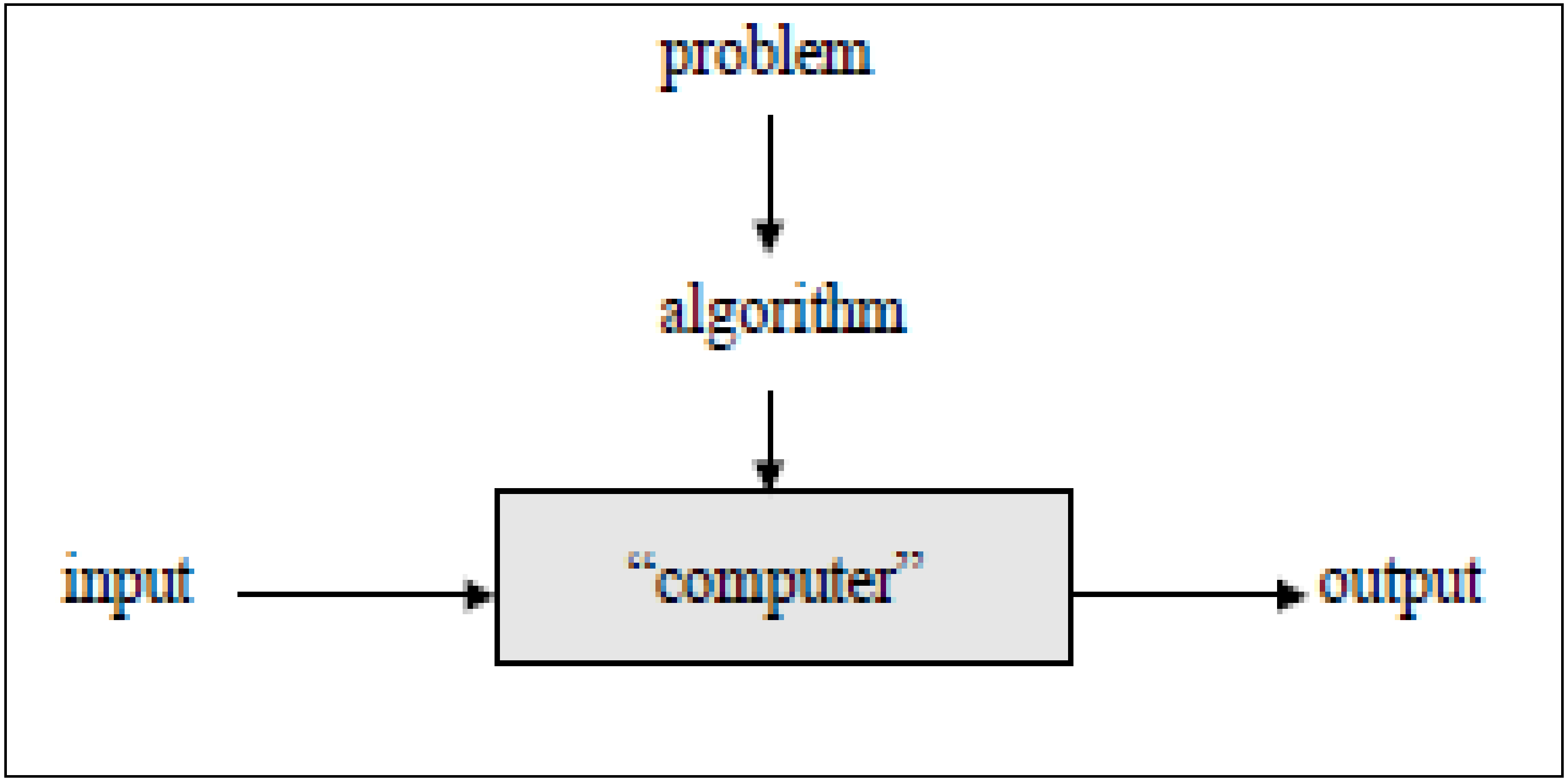
Unit I

Introduction

- Algorithm
  - An **algorithm is any well-defined computational procedure that take** some value, or set of values, as **input and produces some value, or set of values, as output. An algorithm is thus a sequence of computational steps that transform the** input into the output.



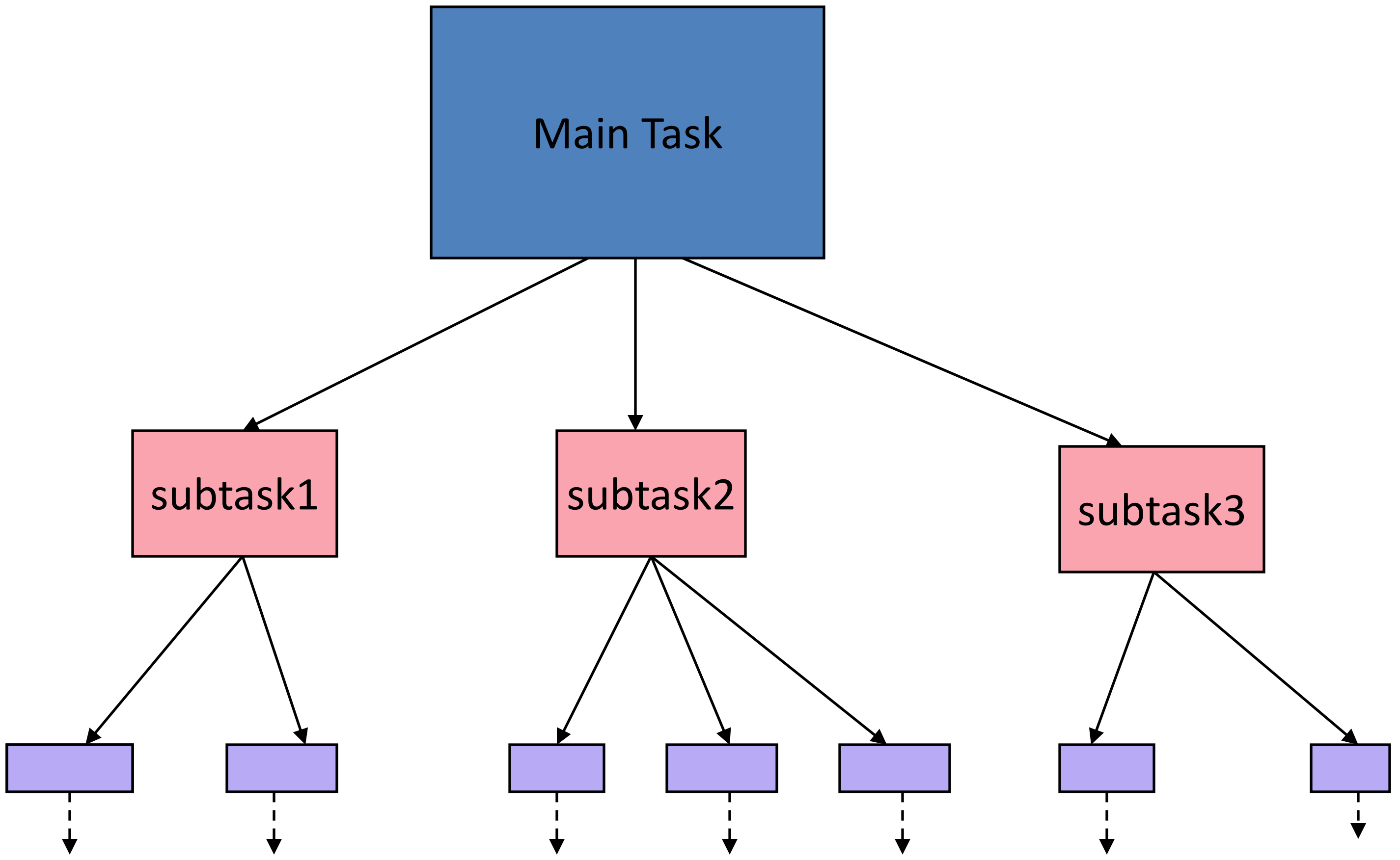
Algorithmic Solution

- With the definition, we can identify five important characteristics of algorithms :
  - Algorithms are well-ordered.
  - Algorithms have unambiguous operations.
  - Algorithms have effectively computable operations.
  - Algorithms produce a result.
  - Algorithms halt in a finite amount of time.

Properties of an Algorithm

- **Finiteness:** - an algorithm terminates after a finite numbers of steps.
- **Definiteness:** - each step in algorithm is unambiguous. This means that the action specified by the step cannot be interpreted (explain the meaning of) in multiple ways & can be performed without any confusion.
- **Input:-** an algorithm accepts zero or more inputs
- **Output:-** it produces at least one output.
- **Effectiveness:-** it consists of basic instructions that are realizable. This means that the instructions can be performed by using the given inputs in a finite amount of time.

Top Down Design



Write an algorithm to find the largest of a set of numbers. You do not know the number of numbers.

FindLargest
Input: A list of positive integers

```
1.  Set Largest to 0
2.  while (more integers)
     2.1  if (the integer is greater than Largest)
              then
                2.1.1  Set largest to the value of the integer
          End if
    End while
3.  Return Largest
    End
```

Distinct Areas of Study of Algorithms

- Devise an Algoritm
  - Find good algorithms
- Validate a problem
  - Correct answers for all legal inputs
- Analyze an Algorithm
  - Computational time and memory requirements.
- Test a Program
  - Debugging and profiling

Algorithm Specification

- Comments
  - //
- Blocks
  - {   }
- Identifier
  - Starts with letter
- Assignment
  - Id:=4
- Boolean Values
  - True and false
- Multidimentional array
  - A[i,j]. Starts with zero index
- Input and Output
  - Use instructions as read and write

Algorithm Specification

Selection (test)

- *If <condition> then task1*
- If *<condition>* then *task1* else *task2*
- *Multiple cases*

$$
\begin{array}{l}
\textbf{case} \\
\{ \\
\qquad \textbf{:}\langle \mathit{condition}\ 1\rangle\textbf{:}\ \langle \mathit{statement}\ 1\rangle \\
\qquad\qquad\qquad \vdots \\
\qquad \textbf{:}\langle \mathit{condition}\ n\rangle\textbf{:}\ \langle \mathit{statement}\ n\rangle \\
\qquad \textbf{:else:}\ \langle \mathit{statement}\ n+1\rangle \\
\}
\end{array}
$$

Algorithm Specification

Repetition

- While/Repeat

```
While (cond) do
   ⁞
end while
```

- Do while /Repeat

```
Do
 ⁞
while (cond)
```

Algorithm Specification

- One procedure

  Algorithm $Name$ $(\langle parameter\ list \rangle)$

  - example

```
Algorithm Max(A, n)
// A is an array of size n.
{
    Result := A[1];
    for i := 2 to n do
        if A[i] > Result then Result := A[i];
    return Result;
}
```

