Chapter 7: Design of IIR Digital Filters[i].

Characteristics of Commonly used Analog Filters

IIR digital filters can easily be obtained by beginning with an analog filter and then using a mapping to transform the s-plane to the z-plane. Thus the design of a digital filter is reduced to designing an appropriate analog filter and then performing the conversion from H(s) to H(z), in such a way so as to preserve as much as possible, the desired characteristics of the analog filter.

Analog filter is a well-developed field and many books have been written on the subject. Analog filters are good, robust and have been started since centuries.

We will be describing the importance characteristics of commonly used analog filter and introduce the relevant filter parameters. Our discussion is limited to lowpass filters.

SOME PRELIMINARIES

We discuss two preliminary issues in this section. First, we consider the magnitude squared response specifications, which are more typical of analog (and hence of IIR) filters. These specifications are given on the relative linear scale. Second, we study the properties of the magnitude-squared response.

Let $H_a(j\Omega)$ be the frequency response of an analog filter. Then the lowpass filter specifications on the magnitude-squared response are given by

$$\frac{1}{1+\varepsilon^2} \le |H_a(j\Omega)|^2 \le 1, |\Omega| \le \Omega_p$$
$$0 \le |H_a(j\Omega)|^2 \le \delta_2^2, |\Omega| \ge \Omega_s$$

Where ε is a passband ripple parameter, Ω_p is the passband cutoff frequency in rad/sec, δ_2 is a stopband attenuation parameter, and Ω_s is the stopband cutoff in rad/sec. These specifications are shown in figure 1, from which we observe that $|H_a(j\Omega)|^2$ must satisfy

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2} \; at \; \Omega = \Omega_p$$
$$|H_a(j\Omega)|^2 = \delta_2^2 \; at \; \Omega = \Omega_s$$

The manner in which specifications of a lowpass filter are given to the engineer is illustrated in figure 1(b).

The attenuation α, can be given as,

$\alpha(\Omega) = -20\log|H_a(j\Omega)|$ measured in dB which makes,
$|H_a(j\Omega)| = 10^{-\alpha(\Omega)/20}$

For the passband extending from $\Omega = 0$ to $\Omega = \Omega_p$ the attenuation should not be larger than α_{max}. from Ω_p to Ω_s we have a transition band. Then the specifications indicate that from Ω_s on for all higher frequencies the attenuation should not be less than α_{min}.

