Integral Calculus

1. **Properties of definite integral :**

   When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the following rules -

   1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
   2. $\int_a^a f(x)dx = 0$
   3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
   4. $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
   5. $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$
   6. Let f be a continuous function on $[-a, a]$,

      a. if f is even (i.e $f(-x) = f(x)$), then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$

      b. if f is odd (i.e $f(-x) = -f(x)$), then $\int_{-a}^a f(x)dx = 0$

2. **Fundamental Theorem of Calculus Part 1 :** If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$.

$$F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

3. **Fundamental Theorem of Calculus: Part 2 : The Evaluation Theorem** If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

4. **Summary of Important Results :**

   1. The net change in a function $F(x)$ over an interval $a \leq x \leq b$ is the integral of its rate of change :

$$F(b) - F(a) = \int_a^b F'(x)dx$$

   If an object with position function $s(t)$ moves along a coordinate line, its velocity is $v(t) = s'(t)$. Then the integral of $v(t)$ gives us total displacement over the time interval $t_1 \leq t \leq t_2$. Further the integral of $|v(t)|$ gives us total distance traveled over the time interval.

   2. Area of the region bounded by the graph of a function $y = f(x)$ and the x- axis is the sum of the absolute values of the definite integrals over all the subintervals where function does not change the sign.

   We know that area of a region bounded by nonnegative function $f(x)$ on $[a, b]$ and x - axis is $\int_a^b f(x)dx$. If the function takes on both positive as well as negative values, the region is divided into subregions which are either above or below x-axis.

The definite integral gives negative of area for the region below x-axis. So we have to add absolute values of definite integrals to get total area. Thus the procedure is as follows -

(i) Subdivide $[a, b]$ at the zeros of f.

(ii) Integrate f over each subinterval.

(iii) Add the absolute values of the integrals.

3. If f is integrable on $[a, b]$, then average value on $[a, b]$ or mean is

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

4. A continuous function attains its average value.

5. **Area between the curves :** If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves** $y = f(x)$ **and** $y = g(x)$ **from** a **to** b **is the integral of** $f - g$ **from** a **to** b :

$$\int_a^b f(x) - g(x)dx$$

6. **Volumes using cross sections : Slicing by parallel planes:** The **Volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)dx$$

The procedure of finding volume by slicing :

1. Sketch the solid and a typical cross-section.
2. Find a formula for $A(x)$, the area of a typical cross-section.
3. Find the limits of integration.
4. Integrate $A(x)$ to find the volume.

7. **Volumes of solids of revolution : Disk Method :** The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution.

Here the cross sectional area $A(x)$ is the area of the disk of a radius $R(x)$, the distance of the boundary of the plane region and the axis of revolution.

Volume by disk method $= \int_a^b \pi[R(x)]^2 dx$

We can apply this method only if one boundary of the region is the axis of revolution.

8. **Volume of solid of revolution : Washer Method :** If the axis of revolution is not one of the boundaries of the region which we want to revolve, then the cross sections perpendicular to the axis of revolution are washers instead of disks.

The area of washer is

$AreaA(x) = \pi([R(x)]^2 - [r(x)]^2)$.

Here $R(x)$ and $r(x)$ are outer and inner radii of washer respectively. Then

Volume by Washer Method $= \int_a^b \pi([R(x)]^2 - [r(x)]^2)dx$.

9. **Arc length :** If f is a continuous function on $[a, b]$, then the length (arc length) of the curve $y = f(x)$ from $A(a, f(a))$ and $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

10. **Area of surface of revolution :** Consider the surface generated by revolving the graph of a function $y = f(x), a \leq x \leq b$ about the x-axis. $A = \int_a^b 2\pi \times f(x)\sqrt{1 + f'(x)^2} dx$

11. **Improper Integrals** We have seen definite integrals with two properties - 1. The domain of the integration $[a, b]$ is finite. 2. The range of the integrand is finite. In practice, we have to deal with the integrals where one or both of these conditions can not be satisfied. Such integrals are called improper integrals.

    Improper Integrals of Type I : Integrals with infinite limits are called improper integrals of type I.

    1. If $f(x)$ is continuous on $[a, \infty)$, then

    $$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

    2. If $f(x)$ is continuous on $(-\infty, b]$, then

    $$\int_{-\infty}^b f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$$

    3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

    $$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$$

    In each case, if the limit is finite, we say that the improper integral converges and the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

    Theorem : The integral $\int_1^\infty \frac{dx}{x^p}$ This integral converges if $p > 1$ and diverges if $p \leq 1$.

    Improper integrals of Type II : Integrals of functions that become infinite at a point within the interval of integration are called improper integrals of type II.