Example: Selection Sort Algorithm

- First Attempt :
  - Find the smallest from unsorted list and place next to sorted list

for $i := 1$ **to** n **do**
{
    Examine $a[i]$ to $a[n]$ and suppose
    the smallest element is at $a[j]$;
    Interchange $a[i]$ and $a[j]$;
}

  - Smallest element is a[j], then Interchange a[i], a[j]

$$t := a[i];\ a[i] := a[j];\ a[j] := t;$$

Selection Sort

```
Algorithm SelectionSort(a, n)
// Sort the array a[1 : n] into nondecreasing order.
{
    for i := 1 to n do
    {
        j := i;
        for k := i + 1 to n do
            if (a[k] < a[j]) then j := k;
        t := a[i]; a[i] := a[j]; a[j] := t;
    }
}
```

Recursive Algorithms

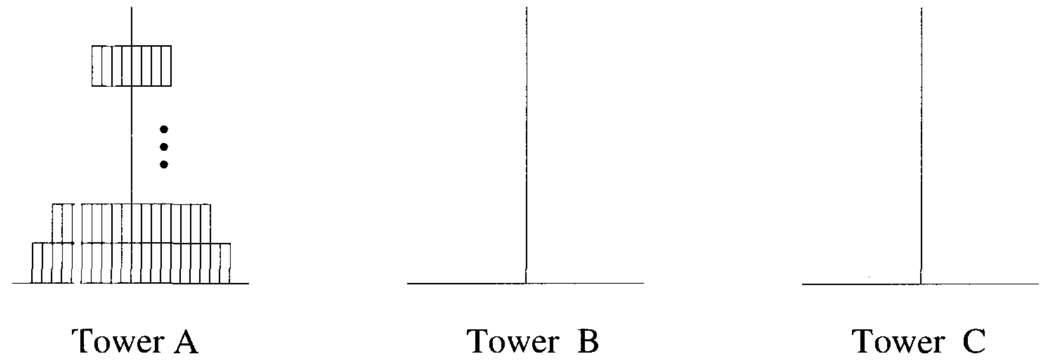
- Direct Recursion
  - Function Calling itself
- Indirect Recursion
  - Function A Calling B and B calling A again.

  - Example: Binomial Theorem

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1} = \frac{n!}{m!(n-m)!}$$

Recursive Algorithms

- Examples
  - Towers of Honoi




```
Algorithm TowersOfHanoi(n, x, y, z)
// Move the top n disks from tower x to tower y.
{
    if (n ≥ 1) then
    {
        TowersOfHanoi(n − 1, x, z, y);
        write ("move top disk from tower", x,
        "to top of tower", y);
        TowersOfHanoi(n − 1, z, y, x);
    }
}
```

Time and Space Complexity

- **Time complexity**
  - **Time complexity** of an algorithm quantifies the amount of **time** taken by an algorithm to run as a function of the length of the input.
- **Space complexity**
  - **Space complexity** of an algorithm quantifies the amount of **space** or memory taken by an algorithm to run as a function of the length of the input.

Count method to calculate time complexity

- introduce a new variable, ***count, into a program by*** identifying Program Step and by Increment the value of ***count by appropriate amount with respect*** to a statement in the original program executes.
- Program Step - A synthetically meaningful segment of a program that requires execution time which is independent on the instance characteristics.

Count method to calculate time complexity…

- The number of steps in any program depends on the kind of statements. For example –
  - – **Comments count as zero steps**
  - – **An assignment statement which does not involves any calls to other algorithm is counted as one step.**
  - – **For looping statements the step count equals to the number of step counts assignable to goal value expression. And should be incremented by one within a block and after completion of block.**
  - – **For conditional statements the step count should incremented by one before condition statement.**
  - – **A return statement is counted as one step and should be write before return statement.**

- **For example –**
  - **Algorithm Sum(a,n)**

```
{ s:=0;
    for i:=1 to n do
    {
            s:=s+a[i];
    }
  return s;
}
```

Algorithm Sum(a,n) // After Adding Count

```
{       s:=0;
count:=count+1; // for assignment statement execution
for i:=1 to n do
{
        count:=count+1; //for For loop Assignment
        s:=s+a[i];
        count:=count+1;// for addition statement execution
}
count:=count+1; // for last time of for
count:=count+1; // for the return
return s;
}
```

- **Algorithm Sum(a,n) //Simplified version for algorithm Sum**

```
{
     for i:=1 to n do
    {
            count:=count+2;
    }
    count:=count+3;
}
```

Form above example,

Total number of program steps= 2n + 3, where n is the loop counter.

- **Algorithm RSum(a,n)**

```
{
    if n ≤ 0 then
    return a[n];
    else
    return RSum(a, n-1) + a[n];
}
```

- **Algorithm RSum(a,n)**

```
{
        count:=count + 1; // for the if condition
        if n ≤ 0 then
        {
                 count:=count + 1;// for the return statement
                return a[n];
        }
        else
        {
                 count:=count + 1; // for the addition, function invoked
& return
                return RSum(a, n-1) + a[n];
        }
}
```

- Therefore we can write,

- tRSum(n) = 2 if n=0 and
- tRSum(n) = 2+ tRSum(n-1) if n>0

  = 2+ 2+ tRSum(n – 2)

  = 2(2)+ tRSum(n – 2)

  =3 (2) + tRSum(n – 3)

  :

  :

  = n(2) + tRSum(0)

  =2n+2

So, the step count for RSum algorithm is 2n+2.

- **Algorithm RSum(a,n)** **//Simplified version of algorithm Rsum** with counting's only

```
{
    count:=count + 1;
    if n ≤ 0 then
        Count:=count + 1;
    else
        count:=count + 1;
}
```

Prof

Table method to calculate time complexity

- build a ***table in which we list the total number of steps*** contributed by each statement. This table contents three columns –
  - ***Steps per execution (s/e)- contents count value by which count*** increases after execution of that statement.
  - ***Frequency – is the value indicating total number of times*** statement executes
  - ***Total steps – can be obtained by combining s/e and frequency.***
  - ***Total step count can be calculated by adding total steps*** contribution values.

Statement	s/e	Frequency	Total steps
Algorithm Sum(a,n)	0	1	0
{	0	1	0
s:=0;	1	1	1
for i:=1 to n do	1	n+1	n+1
s:=s+a[i];	1	n	n
return s;	1	1	1
}	0	1	0

Total step count = 2n+3

Analyzing Algorithm

- Phases of Analysis
  - Priori Analysis
    - The bounds of time are obtained by formalating a function based on theroy.
    - Independent of programming languages and machine structures. Ex, O-notation.
  - Posteriori Analysis
    - Depends on programming language and machine structure.
    - Time and space is recorded during execution.
    - More is the number of input more is the time taken.

      Ex. insertion sort

Types of Analysis

- Worst case
  - Provides an upper bound on running time
  - An absolute guarantee that the algorithm would not run longer, no matter what the inputs are
- Best case
  - Provides a lower bound on running time
  - Input is the one for which the algorithm runs the fastest

$$Lower\ Bound \leq Running\ Time \leq Upper\ Bound$$

- Average case
  - Provides a prediction about the running time
  - Assumes that the input is random

How do we compare algorithms?

- We need to define a number of objective measures.
  - Compare execution times?
    - ***Not good***: times are specific to a particular computer !!
  - Count the number of statements executed?
    - ***Not good***: number of statements vary with the programming language as well as the style of the individual programmer.

Ideal Solution

- Express running time as a function of the input size *n* (i.e., *f(n)*).
- Compare different functions corresponding to running times.
- Such an analysis is independent of machine time, programming style, etc.

Example

- Associate a "cost" with each statement.
- Find the "total cost" by finding the total number of times each statement is executed.

Algorithm 1

	Cost
arr[0] = 0;	c_1
arr[1] = 0;	c_1
arr[2] = 0;	c_1
...	...
arr[N-1] = 0;	c_1

$c_1+c_1+...+c_1 = c_1 \times N$

Algorithm 2

	Cost
for(i=0; i<N; i++)	c_2
arr[i] = 0;	c_1

$(N+1) \times c_2 + N \times c_1 =$

$(c_2 + c_1) \times N + c_2$

Another Example

- ***Algorithm 3*** *Cost*

```
sum = 0;                          c1
for(i=0; i<N; i++)                c2
  for(j=0; j<N; j++)              c2
      sum += arr[i][j];           c3
                             ------------
```

$$c_1 + c_2 \, x \, (N+1) + c_2 \, x \, N \, x \, (N+1) + c_3 \, x \, N^2$$

Asymptotic Notation

- O notation: asymptotic "less than":
  - f(n)=O(g(n)) implies: f(n) "≤" g(n)
- Ω notation: asymptotic "greater than":
  - f(n)= Ω (g(n)) implies: f(n) "≥" g(n)
- Θ notation: asymptotic "equality":
  - f(n)= Θ (g(n)) implies: f(n) "=" g(n)

Big-O Notation

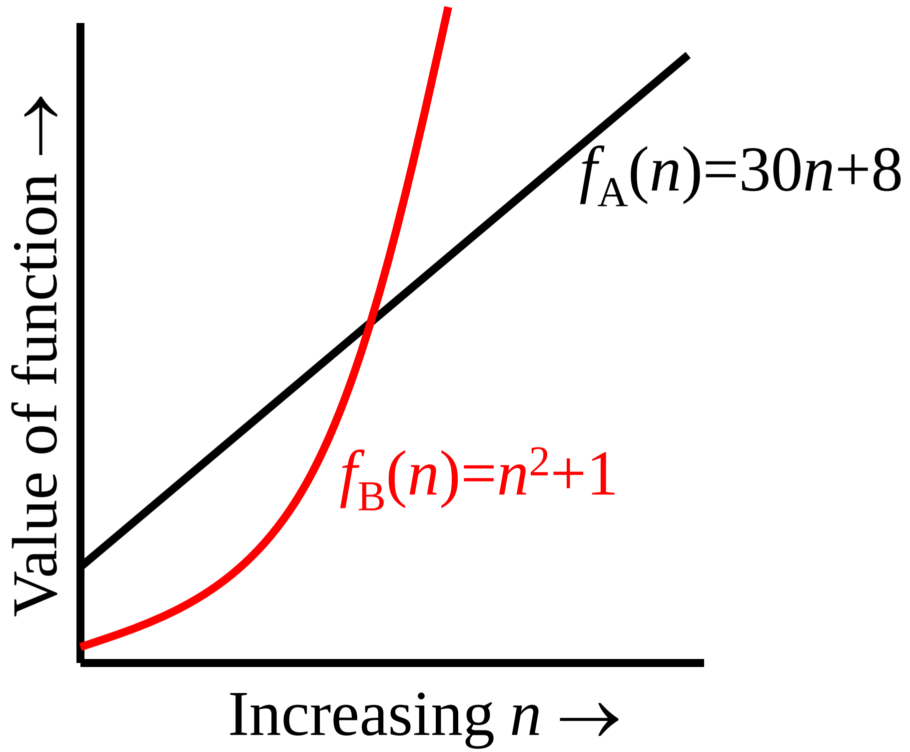
- We say $f_A(n)$=30n+8 is *order n*, or O (n)
  It is, at most, roughly *proportional* to n.
- $f_B(n)=n^2+1$ is *order* n^2, or O(n^2). It is, at most, roughly proportional to n^2.
- In general, any O(n^2) function is faster-growing than any O(n) function.

