Question Bank : Unit 3

CO1

1. Define Absolute extreme values, local extreme values, critical points, point of inflection.

2. State Rolle's theorem, LMVT, CMVT, Taylor's Theorem.

3. Say True/ False. Justify your answer.

   (a) If a function has global maximum then it has local maximum.
   (b) If a function is continuous then it has global minimum.
   (c) If a function is defined on a closed interval then it has global extrema.
   (d) If a function is such that its derivative is not zero at any point then it has no global extrema.
   (e) If local maximum = local minimum then function is constant.

CO2

4. Find the absolute maximum and minimum values of each function on the given interval.

   $(a) f(x) = \frac{-1}{x+3},\ -2 \le x \le 3$ $\quad$ $(b) f(x) = \sqrt[3]{x},\ -1 \le x \le 8$

   $(c) f(x) = -3x^{\frac{2}{3}},\ -1 \le x \le 1$ $\quad$ $(d) f(x) = \sqrt{4 - x^2},\ -\sqrt{2} \le x \le 1$

   $(e) f(\theta) = sin\theta,\ -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$ $\quad$ $(f) f(t) = 2 - |t|\,,\ -1 \le t \le 3.$

   $(g) f(x) = 1 + 12\,|x| - 3x^2,\ [-2,\ 5]$

5. Find the set of critical points.

   $(a) f(x) = x^{\frac{2}{3}}(x+2),\ \ (b) f(x) = x^2\sqrt{3 - x},$

   $$(c) f(x) = x\,|x| - x, \quad (d) f(x) = \begin{cases} 3 - x & if\ x < 0 \\ 3 + 2x - x^2 & if\ x \ge 0 \end{cases}$$

6. Attempt the following.
   (a) Sketch the graph of a function that has a local maximum at 2 and is differentiable at 2.
   (b) Sketch the graph of a function that has a local maximum at 2 and is continuous but not differentiable at 2.
   (c) Sketch the graph of a function that has a local maximum at 2 and is not continuous at 2.
   (d) Sketch the graph of a function on $[-1,\ 2]$ that has an absolute maximum but no local maximum.
   (e) Sketch the graph of a function on $[-1,\ 2]$ that has a local maximum but no absolute maximum.
   (f) Sketch the graph of a function on $[-1,\ 2]$ that has an absolute maximum but no absolute minimum.
   (g) Sketch the graph of a function on $[-1,\ 2]$ that is discontinuous but has both an absolute maximum and an absolute minimum.
   (h) Sketch the graph of a function that has two local maxima, one local minimum and no absolute minimum.
   (i) Sketch the graph of a function that has three local minima, one local maximum and seven critical points.

7. Find the intervals on which the function is increasing and decreasing. Then identify the function's local extreme values, if any, saying where they are taken on.
   (a) $f(x) = 2x^3 - 18x$ (b) $f(x) = x^4 - 8x^2 + 16$

   (c) $f(x) = x\sqrt{8 - x^2}$ (d) $f(x) = \frac{x^2-3}{x-2},\ x \neq 2$
   (e) $f(x) = x^{1/3}(x + 8)$ (f) $k(x) = x^3 + 3x^2 + 3x + 1$

   (g) $f(x) = ln(x^4 + 27)$ (h) $f(\theta) = 2\cos\theta - \cos 2\theta$, for $0 \leq \theta \leq 2\pi$.

8. Show that the following functions have exactly one root in the given interval.
   (a) $f(x) = x^3 + \frac{4}{x^2} + 7; \quad (-\infty, 0]$
   (b) $g(t) = \sqrt{t} + \sqrt{1+t} - 4; \quad [0, \infty)$

9. Find all values of c, that satisfy the conclusion of LMVT for the given functions and interval.

(a) $f(x) = x^2 + 2x - 1; \quad [0, 1]$
(b) $f(x) = x + \frac{1}{x}; \quad [1/2, 2]$

10. Find all values of c, that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.
(a) $f(x) = x, \quad g(x) = x^2, \quad (a, b) = (-2, 0)$
(b) $f(x) = x, \quad g(x) = x^2, \quad (a, b)$ arbitrary interval
(c) $f(x) = x^3/(3 - 4x), \quad g(x) = x^2, \quad (a, b) = (0, 3)$

CO3

11. Prove the following inequalities.
(a) $|\tan^{-1} x - \tan^{-1} y| \leq |x - y|$ for all x and y in $\mathcal{R}$

(b) $\frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$ for $0 < a < 1$ and $0 < b < 1$.

(c) $e^x > 1 + x + \frac{x^2}{2}$ for all $x > 0$.

(d) $\tan x > x$ for $0 < x < \pi/2$.

(e) $\frac{x}{1+x} < \ln(1 + x) < x$ for $x > 0$.

12. Sketch the graph of a continuous function $y = f(x)$ satisfying all the five conditions given below simultaneously
(a) $f(2) = 2$, (b) $0 < f'(x) < 1$ for $x < 2$,
(c) $\lim_{x \to 2^-} f'(x) = 1^-$, (d) $-1 < f'(x) < 0$ for $x > 2$, and
(e) $\lim_{x \to 2^+} f'(x) = -1^+$

13. Sketch the graph of a continuous function $y = h(x)$ satisfying all the four conditions given below simultaneously
(a) $h(0) = 0$, (b) $-2 \leq h(x) \leq 2$ for all x
(c) $\lim_{x \to 0^-} h'(x) = \infty$, and (d) $\lim_{x \to 0^+} h'(x) = \infty$

14. Find a value of c that makes the function

$$f(x) = \begin{cases} \frac{9x - 3\sin 3x}{5x^3} & \text{if} x \neq 0 \\ c & \text{if} x = 0 \end{cases}$$

continuous at $x = 0$.

15. For the following functions, (i) Determine the critical points (ii) Determine the intervals on which the function is increasing and decreasing (iii) determine the points of inflection (iv) Determine the intervals on which the graph of the function is concave up nd concave down (v) sketch the graph of the function and hence state all the extreme values of the function.

    (a) $y = 6 - 2x - x^2$ (b) $y = 1 - 9x - 6x^2 - x^3$ (c) $y = x^4 + 2x^3$ (d) $y = x^{2/3}(x - 5)$

CO4

16. State and prove Rolle's Theorem
17. State and prove LMVT
18. State and prove CMVT
19. Assume that f is continuous on $[a,\ b]$ and differentiable on $(a,\ b)$. Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f^{'} \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

20. State true or false and justify your answers.
    (a) If $f(x)$ is increasing then $f'(x) > 0$ for all x.
    (b) If $f'(x) > 0$ for all x then $f(x)$ is increasing function.
    (c) $f(x) = \sin x - x$ increases in every interval.

CO5

21. One tower is 50 ft high and another is 30 ft high. The towers are 150 ft apart from each other. A guy wire is to run from point A to the top of each tower. Locate the point A so that the total length of guy wire is minimum.(Ans: 170)

22. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window is to be p feet, find the dimensions of the window which will admit

the maximum light.(Ans: width$=\frac{4p}{3\pi+8}$ and height $=\frac{p(\pi+4)}{3\pi+8}$.)

23. A piece of wire of length L is to be divided into two parts, one of which is to be bent into a circle and the other into a square. Find the maximum and the minimum total area enclosed by the two pieces.

24. Estimate $\sin 34^\circ$ upto six places of decimal using Taylor approximations at (i) 0, (ii)$\frac{\pi}{6}$. Compare.

25. Estimate the following values correct upto 4 decimal places:
(a) $tan^{-1}(1.1)$
(b) $\sqrt{9.2}$