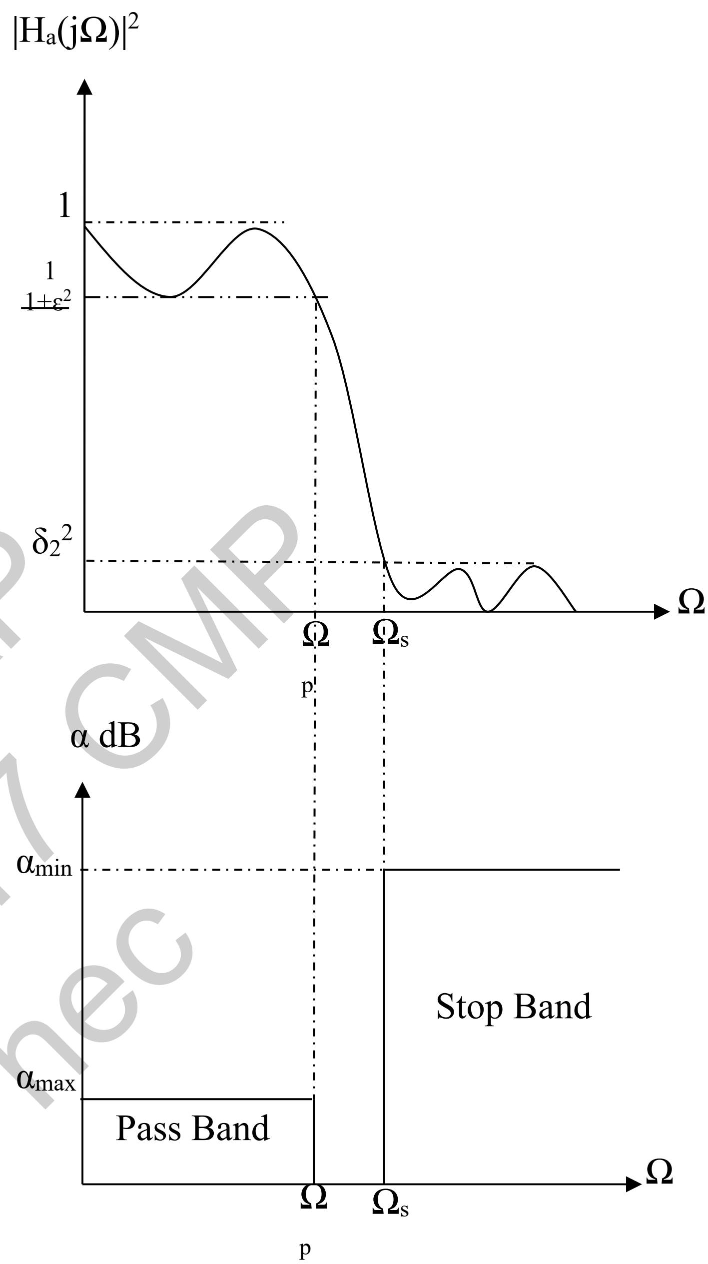


Figure (a) Magnitude squared response of analog filter.
(b) Attenuation curve

The parameters ε and δ_2 are related to parameters α_{max} and α_{min} respectively, of the dB scale. These relations are given by,

$$\alpha_{max} = -20\log(\sqrt{\frac{1}{1+\varepsilon^2}} => \varepsilon = \sqrt{10^{0.1\alpha_{max}} - 1}$$

$$\& \; \alpha_{min} = -20log\delta_2 => \delta_2 = 10^{-\frac{\alpha_{min}}{20}}$$

$$\delta = \sqrt{10^{0.1\alpha_{min}-1}}, where\ \delta^2 = \frac{1}{\delta_2^2} - 1$$

Butterworth lowpass filter (Maximally flat response)

This filter is characterized by the property that its magnitude response is flat in both passband and stopband. The magnitude-squared response for an N^{th} order lowpass filter is given by:

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} -- -(i)$$

Where N is the order of the filter and Ω_c is the cutoff frequency in rad/sec. The plot of the magnitude-squared response is shown below:

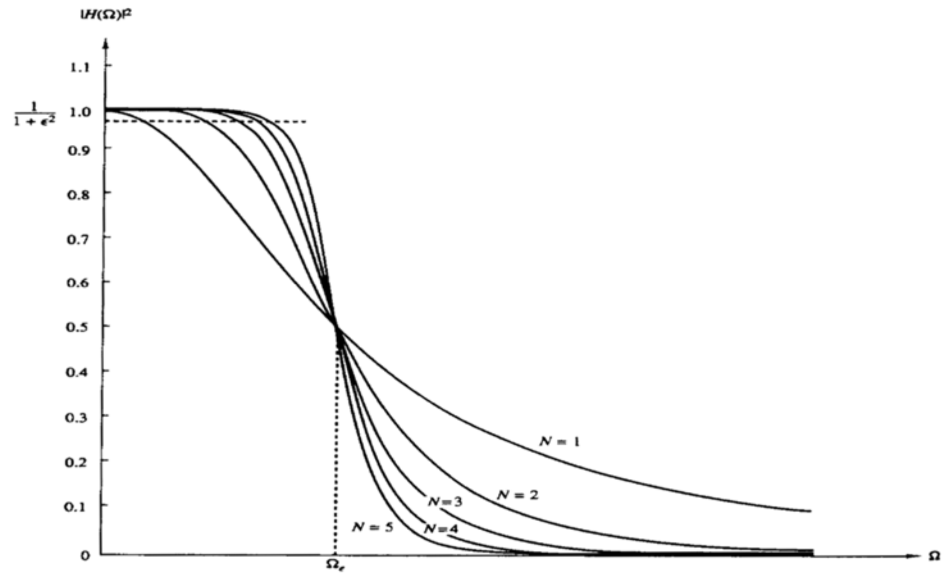


Fig: Frequency response of Butterworth filters.

From this plot we can observe the following properties:

- At Ω = 0, $|H_a(j0)|^2 = 1$ for all N.
- At Ω = Ω_c, $|H_a(j\Omega_c)|^2 = 0.5$ for all N, which implies a 3 dB attenuation at Ω_c.
- $|H_a(j\Omega)|^2$ is a monotonically decreasing function of Ω.
- $|H_a(j\Omega)|^2$ approaches an ideal lowpass filter as N tends to infinity(∞).
- $|H_a(j\Omega)|^2$ is maximally flat at Ω = 0 since derivatives of all orders exist and are equal to zero.

Now,

$$|H_a(j\Omega)|^2 = \frac{1}{1+\left(\Omega/\Omega_c\right)^{2N}} = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}$$

Where, Ω is the normalized frequency.

