Chapter 3: Review of Z-Transform

Introduction:

Transform techniques are an important tool in the analysis of signals and linear time invariant (LTI) systems.
The Z-transform plays the same role in the analysis of discrete-time signals & LTI systems as the Laplace transform does in the analysis of continuous-time signals & LTI systems. Laplace transform can be developed as an extension of the continuous-time Fourier transform. This extension was motivated in part by the fact that it can be applied to a broader class of signals than the Fourier transform does not converge but the Laplace transform does. The Laplace transform allowed us, for example to perform transform analysis of unstable systems & to develop additional insights and tools for LTI system analysis.

The Direct Z-Transform:

The Z-transform of a discrete-time signal x[n] is defined as an infinite sum or infinite power series

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where z is a continuous complex variable. This expression is generally referred to as the two-sided z-transform.
For convenience, the z-transform of a signal x[n] is denoted by

$$X(z) \equiv Z\{x[n]\}$$

Whereas the relationship between x[n] and X(z) is indicated by

$$x[n] \xleftrightarrow{z} X(z)$$

Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges. The region of convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value. Thus any time we cite a z-transform we should also indicate its ROC

Find z-transform of following signals:

1. $x_1[n] = \{1, 2, 1, 3, 4\}$ [Causal Signal]
2. $x_2[n] = \{1, 2, 1, 3, 4\}$ [Anti-causal signal]
3. $x_3[n] = \{1, 2, 1, 3, 4\}$ [Non-causal signal]

Z-Transform of some elementary signals:

1. x[n] = δ[n].

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = z^0 = 1 \qquad ROC: All\ z.$$

2. x[n] = δ[n-k].

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n-k]z^{-n} = z^{-k} \qquad ROC: All\ z\ except\ at\ z = 0.$$

3. x[n] = u[n]

$$X(z) = \sum_{n=-\infty}^{\infty} u[n]z^{-n}$$
$$= \sum_{n=0}^{\infty} z^{-n}$$
$$= \frac{1}{1-z^{-1}}\ if\ |z^{-1}| < 1$$
$$= \frac{1}{1-z^{-1}}, \qquad ROC: |z| > 1$$

4. x[n] = u[-n-1]

$$X(z) = \sum_{n=-\infty}^{\infty} u[-n-1]z^{-n}$$
$$= \sum_{n=-\infty}^{-1} z^{-n}$$
$$= \sum_{n=1}^{\infty} z^{n}$$
$$= \frac{z}{1-z}\ if\ |z| < 1$$
$$= -\frac{1}{1-z^{-1}}, \qquad ROC: |z| < 1$$

Analysis of ROC:

We can express the complex variable z in polar form as $z = re^{j\theta}$.
Then,

$$X(z)|_{z=re^{j\theta}} = \sum_{n=-\infty}^{\infty} x[n]\left(re^{j\theta}\right)^{-n}$$
$$= \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\theta n}$$

The region of convergence (ROC) of X(z) is the set of all values of z for which X(z) attains a finite value. In the ROC of X(z),

$$|X(z)| < \infty, But$$
$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\theta n}\right|$$
$$\le \sum_{n=-\infty}^{\infty} \left|\{x[n]r^{-n}\}e^{-j\theta n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]r^{-n}|$$

