Matrix Operations

Practice Examples

Dot Product

Definition: Let x = [x_1,x_2,.......x_n] and y = y_1,y_2,.......y_n] be two vectors in R^n. The dot product of x and y is given by

$$x \bullet y = x_1y_1 + x_2y_2 + ... + x_ny_n = \sum_{k=1}^{n} x_k y_k$$

For example, if x = [2, 4, 3] and y = [1, 5, -2], then x · y = (2)(1) + (-4)(5) + (3)(-2) = -24. Notice that the dot product involves two vectors and the result is a scalar, whereas scalar multiplication involves a scalar and a vector and the result is a vector. Also, the dot product is not defined for vectors having different numbers of coordinates. The next theorem states some elementary results involving the dot product.

Orthogonal vector:

Definition: two vectors x and y in R^n are orthogonal (perpendicular) if and only if x .y = 0

The vectors x = [2,5] and y = [10,4] are orthogonal in R^2 because x **.** y = 0

Solving system of linear equations

Example: Consider the linear system

$$\begin{cases} 4w - 2x + y - 3z = 5 \\ 3w + x \quad\quad + 5z = 12 \end{cases}.$$

Letting

$$\mathbf{A} = \begin{bmatrix} 4 & -2 & 1 & -3 \\ 3 & 1 & 0 & 5 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 5 \\ 12 \end{bmatrix},$$

we see that the system is equivalent to **AX = B**, or,

$$\begin{bmatrix} 4 & -2 & 1 & -3 \\ 3 & 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4w - 2x + y - 3z \\ 3w + x + 5z \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}.$$

This system can also be represented by the augmented matrix

$$[\mathbf{A}|\mathbf{B}] = \left[\begin{array}{cccc|c} 4 & -2 & 1 & -3 & 5 \\ 3 & 1 & 0 & 5 & 12 \end{array}\right].$$

Gaussian Elimination

Many methods are available for finding the complete solution set for a given linear system. The first one we present, Gaussian elimination, involves systematically replacing most of the coefficients in the system with simpler numbers (1's and 0's) to make the solution apparent. In Gaussian elimination,we begin with the augmented matrix for the given system, and then examine each column in turn from left to right. In each column, if possible we choose a special entry, called a pivot entry, convert that pivot entry to "1" and then perform further operations to zero out the entries below the pivot. The pivots will be "staggered" so that as we proceed from column to column, each new pivot occurs in a lower row.

Let us solve the following system of linear equations:

$$\begin{cases} 5x - 5y - 15z = 40 \\ 4x - 2y - 6z = 19 \\ 3x - 6y - 17z = 41 \end{cases}.$$

The augmented matrix associated with this system is

$$\left[\begin{array}{ccc|c} 5 & -5 & -15 & 40 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right].$$

We now perform row operations on this matrix to give it a simpler form, proceeding through the columns from left to right. Starting with the first column, we choose the (1, 1) position as our first pivot entry. We want to place a 1 in this position. The row containing the current pivot is often referred to as the pivot row, and so row 1 is currently our pivot row. Now, when placing 1 in the matrix, we generally use a type (I) operation to multiply the pivot row by the reciprocal of the pivot entry. In this case, we multiply each entry of the first row by 1/5 .

type (I) operation: (1) ← 1/ 5 (1)

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 4 & -2 & -6 & 19 \\ 3 & -6 & -17 & 41 \end{array}\right].$$

For reference, we circle all pivot entries as we proceed. Next we want to convert all entries below this pivot to 0. We will refer to this as "targeting" these entries. As each entry is changed to 0, it is called the target, and its row is called the target row. To change a target entry to 0, we always use the following type (II) row operation:

(II): target row ← (-target value) X (pivot row) + target row\

For example, to zero out (target) the (2, 1) entry, we use the type (II) operation (2) ← (-4) X (1) + (2) . (That is, we add (-4) times the pivot row to the target row.) To perform this operation, we first do the following side calculation:

$(-4) \times (\text{row}1)$	-4	4	12	-32
$(\text{row}2)$	4	-2	-6	19
(sum)	0	2	6	-13

The resulting sum is now substituted in place of the old row 2, producing type (II) operation:
(2) ← (- 4) X (1) + (2)

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 0 & 2 & 6 & -13 \\ 3 & -6 & -17 & 41 \end{array}\right].$$

Note that even though we multiplied row1 by 4 in the side calculation, row 1 itself was not changed in the matrix. Only row 2, the target row, was altered by this type (II) row operation. Similarly, to target the (1, 3) position (that is, convert the (1, 3) entry to 0), row 3 becomes the target row, and we use another type (II) row operation. We replace row 3 with (- 3) X (row 1) + (row 3). This gives type (II) operation: (3) ← (- 3) X (1) + (3)

type(II) operation: (3) ← (- 3) X (1) + (3)

Side Calculation

$(-3) \times (\text{row}1)$	-3	3	9	-24
$(\text{row}3)$	3	-6	-17	41
(sum)	0	-3	-8	17

Resulting Matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 0 & 2 & 6 & -13 \\ 0 & -3 & -8 & 17 \end{array}\right]$$

Now, the last matrix is associated with the linear system

$$\begin{cases} x - y - 3z = 8 \\ 2y + 6z = -13. \\ -3y - 8z = 17 \end{cases}$$

Note that x has been eliminated from the second and third equations, which makes this system simpler than the original. However, as we will prove later, this new system has the same solution set.

Our work on the first column is finished, and we proceed to the second column. The pivot entry for this column must be in a lower row than the previous pivot, so we choose the (2, 2) position as our next pivot entry. Thus, row 2 is now the pivot row. We first perform a type (I) operation on the pivot row to convert the pivot entry to 1. Multiplying each entry of row 2 by 1/2 (the reciprocal of the pivot entry), we obtain type (I) operation: (2) ← ½ (2)

Resulting matrix =

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 0 & 1 & 3 & -\frac{13}{2} \\ 0 & -3 & -8 & 17 \end{array}\right].$$

We now use a type (II) operation to target the (3, 2) entry. The target row is now row 3.

type (II) operation: (3) ← 3 X (2) + (3)

Side Calculation

$$\begin{array}{rrrr|r} (3)\times(\text{row2}) & 0 & 3 & 9 & -\frac{39}{2} \\ (\text{row3}) & 0 & -3 & -8 & 17 \\ \hline (\text{sum}) & 0 & 0 & 1 & -\frac{5}{2} \end{array}$$

Resulting Matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -3 & 8 \\ 0 & 1 & 3 & -\frac{13}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{array}\right]$$

The last matrix corresponds to

$$\begin{cases} x - y - 3z = 8 \\ \quad\;\; y + 3z = -\frac{13}{2} \\ \qquad\quad\;\; z = -\frac{5}{2} \end{cases}.$$

Notice that y has been eliminated from the third equation. Again, this new system has exactly the same solution set as the original system. At this point, we know from the third equation that z = -5/2 .
Substituting this result into the second equation and solving for y, we obtain y + 3 (-5/2)= -13/2 , and hence, y = 1. Finally, substituting these values for y and z into the first equation, we obtain x - 1 - 3 (-5/2) = 8, and hence x = 3/2 . This process of working backward through the set of equations to solve for each variable in turn is called **back substitution**.

Thus, the final system has a unique solution — the ordered triple (3/2 , 1,-5/2) . However, you can check by substitution that (3/2 , 1,-5/2) is also a solution to the original system. In fact, Gaussian elimination always produces the complete solution set, and so (3/2 , 1,-5/2) is the unique solution to the original linear system.

