7

LINK FRAME ASSIGNMENT AND FORWARD KINEMATICS

7.1 INTRODUCTION

The frame assignment problem for typical manipulators was analyzed in detail in Chapter 6. This chapter deals with the derivation of the general transformation which relates the frames attached to two neighbouring links. The forward and the inverse kinematics problems for a given robot can be formulated using these transformations. The forward kinematics problem deals with the determination of the position and orientation of the gripper frame when the joint variables are numerically specified. The inverse kinematics problem deals with the determination of the joint variables, when the position and orientation of the gripper frame are given.

7.2 DERIVATION OF LINK TRANSFORMATION MATRIX

Consider the two neighbouring frames, $\{i-1\}$ and $\{i\}$. The transformation between the two frames would be a function of four link parameters, viz. α_{i-1}, the link twist, a_{i-1}, the link length, θ_i, the joint angle, and d_i, the joint offset.

Further, for most of the robots, only one of these four parameters is a variable and the other three are constants, fixed by the mechanical design. To obtain the transform $[{}^{i-1}_{i}T]$, we break up the problem into four sub-problems. Each of the four transformations will be a function of one link parameter only.

Consider a typical robot link $(i-1)$, connected to the link $\{i\}$ at the joint (i). Intermediate frames $\{P\}\{Q\}\{R\}$ are assigned for the sake of convenience. It is enough if only the x and the z axes are considered because (x, y, z) form a right handed cartesian system. (Fig. 7.1).

1. The frame $\{R\}$ differs from the frame $\{i-1\}$ only by a rotation of α_{i-1} about the x_{i-1} axis. By the definition of the link twist, it is seen that the z axes of the frames $\{R\}$ and $\{i\}$ are parallel. The x axes of the frames $\{R\}$ and $\{i-1\}$ are identical.
2. The frame $\{Q\}$ differs from the frame $\{R\}$ by a translation a_{i-1} along the X_R axis. The x axes of the frames $\{Q\}$ and $\{R\}$ are along the same line and the z axes of the frames $\{R\}$ and $\{Q\}$ are parallel to the z axis of the frame $\{i\}$. Frame $\{P\}$ differs from frame $\{Q\}$ by a rotation of θ_i about the z_i axis. The z axes of $\{Q\}$ and $\{P\}$ coincide.

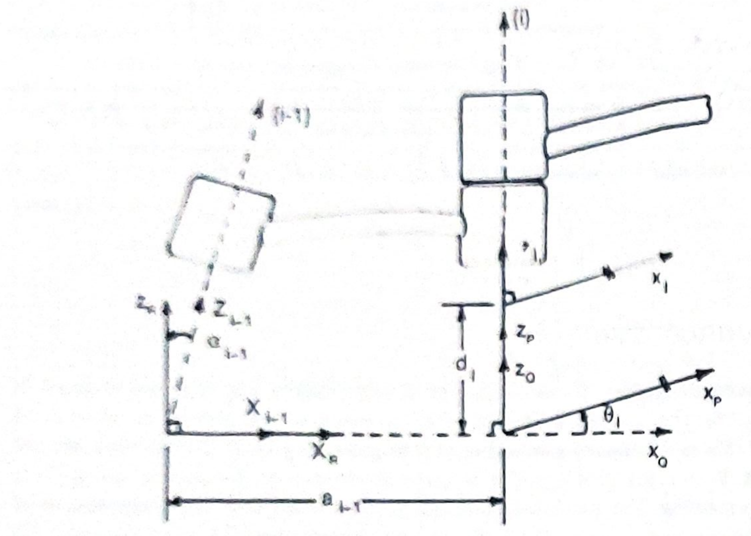

Fig. 7.1 Intermediate Frames for Links

3. The frame $\{i\}$ differs from frame $\{P\}$ by a translation equal to d_i along the z_i axis. Further, x_i is parallel to x_p. In this manner, it is possible to move from frame $\{i-1\}$ to frame $\{i\}$ through the intermediate frames R, Q and P.