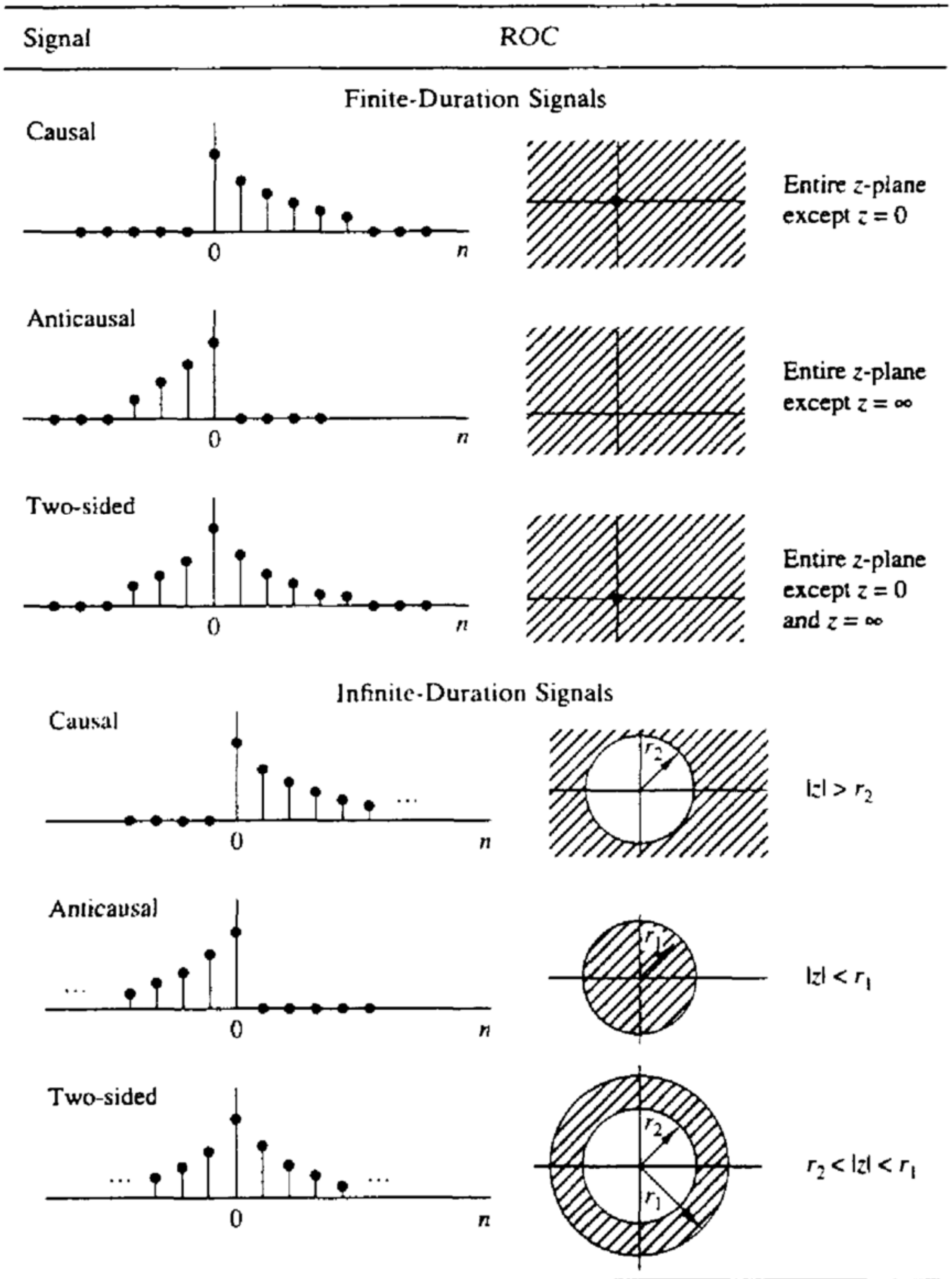


Figure : Characteristic families of signals with their corresponding ROC.

Hence $|X(z)|$ is finite if the sequence $x[n]r^{-n}$ is absolutely summable.
To elaborate again,

$$|X(z)| \le \sum_{n=-\infty}^{-1} |x[n]r^{-n}| + \sum_{n=0}^{\infty} |x[n]r^{-n}|$$

$$\le \sum_{n=1}^{\infty} |x[-n]r^{n}| + \sum_{n=0}^{\infty} |x[n]r^{-n}|$$

The first series, a non-causal sequence, converges for $|z| < r_2$, and the second series, a causal sequence, converges for $|z| > r_1$, resulting in an annular region of convergence.

Properties of Z-transform:

1. Linearity:

If $x_1[n] \overset{z}{\longleftrightarrow} X_1(z)$ & $x_2[n] \overset{z}{\longleftrightarrow} X_2(z)$, then

$$x[n] = ax_1[n] + bx_2[n] \overset{z}{\longleftrightarrow} aX_1(z) + bX_2(z)$$

For any constants a and b. ROC is intersection of the ROC of $X_1(z)$ and $X_2(z)$.

