INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

MECHANICAL ENGINEERING

DIFFERENT FORMULATIONS OF MECHANICS

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Different Formulatuions of Mechanics

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Contents

1 Gauss' Principle of Least Constraint

Gauss' principle of least constraint says that "A contact group's acceleration is closest possible acceleration to its unconstrained acceleration." It defines 'G' as:

$$G = \frac{\sum_{k=1}^{N} m_k \left| \frac{d^2 \boldsymbol{r}_k}{dt^2} - \frac{\boldsymbol{F}_k}{m_k} \right|^2}{2} \tag{1}$$

$$= \frac{(\boldsymbol{a} - \boldsymbol{a_u})^T \boldsymbol{M} (\boldsymbol{a} - \boldsymbol{a_u})}{2} \tag{2}$$

$$= \frac{||\boldsymbol{a} - \boldsymbol{a_u}||^2 M}{2} \tag{3}$$

(: $\boldsymbol{M}$ is symmetric)

Now 'G' must be minimum.

: The above expression takes the form similar to Kinetic energy. Therefore we say that the kinetic distance between unconstrained a_u and actual acceleration (a) must be minimum.

The types of constraints which might be applied to a system fall naturally into two types, holonomic and nonholonomic. A holonomic constraint is one which can be integrated out of the equations of motion. For instance, if a certain generalized coordinate is fixed, its conjugate momentum is zero for all time, so we can simply consider the problem in the reduced set of unconstrained variables. We need not be conscious of the fact that a force of constraint is acting upon the system to fix the coordinate and the momentum. An analysis of the two dimensional motion of an ice skater need not refer to the fact that the gravitational force is exactly resisted by the stress on the ice surface fixing the vertical coordinate and velocity of the ice skater. We can ignore these degrees of freedom. Nonholonomic constraints usually involve velocities. These constraints are not integrable. In general a nonholonomic constraint will do work on a system. Thermodynamic constraints are invariably nonholonomic.

Now, Gauss' principle; similar to Udwadia Kalaba equation(discussed later) solves the constraint in the form:

$$A(q,t)q^{\cdot\cdot} = b(q,t)$$

[holonomic and most of non holonomic constraint can be brought to this format;discussed in UW equation.]

We can find a general constraint in the form:

$$g(q, \dot{q}, t) = 0$$

Both holonomic and non holonomic constraint can be written in the above form as:

$$n(q, \ddot{q}, t)\ddot{q} = s(q, \ddot{q}, t)$$

We refer to this equation as the differential constraint equation and it plays a fundamental role in Gauss' Principle of Least Constraint. It is the equation for a plane which we refer to as the constraint plane. "n" is the vector normal to the constraint plane.

The constraint function tells us that the only accelerations which do continuously satisfy the constraint, are those which terminate on the constraint plane. To obtain the constrained acceleration we must project the unconstrained acceleration back into the constraint plane. Gauss' principle states that the trajectories actually followed are those which deviate as little as possible, in a least squares sense, from the unconstrained Newtonian trajectories. The projection which the system actually follows is the one which minimizes the magnitude of the Jacobi frame constraint force. This means that the force of constraint must be parallel to the normal of the constraint surface.

Thus, Gauss' principle takes the form:

$$m\ddot{q_i} = F_i - \lambda n$$

and therefore we solve for

$$\lambda$$

which is known as Gaussian multiplier as :