More Examples …

- $n^4 + 100n^2 + 10n + 50$ is $O(n^4)$
- $10n^3 + 2n^2$ is $O(n^3)$
- $n^3 - n^2$ is $O(n^3)$
- constants
  - 10 is $O(1)$
  - 1273 is $O(1)$

Visualizing Orders of Growth

- On a graph, as you go to the right, a faster growing function eventually becomes larger...



Back to Our Example

Algorithm 1

	Cost
arr[0] = 0;	c_1
arr[1] = 0;	c_1
arr[2] = 0;	c_1
...	
arr[N-1] = 0;	c_1

$c_1+c_1+\ldots+c_1 = c_1 \times N$

Algorithm 2

	Cost
for(i=0; i<N; i++)	c_2
arr[i] = 0;	c_1

$(N+1) \times c_2 + N \times c_1 =$

$(c_2 + c_1) \times N + c_2$

- Both algorithms are of the same order: *O(N)*

Example (cont'd)

Algorithm 3	***Cost***
sum = 0;	c_1
for(i=0; i<N; i++)	c_2
for(j=0; j<N; j++)	c_2
sum += arr[i][j];	c_3

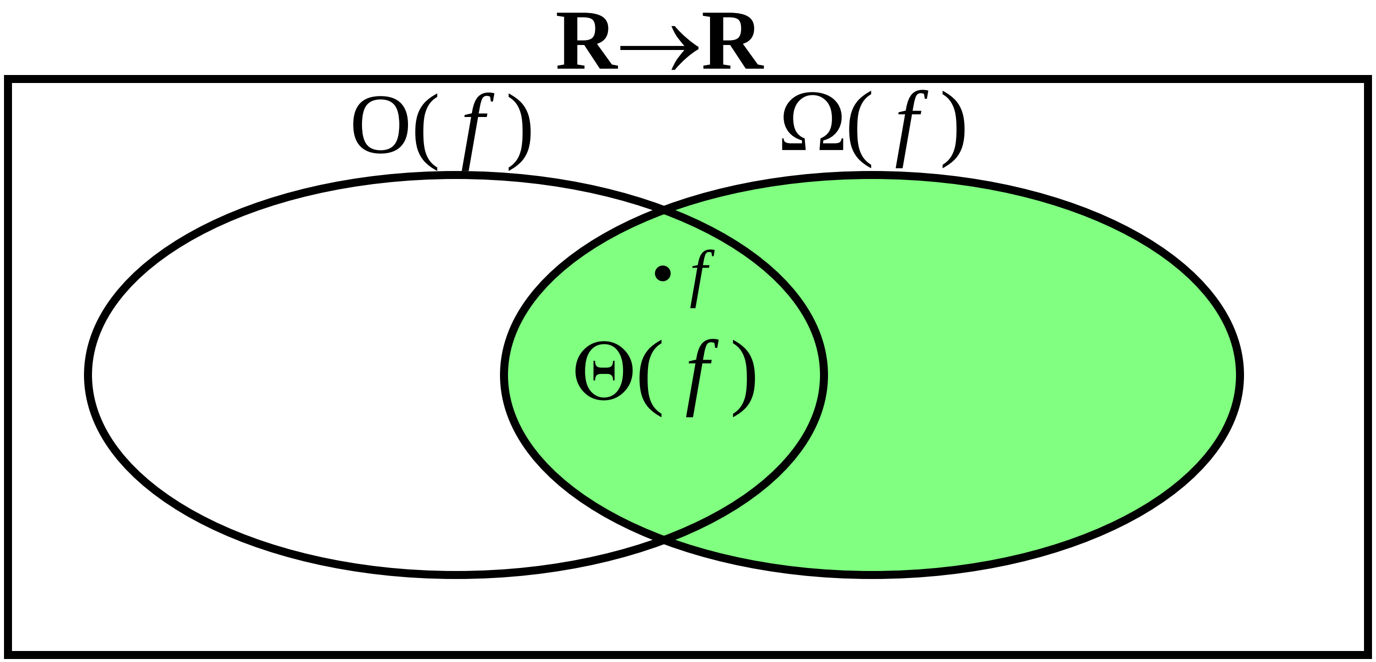
$$c_1 + c_2 \; x \; (N+1) + c_2 \; x \; N \; x \; (N+1) + c_3 \; x \; N^2 = O(N^2)$$

Review: Asymptotic Performance

- *Asymptotic performance*: How does algorithm behave as the problem size gets very large?
  - Running time
  - Memory/storage requirements
- Remember that we use the RAM model:
  - All memory equally expensive to access
  - No concurrent operations
  - All reasonable instructions take unit time
    - Except, of course, function calls
  - Constant word size
    - Unless we are explicitly manipulating bits

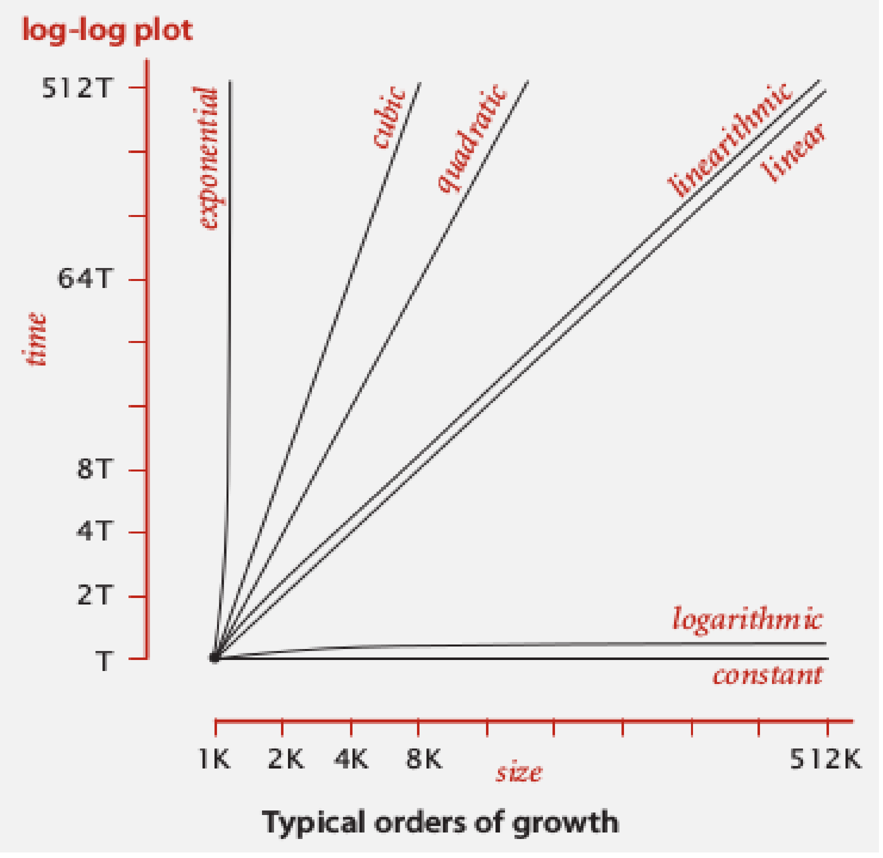
Review: Running Time

- Number of primitive steps that are executed
  - Except for time of executing a function call most statements roughly require the same amount of time We can be more exact if need be

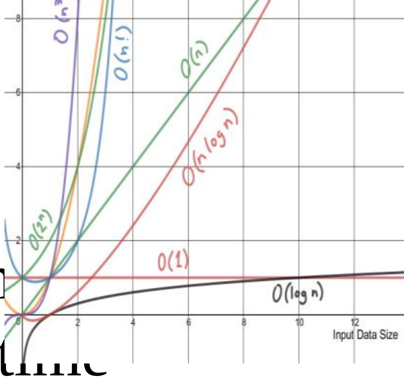


Asymptotic Notations

- Allow us
  - to analyze an algorithm's running time by identifying its behavior as the input size for the algorithm increases.
- This is also known as an a
rate.
- Order of Growth classifica

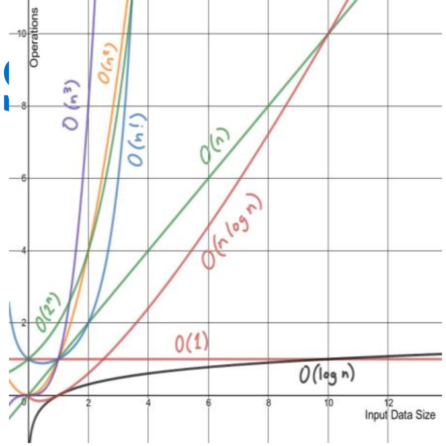


Asymptotic Notations



- Constant
  - No matter the size of the data it receives, the algorith the same amount of time to run. We denote this as a time complexity of O(1).
- Linear
  - The running duration of a linear algorithm is constant. It will process the input in n number of operations O(n).
- Quadratic
  - two nested loops, or nested linear operations, the algorithm process the input n^2 times.
- Logarithmic
  - A logarithmic algorithm is one that reduces the size of the input at every step. We denote this time complexity as O(log n). Example : binary search algorithm

Asymptotic Notations

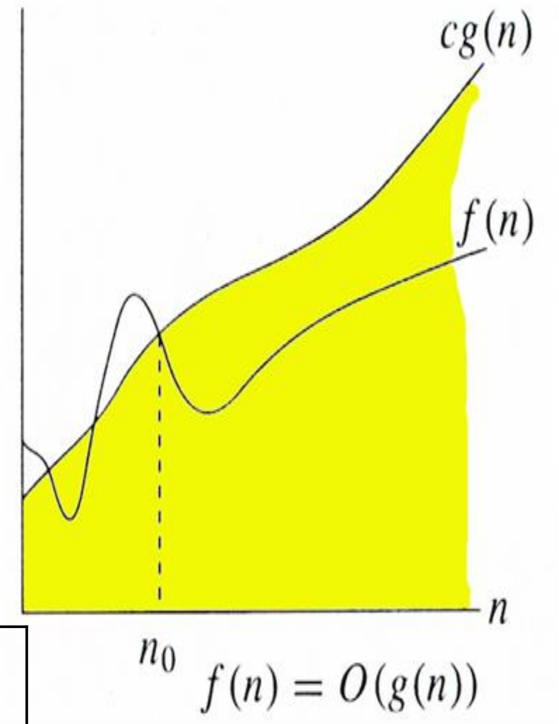


- Quasilinear
  - the time complexity O(n log n) can describe a data structure where each operation takes O(log n) time. example :quick sort, a divide-and-conquer algorithm.
- Non-polynomial time complexity
  - An algorithm with time complexity O(n!) often iterates through all permutations of the input elements. example brute-force search seen in the travelling salesman problem
- Exponential
  - An exponential algorithm often also iterates through all subsets of the input elements. It is denoted O(2n). The larger the data set, the more steep the curve becomes. a brute-force attack.