Find z-transform of $x[n] = (\cos\omega_0 n)u[n]$. (*hint: use Euler's identity*)

Ans:

$$X(z) = \frac{(1 - z^{-1}\cos\omega_0)}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \quad ROC: |z| > 1$$

2. Time Shifting:

If $x[n] \overset{z}{\longleftrightarrow} X(z)$ then $x[n-k] \overset{z}{\longleftrightarrow} z^{-k}X(z)$

ROC of $z^{-k}X(z)$ is the same as that of X(z) except for z = 0 if k>0 and z = ∞ if k<0.

The properties of linearity and time shifting are the key features that make the z-transform extremely useful for the analysis of discrete-time LTI systems.

3. Scaling in z-domain:

If $x[n] \overset{z}{\longleftrightarrow} X(z)$ ROC: $r_1 < |Z| < r_2$

Then $a^n x[n] \overset{z}{\longleftrightarrow} X(a^{-1}z)$ ROC: $|a|r_1 < |Z| < |a|r_2$

for any constant a, real or complex.

Proof:

$$Z\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n](a^{-1}z)^{-n} = X(a^{-1}z)$$

ROC is $r_1 < |a^{-1}z| < r_2 \Rightarrow |a|r_1 < |z| < |a|r_2$

Eg:- $x[n] = a^n(\cos\omega_0 n)u[n]$

$$We\ know, \quad Z\{\cos\omega_0 n\, u[n]\} = \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}, \ ROC: |z| > 1$$

$$Then, Z\{a^n(\cos\omega_0 n)\, u[n]\} = \frac{1 - az^{-1}\cos\omega_0}{1 - 2az^{-1}\cos\omega_0 + (az^{-1})^2}, \ ROC: |z| > |a|$$

4. Time Reversal:

If $x[n] \overset{z}{\longleftrightarrow} X(z)$ ROC: $r_1 < |Z| < r_2$

Then, $x[-n] \stackrel{z}{\longleftrightarrow} X(z^{-1})$ $ROC: \frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof:

$$Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n](z^{-1})^{-n} = X(z^{-1})$$

$$ROC: r_1 < |z^{-1}| < r_2 ==> \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

Eg:- x[n] = a^{-n}u[-n]

$$Ans: \frac{1}{1-az}, Roc: |z| < |a|$$

5. Differentiation in the z-domain:

If $x[n] \stackrel{z}{\longleftrightarrow} X(z)$ ROC: $r_1 < |Z| < r_2$

Then, $nx[n] \stackrel{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}$ $ROC: r_1 < |z| < r_2$

Proof:-

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Differentiating both sides w.r.t. z

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x[n](-n)z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty}\{nx[n]\}z^{-n} = -z^{-1}Z\{nx[n]\}$$

Eg. Determine the signal x[n] whose z-transform is given by

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

$$Ans: nx[n] = (-1)^{n+1}a^n u[n-1]$$

x[n] = n.a^nu[n]

6. Convolution of two Sequences:

$$If, \ x_1[n] \stackrel{z}{\longleftrightarrow} X_1(z) \ \& \ x_1[n] \stackrel{z}{\longleftrightarrow} X_1(z)$$

$$then, x[n] = x_1[n] * x_2[n] \stackrel{z}{\longleftrightarrow} X(z) = X_1(z)X_2(z) \qquad ROC: R_{x_1} \cap R_{x_2}$$

Proof:

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

$$X(z) = Z\{x_1[n] * x_2[n]\} = \sum_{n=-\infty}^{\infty} \{x_1[n] * x_2[n]\} z^{-n} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] z^{-n}$$

Upon interchanging the order of summation

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n}$$

Now, applying time shifting property

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k] z^{-k} X_2(Z) = X_1(z) X_2(z)$$

Computation of the convolution of two signals $x_1[n]$ & $x_2[n]$ using the z-transform:

(1) Find $X_1(z)$ & $X_2(z)$.
(2) Multiply $X_1(z)$ & $X_2(z)$.
(3) Find the inverse z-transform of multiplication.