The order of the filter required to meet an attenuation δ_2 at a specified frequency Ω_s is easily determined from above equation. Thus at $\Omega = \Omega_s$ we have

$$\frac{1}{1+\varepsilon^2\left(\Omega_s/\Omega_p\right)^{2N}} = \delta_2^2$$

Simplifying above equation we get

$$N = \frac{log_{10}\left(\delta/\varepsilon\right)}{log_{10}\left(\Omega_s/\Omega_p\right)}$$

Where,

$$\varepsilon^2 = 10^{0.1\alpha_{max}} - 1 \; and \; \delta^2 = 10^{0.1\alpha_{min}} - 1 = \frac{1}{{\delta_2}^2} - 1$$

Thus the butterworth filter is completely characterized by the parameters N, δ_2, ε and the ratio Ω_s/Ω_p. The transfer function with Butterworth response is given by

$$H_a(s) = \frac{1}{B_N(s)}$$

Where,

$$B_N(s) = 1 \; or \; (s+1) \times \prod (s^2 + 2cos\psi_k s + 1)$$

Two simple rules permit us to determine ψ_k

1. a) If N is odd, then there is a pole at $\psi=0^0$;
   b) If N is even, then there are poles at $\psi=\pm 90^0/N$
2. Poles are separated by $\psi=180^0/N$

Example: for N=2, poles are at $\psi=\pm 90^0/N=\pm 45^0$

$$B_2(s) = (s^2 + 2cos45s + 1) = s^2 + 1.414s + 1$$

The table of the transfer function for different order of the Butterworth filter is given as:

Order of Filter N	**Transfer function H(s) = 1/A(s) where A(s)**
1	(s+1)
2	(s^2+1.414s+1)
3	(s^2+s+1)(s+1)
4	(s^2+0.766s+1)(s^2+1.848s+1)
5	(s^2+0.618s+1)(s^2+1.618s+1)(s+1)
6	(s^2+0.518s+1)(s^2+4.414s+1)(s^2+1.932s+1)
7	(s^2+1.802s+1)(s^2+1.247s+1)(s^2+0.445s+1)(s+1)

If -3 dB frequency, or cut-off or center frequency is given we use the following formula,

$$N = \frac{log_{10}\left({}^{1}/_{\delta_2^2} - 1\right)}{2log_{10}\left({}^{\Omega_s}/_{\Omega_c}\right)} = \frac{\log(10^{0.1\alpha_{min}} - 1)}{2log_{10}\left({}^{\Omega_s}/_{\Omega_c}\right)}$$

To find Ω_c from N, we have,

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \left({}^{\Omega}/_{\Omega_c}\right)^{2N}}$$

Simplifying above equation we get,

$$\frac{\Omega_s}{\Omega_c} = (10^{0.1\alpha_{min}} - 1)^{1/2N} = \delta^{1/N}$$

$$\therefore \Omega_c = \frac{\Omega_s}{\delta^{1/N}}$$

Matching the frequency response exactly at stopband.

If we were instead to match the frequency response at passband, we would obtain

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}}$$

In principle, any value of the critical frequency that satisfies

$$\frac{\Omega_p}{\varepsilon^{1/N}} \leq \Omega_c \leq \frac{\Omega_s}{\delta^{1/N}}$$

would be valid.

Chebyshev Lowpass filters (Equi-ripple Magnitude response)

There are two types of Chebyshev filters. The Chebyshev-I filters have equiripple response in the passband, while the Chebyshev-II filters have equiripple response in the stopband. Butterworth filters have monotone response in both bands. It is noted that by choosing a filter that has an equiripple rather than a monotonic behavior, we can obtain lower-order filter. Therefore Chebyshev filters provide lower order than Butterworth filters for the same specifications.

The magnitude-squared response of a Chebyshev-I filter is

$$|\mathrm{H_a}(\mathrm{j}\Omega)|^2 = \frac{1}{1 + \epsilon^2\ \mathrm{T}_\mathrm{N}^2\left(\Omega/\Omega_\mathrm{c}\right)}$$

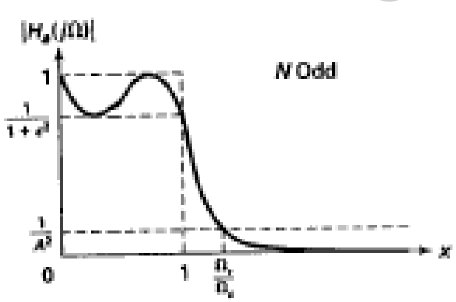
Where N is the order of the filter, ε is the passband ripple factor, and $T_N(x)$ is the N^{th} order Chebyshev polynomial given by

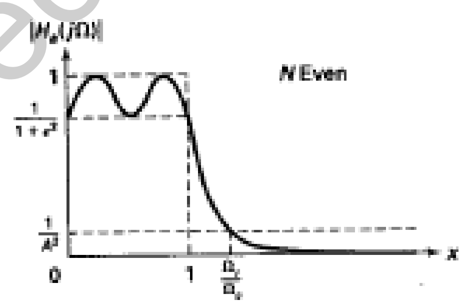
$$T_N(x) = \begin{cases} \cos\left(N\cos^{-1}(x)\right), & 0 \leq x \leq 1 \\ \cosh\left(\cosh^{-1}(\mathrm{x})\right), & 1 < x < \infty \end{cases}$$

Where $x = \Omega/\Omega_c$.