Introduction to Gauss-Jordan Row Reduction

In the Gaussian elimination method, we created the augmented matrix for a given linear system and systematically proceeded through the columns from left to right, creating pivots and targeting (zeroing out) entries below the pivots. Although we occasionally skipped over a column,we placed pivots into successive rows, and so the overall effect was to create a staircase pattern of pivots, as in such matrices are said to be in row echelon form. However, we can extend the Gaussian elimination method further to target (zero out) the entries above each pivot as well, as we proceed from column to column. This extension is called the GaussJordan row reduction method, sometimes simply referred

Example:
We will solve the following system of equations using the Gauss-Jordan method:

$$\begin{cases} 2x_1 + \ x_2 + \ 3x_3 \qquad = 16 \\ 3x_1 + 2x_2 \qquad + \ x_4 = 16 \\ 2x_1 \qquad + 12x_3 - 5x_4 = \ 5 \end{cases}.$$

This system has the corresponding augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 1 & 3 & 0 & 16 \\ 3 & 2 & 0 & 1 & 16 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right].$$

As in Gaussian elimination, we begin with the first column and set row 1 as the pivot row. The following operation places 1 in the (1, 1) pivot position:

Row Operation	Resulting Matrix
(I): $\langle 1\rangle \leftarrow \frac{1}{2}\langle 1\rangle$	$\left[\begin{array}{cccc\|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 & 8 \\ 3 & 2 & 0 & 1 & 16 \\ 2 & 0 & 12 & -5 & 5 \end{array}\right].$

The next operations target (zero out) the entries below the (1, 1) pivot.

Row Operations	Resulting Matrix
(II): $\langle 2\rangle \leftarrow (-3)\langle 1\rangle + \langle 2\rangle$ (II): $\langle 3\rangle \leftarrow (-2)\langle 1\rangle + \langle 3\rangle$	$\left[\begin{array}{cccc\|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 & 8 \\ 0 & \frac{1}{2} & -\frac{9}{2} & 1 & -8 \\ 0 & -1 & 9 & -5 & -11 \end{array}\right]$

Proceeding to the second column, we set row 2 as the pivot row. The following operation places a 1 in the (2, 2) pivot position.

Row Operation	Resulting Matrix
(I): $\langle 2\rangle \leftarrow 2\langle 2\rangle$	$\left[\begin{array}{cccc\|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 & 8 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & -1 & 9 & -5 & -11 \end{array}\right]$

The next operations target the entries above and below the (2, 2) pivot.

Row Operations	Resulting Matrix
(II): $\langle 1\rangle \leftarrow -\frac{1}{2}\langle 2\rangle + \langle 1\rangle$ (II): $\langle 3\rangle \leftarrow 1\langle 2\rangle + \langle 3\rangle$	$\left[\begin{array}{cccc\|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & -3 & -27 \end{array}\right]$

We cannot place a nonzero pivot in the third column, so we proceed to the fourth column and set row 3 as the pivot row. The following operation places 1 in the (3, 4) pivot position.

Row Operation	Resulting Matrix
(I): $\langle 3\rangle \leftarrow -\frac{1}{3}\langle 3\rangle$	$\left[\begin{array}{cccc\|c} 1 & 0 & 6 & -1 & 16 \\ 0 & 1 & -9 & 2 & -16 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$

The next operations target the entries above the (3, 4) pivot.

Row Operations	Resulting Matrix
(II): $\langle 1\rangle \leftarrow 1\langle 3\rangle + \langle 1\rangle$ (II): $\langle 2\rangle \leftarrow -2\langle 3\rangle + \langle 2\rangle$	$\left[\begin{array}{cccc\|c} 1 & 0 & 6 & 0 & 25 \\ 0 & 1 & -9 & 0 & -34 \\ 0 & 0 & 0 & 1 & 9 \end{array}\right]$

Since we have reached the augmentation bar, we stop. (Notice the staircase pattern of pivots in the final augmented matrix.) The corresponding system for this final matrix is

$$\begin{cases} x_1 \quad + 6x_3 \quad = 25 \\ \quad x_2 - 9x_3 \quad = -34 \\ \qquad\qquad x_4 = 9 \end{cases}.$$

The third equation gives $x_4 = 9$. Since the third column is not a pivot column, the independent variable x_3 can take on any real value, say c. The other variables x_1 and x_2 are now determined to be $x_1 = 25$ - 6c and $x_2 = -34$ + 9c.