This procedure is, in many cases, computationally easier than the direct evaluation of the convolution summation.

7. Correlation of two Sequences:

$$If,\ x_1[n] \xleftrightarrow{z} X_1(z)\ \ \&\ \ x_1[n] \xleftrightarrow{z} X_1(z)$$

$$then, r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \xleftrightarrow{z} R_{x_1x_2}(z) = X_1(z) X_2(z^{-1})$$

$$ROC{:}\ -ROC\ of\ X_1(z)\ \cap ROC\ of\ X_2(z^{-1})$$

Proof:

$$r_{x_1x_2}(l) = x_1[l] * x_2[-l]$$

Using convolution property

$$R_{x_1x_2}(z) = Z\{x_1(l)\}.Z\{x_2(l)\}$$

Using time-reversal property

$$R_{x_1x_2}(z) = X_1(z) X_2(z^{-1})$$

8. Conjugate of a Complex Sequence:

$$x^*[n] \xleftrightarrow{z} X^*(z^*)\quad ROC{:}\ ROC\ X(z)$$

9. Multiplication of two sequences:

$$If,\ \ x_1[n] \overset{z}{\longleftrightarrow} X_1(z)\ \ \&\ \ x_1[n] \overset{z}{\leftrightarrow} X_1(z)$$

$$then, x[n] = x_1[n]x_2[n] \overset{z}{\leftrightarrow} X(z) = \frac{1}{2\pi j}\oint_c X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$$

Where c is a closed contour that encloses the origin & lies within the ROC common to both $X_1(v)$ & $X_2(1/v)$. ROC: $r_{1l}r_{2l} < |z| < r_{1u}r_{2u}$

10. Parseval's Relation:

If $x_1[n]$ and $x_2[n]$ are complex-valued sequences, then

$$\sum_{n=-\infty}^{\infty} x_1[n]x_2^*[n] = \frac{1}{2\pi j}\oint_c X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

Provided that $r_{1l}r_{2l} < 1 < r_{1u}r_{2u}$, where $r_{1l} < |z| < r_{1u}$ and $r_{2l} < |z| < r_{2u}$ are the ROC of $X_1(z)$ and $X_2(z)$.

11. Initial Value Theorem:

If x[n] is causal [i.e. x[n] = 0 for n < 0], then

$$x[0] = \lim_{z \to \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} x[0] + x[1]z^{-1} + x[2]z^{-2} + --- = x[0]$$

12. Final Value Theorem:

If x[n] is causal, then

$$\lim_{n\to\infty} x[n] = \lim_{z\to\infty}(z-1)X(z)$$

Note: All the poles of X(z) should lie inside the unit circle.

Rational Z-Transforms:

Poles & Zeros:

The zeros of a z-transform X(z) are the values of z for which X(z) = 0. The poles of a z-transform are the values of z for which X(z) = ∞. If X(z) is a rational function then,

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1z^{-1} + \cdots + b_Mz^{-M}}{a_0 + a_1z^{-1} + \cdots + a_Nz^{-N}} = \frac{\sum_{k=0}^{M} b_kz^{-k}}{\sum_{k=0}^{N} a_kz^{-k}}$$

If $a_0 \neq 0$ and $b_0 \neq 0$, we can avoid the negative powers of z by factoring out the terms b_0z^{-M} & a_0z^{-N} as follows:

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 z^{-M}}{a_0 z^{-N}} \cdot \frac{z^M + \left(\frac{b_1}{b_0}\right) z^{M-1} + \cdots + b_M/b_0}{z^N + \left(\frac{a_1}{a_0}\right) z^{M-1} + \cdots + a_M/a_0}$$

Since N(z) & D(z) are polynomials in z, they can be expressed in factored form as:

$$X(z) = \frac{N(z)}{D(z)} z^{-M+N} \cdot \frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)} = Gz^{N-M} \cdot \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$$

Where, $G \equiv b_0/a_0$. Thus X(z) has

(a) M finite zeros at $z = z_1, z_2, z_3, \ldots, z_M$ (the roots of the numerator polynomials).