The equiripple response of the Chebyshev filters is due to this polynomial $T_N(x)$. Its key properties are (a) for $0 < x < 1$, $T_N(x)$ oscillates between -1 and 1, and (b) for $1 < x < \infty$, $T_N(x)$ increases monotonically to ∞.

There are two possible shapes of $|\mathrm{H_a}(\mathrm{j}\Omega)|^2$, one for N odd and one for N even as shown below. Note that $x = \Omega/\Omega_c$ is the normalized frequency.





From the above two response plots we observe the following properties:

- At x = 0 (or Ω = 0):

$$|\mathrm{H_a}(\mathrm{j}0)|^2 = 1, \quad \text{for N odd.}$$

$$|\mathrm{H_a}(\mathrm{j}0)|^2 = \frac{1}{1+\epsilon^2} \quad for\ N\ even.$$

- At x = 1 (or $\Omega = \Omega_c$):

$$|\mathrm{H_a}(\mathrm{j}1)|^2 = \frac{1}{1+\epsilon^2} \quad for\ all\ N.$$

- For $0 \le x \le 1$ (or $0 \le \Omega \le \Omega_c$), $|\mathrm{H_a}(\mathrm{j}\Omega)|^2$ oscillates between 1 and $\frac{1}{1+\epsilon^2}$.
- For $x > 1$ (or $\Omega > \Omega_c$), $|\mathrm{H_a}(\mathrm{j}\Omega)|^2$ decreases monotonically to 0.

At $x=\Omega_r$, $|H_a(jx)|^2 = \delta_2^{2}$.
Design Equations:
Given Ω_p, Ω_s, α_{max}, α_{min}; three parameters are required to determine a chebyshev-filter:
ε, Ω_c & N

$$N = \frac{cosh^{-1}\left(\delta/_{\varepsilon}\right)}{cosh^{-1}\left(\Omega_s/_{\Omega_p}\right)}$$

Elliptic filters:

Elliptic (or Cauer) filters exhibit equiripple behavior in both the passband and the stopband, as illustrated in figure below for N odd and N even. This class of filters contains both poles and zeros and is characterized by the magnitude squared frequency response

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N\left(\frac{\Omega}{\Omega_p}\right)}$$

Design of IIR Filters:

$$\mathrm{H(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

IIR filters can be obtained from analog filters by using mapping function. They have infinite impulse response and contain poles, so they are unstable.

Firstly Analog filters is designed from the desired specifications then digital filters is obtained by mapping s-domain to z-domain.

H(s) ---(mapping)→ H(z)

Analog filter design is a mature and well developed field, so it is not surprising that we begin the design of a digital filter in the analog domain and then convert the design into the digital domain.

An analog filter can be described by its system function,

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k\, s^k}{\sum_{k=0}^{N} \alpha_k\, s^k}$$

Where $\{\alpha_k\}$ and $\{\beta_k\}$ are the filter coefficients, or by its impulse response, which is related to $H_a(s)$ by the Laplace transform

$$H_a(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

Alternatively, the analog filter having the rational system function H(s) can be described by the linear constant-coefficient differential equation

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

Where x(t) denotes the input signal and y(t) denotes the output of the filter.

Each of these three equivalent characterizations of an analog filter leads to alternative methods for converting the filter into the digital domain. We recall that an analog linear time-invariant system with system function H(s) is stable if all its poles lie in the left half of the s-plane. Consequently, if the conversion technique is to be effective, it should possess the following desrirable properties:

1. The jΩ axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains.
2. The left-half plane (LHP) of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

IIR Filter Design:

H(s) –(inverse Laplace transform)→ **h_a(t)** –(sampling)→ **h[n]** –(Z-transform)→ **H(z)**

Techniques:

1. By approximation of derivative method (**not needed in our course)**
2. By Impulse invariance method. (**imp.**)
3. By Bilinear transformation method. (**imp.**)
4. By Matched z-Transform method (**only for short notes**)

Impulse Invariance Method:

In the impulse invariance method, the objective is to design an IIR filter having a unit sample response h[n] that is the sampled version of the impulse response of the analog filter. That is

$$h[n] \equiv h(nT) \;\; n = 0,1,2,3,\ldots$$

where T is the sampling interval.