Then the complete solution set is {(25 - 6c, 9c - 34, c, 9) | c ∈ R }

Linear dependence and linear independence

Definition: Let S $\{v_1,\ldots,v_n\}$ be a finite nonempty subset of a vector space V. Then S is **linearly dependent** if and only if there exist real numbers $a_1,\ldots,a_n$, not all zero, such that $a_1 v_1 + \cdots + a_n v_n = 0$.That is,S is linearly dependent if and only if the zero vector can be expressed as a nontrivial linear combination of the vectors in S.

S is **linearly independent** if and only if it is not linearly dependent.
The empty set,{ }, is linearly independent.

Let S_1= {[3,1, 4]}. Since S_1 contains a single vector and this vector is nonzero, S_1 is a linearly independent subset of R^3. On the other hand, S_2= {[0, 0, 0, 0]} is a linearly dependent subset of R^4.

Method to Test for Linear Independence using Row Reduction (Independence Test Method)

Let S be a finite nonempty set of vectors in R^n. To determine whether S is linearly independent, perform the following steps:

Step 1: Create the matrix A whose columns are the vectors in S.
Step 2: Find B, the reduced row echelon form of A.
Step 3: If there is a pivot in every column of B, then S is linearly independent. Otherwise, S is linearly dependent

Example:
Consider the following subset of M_{22}:

$$S = \left\{ \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 6 & -1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} -11 & 3 \\ -2 & 2 \end{bmatrix} \right\}.$$

We determine whether S is linearly independent using the Independence Test Method. First, we represent the 2x2 matrices in S as 4-vectors. Placing them in a matrix, using each 4-vector as a column, we get.

$$\begin{bmatrix} 2 & -1 & 6 & -11 \\ 3 & 0 & -1 & 3 \\ -1 & 1 & 3 & -2 \\ 4 & 1 & 2 & 2 \end{bmatrix}, \text{ which reduces to } \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is no pivot in column 4. Hence, S is linearly dependent.

Rank of a matrix

You can think of an r x c matrix as a set of r row vectors, each having c elements; or you can think of it as a set of c column vectors, each having r elements.

The rank of a matrix is defined as

a) the maximum number of linearly independent column vectors in the matrix or

b) the maximum number of linearly independent row vectors in the matrix.

Both definitions are equivalent.

For an r x c matrix,

- If *r* is less than *c*, then the maximum rank of the matrix is *r*.
- If *r* is greater than *c*, then the maximum rank of the matrix is *c*.

The rank of a matrix would be zero only if the matrix had no elements. If a matrix had even one element, its minimum rank would be one.

How to find the Matrix Rank

In this section, we describe a method for finding the rank of any matrix. This method assumes familiarity with echelon matrices and echelon transformations.

The maximum number of linearly independent vectors in a matrix is equal to the number of non-zero rows in its row echelon matrix. Therefore, to find the rank of a matrix, we simply transform the matrix to its row echelon form and count the number of non-zero rows.

Consider matrix **A** and its row echelon matrix, **Aref**.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A} \qquad\qquad \mathbf{A_{ref}}$$

Because the row echelon form **Aref** has two non-zero rows, we know that matrix **A** has two independent row vectors; and we know that the rank of matrix **A** is 2.

You can verify that this is correct. Row 1 and Row 2 of matrix **A** are linearly independent. However, Row 3 is a linear combination of Rows 1 and 2. Specifically, Row 3 = 3*(Row 1) + 2*(Row 2). Therefore, matrix **A** has only two independent row vectors.

Reference: **Elementary Linear Algebra,** Textbook by David Hecker and Stephen Andrilli