Order of Growth classification

order of growth	name	typical code framework	description	example	T(2N) / T(N)
1	constant	`a = b + c;`	statement	add two numbers	1
log N	logarithmic	`while (N > 1) { N = N / 2; ... }`	divide in half	binary search	~ 1
N	linear	`for (int i = 0; i < N; i++) { ... }`	loop	find the maximum	2
N log N	linearithmic	[see mergesort lecture]	divide and conquer	mergesort	~ 2
N^2	quadratic	`for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) { ... }`	double loop	check all pairs	4
N^3	cubic	`for (int i = 0; i < N; i++) for (int j = 0; j < N; j++) for (int k = 0; k < N; k++) { ... }`	triple loop	check all triples	8
2^N	exponential	[see combinatorial search lecture]	exhaustive search	check all subsets	T(N)

Asymptotic Notations- Big O Notation
{f(n) = O(g(n))}



f(n) <= cg(n) for all n >= n0

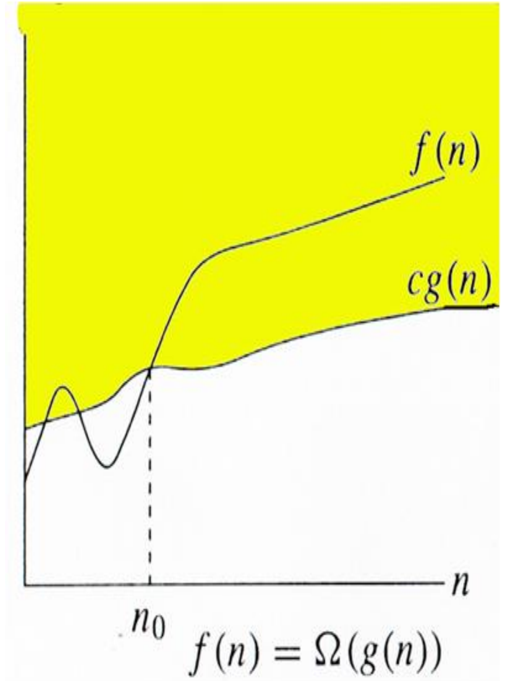
there exist positive constants **c>0 and n0 >=1**

Example:

f(n)=3n+5	**f(n)=27n²+16n**	**f(n)= $2n^{3+}6n^2+2n$**
$3n+5 \leq$ $3n+n \leq 4n$ $n \geq 5$	**$27n^2+16n \leq 27n^2+n^2$** **$\{n \leq n^2\}$** **$27n^2+16n \leq 28n^2$**	
f(n) =**O(n)**	f(n) =O(**n²**)	

Asymptotic Notations- Big Omega (Ω)Notation

$\{f(n)\} = \Omega\ (g(n))$



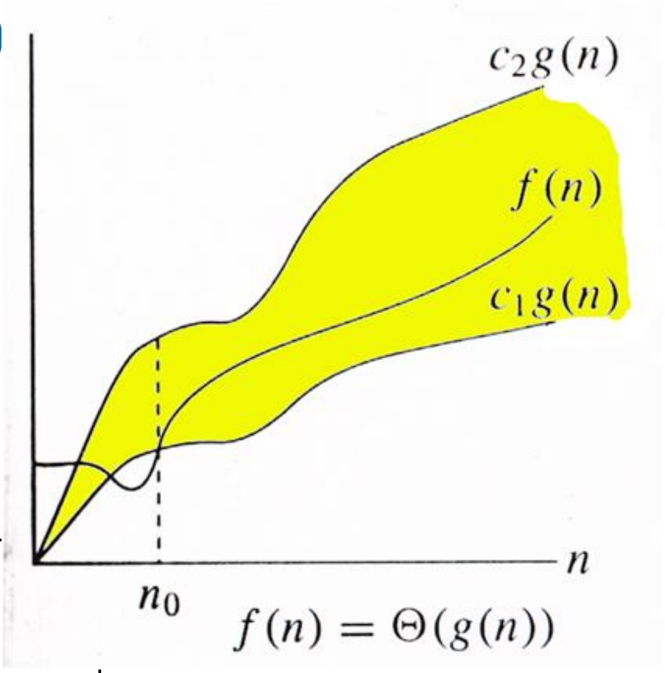
cg(n) <= f(n) for all n >= n0
there exist positive constants **c>0 and n0 >=1**

Example:

$f(n)=3n+5$	**$f(n)=27n^2+16n$**	**$f(n)= 2n^3+6n^2+2n$**
$3n \leq 3n+5$	$27n^2 \leq 27n^2+16n$	
$f(n) = \Omega\ (n)$	$f(n) = \Omega\ (n^2)$	

Asymptotic Notations- Big Theta (Θ)Notation

$\{f(n)\} = \Theta(g(n))$



c1g(n) <= f(n) <= c2g(n) for all n >= n0
there exist positive constants **c1,c2>0 and n0 >=1**

Example:

f(n)=3n+5	**f(n)=$27n^2+16n$**	**f(n)= $2n^3+6n^2+2n$**
$3n \le 3n+n \le 4n$ C1=3 & c2=4	**$27n^2+16n \le 27n^2+n$ $\{n \le n^2\}$** **$27n^2 \le 27n^2+16n \le 28n^2$**	
f(n) =Θ (n)	f(n) =Θ (n^2)	

How to Determine Complexities

- In general, how can you determine the running time of a piece of code? The answer is that it depends on what kinds of statements are used.

Sequences of Statements
  Statement 1;
  Statement 2;
  ………
  Statement k;

Complexity

Total Time = time(statement1) + time(statement2) +… …………………… + time(statement k)

If – then-else statement
  If (Condition) {
    Sequences of statements 1
}
Else {
    Sequences of Statement 2
}

Complexity

Here, either sequence 1 will execute, or sequence 2 will execute.
Therefore, the worst-case time is the slowest of the two possibilities:
  max(time(sequence 1), time(sequence 2))

How to Determine Complexities

for loops

```
for (i = 0; i < N; i++) {
    {
        sequence of statements
    }
}
```

Complexity

The loop executes N times, so the sequence of statements also executes N times.
which is **O(N)** overall.

Nested loops

```
for (i = 0; i < N; i++) {
   for (j = 0; j < M; j++) {
      sequence of statements
   }
}
```

Complexity

The outer loop executes N times. Every time the outer loop executes, the inner loop executes M times
Thus, the complexity is **O(N * M)**

```
x = 0;
  A[n] = some array of length n;
        while (x != A[i])
{
              i++;
}
```

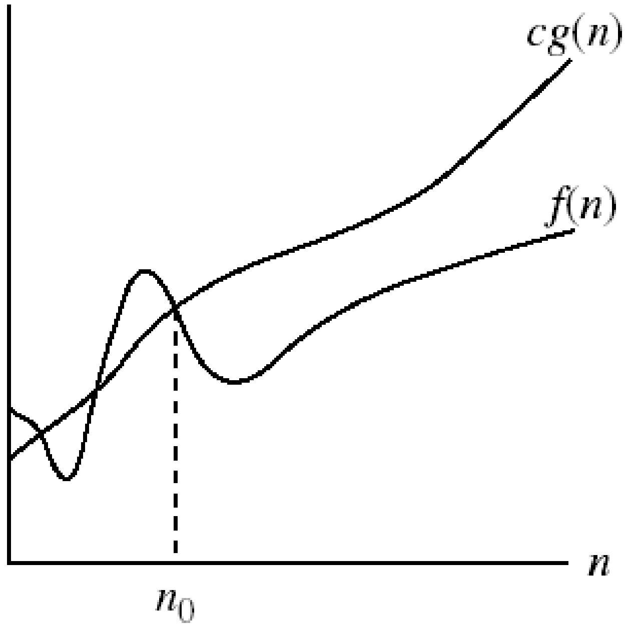
Complexity

The loop executes N times, so the sequence of statements also executes N times.
which is **O(N)** overall.

Asymptotic notations

- *O-notation*

$O(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that
$0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.



$g(n)$ is an ***asymptotic upper bound*** for $f(n)$.

