

Chapter: Electromagnetic wave

Maxwell's Equations in integral form:

1. Gauss's law in electrostatics, $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$. **Its significance:** Existence of monopole of electric charge. Electric charge distribution induces electric field.
2. Gauss's law in magnetism, $\oint \vec{B} \cdot d\vec{A} = 0$. **Its significance:** non-existence of monopole of magnet.
3. Faraday's law of induction, $\oint \vec{E} \cdot d\vec{l} = \frac{-d\phi_B}{dt}$. **Its significance:** Change in magnetic field induces electric field.
4. Ampere – Maxwell law, $\oint \vec{B} \cdot d\vec{l} = \mu_0 [I + I_d] = \mu_0 \left[I + \epsilon_0 \frac{d\phi_E}{dt} \right]$. **Its significance:** Magnetic field is induced either by steady current or by displacement current (Rate of Change in electric field).

Some mathematical and physical concepts:

- Volume charge density, $\rho = \frac{dq}{dv} \Rightarrow dq = \rho dv \Rightarrow \text{total charge, } q = \oint \rho dv$
- Current density, $\vec{J} = \frac{dI}{dA} \Rightarrow dI = \vec{J} \cdot d\vec{A} \Rightarrow \text{total current, } I = \oint \vec{J} \cdot d\vec{A}$
- Electric flux, $\phi_E = \oint \vec{E} \cdot d\vec{A}$
- Magnetic flux, $\phi_B = \oint \vec{B} \cdot d\vec{A}$
- Del operator, $\nabla = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$, $\nabla \cdot \nabla = \nabla^2$
- Divergence of a vector: $\nabla \cdot \vec{E}$
- Curl of a vector: $\nabla \times \vec{E}$
- Gauss's divergence theorem: Surface integral of a vector = volume integral of divergence of that vector. i.e. $\oint \vec{E} \cdot d\vec{A} = \oint (\nabla \cdot \vec{E}) dv$
- Stoke's theorem: Line integral of a vector = Surface integral of curl of that vector. i.e. $\oint \vec{E} \cdot d\vec{l} = \oint (\nabla \times \vec{E}) \cdot d\vec{A}$
- Vector product of three vectors, $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\vec{c}$
i.e. $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla)\vec{E}$

Maxwell's Equations in differential form:

1. First equation, $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

We have, Gauss's law in electrostatics: $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$

Using Gauss's divergence theorem, $\oint \vec{E} \cdot d\vec{A} = \oint (\nabla \cdot \vec{E}) dv$ in L.H.S

And Charge, $q = \oint \rho dv$ in RHS. Where, ρ is volume charge density.

We get, $\oint (\nabla \cdot \vec{E}) dv = \oint \frac{\rho}{\epsilon_0} dv$

Comparing the terms inside the integration we see, $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$. **This is first equation.**

2. Second equation, $\nabla \cdot \vec{B} = 0$:

We have, Gauss's law in magnetism: $\oint \vec{B} \cdot \vec{dA} = 0$

Using Gauss's divergence theorem, $\oint \vec{B} \cdot \vec{dA} = \oint (\nabla \cdot \vec{B}) dv$ in L.H.S

We get, $\oint (\nabla \cdot \vec{B}) dv = 0$

i.e. $\nabla \cdot \vec{B} = 0$. **This is second equation.**

3. Third equation, $\nabla \times \vec{E} = \frac{-d\vec{B}}{dt}$

We have, Faraday's law of induction, $\oint \vec{E} \cdot \vec{dl} = \frac{-d\phi_B}{dt}$

Using Stoke's theorem, $\oint \vec{E} \cdot \vec{dl} = \oint (\nabla \times \vec{E}) \cdot \vec{dA}$

Also, Magnetic flux, $\phi_B = \oint \vec{B} \cdot \vec{dA}$

We get, $\oint (\nabla \times \vec{E}) \cdot \vec{dA} = \frac{-d}{dt} \oint \vec{B} \cdot \vec{dA}$

$\Rightarrow \oint (\nabla \times \vec{E}) \cdot \vec{dA} = \oint \frac{-d\vec{B}}{dt} \cdot \vec{dA} \Rightarrow \nabla \times \vec{E} = \frac{-d\vec{B}}{dt}$ **This is third equation.**

4. Fourth equation, $\nabla \times \vec{B} = \mu_0 \left[\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt} \right]$

We have, Ampere - Maxwell's equation, $\oint \vec{B} \cdot \vec{dl} = \mu_0 \left[I + \epsilon_0 \frac{d\phi_E}{dt} \right]$

Where, I is steady current and ϕ_E is electric flux.