(b) M finite poles at $z = p_1, p_2, p_3, \ldots, p_N$ (the roots of the denominator polynomials).

(c) |N-M| zeros (if N > M) or poles (if N < M) at z = 0.

Note:

- A zero exists at $z = \infty$ if $X(\infty) = 0$ & a pole exists at $z = \infty$ if $X(\infty) = \infty$.
- If we count the poles and zeros at zero & infinity, we find that X(z) has exactly the same number of poles and zeros.
- For pole-zero plot we denote the location of poles by '×' (cross) sign and location of zeros by 'o' (circle) sign.

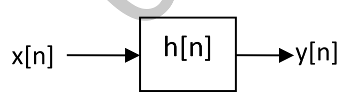
Obviously, by definition, the ROC of a z-transform should not contain any poles.

Pole location and time-domain behavior for causal signals

1. Causal real signals with simple real poles or simple complex-conjugate pairs of poles, which are inside or on the unit circle, are always bounded in amplitude.
2. A signal with a pole (or a complex-conjugate pair of poles) near the origin decays more rapidly than one associated with a pole near (but inside) the unit circle.
3. A double real pole on the unit circle results in an unbounded signal.
4. If a pole of a system is outside the unit circle, the impulse response of the system becomes unbounded and consequently, the system is unstable.

The System Function of an LTI System:

We know, for an LTI system



y[n] = x[n]*h[n]

Taking z-transform, Y(z) = X(z)H(z) => H(z) = Y(z)/X(z)

If we know the x[n] and we observe the output y[n] of the system, we can determine the unit sample response. It is clear that H(z) represents the z-domain characterization of a system, whereas h[n] is the corresponding time-domain characterization of the system. The transform function H(z) is called the system function.

When the system is described by an LCCD equation,

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

Taking z-transform

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Simplifying\ we\ get, \qquad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Hence, and LTI system described by a LCCD equation has a rational system function.

Case I: If $a_k = 0$ for $1 \le k \le N$

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

In this case, H(z) contains M zeros, whose values are determined by system parameters $\{b_k\}$ & an M^{th} order pole at z=0 (trivial poles). Hence the system is called all-zero system. Such a system has finite duration impulse response & is called FIR system.

Case II: If $b_k = 0$ for $1 \le k \le M$,

$$H(z) = \frac{b_0}{\sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=1}^{N} a_k z^{N-k}},\ a_0 \triangleq 1$$

In this case H(z) consists of N poles, whose values are determined by the system parameters $\{a_k\}$ & and N^{th} order zero at z = 0. The corresponding system is called all-pole system. Due to presence of poles, the impulse response of such a system is infinite in duration & hence it is an IIR system.

Transfer Function Representation:

$$H(z) = b_0 z^{N-M} \left(\prod_{l=1}^{N} (z - z_l) \right) / \left(\prod_{k=1}^{N} (z - p_k) \right)$$

If the ROC of H(z) includes a unit circle ($z = e^{j\omega}$), then we can evaluate H(z) on unit circle, resulting in a frequency response function or transfer function $H(e^{j\omega})$

$$H(e^{j\omega}) = b_0 e^{j(N-M)\omega} \left(\prod_{l=1}^{N} (e^{j\omega} - z_l) \right) / \left(\prod_{k=1}^{N} (e^{j\omega} - p_k) \right)$$

Here magnitude response function,

$$|H(e^{j\omega})| = |b_0| \times \frac{|e^{j\omega} - z_1||e^{j\omega} - z_2| \ldots . |e^{j\omega} - z_M|}{|e^{j\omega} - p_1||e^{j\omega} - p_2| \ldots . |e^{j\omega} - p_N|}$$

The phase response function,

$$Arg\left(H(e^{j\omega})\right) = \{0\ or\ \pi\} + (N-M)\omega + \sum_{k=1}^{M} Arg(e^{j\omega} - z_k) \ - \sum_{k=1}^{N} Arg(e^{j\omega} - p_k)$$

Inversion of Z-Transform:

1. **Direct evaluation by Contour integration (not in syllabus).**
2. **Expansion into a series of terms in the variables z and z^{-1}.**
3. **Partial fraction expansion and table lookup.**

Long-Division Method:

Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

When (a) ROC: |z| > 1. (b) ROC: |z| < 0.5.