Examples – ('=' symbol readed as 'is' instead of 'equal')

- The function $3n+2 = O(n)$ as $3n+2 \leq 4n$ for all $n \geq 2$
- The function $3n+3 = O(n)$ as $3n+3 \leq 4n$ for all $n \geq 3$
- The function $100n+6 = O(n)$ as $100n+6 \leq 101n$ for all $n \geq 6$
- The function $10n2+4n+2 = O(n\text{^}2)$ as $10n\text{^}2+4n+2 \leq 11n2$ for all $n \geq 5$
- The function $1000n2+100n-6 = O(n2)$ as $1000n2+100n-6 \leq 1001n2$ for all $n \geq 100$
- The function $6*2n+n2 = O(2n)$ as $6*2n +n2 \leq 7*2n$ for all $n \geq 4$
- The function $3n+3 = O(n2)$ as $3n+3 \leq 3n2$ for all $n \geq 2$

Tabular Method

n	Function f(n)	compare	c. g(n)
	10n^2+4n+2		11n^2
1	10+4+2=16	>	11
2	40+8+2=50	>	44
3	90+12+2=104	>	99
4	160+16+2=178	>	176
5	250+20+2=272	<	275
6	360+24+2=386	<	396

- Consider the job offers from two companies. The first company offer contract that will double the salary every year. The second company offers you a contract that gives a raise of Rs. 1000 per year. This scenario can be represented with Big-O notation as –
  - For first company, New salary = Salary X 2^n (where n is total service years)
  - Which can be denoted with Big-O notation as O(2^n)
  - For second company, New salary = Salary +1000n (where n is total service years)
  - Which can be denoted with Big-O notation as O(n)

O(1)

- Describes an algorithm that will always execute in the same time (or space) regardless of the size of the input data set i.e. a computing time is a constant time.

```
int IsFirstElementNull(char String[])
{
     if(strings[0] == ‘\0’)
    {
          return 1;
    }
    return 0;
}
```

O(N): is called *linear time,*

- Describe an algorithm whose performance will grow linearly and in direct proportion to the size of the input data set.

```
int ContainsValue(char String[], int no, char ch)
{
    for( i = 0; i < no; i++)
    {
        if(string[i] == ch)
        {
            return 1;
        }
    }
    return 0;
}
```

$O(N^K)$: (k fixed) refers to *polynomial time; (if k=2, it is called quadratic time, k=3, it is called cubic time),*

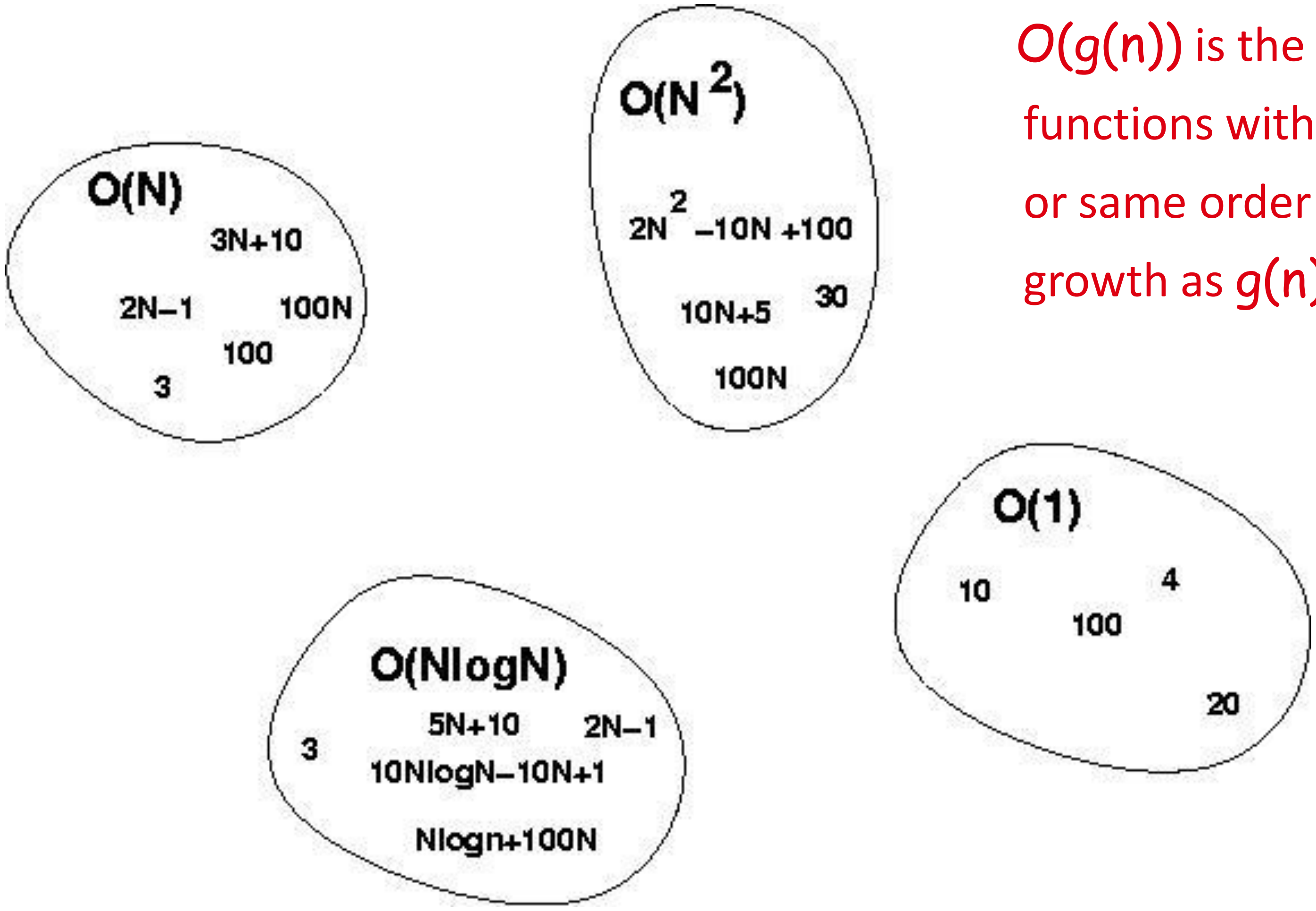
- which represents an algorithm whose performance is directly proportional to the square of the size of the input data set.
  - This is common with algorithms that involve nested iterations over the data set. Deeper nested iterations will result in $O(N^3), O(N^4)$ etc.

```
bool ContainsDuplicates(String[] strings)
{
    for(int i = 0; i < strings.Length; i++)
    {
        for(int j = 0; j < strings.Length; j++)
        {
            if(i == j) // Don't compare with self
            continue;
            if(strings[i] == strings[j])
            return true;
        }
        return false;
    }
}
```

$O(2^N)$: is called ***exponential time,***

- Denotes an algorithm whose growth will double with each additional element in the input data set.
  - The execution time of an $O(2^N)$ function will quickly become very large.

Big-O Visualization



O(*g*(*n*)) is the set of functions with smaller or same order of growth as *g*(*n*)

An Example: Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
           A[j+1] = A[j]
           j = j - 1
      }
      A[j+1] = key
  }
}
```

Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
     key = A[i]
     j = i - 1;
     while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
     }
     A[j+1] = key
  }
}
```

What is the precondition for this loop?

Insertion Sort

```
InsertionSort(A, n) {
  for i = 2 to n {
      key = A[i]
      j = i - 1;
      while (j > 0) and (A[j] > key) {
          A[j+1] = A[j]
          j = j - 1
      }
      A[j+1] = key
  }
}
```

How many times will this loop execute?

Insertion Sort

Statement	Effort
`InsertionSort(A, n) {`	
`for i = 2 to n {`	$c_1 n$
`key = A[i]`	$c_2(n-1)$
`j = i - 1;`	$c_3(n-1)$
`while (j > 0) and (A[j] > key) {`	$c_4 T$
`A[j+1] = A[j]`	$c_5(T-(n-1))$
`j = j - 1`	$c_6(T-(n-1))$
`}`	0
`A[j+1] = key`	$c_7(n-1)$
`}`	0
`}`	

$T = t_2 + t_3 + \ldots + t_n$ where t_i is number of while expression evaluations for the i^{th} for loop iteration

David Luebke

Analyzing Insertion Sort

- $T(n) = c_1 n + c_2(n-1) + c_3(n-1) + c_4 T + c_5(T - (n-1)) + c_6(T - (n-1)) + c_7(n-1)$
  $= c_8 T + c_9 n + c_{10}$
- What can T be?
  - Best case -- inner loop body never executed
    - $t_i = 1 \rightarrow T(n)$ is a linear function
  - Worst case -- inner loop body executed for all previous elements
    - $t_i = i \rightarrow T(n)$ is a quadratic function
  - Average case
    - ???

Upper Bound Notation

- We say InsertionSort's run time is $O(n^2)$
  - Properly we should say run time is *in* $O(n^2)$
  - Read O as "Big-O" (you'll also hear it as "order")
- In general a function
  - f(n) is O(g(n)) if there exist positive constants c and n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$
- Formally
  - $O(g(n)) = \{ f(n) \colon \exists$ positive constants c and n_0 such that $f(n) \leq c \cdot g(n)\ \forall\ n \geq n_0$

Insertion Sort Is $O(n^2)$

- **Proof**
  - Suppose runtime is $an^2 + bn + c$
    - If any of a, b, and c are less than 0 replace the constant with its absolute value
  - $an^2 + bn + c \leq (a + b + c)n^2 + (a + b + c)n + (a + b + c)$
  - $\leq 3(a + b + c)n^2$ for $n \geq 1$
  - Let $c' = 3(a + b + c)$ and let $n_0 = 1$
- **Question**
  - Is InsertionSort $O(n^3)$?
  - Is InsertionSort $O(n)$?

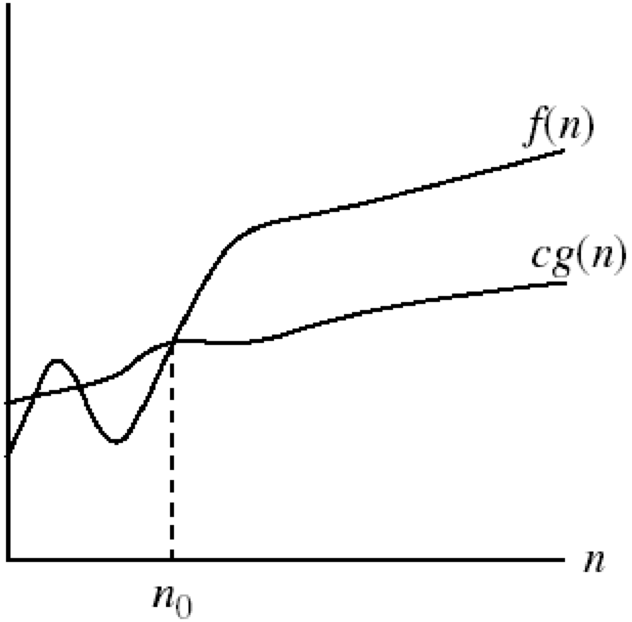
Omega notation (Ω)

- specifically describes the **best-case scenario,** and can be used to describe the minimum execution time (asymptotic lower bounds) required or the space used (e.g. in memory or on disk) by an algorithm.
- The function *f (n) = Ω(g(n)) (read as 'f of n is* said to be Omega of g of n') if and only if there exists a real, positive constant *C and a positive* integer *n0 such that, f (n) ≥ C*g(n) for all n≥ n0*
- Here, *n0 must be greater than 0.*

Asymptotic notations (cont.)

- Ω - *notation*

$$\Omega(g(n)) = \{f(n) : \text{there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\} .$$



$\Omega(g(n))$ is the set of functions with larger or same order of growth as $g(n)$

$g(n)$ is an ***asymptotic lower bound*** for $f(n)$.