We know when a continuous time signal $x_a(t)$ with spectrum $X_a(F)$ is sampled at a rate F_s = 1/T samples per second, the spectrum of the sampled signal is the periodic repetition of the scaled spectrum $F_sX_a(F)$ with period F_s. Specifically, the relation is

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a[(f-k)F_s] \;\; ---(1)$$

Where f = F/F_s is the normalized frequency. Aliasing occurs if the sampling rate F_s is less than twice the higest frequency contained in $X_a(F)$.

Expressed in the context of sampling the impulse response of an analog filter with frequency response $H_a(F)$, the digital filter with unit sample response h(n) = $h_a(nT)$ has the frequency response

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k)F_s]$$

Or, equivalentely,

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s] \, ---(2)$$

Or,

$$H(\Omega T) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

Mapping relates the z-transform of h[n] to the Laplace transform of $h_a(t)$ by (generalization of above equation):

$$H(z)|_{z=e^{sT}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi k}{T}\right) --- (3)$$

Where

$$H(z) = \sum_{n=0}^{\infty} h[n]\, z^{-n} --- (4)$$

$$H(z)|_{z=e^{sT}} = \sum_{n=0}^{\infty} h[n]\, e^{-sTn} --- (5)$$

Note that when s = jΩ, eq^n(3) reduces to (2), where the factor of j in $H_a(\Omega)$ is suppressed in our notation.

The Mapping Relation is given by:

$$z = e^{sT}$$

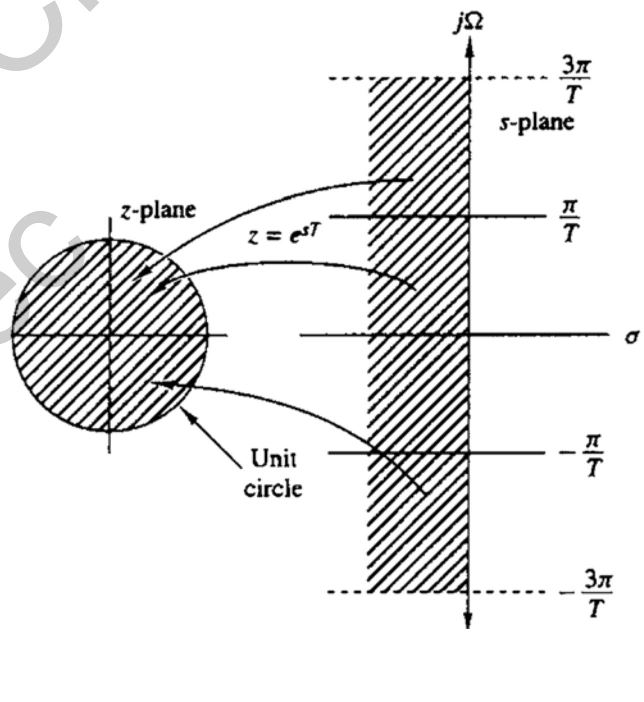
If we substitute s = σ + jΩ and express the complex variable z in polar form as $z = r\,e^{j\omega}$, then

$$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T}$$

Clearly, we must have

$$r = e^{\sigma T} \quad and \quad \omega = \Omega T$$

Consequently, σ < 0 implies that 0 < r < 1 and σ > 0 implies that r > 1. When σ = 0, we have r = 1. Therefore, the left half in s-plane is mapped inside the unit circle in z-plane and the right half in s-plane is mapped outside the unit circle in z.



Since

$$\omega = \Omega T, \qquad \frac{(2k-1)\pi}{T} \le \Omega \le \frac{(2k+1)\pi}{T}$$

$$maps\ into\ the\ interval\ \ -\pi \le \omega \le \pi, where\ k\ is\ an\ integer.$$

Thus the mapping from the analog frequency Ω to the frequency variable ω in the digital domain in many-to-one, which simply reflects the effects of aliasing due to sampling. The Impulse invariance method is in-appropriate for designing highpass filters due to the spectrum aliasing that results from the sampling process.