A point S specified w.r.t. $\{i\}$ can be represented by

$$\begin{bmatrix} {}^iS \\ \hline 1 \end{bmatrix} = \begin{bmatrix} {}^iS_x \\ {}^iS_y \\ {}^iS_z \\ 1 \end{bmatrix} \tag{7.1}$$

The same point can be represented w.r.t. $\{P\}$ as

$$\begin{bmatrix} {}^PS \\ \hline 1 \end{bmatrix} = [{}^P_iT] \begin{bmatrix} {}^iS \\ \hline 1 \end{bmatrix} \tag{7.2}$$

Similarly

$$\begin{bmatrix} {}^QS \\ \hline 1 \end{bmatrix} = [{}^Q_PT] \begin{bmatrix} {}^PS \\ \hline 1 \end{bmatrix} \tag{7.3}$$

$$= [{}^Q_PT][{}^P_iT] \begin{bmatrix} {}^iS \\ \hline 1 \end{bmatrix} \tag{7.4}$$

and

$$\begin{bmatrix} {}^{i-1}S \\ \hline 1 \end{bmatrix} = [{}^{i-1}_RT][{}^R_QT][{}^Q_PT][{}^P_iT] \begin{bmatrix} {}^iS \\ \hline 1 \end{bmatrix} \tag{7.5}$$

Symbolically the transformation from $\{i\}$ to $\{i-1\}$ can be written as

$$[{}^{i-1}_iT] = \text{Rot}(x_{i-1}, \alpha_{i-1}) \cdot \text{Trans}(x_{i-1}, a_{i-1}) \cdot \text{Rot}(z_i, \theta_i) \cdot \text{Trans}(z_i, d_i) \tag{7.6}$$

Eq. (7.6) means the transformation due to a rotation of α_{i-1} about the axis x_{i-1}, transformation due to a shift of a_{i-1} along x_{i-1}, transformation due to a rotation of θ_i about z_i and transformation due to a shift along the z_i axis by a distance d_i.

Clearly,

$$[{}^{i-1}_iT] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_{i-1} & -S\alpha_{i-1} & 0 \\ 0 & S\alpha_{i-1} & C\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C\theta_i & -S\theta_i & 0 & 0 \\ S\theta_i & C\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.7}$$

where C represents cosine and S represents sine function.

Multiplying, (7.7) can be written as

$$[{}^{i-1}_iT] = \begin{bmatrix} C\theta_i & -S\theta_i & 0 & a_{i-1} \\ S\theta_i\, C\alpha_{i-1} & C\theta_i\, C\alpha_{i-1} & -S\alpha_{i-1} & -S\alpha_{i-1}d_i \\ S\theta_i\, S\alpha_{i-1} & C\theta_i\, S\alpha_{i-1} & C\alpha_{i-1} & C\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 1

For the link parameter Table 7.1, obtain the transform $[{}^0_1T], [{}^1_2T], [{}^2_3T], [{}^0_3T]$.

Table 7.1

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1*
2	90°	0	d_2*	0°
3	0	0	L_2	θ_3*

Solution

Link 1 to link 0:

$$\alpha_0 = 0;\ a_0 = 0;\ \theta_1 = \theta_1;\ d_1 = 0$$

Clearly,

$$[{}^0_1T] = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Link 2 to link 1:

$$\alpha_1 = 90°;\ a_1 = 0;\ \theta_2 = 0;\ d_2 = d_2$$

$$[{}^1_2T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -d_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
[{}^2_3T] = \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Further, $[{}^0_3T] = [{}^0_1T][{}^1_2T][{}^2_3T]$; Clearly $[{}^0_3T]$ is a function of all the variables: θ_1, d_2, θ_3.