Solution: (a) Since the ROC is the exterior of a circle, we expect x[n] to be a causal signal. Thus we seek a power series expansion in negative powers of z.

$$X(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}$	1 $1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}$	$1 - \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + ---$
	$\frac{3}{2}z^{-1} - \frac{1}{2}z^{-2}$ $\frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} + \frac{3}{4}z^{-3}$	

$$X(z) = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + ---$$

$$\therefore x[n] = \left\{\underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, ---\right\}.$$

(b) In this case ROC is the interior of a circle. The signal x[n] is anticausal. So, we seek a power expension in powers of z, so we perform the long division in the following way:

$$X(z) = 2z^2 + 6z^3 + 14z^4 + ---$$

$$\therefore x[n] = \left\{---, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\right\}$$

This method is used only if one wished to determine the values of the first few samples of the signal.

The Inverse Z-Transform by Partial Fraction Expansion:

In the table lookup method, we attempted to express the function X(z) as a linear combination.

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + --- + \alpha_k X_k$$

Then the inverse z-transform of X(z), can be found using the linearity property as,

$$x[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n] + --- + \alpha_k x_k[n]$$

This approach is useful if X(z) is a rational function,

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} --- (A)$$

$$= \frac{b_0 + b_1 z^{-1} + --- + b_M z^{-M}}{1 + a_1 z^{-1} + --- + a_N z^{-N}}, \quad a_0 \cong 1$$

Note: If $a_0 \neq 1$, divide N(z) & D(z) by a_0.

A rational function of the form (A) is called proper, if $a_N \neq 0$ and M < N (finite zeros is less than the number of finite poles).

An improper rational function (M ≥ N) can always be written as the sum of a polynomial & a proper rational function.

$$Let\ X(a) = \frac{b_0 + b_1 z^{-1} + \ ---\ + b_M z^{-M}}{1 + a_1 z^{-1} + \ ---\ + a_N z^{-N}}$$

Where $a_N \neq 0$ and M < N

Eliminating negative powers of z

$$X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \ ---\ + b_M z^{N-M}}{z_N + a_1 z^{N-1} + \ ---\ + a_N}$$

Since N > M, the function

$$\frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \ ---\ + b_M z^{N-M-1}}{z_N + a_1 z^{N-1} + \ ---\ + a_N}$$

Is also always proper.

Now the next step is to express X(z)/z as a sum of simple fractions.

Two cases:

(I) Distinct Poles:

Suppose that the poles p_1, p_2, ---, p_N are all different (distinct)

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ ---\ + \frac{A_N}{z-p_N}$$

The problem is to determine the coefficients A_1, A_2, ---, A_N.

$$A_k = \left.\frac{(z-p_k)X(z)}{z}\right|_{z\,=\,p_k},\quad k = 1,2,3,--,N$$

(II) Multiple order poles:

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{(z-p_2)^2}$$

$$A_1 = \left.\frac{(z-p_1)X(z)}{z}\right|_{z\,=\,p_1}$$

$$A_3 = \left.\frac{(z-p_2)^2 X(z)}{z}\right|_{z\,=\,p_2}$$

$$A_2 = \left.\frac{d}{dz}\frac{(z-p_2)^2 X(z)}{z}\right|_{z\,=\,p_2}$$

In terms of z^{-1}

$$X(z) = \frac{A_1}{1-p_1 z^{-1}} + \frac{A_2}{1-p_2 z^{-1}} + \frac{A_3}{(1-p_2 z^{-1})^2}$$

$$A_1 = (1-p_1 z^{-1})X(z)\Big|_{z\,=\,p_1}$$

$$A_3 = (1-p_2 z^{-1})^2 X(z)\Big|_{z\,=\,p_2}$$

$$A_2 = \frac{d}{dz}(1-p_2 z^{-1})^2 X(z)\Big|_{z\,=\,p_2}$$

Determine the z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

If

(a) ROC: |z| > 1

(b) ROC: |z| < 0.5

(c) ROC: 0.5 < |z| < 1.