Using Stoke's theorem, $\oint \vec{B} \cdot \vec{dl} = \oint (\nabla \times \vec{B}) \cdot \vec{dA}$

Also, current, $I = \oint \vec{J} \cdot \vec{dA}$, with \vec{J} being current density.

And Electric flux, $\phi_E = \oint \vec{E} \cdot \vec{dA}$

Then, Ampere- Maxwell's equation become,

$$\oint (\nabla \times \vec{B}) \cdot \vec{dA} = \mu_0 \left[\oint \vec{J} \cdot \vec{dA} + \epsilon_0 \frac{d}{dt} \oint \vec{E} \cdot \vec{dA} \right]$$

$$\Rightarrow \oint (\nabla \times \vec{B}) \cdot \vec{dA} = \oint \left[\mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \right] \cdot \vec{dA}$$

$$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \text{ .Required fourth equation.}$$

Maxwell's Equations for Electromagnetism:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = \frac{-d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

EM wave in free space:

In free space, current density, $J = 0$, and charge density, $\rho = 0$

Then, Maxwell's equations become

$$\nabla \cdot \vec{E} = 0 \dots (i)$$

$$\nabla \cdot \vec{B} = 0 \dots (ii)$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \dots (iii)$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \dots (iv)$$

Taking Curl of equation (iii),

$$\begin{aligned}\nabla \times (\nabla \times \vec{E}) &= -\nabla \times \frac{d\vec{B}}{dt} \\ \Rightarrow \nabla \times (\nabla \times \vec{E}) &= -\frac{d}{dt}(\nabla \times \vec{B})\end{aligned}$$

$$\text{We have, } \nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla)\vec{E}$$

$$\text{And from (iv), } \nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

$$\text{Then we get, } \nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla)\vec{E} = -\frac{d}{dt}\left[\mu_0 \epsilon_0 \frac{d\vec{E}}{dt}\right]$$

$$\text{From (i), } \nabla \cdot \vec{E} = 0 \text{ and } \nabla \cdot \nabla = \nabla^2$$

$$\Rightarrow -\nabla^2 \vec{E} = -\mu_0 \epsilon_0 \frac{d^2 \vec{E}}{dt^2}$$

$$\Rightarrow \nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{d^2 \vec{E}}{dt^2} \dots (v)$$

Similarly, taking curl of equation (iv) and proceeding as above, we get

$$\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{d^2 \vec{B}}{dt^2} \dots (vi)$$

Equations (v) and (vi) are the equations of EM wave in free space.

The solutions of these equations are, $\vec{E} = E_0 \sin(kx - \omega t) \dots (vii)$

$$\text{And, } \vec{B} = B_0 \sin(kx - \omega t) \dots (viii)$$

Comparing (v) and (vi) with general equation of wave, $\nabla^2 y = \frac{1}{v^2} \frac{d^2 y}{dt^2}$, v is speed of the wave.

We get, $\frac{1}{v^2} = \mu_0 \epsilon_0 \Rightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

$\Rightarrow v = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3 \times 10^8 \text{ m/s}$. Which shows the speed of EM wave in free space is speed of light (c).

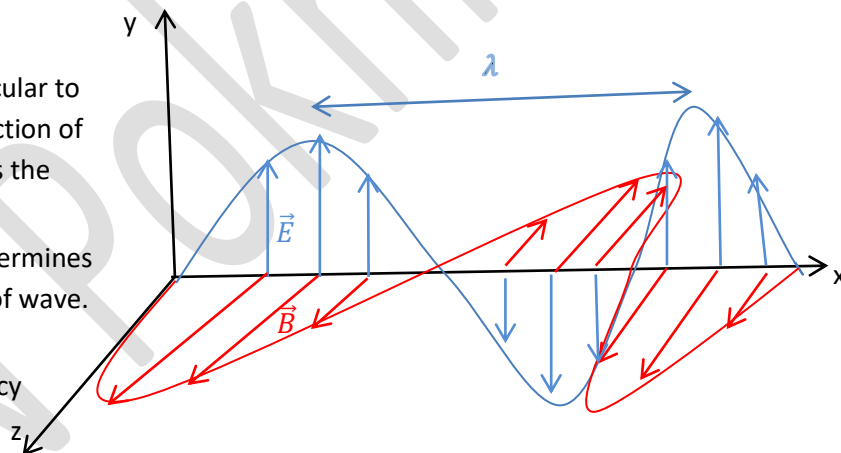
Electromagnetic wave:

Consider a magnetic field with sinusoidal variation. It induces a perpendicular electric field (as due to Faraday's law of induction) that have a sinusoidal variation too. Again, this electric field induces perpendicular magnetic field (due to Maxwell's law of induction) and so on.