Examples –

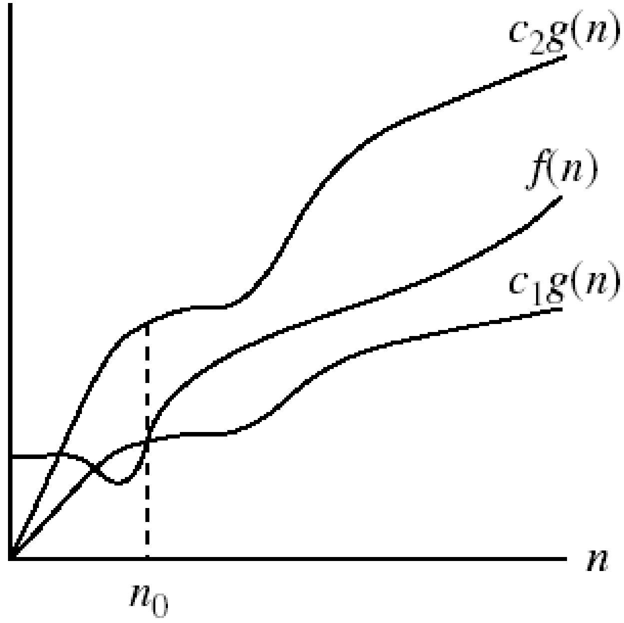
- The function 3n+2 = Ω(n) as 3n+2 ≥ 3n for all n≥1
- The function 3n+3 = Ω(n) as 3n+3 ≥ 3n for all n≥1
- The function 100n+6 = Ω(n) as 100n+6 ≥ 100n for all n≥1
- The function 10n2+4n+2 = Ω(n2) as 10n2+4n+2 ≥ n2 for all n≥1
- The function 6*2n+n2 = Ω(2n) as 6*2n +n2 ≥ 2n for all n≥1
- The function 3n+3 = Ω(1)
- The function 10n2+4n+2 = Ω(n)
- The function 10n2+4n+2 = Ω(1)

Theta notation (Θ)

- Theta specifically describes the **average-case** scenario, and can be used to describe the average execution time required or the space used by an algorithm.
- A description of a function in terms of Θ notation usually only provides an tight bound on t
- The function *f (n) = Θ(g(n)) (read as 'f of n is* said to be Theta of g of n') if and only if there exists a positive constants *C1, C2, and a positive* integer *n0 such that,* $C1*g(n) \le f(n) \le C2*g(n)$ *for all* $n \ge n0$

Asymptotic notations (cont.)

$$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1, c_2, \text{ and } n_0 \text{ such that } 0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\} .$$



$\Theta(g(n))$ is the set of functions with the same order of growth as $g(n)$

$g(n)$ is an ***asymptotically tight bound*** for $f(n)$.

Examples

- The function $3n+2 = \Theta(n)$ as

  $3n+2 \geq 3n$ for all $n \geq 2$ and

  $3n+2 \leq 4n$ for all $n \geq 2$

So, $c1=3$, $c2=4$ and $n0=2$.

- The function $3n+3 = \Theta(n)$
- The function $10n^2+4n+2 = \Theta(n^2)$
- The function $6 * 2^n + n^2 = \Theta(2^n)$

Next Lecture

- Recurrence Relations

Recurrence Relations

- Definition: Given a recursive algorithm a recurrence relation for the algorithm is an equation that gives the run time on an input size in terms of the run times of smaller input sizes.
- When iterative formulas for $T(n)$ are difficult or impossible to obtain, one can use either
  - a recursion tree method,
  - an iteration method , or
  - a substitution method with Induction to get $T(n)$ or a bound $U(n)$of $T(n)$, where $T(n)= \Theta(U(n))$.

Recursion Tree

- A recursion tree is a tree generated by tracing the execution of a recursive algorithm.

Recurrence Relations

- A *recurrence relation for the sequence* $\{a_n\}$ *is an equation that expresses* a_n *in terms of one or* more of the previous terms of the sequence, namely, $a_0, a_1, \ldots a_n$ *for all integers* n *with* $n \geq n_0$*, where* n_0 *is a nonnegative integer. A sequence is called a solution of a recurrence* relation if its terms satisfy the recurrence relation.

- Let $\{a_n\}$ *be a sequence that satisfies the recurrence relation* $a_n = a_{n-1} + a_{n-2}$ *for n = 1, 2, 3, . . . ,* and suppose that $a_0 = 2$ *. What are* $a_1, a_2, and\ a_3$?

Solution: We see from the recurrence relation that

$a_1 = a_0 + 3 = 2 + 3 = 5.$

$a_2 = 5 + 3 = 8$ *and*

$a_3 = 8 + 3 = 11$

- Let $\{a_n\}$ *be a sequence that satisfies the recurrence relation* $a_n = a_{n-1} - a_{n-2}$ *for n = 2, 3, 4, . . . , and suppose that* $a_0 = 3$ *and* $a_1 = 5$*. What are* $a_2,\ and\ a_3$?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$

and $a_3 = a_2 - a_1 = 2 - 5 = -3$

We can find a4, a5, and each successive term in a similar way.

Creating Recurrence Relation

```
Fib(a)                                   T(n)
{
   if(a==1 || a==0)                      1
   return 1;
   return Fib(a-1) + Fib(a-2);           T(n-1)+T(n-2)
 }
```

(comparison, comparison, addition) and also calls itself recursively.

$$f_n = f_{n-1} + f_{n-2}$$

The *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, *is defined by the initial conditions* $f_0 = 0$, $f_1 = 1$, and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$.

- Find the Fibonacci numbers f_2, f_3, f_4, f_5, *and* f_6.

Solution:. Because the initial conditions tell us that $f_0 = 0$ *and* $f_1 = 1$, *using the recurrence relation in the definition we find that*

- $f_2 = f_1 + f_0 = 1 + 0 = 1,$
- $f_3 = f_2 + f_1 = 1 + 1 = 2,$
- $f_4 = f_3 + f_2 = 2 + 1 = 3,$
- $f_5 = f_4 + f_3 = 3 + 2 = 5,$
- $f_6 = f_5 + f_4 = 5 + 3 = 8.$

Solving Recurrence Relations

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms (Expressing F_n as some combination of F_i with $i<n$).

Recurrence Relations

Substitution Method

- The ***substitution method for solving recurrences comprises two steps:***
  - Guess the form of the solution.
  - Use mathematical induction to find the constants and show that the solution works.
- We substitute the guessed solution for the function when applying the inductive hypothesis to smaller values; hence the name “substitution method.”
- This method is powerful, but we must be able to guess the form of the answer in order to apply it.

Example

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \, ,$$

- We guess that the solution is $T(n) = O(n \lg n)$. The substitution method requires us to prove that $T(n) \le cn \lg n$ for an appropriate choice of the constant c > 0. We start by assuming that this bound holds for all positive m < n, in particular for $m = \lfloor n/2 \rfloor$, yielding

$$T(\lfloor n/2 \rfloor) \le c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor).$$

- Substituting into the recurrence yields

$$\begin{aligned}
T(n) &\le 2(c \lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n \\
&\le cn \lg(n/2) + n \\
&= cn \lg n - cn \lg 2 + n \\
&= cn \lg n - cn + n \\
&\le cn \lg n \, ,
\end{aligned}$$

where the last step holds as long as $c \ge 1.$

$$T(n) = \begin{cases} 3T(n-1), & \text{if } n>0, \\ 1, & \text{otherwise} \end{cases}$$

Let us solve using substitution.

$$T(n) = 3T(n-1) = 3(3T(n-2))$$

$$= 3^2T(n-2)$$

$$= 3^3T(n-3) \ldots \ldots$$

$$= 3^nT(n-n)$$

$$= 3^nT(0)$$

$$= 3^n$$

This clearly shows that the complexity of this function is $O(3^n)$.

How to solve linear recurrence relation
Homogeneous Recurrence Relations

- Suppose, a two ordered linear recurrence relation is $F_n = AF_{n-1} + BF_{n-2}$
  - where A and B are real numbers.
- The characteristic equation for the above recurrence relation is –

  $$x^2 - Ax - B = 0$$

- Three cases may occur while finding the roots –
- **Case 1** – If this equation factors as $(x - x_1)(x - x_1) = 0$ and it produces two distinct real roots x_1 and x_2, then $\boxed{F_n = ax_1^n + bx_2^n}$ is the solution. [Here, a and b are constants]
- **Case 2** – If this equation factors as $(x - x_1)^2$
- and it produces single real root x_1, then $F_n = ax_1^n + bnx_1^n$ is the solution.
- **Case 3** – If the equation produces two distinct complex roots, x_1 and x_2 in polar form $x_1 = r\angle\Theta$ and $x_2 = r\angle(-\Theta)$, then following is the solution

  $$\boxed{F_n = r^n(a\cos(n\theta) + b\sin(n\theta))}$$

Problem 1

Solve the recurrence relation $\boxed{F_n = 5F_{n-1} - 6F_{n-2}}$

where $F_0 = 1$ and $F_1 = 4$

Solution

- The characteristic equation of the recurrence relation is – $x^2 - 5x + 6 = 0$

So, $(x-3)(x-2) = 0$

Hence, the roots are –

$$x_1 = 3$$

and $x_2 = 2$

The roots are real and distinct. So, this is in the form of case 1

- Hence, the solution is –

$$F_n = ax_1^n + bx_2^n$$

- Here, $F = a3^n + b2^n$ (*As x*1=3 *and x*2=2)
- Therefore,

$$1 = F_0 = a3^0 + b2^0 = a + b$$

$$4 = F_1 = a3^1 + b2^1 = 3a + 2b$$

- Solving these two equations, we get *a*=2 and *b*=−1
- Hence, the final solution is –

$$F_n = 2.3^n + (-1).2^n = 2.3^n - 2^n$$

Problem 2

Solve the recurrence relation $F_n = 10F_{n-1} - 25F_{n-2}$

where $F_0 = 3$ and $F_1 = 17$

- **Solution**
- The characteristic equation of the recurrence relation is –

$$x^2 - 10x - 25 = 0$$

- So $(x-5)^2 = 0$
- Hence, there is single real root $x_1 = 5$
- As there is single real valued root, this is in the form of case 2
- Hence, the solution is –

$$F_n = ax_1^n + bnx_1^n$$

- $3 = F_0 = a.5^0 + b.0.5^0$ & $17 = F_1 = a.5^1 + b.1.5^1$

- Solving these two equations, we get a=3
- and b=2/5
- Hence, the final solution is $F_n = 3.5^n + (2/5).n.\cancel{2}5^n$