7.3 DESCRIPTION OF AN INDUSTRIAL ROBOT

Figure 7.2 shows a typical industrial robot, very similar to the Unimate PUMA 560 (TM). The frame assignment for the various links are shown in Fig. 7.3. Frame {1} corresponds to a revolute joint causing a rotation in the horizontal plane about a vertical axis. Link-2 and Link-3 are simple revolute joints which can be tilted up or down. Link-4 is a rotary joint. Link-5 is a swivel joint and Link-6 is a rotary joint. DC servomotors are used in the various position control systems. The 6-axis robot can be driven by a computer such that the tool tip can take an arbitrary position and orientation. It may be seen from the elevation that Link-3 swivels about an axis which is located slightly off the centre by an amount $= a_3$. (Fig. 7.3).

1. The base frame is defined by (x_0, y_0, z_0) with z_0 being vertical. The origin of {1} and {0} are chosen to be coincident. The axes z_0 and z_1 are also coincident axes. The angle made by x_1 with respect to x_0 is θ_1.
2. The origin of {2} and {1} can also be chosen as coincident. The axis z_2 is in the horizontal plane. x_1 is perpendicular to both z_1 and z_2. For the direction chosen for z_2, it is clear that $\alpha_1 = -90°$.
3. The origin of {3} is located on the member (3) at the pivot. The joint axis z_3 is the pivotal axis about which the link-3 revolves. The axis x_2 is perpendicular to both z_2 and z_3. The link twist is zero and the link length is a_2. Further x_2 and x_3 make an offset of d_3 on the axis z_3.
4. The joint axis-4 is z_4. The axis x_3 is chosen to be perpendicular to both z_3 and z_4. For the particular axes assignment chosen, the link twist $\alpha_3 = -90°$. The distance between z_3 and z_4 along x_3 is the link length a_3. Also x_3 and x_4 make an offset d_4 along z_4.
5. The origin of {4} is located at the swivel joint where z_4 and z_5 intersect. The origin of {5} is coincident with the origin of {4}. z_5 is the swivel axis

