfrac{92 + 2u_1}{4} = \dfrac{46 + u_1}{2}$ — (VI)

from (III):

$u_3 = \dfrac{73 + u_1}{2}$ — (VII)

solving (V), (VI) & (VII) using Gauss Seidel Method:

put, $u_2 = u_3 = 0$

Page:

from (V):

$u_1 = \dfrac{65}{4} = 16.25$

$u_2 = \dfrac{46 + 16.25}{2} = 31.125$

$u_3 = \dfrac{73 + 16.25}{2} = 44.625$

iterative table:

it no	u_1 from (V)	u_2 from (VI)	u_3 from (VII)
1	16.25	31.125	44.625
2	35.18	40.590	54.69
3	39.92	42.96	56.46
4	41.205	~~43.558~~ 43.553	~~53.918~~ 57.053
5	~~41.401~~ 41.402	43.701	57.201

$\because u_1 = u_4 = 41.402$

$u_2 = 43.701$

$u_3 = 57.201$ //

One Dimensional Heat Equation (Schmidt Method)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{—(1)}$$

Boundary condition: $u(0,t) = u(L,t) = 0$

initial condition: $u(x,0) = f(x)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}\left(u(i+1,j) + u(i-1,j) - 2u(i,j)\right)$$

$$\frac{\partial u}{\partial t} = \frac{1}{k}\left(u(i,j+1) - u(i,j)\right)$$

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

from (1):

$$\frac{1}{k}\left[u(i,j+1) - u(i,j)\right] = \frac{c^2}{h^2}\left[u(i+1,j) + u(i-1,j) - 2u(i,j)\right]$$

or, $$u(i,j+1) - u(i,j) = \frac{c^2 k}{h^2}\left[u(i+1,j) + u(i-1,j) - 2u(i,j)\right]$$

put, $\boxed{\alpha = \frac{c^2 k}{h^2}}$

$$u(i,j+1) = \alpha\, u(i+1,j) + \alpha\, u(i-1,j) \cancel{- \alpha u(i,j)} + (1-2\alpha)\, u(i,j) \quad \#$$

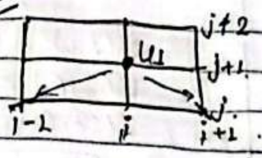
$0 < \alpha \le 1/2$

when, $\alpha = 1/2$

$$\boxed{u(i,j+1) = \frac{1}{2}\left[u(i+1,j) + u(i-1,j)\right] \qquad \alpha = \frac{c^2 k}{h^2}} \quad \#$$

$\alpha = \frac{c^2 k}{h^2}$

(imp)



Example: solve: $\overset{c^2}{2} u_{xx} = u_t$, $0 \le t \le 1.0$, $0 \le x \le 4$

Given that $u(x,0) = 50(4-x)$ //initial

and Boundary Condition $u(0,t) = u(4,t) = 0$

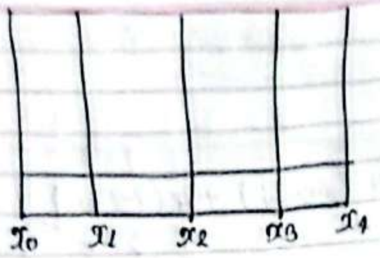
$h = 1$; $\alpha = 1/2$

Soln: since, $\alpha = c^2k/h^2$

$$k = \frac{\alpha h^2}{c^2} = \frac{1 \times 1^2}{2 \times 2} = \frac{1}{4} = 0.25$$

∴

$x_0 = 0$,	$t_0 = 0$
$x_1 = x_0 + h = 1$	$t_1 = 0 + 0.25 = 0.25$
$x_2 = 1 + h = 2$	$t_2 = 0.25 + 0.25 = 0.50$
$x_3 = 3$	$t_3 = 0.75$
$x_4 = 4$	$t_4 = 1.0$



$u(x,0) = 50(4-x)$
$= 50(4-1)$
$u(1,0) = 50\times 3$
$u(2,0) = 50\times 2$
$u(3,0) = 50$

t \ x	x_0	x_1	x_2	x_3	x_4
$t_0=0$	0	150	100	50	0
$t_1=0.25$	0	50	100	50	0
$t_2=0.50$	0	50	50	50	0
$t_3=0.75$	0	25	50	25	0
$t_4=0.10$	0	25	25	25	0

$\boxed{\text{diagonal}}$

by Schimidt formula,

$$u(i,j+1) = \frac{1}{2}\left[u(i+1,j) + u(i-1,j)\right]$$

Q. using Schmidt formula, solve $u_t = u_{xx}$ with the boundary condition
$u(0,t) = u(1,t) = 0$
take $h = 0.2$ and $\alpha = 1/2$, $0 \le x \le 1$,
$0 \le t \le 0.1$

Soln: initial condition: $u(x,0) = \sin \pi x$

Solve: $\alpha = \dfrac{c^2 k}{h^2}$

~~$K = \dfrac{1\times 1/2}{0.2^2}$~~

$\sin \pi x$

$$k = \frac{\frac{1}{2}\times 0.2^2}{1}$$
$$= 0.020$$

$\because$

$x_0 = 0$
$x_1 = 0+0.20 = 0.20$
$x_2 = 0.40$
$x_3 = 0.60$
$x_4 = 0.80$
$x_5 = 1.0$

$t_0 = 0$
$t_1 = 0+0.02 = 0.02$
$t_2 = 0.02+0.02 = 0.04$
$t_3 = 0.06$
$t_4 = 0.08$
$t_5 = 0.1$

t \ x	$x_0=0$	$x_1=0.20$	$x_2=0.40$	$x_3=0.60$	$x_4=0.80$	$x_5=1.0$
$t_0=0$	0	0.588	0.951	0.951	0.588	0
$t_1=0.02$	0	0.476	0.770	0.770	0.476	0
$t_2=0.04$	0	0.385	0.623	0.623	0.385	0
$t_3=0.06$	0	0.312	0.504	0.504	0.312	0
$t_4=0.08$	0	0.156	0.408	0.408	0.156	0
$t_5=0.10$	0	0.204	0.282	0.282	0.204	0