Solution: -→

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{1}{1 - z^{-1} - 0.5z^{-1} + 0.5z^{-2}} = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$$
$$= \frac{A}{1 - z^{-1}} + \frac{B}{1 - 0.5z^{-1}}$$

Solving, we get A = 2 and B = -1.

$$Ans\ (a): x[n] = 2u[n] - 0.5^n u[n]$$
$$Ans\ (b): x[n] = -2u[-n-1] + 0.5^n u[-n-1]$$
$$Ans\ (c): x[n] = -2u[-n-1] - 0.5^n u[n]$$

Causality and Stability:

Causality:-

We know, a causal LTI system is one whose unit sample response satisfies the condition

h[n] = 0 n<0

We also know that the ROC of the z-transform of a causal sequence is the exterior of a circle. Consequently, a linear time invariant system is a causal if and only if the ROC of the system function is the exterior of a circle of radius r<∞, including the point z=∞.

Stability:-

A necessary and sufficient condition for a LTI system to be BIBO stable is,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

In turn, this condition implies that H(z) must contain the unit circle within its ROC.
Indeed since

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

It follows that,

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]z^{-n}|$$
$$\leq \sum_{n=-\infty}^{\infty} |h[n]||z^{-n}|$$

When evaluated on the unit circle i.e. |z| = 1

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h[n]|$$

Hence, if the system is BIBO stable, the unit circle is contained in the ROC of H(z). The converse is also true. Therefore, a linear time invariant system is BIBO stable if and only if the ROC of the system function includes the unit circle.
A causal linear time invariant system (ROC: |z|>r<1) is BIBO stable if and only if all the poles of H(z) are inside the unit circle.

 # Example

A LTI system is characterized by the system function,

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

Specify the ROC of H(z) and determine h[n] for the following conditions:

(a) The system is stable.
(b) The system is causal.
(c) The system is anticausal.

Solution →

The given system is

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} = \frac{1}{1 - {}^{1}\!/_{2}\, z^{-1}} + \frac{2}{1 - 3z^{-1}}$$

The system has poles at z = ½ and z = 3.

(a) Since the system is stable, its ROC must include the unit circle & hence it is ½<|z|<3. Consequently h[n] is non-causal and is given as,

$\quad h[n] = (½)^n u[n] – 2(3)^n u[-n-1].$

(b) Since the system is causal, its ROC is |z| > 3. In this case,

$\quad h[n] = (½)^n u[n] + 2(3)^n u[n].$

This system is unstable.

(c) If the system is anticausal, its ROC is |z| < 0.5 hence

$\quad h[n] = -[(½)^n + 2(3)^n]u[-n-1].$

In this case the system is unstable.

Partial Fraction Expansion for multiple order poles:

$$\frac{X(z)}{z} = \sum_{p} \frac{A_p}{Z - Z_p} + \sum_{q} \sum_{k=1}^{M_q} \frac{B_{qk}}{\left(z - z_q\right)^k}$$

Where z_p is the p^{th} single pole and z_q is the q^{th} multiple pole of the order M_q. The constants A_p and B_{qk} are given as follows:

$$A_p = \left(z - z_p\right) X(z) \Big|_{z\, =\, z_p}$$

$$B_{qk} = \frac{1}{\left(M_q - k\right)!} \cdot \frac{d^{M_q - k}}{dz^{M_q - k}} \left(z - z_q\right)^{M_q} X(z) \Big|_{z\, =\, z_q}$$

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7	$(\cos \omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
8	$(\sin \omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\|z\| > 1$
9	$(a^n \cos \omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$
10	$(a^n \sin \omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\|z\| > \|a\|$

Figure : Common Z-transform pairs.

Reference:

1. J. G. Proakis, D. G. Manolakis, “Digital Signal Processing, Principles, Algorithms and Applications”, 3rd Edition, Prentice-hall, 2000. Chapter 3.