To explore further the effect of the impulse invariance design method on the characteristics of the resulting function of the analog filter in partial-fraction form. On the assumption that the poles of the analog filter are distinct, we can write

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

Where $\{p_k\}$ are the poles of the analog filter and $\{c_k\}$ are the coefficients in the partial-fraction expansion. Consequently,

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t}, \quad t \geq 0$$

If we sample $h_a(t)$ periodically at t = nT, we have

$$h[n] = h_a(nT) = \sum_{k=1}^{N} c_k e^{p_k T n}$$

Now, with the above substitution, the system function of the resulting digital IIR filter becomes

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{k=1}^{N} c_k e^{p_k T n} \right) z^{-n} = \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n$$

The inner summation in above equation converges because $p_k<0$ and yields

$$\sum_{n=0}^{\infty} (e^{p_k T} z^{-1})^n = \frac{1}{1 - e^{p_k T} z^{-1}}$$

Therefore, the system function of the digital filter is

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

We observe that the digital filter has poles at $z_k = e^{p_k T}$, k=1, 2, … , N.
Comparing the analog filter H(s) and H(z), we can observe that the mapping relation is given by

Mapping

Analog Domain H(s)	Digital Domain H(z)
$\frac{1}{s-p_k}$	$\frac{1}{1-e^{p_kT}z^{-1}}$
$\frac{s+a}{(s+a)^2+b^2}$	$\frac{1-e^{-aT}cos(bT)z^{-1}}{1-2e^{-aT}cos(bT)z^{-1}+e^{-2aT}z^{-2}}$
$\frac{b}{(s+a)^2+b^2}$	$\frac{e^{-aT}sin(bT)z^{-1}}{1-2e^{-aT}cos(bT)z^{-1}+e^{-2aT}z^{-2}}$

Example: Convert the analog filter with system function

$$\boldsymbol{H(s) = \frac{s+0.1}{s^2+0.2s+9.01}}$$

Into a digital IIR filter by means of the impulse invariance method.
Solution: Given

$$H(s) = \frac{s+0.1}{s^2+0.2s+9.01} = \frac{s+0.1}{(s+0.1)^2+3^2}$$

We note that the analog filter has a zero at s = -0.1 and a pair of complex conjugate poles at

$$p_{1,2} = -0.1 \pm j3$$

Now, by method of partial fraction expansion, we obtain

$$H(s) = \frac{s+0.1}{(s+0.1)^2+3^2} = \frac{A}{s+0.1+j3} + \frac{A^*}{s+0.1-j3}$$

$$A = \left[\frac{s+0.1}{s+0.1-j3}\right]_{s=-0.1-j3} = \frac{1}{2} = A^*$$

$$\therefore H(s) = \frac{\frac{1}{2}}{s+0.1+j3} + \frac{\frac{1}{2}}{s+0.1-j3}$$

Mapping from analog to Digital using impulse invariance method, we get

$$H(z) = \frac{\frac{1}{2}}{1-e^{-0.1T}e^{j3T}z^{-1}} + \frac{\frac{1}{2}}{1-e^{-0.1T}e^{-j3T}z^{-1}}$$

IIR Filter Design by the Bilinear Transformation:

Drawbacks of Impulse Invariance methods are:

- Many to one mapping
- Due to which aliasing effect occurs
- Appropriate for lowpass and limited class of bandpass filters

The bilinear transformation is a conformal mapping that transforms the jΩ-axis into the unit circle in the Z-plane only once, thus avoiding aliasing of frequency components.
Let us consider an analog filter with system function,

$$H(s) = \frac{b}{s+a} --- \quad (i)$$

This system is also characterized by the differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t) ----(ii)$$

We integrate the derivative and approximate the integral by trapezoidal formula,

$$y(t) = \int_{t_0}^{t} y'(\tau)d\tau + y(t_0) ----(iii)$$

Where, y'(t) denotes the derivative of y(t)
The approximation of the integral by the trapezoidal formula at t = nT and t_0 = nT-T yields,

$$y(nT) = \frac{T}{2}[y'(nT) + y'(nT - T)] + y(nT - T) --- (iv)$$

But from (ii)

$$y'(nT) = -ay(nT) + bx(nT) ----(v)$$

Now substituting y'(nT) in equation (iv)
y[n] = T/2{-ay[n] + bx[n] + (-ay[n-1] + bx[n-1])} + y[n-1]
where x(nT) = x[n] and y(nT) = y[n]. Simplifying above equation:

$$\left(1 + \frac{aT}{2}\right)y[n] - \left(1 - \frac{aT}{2}\right)y[n-1] = \frac{bT}{2}\big[x[n] + x[n-1]\big]$$

Taking z-transform

$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}(1 + z^{-1})X(z)$$

Consequently, the equivalent digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\frac{bT}{2}\right)(1 + z^{-1})}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right)z^{-1}}$$

Or equivalently,

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + a} ----(VI)$$

Clearly, the mapping from the s-plane to the Z-plane is

$$s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

This is called **Bilinear Transformation**.