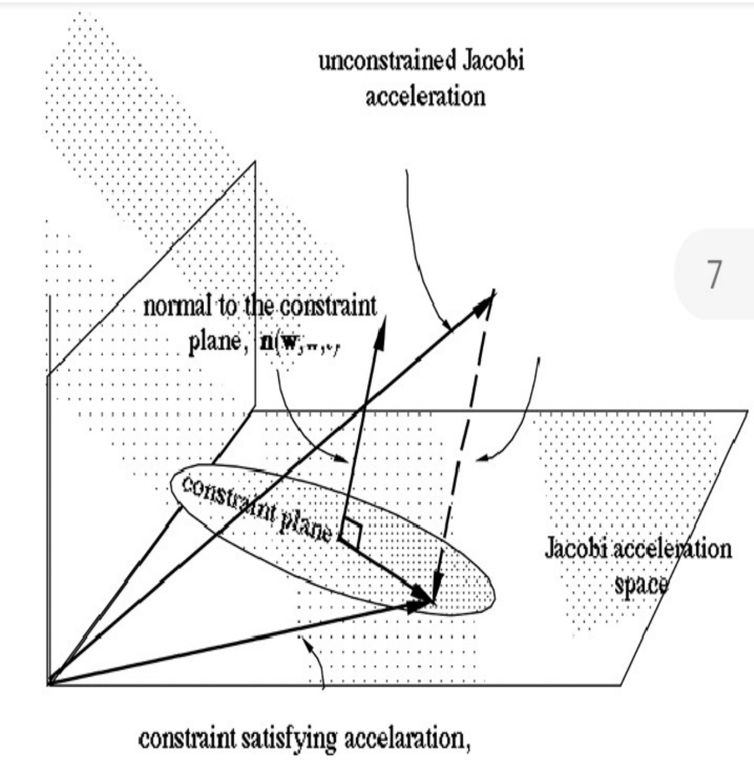
$$\lambda = (n.F_i - ms)/(n.n)$$

The square of the curvature G is a function of accelerations only (the Cartesian coordinates and velocities are considered to be given parameters). Gauss' principle reduces to finding the minimum of G , subject to the constraint. The constraint function must also be written as a function of accelerations, but this is easily achieved by differentiating with respect to time. If C is the acceleration dependent form of the constraint, then the constrained equations of motion are obtained from

$$\frac{\partial(G - \lambda C)}{\partial r^{\cdot\cdot}} = 0$$

Graphical Interpretation

We refer to above equation as the differential constraint equation and it plays a fundamental role in Gauss' Principle of Least Constraint. It is the equation for a plane which we refer to as the constraint plane.



The differential constraint equation places a condition on the acceleration vector for the system. The differential constraint equation says that the constrained acceleration vector must terminate on a hyper-plane in the 3N-dimensional Jacobi acceleration space.

The constraint function tells us that the only accelerations which do continuously satisfy the constraint, are those which terminate on the constraint plane. To obtain the constrained acceleration we must project the unconstrained acceleration back into the constraint plane.

Conclusion

This method can be applied to even non conservative system. Using constraint we have to solve and minimize for 'G'.

If one applies Gauss' principle to the problem of maintaining a constant heat flow, then a comparison with linear response theory shows that the Gaussian equations of motion cannot be used to calculate thermal conductivity (Hoover 1986). The correct application of Gauss' principle is limited to arbitrary holonomic constraints and apparently, to nonholonomic constraint functions which are homogeneous functions of the momenta.

Moreover, Gauss' principle can be applied to those constraint forces which work done is 0.

2 Principle of Least Action

The generalized coordinates are selected after applying constraints and cyclic coordinates are eliminated as they are of no further use.

Principle of Least Potential Energy Potential Energy of a particle tends to remain its most minimum value.(Ex- Concept of 'Virtual Work' in statics)

Principle of Least Kinetic Energy

$$K_{avg} = \frac{\int_{entirepath} K dt}{t_{total}}$$

The above expression must be minimum if potential energy does not changes. It can be seen by plotting simple motion (Euler's Analogy).