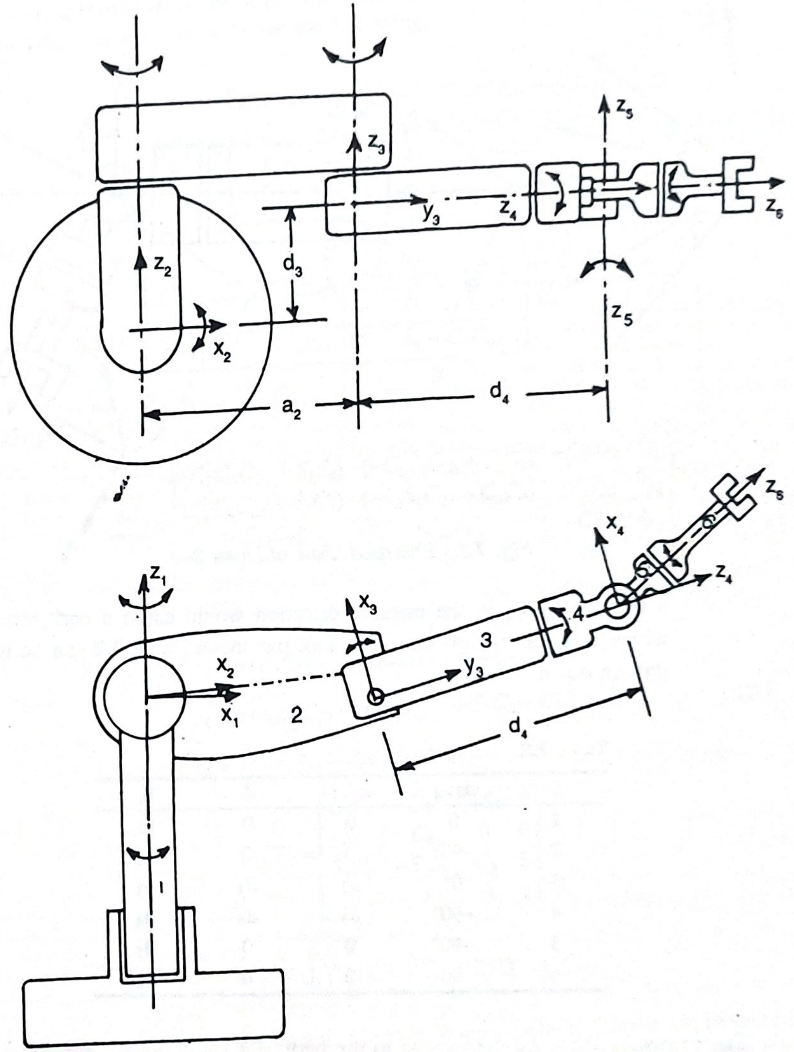

Fig. 7.2 Frame Assignment for 6-deg of Freedom Robot

for the joint-5. x_4 is perpendicular to both z_4 and z_5 and the link twist $\alpha_4 = 90°$. The link length $a_4 = 0$.

6. The Joint-6 is a rotary joint about z_6. The origin of {6} is chosen to be the same as the origin of {5} because z_5 and z_6 intersect. x_5 is perpendicular to both z_5 and z_6. For the chosen direction of x_5, the link twist $\alpha_5 = -90°$. Initially, x_6 is chosen to be parallel to x_5. However, as the robot performs, the angle made by x_6 with respect to x_5 would be θ_6 such that

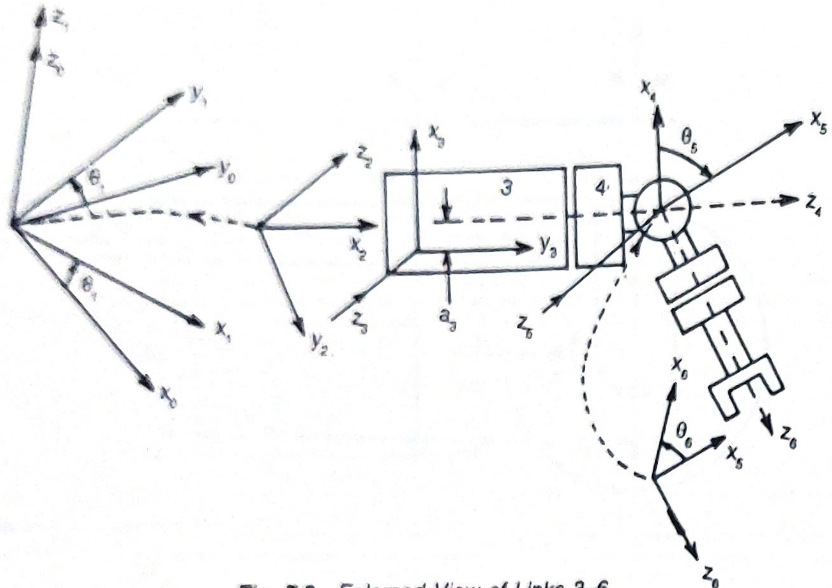

Fig. 7.3 Enlarged View of Links 3–6

a rotation by θ_6 in the positive direction would cause a cork screw to advance in the z_6 direction. The link parameter Table 7.2 can be easily written down.

Table 7.2

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	−90°	0	0	θ_2
3	0°	a_2	d_3	θ_3
4	−90°	a_3	d_4	θ_4
5	+90°	0	0	θ_5
6	−90°	0	0	θ_6

The entries in Table 7.2 correspond to the particular choice of the various frames assigned by us.

Since the axes z_4, z_5 and z_6 intersect we have a common origin for {4}, {5} and {6}. Similarly, {0}, {1} and {2} have a common origin.

1. Using line 1 of Table 7.2, we get

$$ {}^0_1T = \begin{bmatrix} C\theta_1 & -S\theta_1 & 0 & 0 \\ S\theta_1 & C\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.8}$$

where $C\theta_1 = \cos(\theta_1)$; $S\theta_1 = \sin(\theta_1)$.