To investigate the characteristics of the bilinear transformation, let

$$z = re^{j\omega} \qquad and \qquad s = \sigma + j\Omega$$

Then bilinear mapping relation can be expressed as

$$s = \frac{2}{T}\frac{z - 1}{z + 1} = \frac{2}{T}\frac{re^{j\omega} - 1}{re^{j\omega} - 1} = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

Consequently,

$$\sigma = \frac{2}{T}\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} \quad and \quad \Omega = \frac{2}{\mathrm{T}}\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}$$

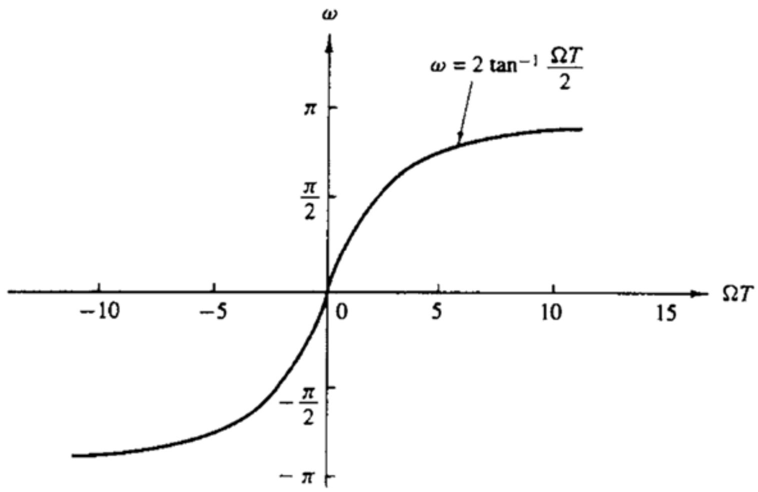


Fig. 1 Mapping between the frequency variables ω and Ω resulting from the bilinear transformation

First, we note that if r < 1, then σ < 0, and if r < 1, then σ < 0. Consequently, the left half of s-plane maps into the inside of the unit circle in the z-plane and the right half of s-plane maps outside of the unit circle. When r = 1, then σ = 0 and

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega} = \frac{2}{T}\tan\frac{\omega}{2}$$

$$or\ equivalently, \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

The relationship in above equation between the frequency variables in the two domains is illustrated in Fig. 5. We observe that the entire range in **Ω** is mapped only once in the range –π ≤ ω ≤ π. However, the mapping is highly nonlinear. We observe a frequency compression or frequency warping, as it is usually called, due to the nonlinearity of the arctangent function.

Frequency Warping:

For an analog filter, the frequency of the filter and the sampling frequency can help find the value of ω. ω = 2πF/Fs = ΩT. Here we see that there is a linear relation between **Ω** and ω. But the relationship we determined through the Bilinear transformation in above equation is non-linear (Fig. 5).

This change in properties when using Bilinear Transformation is referred to as Frequency warping. The non-linear relationship between **Ω** and ω results in a distortion of the frequency axis, as seen in Fig. 5.

Pre-warping:

We can remove the warping problem using a simple technique called pre-warping.

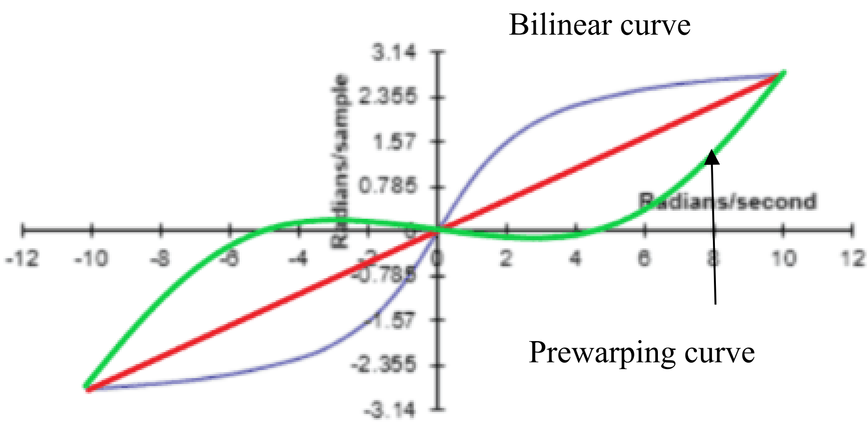


Fig. 2 : Frequency response with pre-warping.

In fig above, to obtain the expected response, which is straight line, we are going to merge the bilinear curve and prewarping curve so that it cancels out and gives us the straight line.