Principle of Least Action

It can be consider similar to Fermat's Principle of least time for ray optics. Now, let us define

$$S = (K - V)_{avg} t_{avg} \tag{4}$$

We will see that dS/dt = 0.

$$\delta \int (K - V) dt = 0 \tag{5}$$

it is so; cause for an object within time t1 and t2; it has to take a specific path when its Lagrangian must be minimum. It gives the idea about the specific path.

therefore

$$\delta \int L dt = 0 \tag{6}$$

Similarity with Lagrangian Equation

Now, Principle of least action is very similar to Lagrangian equation as Lagrangian equation can be derived from it.

$$\delta S = 0$$

$\delta \int_{t_1}^{t_2} L dt = 0$

$\int_{t_1}^{t_2} (\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial q_{\dot{i}}} \delta q_{\dot{i}}) dt = 0$

$\int_{t_1}^{t_2} (\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial q_{\dot{i}}} \frac{\delta(q_i)}{dt}) dt = 0$

Applying By Parts;

$\int_{t_1}^{t_2} (\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \frac{\partial L}{\partial q_{\dot{i}}} \delta q_i) dt = 0$

$\partial L / \partial q_i - d \partial L / dt \partial q_{\dot{i}} = 0 (7)$

The above equation derived is Euler-Lagrange's Equation. Now, PRINCIPLE OF LEAST ACTION or Lagrangian Equation can also derive Hamilton's Equation which we will see later on.

Conclusion

Now, all the above equation are restricted to the conservation of Energy.

3 Hertz' Principle of Least Curvature

This is just a special case of Gauss' Principle; restricted by two conditions: i) No externally applied Force; no interactions. ii) All masses are equal. Therefore ,here 'G', which should be minimize turns out to be:

$$G = m \sum_{k=1}^{N} |\frac{d^2 \boldsymbol{r_k}}{dt^2}|^2 \tag{8}$$

now,in this situation Kinetic Energy must also be conserved

$$KineticEnergy = T = \frac{\sum_{k=1}^{N} |\frac{dr_k}{dt}|^2}{2}$$

also, line element:

ds$^2 = \sum_{k=1}^{N} |dr_k|^2$

Therefore, m(ds/dt)$^2 = 2T$

K= G/2T= $\sum_{k=1}^{N} |\frac{d^2 r_k}{ds^2}|^2(9)$

now; K must be minimum Since; square root of K; is local curvature of trajectory in 3N dimension space; minimization of K is equivalent to finding trajectory of least curvature(geodesic) that is consistent with constraints.

4 Hamilton Equations

Lagrange's Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = 0 \tag{10}$$

We define Canonical Momentum as:

p$_i = \frac{\partial L(q_j, q_j^{\cdot}, t}{\partial q_i^{\cdot}}(11)$

We get:

$$dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial q_i^{\cdot}} dq_i^{\cdot} + \frac{\partial L}{\partial t} dt$$

p$_i = \frac{\partial L}{\partial q_i}$

p$_i^{\cdot} = \frac{\partial L}{\partial q_i}$

dL= p$_i^{\cdot} dq_i + p_i dq_i^{\cdot} + \frac{\partial L}{\partial t} dt(12)$Let us define :

$$H(q, p, t) = q_i^{\cdot} p_i - L(q, q^{\cdot}, t)$$

dH = q$_i^{\cdot} dp_i - p_i^{\cdot} dq_i - \frac{\partial L}{\partial t}(13)$

And thus we get 2n + 1 equations:(2n: Canonical equations are represented by first two equations.)

$\dot{q_i} = \frac{\partial H}{\partial p_i}$

$-\dot{p_i} = \frac{\partial H}{\partial q_i}$

$\partial H/\partial t = -\partial L/\partial t$
(14)

Procedure for constructing H

Steps:

1. L is constructed with generalized coordinates.

2. Conjugate Momenta are defined .

3. H is formulated.

4.

$$\dot{q_i} \tag{15}$$

is expressed as function of (q,p,t).