Problem 3

Solve the recurrence relation $F_n = 2F_{n-1} - 2F_{n-2}$

where $F_0 = 1$ and $F_1 = 3$

- **Solution**
- The characteristic equation is $x^2 - 2x - 2x = 0$
  Hence, the roots are – $x_1 = 1+i$ and $x_2 = 1-i$
- In polar form, $x_1 = r\angle\theta$ And $x_2 = r\angle(-\theta)$
  where $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

- The roots are imaginary. So, this is in the form of case 3.
- Hence,

$$F_n = (\sqrt{2})^n (a\cos(n.\Pi/4) + b\sin(n.\Pi/4))$$

$$1 = F_0 = (\sqrt{2})^0 (a\cos(0.\Pi/4) + b\sin(0.\Pi/4)) = a$$

$$3 = F_1 = (\sqrt{2})^1 (a\cos(1.\Pi/4) + b\sin(1.\Pi/4)) = \sqrt{2}(a/\sqrt{2} + b\sqrt{2})$$

Solving these two equations we get *a*=1 and *b*=2

- Hence, the final solution is –

$$F_n = (\sqrt{2})^n (\cos(n.\Pi/4) + 2\sin(n.\Pi/4))$$

Example: Fibonacci sequence

$$f_n = f_{n-1} + f_{n-2}$$

$$f_0 = 0, \quad f_1 = 1$$

Has solution: $$f_n = \lambda_1 r_1^n + \lambda_2 r_2^n$$

Characteristic roots: $$r_1 = \frac{1+\sqrt{5}}{2} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

$$\lambda_1 = \frac{f_1 - f_0 r_2}{r_1 - r_2} = \frac{1}{\sqrt{5}}$$

$$\lambda_2 = \frac{f_0 r_1 - f_1}{r_1 - r_2} = -\frac{1}{\sqrt{5}}$$

$$f_n = \lambda_1 r_1^n + \lambda_2 r_2^n$$

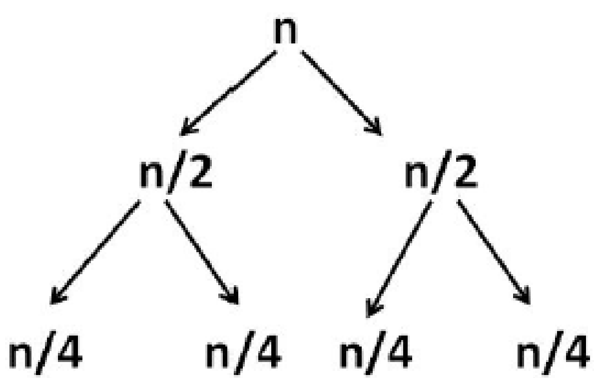
$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

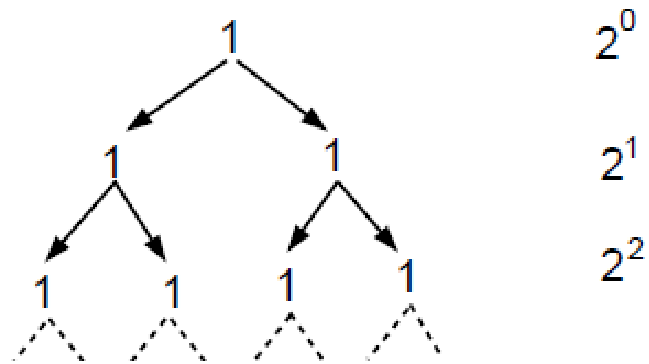
Recursion Tree (Back Subtitution)

$$T(n) = 2T(n/2) + 1$$

$$\text{Here the number of leaves} = 2^{\log n} = n$$

and the sum of effort in each level except leaves is $\sum_{i=0}^{\log_2(n)-1} i = 2^{\log_2(n)} - 1 = n - 1$

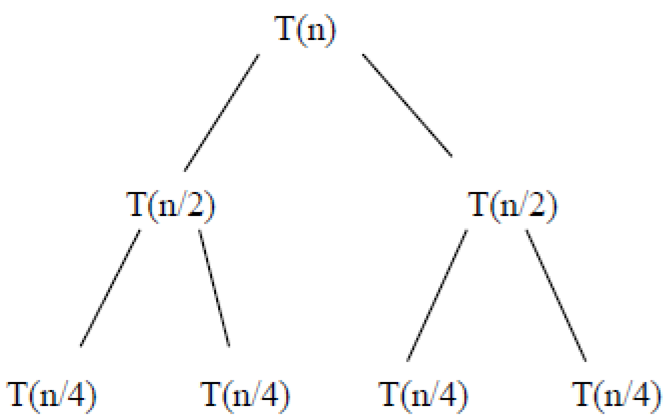


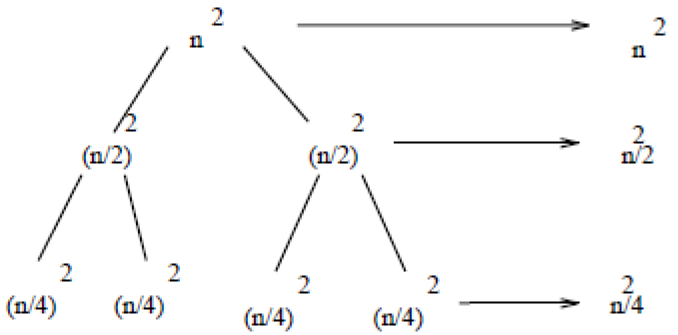


$$\text{total time} = n + n - 1 = 2n - 1 = \Theta(n).$$

Recursion Tree (Back Subtitution)

$$T(n) = 2T(n/2) + n^2$$

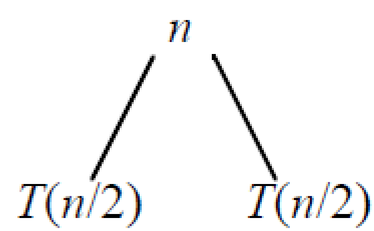


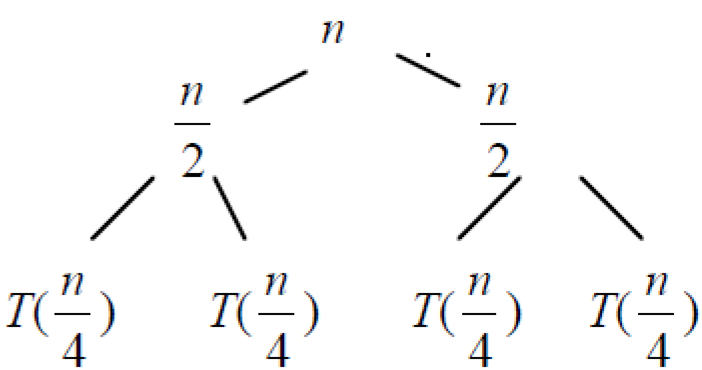


$$T(n) = \sum_{i=0}^{\infty} n^2/2^i = n^2 \sum_{i=0}^{\infty} 1/2^i = \Theta(n^2)$$

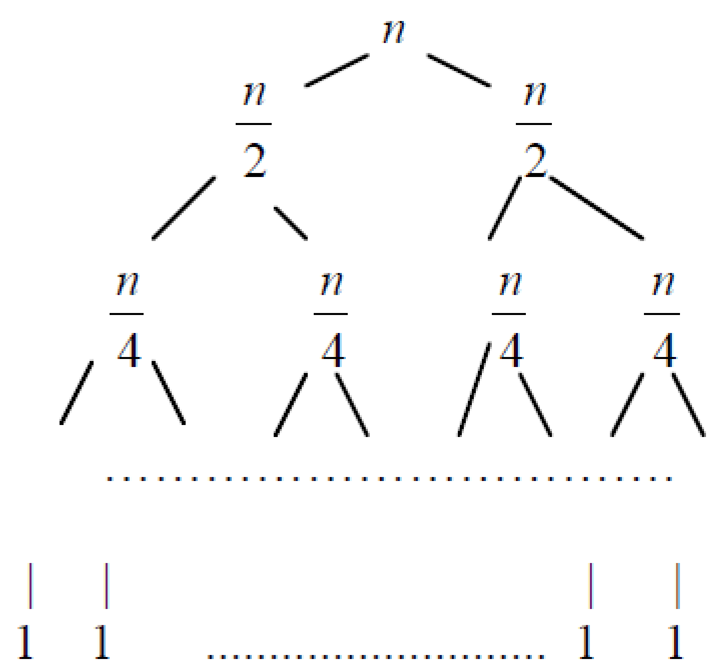
- $T(n) = 2T(n/2) + n$ **for** $k \le n,$ $T(k) = 2T(\frac{k}{2}) + k.$