Example of Bilinear Transformation

Design a single-pole lowpass digital filter with a 3 dB bandwidth of 0.2π, using the bilinear transformation applied to the analog filter

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Where Ω_c is the 3 dB bandwidth of the analog filter.
Solution: The digital filter is specified to have its -3 dB gain at $\omega_c = 0.2\pi$. In the frequency domain of the analog filter $\omega_c = 0.2\pi$ corresponds to

$$\Omega_c = \frac{2}{T}\tan 0.1\pi = \frac{0.65}{T}$$

Thus the analog filter has the system function

$$H(s) = \frac{0.65/T}{s + 0.65/T}$$

This represents our filter design in the analog domain.
Now, we apply the bilinear transformation method to convert the analog filter into the desired digital filter. Thus we obtain

$$H(z) = \frac{0.65/T}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + 0.65/T} = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

Where the parameter T has been divided out.

$$The\ frequency\ response\ of\ the\ digital\ filter\ is\ H(\omega) = \frac{0.245\left(1 + e^{-j\omega}\right)}{1 - 0.509e^{-j\omega}}$$

At $\omega = 0$, $H(0) = 1$ and at $\omega = 0.2\pi$, we have $|H(0.2\pi)| = 0.707$, which is the desired response.

$$\# \boldsymbol{Given\ H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 16}\ and\ \omega_r = \frac{\pi}{2}. Find\ H(z).}$$

Solution: Comparing the given H(s) with the Laplace Transform equation

$$H(s) = \frac{s + a}{(s + a)^2 + \Omega_c^2}$$

We can deduce that Ω_c = 4 rad/sec. The relation between Ω and ω for bilinear transformation is

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Substituting values of Ω_s and ω_r in above equation we get T = 0.5 sec.
Hence

$$H(z) = H(s)\big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

After substituting, we get

$$H(z) = \frac{0.128z^2 + 0.006z - 0.122}{z^2 + 0.0006z + 0.975}$$

The Matched-z Transformation

Another method for converting an analog filter into an equivalent digital filter is to map the poles and zeros of H(s) directly into poles and zeros in the z-plane. Suppose that the system function of the analog filter is expressed in the factored form

$$H(s) = \frac{\prod(s - z_k)}{\prod(s - p_k)}$$

Where $\{z_k\}$ are the zeros and {pk} are the poles of the filter. Then the system function for the digital filter is

$$H(z) = \frac{\sum(1 - e^{z_k T} z^{-1})}{\sum(1 - e^{p_k T} z^{-1})}$$

Where T is the sampling interval. Thus each factor of the form (s-a) in H(s) is mapped into the factor $(1- e^{aT}z^{-1})$. This mapping is called the matched-z transformation.
We observe that the poles obtained from the matched-z transformation are identical to the poles obtained with the impulse invariance method. However, the two techniques result in different zero positions.
To preserve the frequency response characteristics of the analog filter, the sampling interval in the matched-z transformation must be properly selected to yield the pole and zero locations at the equivalent position in the z-plane. Thus aliasing must be avoided by selecting T sufficiently small.

Frequency Transformation

In the Analog Domain

Suppose we have a lowpass filter with passband edge frequency Ω_p and we wish to convert it to another filter with passband edge frequency Ω_p',

Type of transformation	Transformation	Band edge frequencies of new filter
Lowpass	$s \rightarrow \frac{\Omega_p}{\Omega_p'} s$	Ω_p'
Highpass	$s \rightarrow \frac{\Omega_p \Omega_p'}{s}$	Ω_p'

Bandpass	$s \to \Omega_p \dfrac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$	Ω_l, Ω_u
Bandstop	$s \to \Omega_p \dfrac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$	Ω_l, Ω_u

In the Digital Domain

Type of transformation	Transformation	Parameters
Lowpass	$z^{-1} \longrightarrow \dfrac{z^{-1} - a}{1 - az^{-1}}$	ω'_p = band edge frequency of new filter $a = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$
Highpass	$z^{-1} \longrightarrow -\dfrac{z^{-1} + a}{1 + az^{-1}}$	ω'_p = band edge frequency new filter $a = -\dfrac{\cos[(\omega_p + \omega'_p)/2]}{\cos[(\omega_p - \omega'_p)/2]}$
Bandpass	$z^{-1} \longrightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = -2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \cot \dfrac{\omega_u - \omega_l}{2} \tan \dfrac{\omega_p}{2}$
Bandstop	$z^{-1} \longrightarrow -\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$	ω_l = lower band edge frequency ω_u = upper band edge frequency $a_1 = -2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$ $K = \tan \dfrac{\omega_u - \omega_l}{2} \tan \dfrac{\omega_p}{2}$

References:

1. J. G. Proakis, D. G. Manolakis, "Digital Signal Processing, Principles, Algorithms and Applications", 3rd Edition, Prentice-hall, 2000.
2. S. Sharma, "Digital Signal Processing", Third Revised Edition, S.K. Kataria & Sons, 2007.

[i] Compiled By: Rupesh Dahi Shrestha, Assistant Professor, Dept. of Electrical and Electronics, nec, 2011