Using line 2 of the Table 7.2 we get

$$[{}^1_2T] = \mathrm{Rot}(\alpha_1)\ \mathrm{Trans}\,(a_1)\cdot \mathrm{Rot}\,(\theta_2)\cdot \mathrm{Trans}\,(d_2)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha_1 & -S\alpha_1 & 0 \\ 0 & S\alpha_1 & C\alpha_1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & 0 \\ S\theta_2 & C\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using $\sin \alpha_1 = -1$ and $\cos \alpha_1 = 0$ we get

$$[{}^1_2T] = \begin{bmatrix} C\theta_2 & -S\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -S\theta_2 & -C\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.9}$$

Using line 3 we get

$$[{}^2_3T] = [\mathrm{Rot}\,(\alpha_2)\cdot \mathrm{Trans}\,(a_2)\cdot \mathrm{Rot}\,(\theta_3)\cdot \mathrm{Trans}\,(d_3)]$$

$$= \begin{bmatrix} 1 & 0 & 0 & a_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & 0 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 & a_2 \\ S\theta_3 & C\theta_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.10}$$

Similarly,

$$[{}^3_4T] = \begin{bmatrix} C\theta_4 & -S\theta_4 & 0 & a_3 \\ 0 & 0 & 1 & d_4 \\ -S\theta_4 & -C\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.11}$$

$$[{}^4_5T] = \begin{bmatrix} C\theta_5 & -S\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S\theta_5 & C\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.12}$$

$$[{}^5_6T] = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -S\theta_6 & -C\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.13}$$

Multiplying $[{}^1_2T]$ by $[{}^2_3T]$ we get

$$[{}^1_3T] = \begin{bmatrix} C_{23} & -S_{23} & 0 & a_2C_2 \\ 0 & 0 & 1 & d_3 \\ -S_{23} & -C_{23} & 0 & -a_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.14}$$

where $C_{23} = \cos(\theta_2 + \theta_3)$

Further, $$[{}^{3}_{4}T][{}^{4}_{5}T][{}^{5}_{6}T] = [{}^{3}_{6}T]$$

Hence

$$[{}^{3}_{6}T] = \left[\begin{array}{c|c|c|c} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & -C_4S_5 & a_3 \\ \hline S_5C_6 & -S_5S_6 & C_5 & d_4 \\ \hline -S_4C_5C_6 - C_4S_6 & S_4C_5S_6 - C_4C_6 & S_4S_5 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{7.15}$$

Now, $[{}^{0}_{6}T] = [{}^{0}_{1}T][{}^{1}_{3}T][{}^{3}_{6}T]$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.16}$$

Then, the following 12 equations (7.17) are valid:

$$\begin{aligned}
r_{11} &= C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + S_1[S_4C_5C_6 + C_4S_6] \\
r_{21} &= S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - C_1[S_4C_5C_6 + C_4S_6] \\
r_{31} &= -S_{23}(C_4C_5C_6 - S_4S_6) - C_{23}S_5C_6 \\
r_{12} &= C_1[C_{23}(-C_4C_5S_6 - S_4C_6) + S_{23}S_5S_6] + S_1[C_4C_6 - S_4C_5S_6] \\
r_{22} &= S_1[C_{23}(-C_4C_5S_6 - S_4C_6) + S_{23}S_5S_6] - C_1[C_4C_6 - S_4C_5S_6] \\
r_{32} &= -S_{23}[-C_4C_5S_6 - S_4C_6] + [C_{23}S_5S_6] \\
r_{13} &= -C_1[C_{23}C_4S_5 + S_{23}C_5] - S_1S_4S_5 \\
r_{23} &= -S_1[(C_{23}C_4S_5 + S_{23}C_5)] + C_1S_4S_5 \\
r_{33} &= S_{23}C_4S_5 - C_{23}C_5 \\
P_x &= C_1[a_2C_2 + a_3C_{23} - d_4S_{23}] - d_3S_1 \\
P_y &= S_1[a_2C_2 + a_3C_{23} - d_4S_{23}] + d_3C_1 \\
P_z &= [-a_3S_{23} - a_2S_2 - d_4C_{23}]
\end{aligned} \tag{7.17}$$

The above procedure indicates a method to compute the position and orientation of frame {6} with respect to the base frame {0} of the robot.