In this way, the two fields continuously induce each other and the resulting fields propagate as a wave. This wave is called EM wave.

Features of EM wave:

- \vec{E} and \vec{B} are always perpendicular to each other and also with direction of propagation of wave. It shows the transverse nature of wave.
- The cross product ($\vec{E} \times \vec{B}$) determines the direction of propagation of wave.
- The oscillation of \vec{E} and \vec{B} are Sinusoidal with same frequency and in phase.



i. e. wave equations are: $\vec{E} = E_0 \sin(kx - \omega t)$

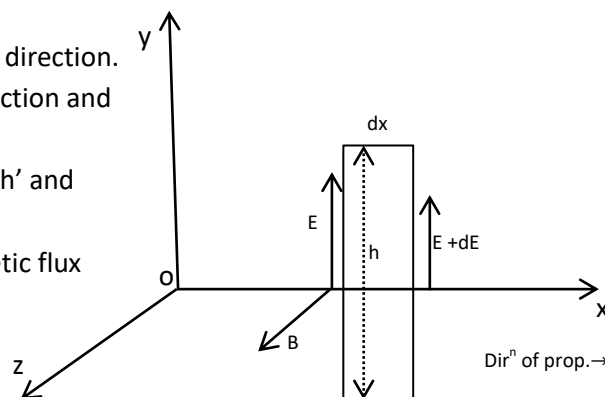
And, $\vec{B} = B_0 \sin(kx - \omega t)$

Here, E_0 and B_0 are amplitudes.

Speed of EM wave (in terms of amplitudes of fields, E_0 and B_0):

Suppose an EM wave propagating along x – direction. Let its electric field oscillating along Y – direction and magnetic field along z – direction. Consider, an imaginary rectangle of height ' h ' and width ' dx '.

During propagation of the wave, the magnetic flux through the rectangle changes, and electric field is induced along y – direction.



By Faraday's law of induction, $\oint \vec{E} \cdot d\vec{l} = \frac{-d\phi_B}{dt}$... (i)

Magnetic flux, $\phi_B = \oint \vec{B} \cdot d\vec{A} = B(hdx) \Rightarrow \frac{d\phi_B}{dt} = hdx \frac{dB}{dt}$

Line integral, $\oint \vec{E} \cdot d\vec{l} = (E + dE)h - Eh = h dE$

Equation (i) then becomes, $hdE = -hdx \frac{dB}{dt}$

$$\Rightarrow \frac{dE}{dx} = -\frac{dB}{dt} \dots (ii)$$

Similarly, when we consider same rectangle with length along z – direction, and applying

Maxwell's law of induction, $\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\phi_E}{dt}$ we get,

$$\frac{dE}{dt} = -\frac{1}{\mu_0 \epsilon_0} \frac{dB}{dx} \dots (iii)$$

Since, $E = E_0 \sin(kx - \omega t)$

$$\frac{dE}{dt} = -\omega E_0 \cos(kx - \omega t)$$

$$\frac{dE}{dx} = kE_0 \cos(kx - \omega t)$$

And, $B = B_0 \sin(kx - \omega t)$ Differentiating w.r.t 'x' and 't',

$$\frac{dB}{dt} = -\omega B_0 \cos(kx - \omega t)$$

$$\frac{dB}{dx} = kB_0 \cos(kx - \omega t)$$

Using respective values in equation (ii), we get,

$$kE_0 \cos(kx - \omega t) = -[-\omega B_0 \cos(kx - \omega t)]$$

$$\Rightarrow kE_0 = \omega B_0$$

$$\Rightarrow \frac{E_0}{B_0} = \frac{\omega}{k} \dots (iv)$$

Since, Angular frequency, $\omega = 2\pi f$ and, wave vector, $k = 2\pi/\lambda$

$$\frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = f\lambda = v \quad (\text{where, } v \text{ is speed of wave})$$

So, equation (iv) gives, speed of EM wave, $v = \frac{E_0}{B_0} (= \frac{E}{B})$

Similarly, from equation (iii), using the values of $\frac{dE}{dt}$ and $\frac{dB}{dx}$ and solving we get,

$$v^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\Rightarrow v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} = 3 \times 10^8 \text{ m/s (speed of light, } c)$$

Energy transport and Pointing vector:

The magnitude of Pointing vector (S) is defined as the rate of energy transport per unit area in a plane EM wave.