$T(n)$ ------→



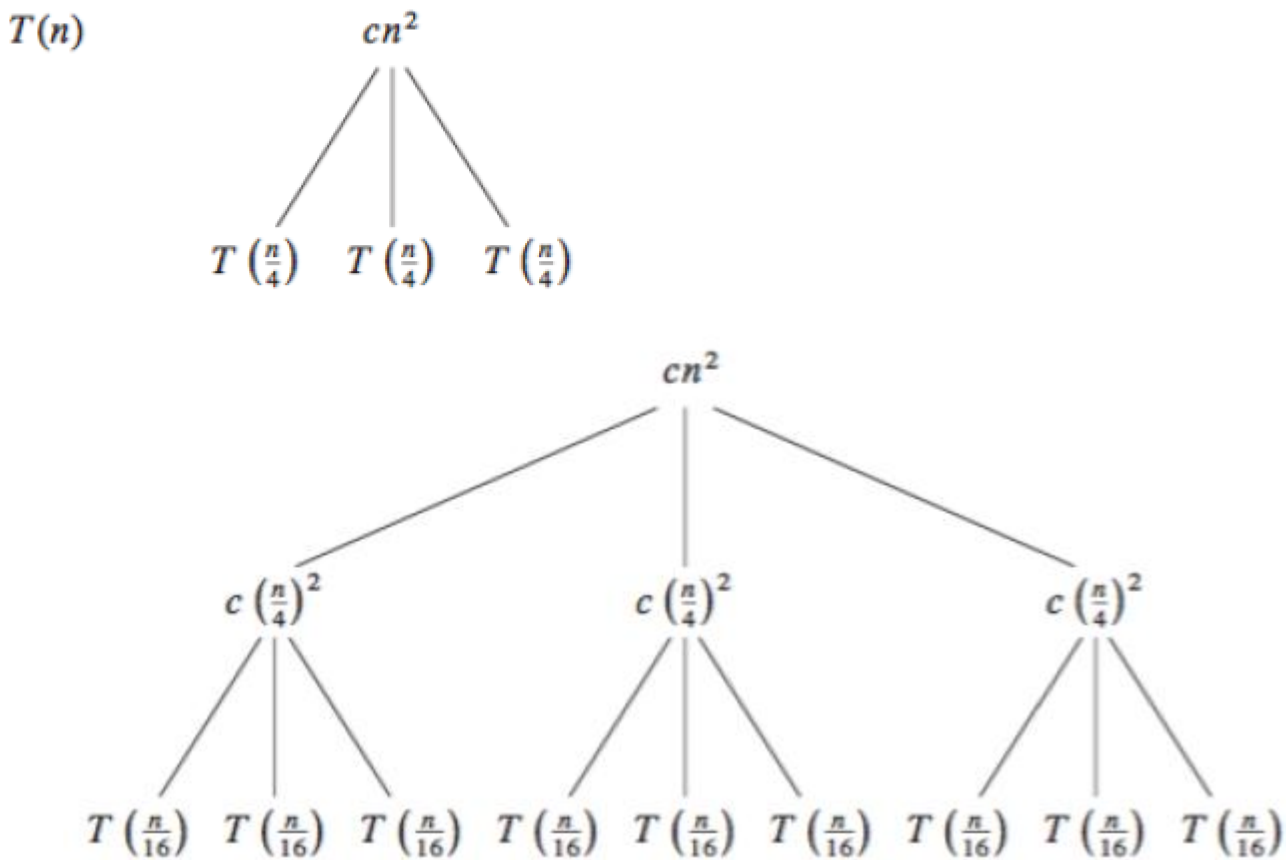


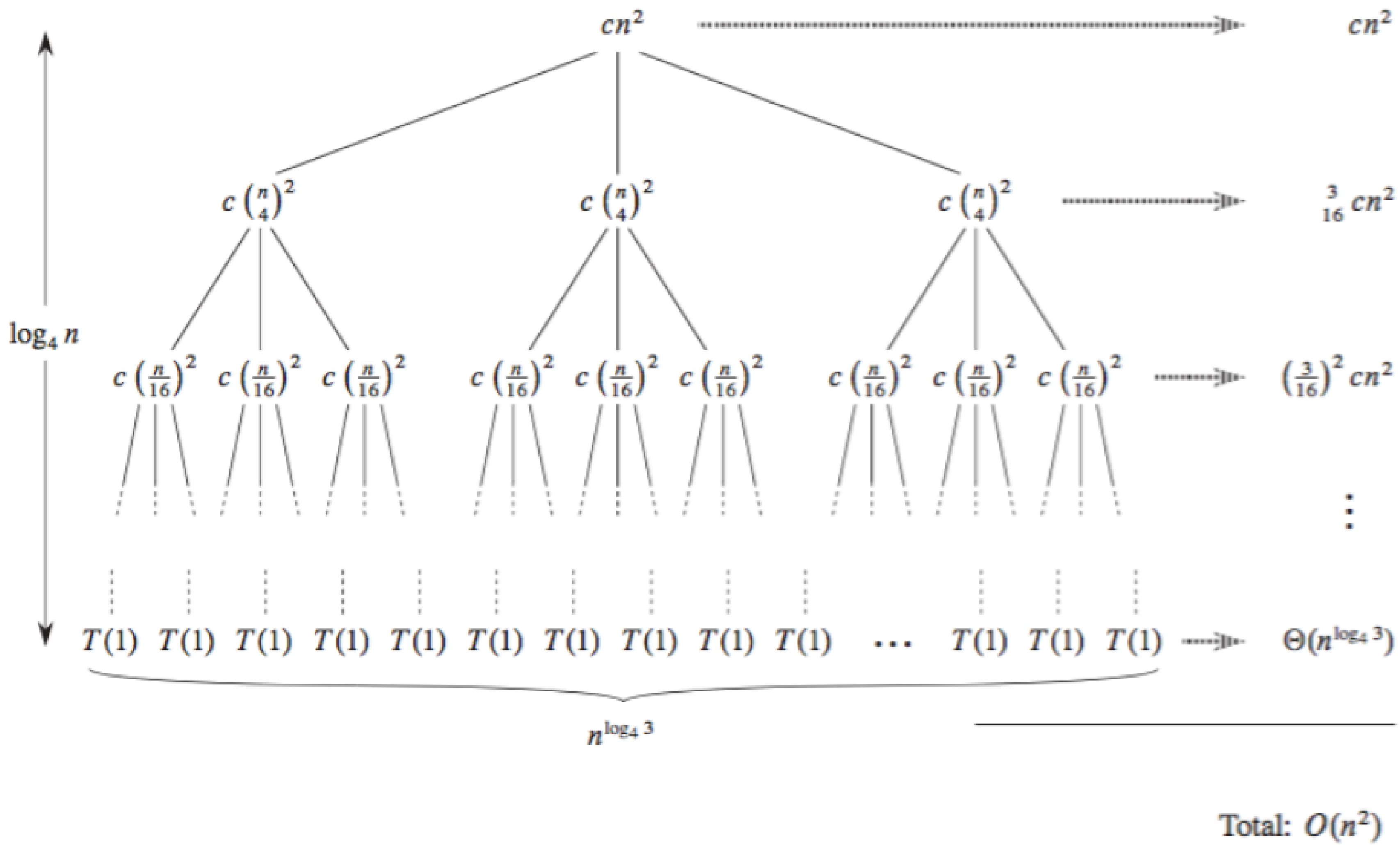
------→



The sum of the values at each level of the tree is n (m nodes times the value $\frac{n}{m}$ in each node). There are $k = \log_2 n$ levels since we can cut the array in half k times. Thus, the sum of the elements in all nodes of the tree is $n \cdot \log_2 n$. Therefore, $T(n) = \Theta(n \cdot \log_2 n)$.

$T(n) = 3T(\lfloor n/4 \rfloor) + \Theta(n^2)$ ie. $T(n) = 3T(n/4) + cn^2.$





cn^2
$c\left(\frac{n}{4}\right)^2$
$c\left(\frac{n}{16}\right)^2$
$\log_4 n$
$T(1)$
$n^{\log_4 3}$
cn^2
$\frac{3}{16}cn^2$
$\left(\frac{3}{16}\right)^2 cn^2$
$\Theta(n^{\log_4 3})$
Total: $O(n^2)$

- Analysis: First we find the height of the recursion tree. Observe that a node at depth i reflects a subproblem of size $n/4^i$. The sub problem size hits n = 1 when $n/4^i = 1$ or $i = \log_4 n$. So the tree has $\log_4 n + 1$ levels
  Now we determine the cost of each level of the tree. The number of nodes at depth is i is 3^i Each node at depth $i = 0, 1, \ldots \log_4 n - 1$ has a cost $c(n/4^i)^2$ of so the total cost of level i is $3^i c(n/4^i)^2 = (3/16)^i cn^2$ However, the bottom level is special. Each of the bottom nodes contribute cost T(1), and there are $3^{\log_4 n} = n^{\log_4 3}$ of them.

So the total cost of the entire tree is

$$\begin{aligned} T(n) &= cn^2 + \frac{3}{16}cn^2 + \left(\frac{3}{16}\right)^2 cn^2 + \cdots + \left(\frac{3}{16}\right)^{\log_4 n - 1} cn^2 + \Theta(n^{\log_4 3}) \\ &= \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) \end{aligned}$$

- The left term is just the sum of a geometric series. So T(n) evaluates to

$$\frac{(3/16)^{\log_4 n} - 1}{(3/16) - 1} cn^2 + \Theta(n^{\log_4 3})$$

- This looks complicated but we can bound it (from above) by the sum of the infinite series

$$\sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) = \frac{1}{1 - (3/16)} cn^2 + \Theta(n^{\log_4 3})$$

- Since functions in $\Theta(n^{\log_4 3})$ are also in $O(n^2)$, this whole expression is $O(n^2)$, Therefore, we can guess that $T(n) = O(n^2)$

Master Theorem

- This method is useful to solve the recurrences of the form, $T(n) = aT(n/b) + f(n)$ where $a \geq 1$ and $b > 1$ and $f(n)$
- This recurrence describes an algorithm that divides a problem of size n into a subproblems, each of size n=b, and solves them recursively.

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

Simplified

- $$T(n) = aT\left(\frac{n}{b}\right) + \theta(n^k \log^p n)$$
- $a \geq 1, b > 1, k \geq 0$ and p is real number
  - Case 1: if $a > b^k$, then $T(n) = \theta(n^{\log_b a})$
  - Case 2: if $a = b^k$ then
    a. If $p > -1$, then $T(n) = \theta(n^{\log_b a} \log^{p+1} n)$
    b. If $p = -1$, then $T(n) = \theta(n^{\log_b a} \log \log n)$
    c. If $p < -1$, then $T(n) = \theta(n^{\log_b a})$
  - Case 3: if $a < b^k$,
    a. If $p \geq 0$ then $T(n) = \theta(n^k \log^p n)$
    b. If $p < 0$ then $T(n) = O(n^k)$

Examples

1. $T(n) = 9T(n/3) + n$

$n^{\log_b a} = n^{\log_3 9} = n^2$
$f(n) = n$
Comparing $n^{\log_b a}$ and $f(n)$
$n = O(n^2)$

Satisfies Case 1 of Master's Theorem
That implies $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$

2. $T(n) = T(2n/3) + 1$

$n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1 = f(n)$
Comparing $n^{\log_b a}$ and $f(n)$
$n^{\log_b a} = \Theta(f(n))$
Satisfies Case 2 of Master's Theorem
That implies $T(n) = \Theta(n^{\log_b a}. \log n) = \Theta(\log n)$

3. $T(n) = 2T(n/2) + n$

$n^{\log_b a} = n^{\log_2 2} = n^1 = n = f(n)$
Comparing $n^{\log_b a}$ and $f(n)$
$n^{\log_b a} = \Theta(f(n))$
Satisfies Case 2 of Master's Theorem
That implies $T(n) = \Theta(n^{\log_b a}. \log n) = \Theta(n \log n)$

4. $T(n) = 3T(n/4) + n \log n$

$n^{\log_b a} = n^{\log_4 3}$
$f(n) = n \log n$
Comparing $n^{\log_b a}$ and $f(n)$
Satisfies Case 3 of Master's Theorem
Checking the regularity condition
$a.f(n/b) < c.f(n)$ (for some constant $c < 1$)
$(3n/4) \log n/4 < c.n \log n$
$(3/4)n[\log n - \log 4] < (3/4)n \log n$ where (c=3/4)
That implies the regularity condition is satisfied
That implies $T(n) = \Theta(f(n)) = \Theta(n \log n)$