5. Eliminate

$$\dot{q} \tag{16}$$

from H to express it as a function of (q,p,t). For this, brains should be applied

Elimination of generalized velocity in general large problems:

In some general large classes problems; L is defined as:

$$L = L_0 + \dot{q_i}a(q,t) + \dot{q_i}2T_L(q,t)$$

in matrix form;

L $= L_0 + \boldsymbol{\dot{q}ta} + \frac{(\boldsymbol{\dot{q}tT\dot{q}})}{2}$

H= $\boldsymbol{\dot{q}p - L}$

H $= \boldsymbol{\dot{q}t(p-a)}$ -$(\boldsymbol{\dot{q}tT\dot{q}})/2 - L_0$

also;

$\boldsymbol{p}$=$\boldsymbol{T\dot{q} + a}$

$\boldsymbol{\dot{q}} = \boldsymbol{T^{-}1(p-a)}$

$\boldsymbol{q}^{\cdot}t = (\boldsymbol{p^t} - \boldsymbol{a^t})\boldsymbol{T^{-}1}$

Therefore; H = $(((\boldsymbol{p^t} - \boldsymbol{a^t})\boldsymbol{T^{-}1}(\boldsymbol{p} - \boldsymbol{a}))/2)$ - $L_0(q,t)(17)$

Cyclic Coordinates and Conservation Theorems:

Cyclic coordinates are those generalized coordinates which doesn't appear explicitly in Lagrangian and therefore, their corresponding conjugate momenta are constant.

$$p_j^{\cdot} = \partial L/\partial q_j = -\partial H/\partial q_j \tag{18}$$

Therefore, the coordinate will also not explicitly present in H.

*) If L and therefore H is not a explicit function of t; then H is a constant of motion.

$$dH/dt = \partial H/\partial t = -\partial L/\partial t \tag{19}$$

*) If equation of transformation that define generalized coordinates:

$$r_m = r_m(q_1, q_2, ...q_n; t) \tag{20}$$

doesn't explicitly deopend on time; if Potential Energy is velocity independent then; H = T + V = E

Conclusion

Now., Hamilton Equations are restricted with conservation of Energy. Mostly, it is convinient to use it when we know that H will be equal to E.

5 Maurpertius' Principle

We assume that Energy is conserved and end points are fixed. Let:

$$S = \int \sum_i p_i dq_i - E(t - t_1) = S_0 - E(t - t_1)$$

now

$\partial S_0 = \partial \int \sum_i p_i dq_i = 0; (21)$

The above expression is Mauerpertius' Principle. S0 is called as Abbreviated Action.

$$NowT = (\sum m_i j q_i^{\cdot} q_j^{\cdot})/2$$

$\text{p}_i = \sum m_i j \dot{q_j}$

$\text{S}_0 = \int \sum_i p_i dq_i = \int \sum_i m_i j \frac{dqj}{dt} dq_i$

E - v(q) $= (\sum m_i j \dot{q_i} \dot{q_j})/2 = \frac{\sum m_i j dq_i dq_j}{2dt^2}$

dt $= (\sum \frac{m_i j dq_i dq_j}{2(E-V)})^{1/2}$

$\text{S}_0 = \int (2(E-V) \sum (m_i j dq_i dq_j))^{1/2}$

if m is known and common;

$\text{S}_0 = \int (2(E-v)m)^{1/2} dl since ((dq_i dq_j)^{1/2} = dl)(22)$

dl represents an element of path.

Derivation of Principle of Least Action from Maurperetius' Principle:

$$\partial S_0 = \partial \int \sum_i p_i dq_i = 0$$

$\partial \int 2T = 0$

$\partial \int T + T = 0$

$\partial \int E - V + T = 0$

$\partial \int T - V = 0$

$\int \partial L = 0(23)$

Difference from Hamilton"s Equation

1. Definition of Action:

$$S = \int L dt \tag{24}$$

Hamilton uses the integral over time varied between fixed t1 and t2 and end points q1 and q2.

Maurpertius' principle uses abbreviated action integral over generlaized coordinates varied along all constant energy paths ending at q1 and q2.

2. Solution they determine:

Hamilton define complete position of particle with time. But, Maurpertius' Principle only defines trajectory of the motion.

