Random Eigenvalue Characterization for Free Vibration of Centrifugally Loaded Euler-Bernoulli Beams

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1. INTRODUCTION & OBJECTIVE

Numerous engineering materials demonstrate considerable spatial randomness in their mechanical and thermal properties [1]. Here, we specifically study the centrifugally loaded Euler-Bernoulli beams, which are used to model important structures like helicopter blades, propellers, turbines, and wind-turbine blades. The randomness stems from intricate interactions among material composition, microstructure, manufacturing processes, and environmental conditions, leading to statistical fluctuations in the properties of these beams, including their natural frequencies and mode shapes. Estimating statistical properties, including the mean and variance of random eigenvalues, is crucial for real-world applications, especially since these structures are prone to resonant phenomena. Consequently, it is essential to obtain precise analytical expressions for these quantities as functions of the statistical variations in beam properties. In this work, we establish a direct mathematical relationship between flexural stiffness and fundamental natural frequency using an inverse problem approach. The probability density function (PDF) of the flexural stiffness is determined by knowing the PDF of the fundamental frequency. The coefficient of variation (COV) for the stiffness distribution is calculated, which is then used to optimize the centrifugally loaded beam profile for a maximum manufacturing tolerance, that is, we determine the optimal value of the parameters such that the COV of flexural stiffness is maximum when compared to the COV of the fundamental natural frequency.

2. METHODS OF ANALYSIS

A mathematical relationship is established between flexural stiffness $EI(x)$ and the pre-specified fundamental natural frequency ω by assuming a quadratic mass distribution $m(x)$, and an assumed fundamental mode shape $\phi(x)$ which satisfies all four boundary conditions in the governing equations for a rotating Euler-Bernoulli beam.

$$m(x) = a_0 + a_1 x + a_2 x^2 = a_0(1 + r_1 x + r_2 x^2) \tag{1}$$

$$EI(x) = 1/10080[28(k(26L^4 + 16L^3x + 6L^2x^2 - 4Lx^3 + x^4) - 2(13L^4 + 26L^3x - 6L^2x^2 - 8Lx^3 + 5x^4)\Omega^2)a_0 + 4(k(142L^5 + 102L^4x + 62L^3x^2 + 22L^2x^3 - 18Lx^4 + 5x^5) - 2(71L^5 + 142L^4x + 3L^3x^2 + 4L^2x^3 - 30Lx^4 + 20x^5)\Omega^2))a_1 + (k(465L^6 + 362L^5x + 259L^4x^2 + 156L^3x^3 + 53L^2x^4 - 50Lx^5 + 15x^6) - 15(31L^6 + 62L^5x + 9L^4x^2 + 12L^3x^3 + L^2x^4 - 10Lx^5 + 7x^6)\Omega^2)a_2] \tag{2}$$

where $k = \omega^2$, a_i-s are mass parameters, Ω is the rotation speed, and L is the length of the beam. Equation (2) can be written as $EI(x) = c(x)k + d(x)$, where $c(x)$ and $d(x)$ are polynomials in x. We consider frequency ω as a random variable denoted by U and flexural stiffness distribution $EI(x)$ as a random field denoted by $V(x)$. Thus, a random field $V(x)$ can be expressed as the function of a random variable $U(x)$, which is $V(x) = c(x)U^2 + d(x)$. Knowing the probability density function (PDF) for the normal distribution of the random variable U, we can formulate the PDF of the $V(x)$. The average coefficient of variation c_T of the random field $V(x)$ for the coefficient of variation $c_V(x)$ over the length of the beam, given by

$$c_T = \frac{\int_0^L c_V(x)dx}{L} \tag{3}$$

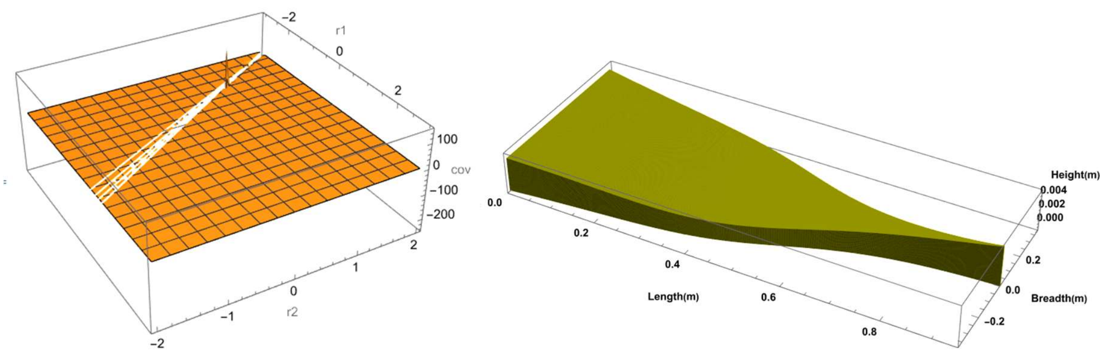
3. RESULTS AND/OR HIGHLIGHTS OF IMPORTANT POINTS

The maximum feasible value of the average coefficient of variation is obtained as 0.44231 in Fig. 1(a), corresponding to the design parameter $r_1 = -1.8145$ and $r_2 = 0.8416$. We determine the value of $a_0 = 27.4512$, $a_1 = 49.808$ and $a_2 = 22.363$ assuming a total mass 10 kg. The optimal mass and flexural stiffness distribution on the optimized design parameter value (a_0, a_1, a_2), fundamental natural frequency $\omega = 140$ rad/sec, length $L = 1$ m, and uniform rotation speed $\Omega = 1100$ RPM is given by

$$m(x) = 27.4512 - 49.808x + 22.363x^2 \tag{4}$$

$$EI(x) = 1314.07 - 5368.76x + 5952.29x^2 + 5270.75x^3 - 15523.3x^4 + 10795.2x^5 - 2438.76x^6 \tag{5}$$

Figure 1(b) shows the structurally optimized rotating beam having a rectangular cross-section and a material density $\rho = 7840$ kg/m^3 and Young's modulus $E = 2 \times 10^{11}$ Pa. The coefficient of variation of the structurally optimized beam of the quadratic mass distribution is 44.23%, which shows an improvement of approximately 5% in comparison to a linear mass distribution [1].



(a) (b)

Figure 1: (a) Variation of the average COV of a random field $V(x)$ as a function of $r_1 = \frac{a_1}{a_0}$ and $r_2 = \frac{a_2}{a_0}$, (b) Breadth and height variation of an optimized rotating cantilever beam.

4. CONCLUSIONS

We derive exact analytical expressions for the flexural stiffness distribution $EI(x)$ of cantilever Euler–Bernoulli beams under centrifugal loading with quadratic mass variation, directly linked to a prescribed natural frequency ω. A probabilistic framework establishes closed-form relations between the probability density functions of $EI(x)$ and ω, offering benchmark solutions for validating stochastic simulations. These distributions yield the COV of stiffness, which is optimized to maximize the average COV to mitigate sensitivity to geometric imperfections, which means that even if there is a significant variation in the geometry due to manufacturing defects or operational wear and tear, its effect on the natural frequency will be insignificant.

5. REFERENCES

[1] K. Sarkar, R. Ganguli, D. Ghosh, and I. Elishakoff, "Random Eigenvalue characterization for free vibration of axially loaded Euler–Bernoulli beams," *AIAA Journal*, vol. 56, no. 9, pp. 3757–3765, Jun. 2018, doi: 10.2514/1.j056942.

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