Problem 1 (d)

Assume that the average case is the most frequent one.
Assume for simplicity that n and k are powers of 2

Let

$$T(n) = \begin{cases} c_1 & , n == 1 \\ 2T(n/2) + c_1 n & , n > 1 \end{cases}$$ usual merge sort

$$P(n) = \begin{cases} c_2 n^2 & , n \leq k \\ 2P(n/2) + c_1 n & , n > k \end{cases}$$ merge sort + insertion sort

with $c_2 < c_1$

The goal is to maximize the difference $T(n) - P(n)$ for large n's.

$T(n)$

0	$c_1 n$	$c_1 n$
1	$c_1 n/2 \quad c_1 n/2$	$c_1 n$
...	--- ---	---
...	...	---
$\log_2 n$	$1 \; 1 \; 1 \; ... \; 1 \; 1$	$c_1 n$

$P(n)$

0	$c_1 n$	$c_1 n$
1	$c_1 n/2 \quad c_1 n/2$	$c_1 n$
...	--- ---	---
$\log_2 \frac{n}{k}$	$c_2 k^2 \quad c_2 k^2 \; ... \; c_2 k^2$	$c_2 k^2 \, 2^{\log_2 \frac{n}{k}}$

this is the last level of the tree since from here on there would be $\log_2 k$ more levels in the usual merge sort and $\log_2 n - \log_2 k = \log_2 \frac{n}{k}$

$$T(n) - P(n) = \sum_{0}^{\log_2 n} c_1 n - \sum_{0}^{\log_2 \frac{n}{k} - 1} c_1 n - c_2 n k$$

$$= \sum_{\log_2 \frac{n}{k}}^{\log_2 n} c_1 n_1 - c_2 n k$$

$$= \left(\log_2 n - \log_2 \frac{n}{k} + 1\right) c_1 n - c_2 n k$$

Call this $f(k) = n\left(c_1 \log_2 2k - c_2 k\right)$

$$f'(k) = n\left(\frac{c_1}{\ln 2\, k} - c_2\right)$$

$f'(k) = 0 \Rightarrow k = \dfrac{c_1}{c_2 \ln 2}$ this is a maximum point

For example $\left.\begin{aligned} c_1 &= 50 \\ c_2 &= 2 \end{aligned}\right\} \Rightarrow k = \dfrac{25}{\ln 2}$