3. Constraints on Variation:

Maurpertius' - q1,q2 Hamilton - q1,q2;t1,t2

Example:

Let initial energy of a ball to be thrown from sea level to cover horizontal distance 'a' be E. Using Maurpertius' Principle we will find Trajectory of its motion:

$$\partial S_0 = \partial \int (2(E - mgy)m)^{1/2} dl = 0 = \partial \int (E - mgy)^{1/2} (1 + y^{\cdot} 2)^{1/2} dy = 0 \quad (25)$$

On integrating and putting limits; we get final trajectory as:

$$y = (C_1 E - 1)/(C_1 mg)) - (C_1 mg(x - a/2)^2)/4 \quad (26)$$

Conclusion

Maurpertius' Equation is also restricted by conservation of energy. Morever; it can't solve motion's equations for us indepedently; but can give us trajectory of equation.

6 Udwadia- Kalaba Equation

Udwadia-Kalaba Equation provides analytical expression of the equation of motion of a constrained mechanical system, where constraints can be holonomic and/or non holonomic.

* Equation of motion of Unconstrained System :-

$$M(q, t) q^{\cdot\cdot} = F(q, q^{\cdot}, t) \quad (27)$$

Constraints:

* Holonomic : constraint of form

$$f(q, t) = 0$$

or reducible to it. 1. Scleronomic: Holonomic of form

$$f(q) = 0$$

or reducible to it. 2. Rheonomic : Holonomic of form

$$f(q,t) = 0$$

or reducible to it. * Non - Holonomic : constraints that are not holonomic.

Now; if, a constraint is of form

$$adx + bdy + cdz = 0$$

then it would be holonomic if and only if:

$$a((\partial b/\partial z) - (\partial c/\partial y)) + b((\partial c/\partial x) - (\partial a/\partial z)) + c((\partial a/\partial y) - (\partial b/\partial x)) = 0 \quad (28)$$

Now; the standard second order form of constraints in Udwadia-Kalaba's Setting :-

$$A(q, q^{\cdot}, t)q^{\cdot}. = b(q, q^{\cdot}, t) \quad (29)$$

Assumption 1 :- 1. Rank of A ¿= 1 2. The constraint is consistent with atleast one solution for q.

Now, Udwadia - Kalaba will solve the constraints in above form only. If first order constraints are given then we will differentiate it to form it to above given expression.

Moore-Penrose Inverse

Let rank of W ¿= 1. then;

$$W = U\Delta V^T$$

$\Delta = [diag(\delta_i)]_{r*r}$

$\delta - singularvaluesofW.$

$\delta_1 >= \delta_2 >= ..... > 0$

MP INVERSE(W) = W$^+$ = $V\Delta^{-1}\hat{T}$

now; if a is 1*n row vector:

a$^+$ = $a^T(aa^T)^{-1}$

if a is n*1 coloumn vector:

a$^+$ = $a^T(a^Ta)^{-1}$(30)

Udawadia - Kalaba Equation :