Supposing P, the tip of the tool is specified w.r.t. the frame {6} as: $\begin{bmatrix} P_{x6} \\ P_{y6} \\ P_{z6} \end{bmatrix}$,

then the same point P could be represented w.r.t the base frame as:

$$\begin{bmatrix} P_{x0} \\ P_{y0} \\ P_{z0} \\ \hline 1 \end{bmatrix} = [{}^{0}_{6}T] \begin{bmatrix} P_{x6} \\ P_{y6} \\ P_{z6} \\ \hline 1 \end{bmatrix} \tag{7.18}$$

In the above Eq., if $\theta_1, \theta_2, \ldots, \theta_6$ and P_{x6}, P_{y6}, P_{z6} are numerically specified, we can compute P_{x0}, P_{y0}, P_{z0} numerically. This is a straight forward problem. The

following results would be of great use while solving the inverse kinematics problem. The reader is advised to derive the following:

(1)

$$[{}^{0}_{3}T] = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & -S_1 & (a_2C_1C_2 - d_3S_1) \\ S_1C_{23} & -S_1S_{23} & C_1 & (a_2S_1C_2 + C_1d_3) \\ -S_{23} & -C_{23} & 0 & (-S_2a_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.19}$$

(2)

$$[{}^{3}_{6}T] = \left[\begin{array}{c|c|c|c} (C_4C_5C_6 - S_4S_6) & (-C_4C_5S_6 - S_4C_6) & (-C_4S_5) & a_3 \\ \hline (S_5C_6) & (-S_5S_6) & C_5 & d_4 \\ \hline (-S_4C_5C_6 - C_4S_6) & (S_4C_5S_6 - C_4C_6) & (S_4S_5) & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \tag{7.20}$$

(3)

$$[{}^{0}_{4}T] = \begin{bmatrix} (C_1C_4C_{23} + S_1S_4) & (-C_1S_4C_{23} + S_1C_4) & -C_1S_{23} & {}^{0}x_4 \\ (S_1C_4C_{23} - S_4C_1) & (-S_4S_1C_{23} - C_1C_4) & -S_1S_{23} & {}^{0}y_4 \\ (-S_{23}C_4) & (S_{23}S_4) & -C_{23} & {}^{0}z_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where,

$$\begin{aligned}
{}^{0}x_4 &= +\{a_3C_1C_{23} - d_4C_1S_{23} + a_2C_1C_2 - d_3S_1\} \\
{}^{0}y_4 &= +\{a_3S_1C_{23} - d_4S_1S_{23} + a_2S_1C_2 + C_1d_3\} \\
{}^{0}z_4 &= +\{-a_3S_{23} - d_4C_{23} - S_2a_2\}
\end{aligned} \tag{7.21}$$

$$[{}^{4}_{6}T] = \begin{bmatrix} C_5C_6 & -C_5S_6 & -S_5 & 0 \\ S_6 & C_6 & 0 & 0 \\ S_5C_6 & -S_5S_6 & C_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.22}$$

FURTHER READING

For further reading on the description of links by matrices, refer to Denavit and Hartenberg [1955] and Paul [1981]. Homogenous coordinate systems are dealt with in Duda and Hart [1973]. Fast computation of trigonometric functions has been reported in Ruoff [1981]. Link frame assignment is dealt with in detail in books by Craig [1986] and Fu, Gonzalez and Lee [1987].