$$\text{i.e. } S = \frac{1}{A} \frac{dU}{dt} \dots (i)$$

Consider propagation of an EM wave in a box of area 'A' and thickness 'dx'. At any instant, energy stored in the box is given by,

$$dU = dU_E + dU_B = \mu_E A dx + \mu_B A dx = (\mu_E + \mu_B) A dx$$

Where, energy density of electric field, $\mu_E = \frac{1}{2} \epsilon_0 E^2$

And, energy density of magnetic field, $\mu_B = \frac{B^2}{2\mu_0}$

$$\Rightarrow dU = \left(\frac{1}{2} \epsilon_0 E^2 + \frac{B^2}{2\mu_0} \right) A dx$$

Using $E/B = c$ where c is speed of light

i.e. $B = E/c$

$$\text{then, } dU = \left(\frac{1}{2} \epsilon_0 E^2 + \frac{E^2}{2\mu_0 c^2} \right) A dx$$

$$\text{Also, } c^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\Rightarrow dU = \left(\frac{1}{2} \epsilon_0 E^2 + \frac{\mu_0 \epsilon_0 E^2}{2\mu_0} \right) A dx = \left(\frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \epsilon_0 E^2 \right) A dx = \epsilon_0 E^2 A dx$$

$$\text{Equation (i), } S = \frac{1}{A} \frac{dU}{dt} = \epsilon_0 E^2 \frac{dx}{dt}$$

$$\text{Since, } \frac{dx}{dt} = c \Rightarrow S = \epsilon_0 E^2 c \dots\dots (ii)$$

$$\Rightarrow S = c \epsilon_0 E \cdot E = c \epsilon_0 E \cdot B \cdot c = c^2 \epsilon_0 EB = \frac{1}{\mu_0 \epsilon_0} \epsilon_0 EB = \frac{E \cdot B}{\mu_0} \dots (iii)$$

$$\text{In vector form, } S = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

$$\text{Equation (iii), } S = \frac{E \cdot B}{\mu_0} = \frac{E_0 B_0}{\mu_0} \sin^2(kx - \omega t)$$

Intensity: Intensity of EM wave is defined as the average of Poynting vector.

$$\text{i.e. Intensity, } I = \frac{E_0 B_0}{\mu_0} [\sin^2(kx - \omega t)]_{av.} = \frac{E_0 B_0}{\mu_0} \frac{1}{2} = \frac{E_0 B_0}{2\mu_0}$$

Radiation pressure:

The force per unit area on an object due to EM wave incident on the object is called radiation pressure.

Show that energy density of Electric field and magnetic field are same for EM wave.

Let a beam of EM radiation having intensity 'I' is incident on an object of area 'A', perpendicular to the path. Suppose, the entire radiation is absorbed by the object. In time interval Δt , the energy intercepted by area 'A' is, $\Delta U = IA\Delta t$, here, I is intensity of radiation

The change in momentum, $\Delta P = \frac{\Delta U}{c} = \frac{IA\Delta t}{c}$ (compare as $\Delta U = mc^2$ and $\Delta P = mc$)

From Newton's law, Force, $F = \frac{\Delta P}{\Delta t} = \frac{IA}{c}$

Therefore, radiation pressure, $P_r = F/A = I/c$

Instead of being absorbed, if the radiation is entirely reflected back, $\Delta P = \frac{2\Delta U}{c}$

Which gives, $P_r = 2I/c$

If the radiation is partly absorbed and partly reflected, the radiation pressure lies between I/c and $2I/c$.

**The charge conservation theorem:
(Continuity equation)**

Show that $\nabla \cdot \vec{J} + \frac{d\rho}{dt} = 0$. Where the symbols have usual meaning.

We have Maxwell's fourth equation, $\nabla \times \vec{B} = \mu_0(\vec{J} + \epsilon_0 \frac{d\vec{E}}{dt})$

Taking Divergence on both sides, $\nabla \cdot (\nabla \times \vec{B}) = \mu_0(\nabla \cdot \vec{J} + \epsilon_0 \frac{d(\nabla \cdot \vec{E})}{dt})$

Since, Divergence of curl of any vector is zero, $\nabla \cdot (\nabla \times \vec{B}) = 0$

Also, $\nabla \cdot \vec{E} = \rho/\epsilon_0$

So, $\nabla \cdot \vec{J} + \epsilon_0 \frac{d(\rho/\epsilon_0)}{dt} = 0$

$\Rightarrow \nabla \cdot \vec{J} + \epsilon_0 \frac{d(\rho)}{\epsilon_0 dt} = 0$

$\Rightarrow \nabla \cdot \vec{J} + \frac{d(\rho)}{dt} = 0 \Rightarrow$ Required Continuity equation.