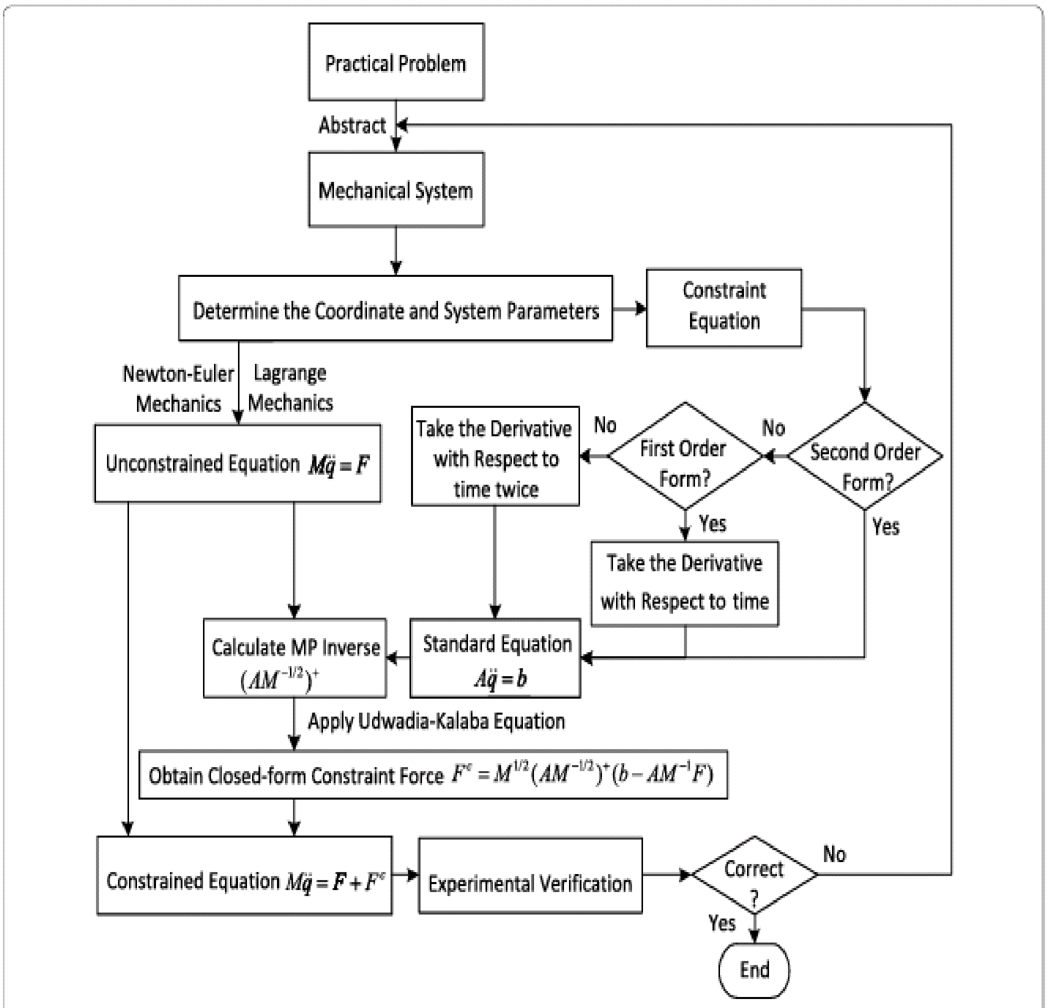


Figure 1 Formulation procedure of Udwadia–Kalaba Equation

Assumption-2 :- M(q,t) is positive definite.

$$M(q,t)q^{\cdot\cdot} = F(q,q^{\cdot},t) + F^c(q,q^{\cdot},t)$$

; F$^c = M^{1/2}[AM^{-1/2}]^{+} * [b - AM^{-}1F]$
(31)

Therefore; no involvement of auxiliary variables.

The above expression can be derived from Gauss' principle; or D Alembert's principle.

The equation follows D Alembert's strategy that constrained force do no virtual work.

BUT; for non- ideal case where constrained force do some work:

$$W_c = C^T(q, q^{\cdot}, t)\partial r(t)$$

M q¨ $= F + M^{-1/2}(AM^{-1/2})(b - AM^{-1}F) + M^{1/2}(I - (AM^{-1/2})^{+}AM^{-1/2})M^{-1/2}C$ (32)

Comparison with Lagrange's Equation :-

1. In U-K equation; Lagrange multiplier

$$'\lambda'$$

is solved numerically when the constraint is of that form; but in Lagrange's Equation, it is not so.

2. Lagrange multiplier is not unique in Lagrange"s Equation; but in U-K equation with matrix A:

$$A^T\lambda \tag{33}$$

is unique.

3. No extra variables required in U-K equation like lambda in Lagrange.

Difference with Gauss' Elimination:

Gauss' Principle can not solve the equation where constrained forces do work; but, the special case of Udwadia- Kalaba Equation can solve for that!

More ever; some of the cases are not be able to solved by Gauss' Elimination which are able to be solved by Udwadia-Kalaba equation.

Conclusion

The major point of using Udwadia - Kalaba Equation is that it can be applied in case of non - conservative forces. It can solve for all non holonomic constraints which can be reduced to its standard form! Also its special case can solve for constraint forces which work done is non zero.