    1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then

    $$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

    2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then

    $$\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$$

    3. If $f(x)$ is discontinuous at c where $a < c < b$, then

    $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

In each case, if the limit is finite, we say that the improper integral converges and the limit is the value of the improper integral. If the limit fails to exist, the improper integral diverges.

Tests for Convergence and Divergence : When we can not evaluate improper integrals directly, we try to determine whether it converges or diverges. If it converges, we can use numerical methods to find approximate value.

Direct Comparison Test : Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$, then

1. $\int_a^\infty f(x)dx$ converges if $\int_a^\infty g(x)dx$ converges.
2. $\int_a^\infty g(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.

Limit Comparison Test : If positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x} \to \infty \frac{f(x)}{g(x)} = L, 0 < L < \infty$$

, then

$$\int_a^\infty f(x)dx \quad \text{and} \quad \int_a^\infty g(x)dx$$

both converge or both diverge.

12. **Reduction formula :**

Theorem : If $I_n = \int \sin^n x dx$, then prove that $I_n = \frac{n-1}{n} I_{n-2} - \frac{\sin^{-1} x \cos x}{n}$

Theorem : If $I_n = \int_0^{\pi/2} \sin^n x dx$, then prove that $I_n = \frac{n-1}{n} I_{n-2}$

Theorem : $\int_0^{\pi/2} \sin^m t \cos^n t = \dfrac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \times \dfrac{\pi}{2}$ if m, n both are even.

$$= \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \text{otherwise.}$$

List of Reduction Formulae:

(a) $\int_0^{\pi/2} \sin^m t \cos^n t = \dfrac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \times \dfrac{\pi}{2}$ if m, n both are even.

$$= \frac{(m-1)(m-3)\cdots(n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \text{otherwise.}$$

(b) $\int_0^{\pi} \sin^m t \cos^n t = 2\int_0^{\pi/2} \sin^m t \cos^n t$ if n is even and zero if n is odd.

(c) $\int_0^{2\pi} \sin^m t \cos^n t = 4\int_0^{\pi/2} \sin^m t \cos^n t$ if m, n both are even and zero otherwise.

13. **Gamma Function :** We define Gamma function as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx; \qquad n > 0$$

It is an improper integral and converges if $n > 0$.

Properties of Gamma Function :

(a) $\Gamma 1 = 1$

(b) $\Gamma(n+1) = n\Gamma n; \quad \Gamma(n+1) = n!$ if n is positive integer.

(c) $\Gamma n = 2\int_0^\infty e^{-x^2} x^{2n-1} dx$

(d) $\Gamma(n+1) = \int_0^\infty e^{-y^{1/n}} dy$

(e) $\Gamma n = \int_0^1 (\ln(1/y))^{n-1} dy$

(f) $\int_0^\infty e^{-kx} x^{n-1} dx = \dfrac{\Gamma n}{k^n}$

14. **Beta Function :** We define beta function as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx; \quad m,n > 0$$

Properties of Beta Function :

(a) $\beta(m,n) = \beta(n,m)$

(b) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$. Hnece $\int_0^{\pi/2} \sin^p\theta \cos^q\theta d\theta = (1/2)\beta(\dfrac{p+1}{2}, \dfrac{q+1}{2})$.

(c) $\beta(m,n) = \int_0^\infty \dfrac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^1 \dfrac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx$

(d) $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$

Exercise

1. The temperature $T(^0F)$ of a room at time t minutes is given by $T = 85 - 3\sqrt{25-t}$ for $0 \le t \le 25$.

   a. Find the room's temperature when $t = 0$, $t = 16$ and $t = 25$.

   b. Find the room's average temperature for $0 \le t \le 25$ (a. 70,76,85. b. 75)

2. Find the area of the regions between the curves and lines - (a) $x - y^3 = 0$ and $x - y = 0$ (Ans: 1/2)

   (b) $y = x^4$ and $y = 8x$ (Ans:48/5)

   (c) $y = x^2$ and $y = -x^2 + 4x$ (Ans: 8/3)

   (d) $x = 2y^2$, $x = 0$, $y = 3$ (Ans: 18)

   (e) $x = y^2 - y$, and $x = 2y^2 - 2y - 6$ (Ans: 125/6)

3. Find the volume of the solid that lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$. (Ans : 8π)

4. Find the volumes of the solids generated by the regions bounded by the given curves about the given axis.

   1. $y = x^3, y = 0, x = 2; x$ axis. (Ans : $128\pi/7$)
   2. $x^2 + y^2 = 3, x = \sqrt{3}, y = \sqrt{3}; y$ axis. (Ans : $\sqrt{3}\pi$)
   3. $y = \sqrt{x}, y = 2, x = 0$; about the line $x = 4$ (Ans : $224\pi/15$)

4. $y = 2x - 1, y = \sqrt{x}, x = 0$; about y axis. (Ans : $7\pi/15$)
5. $x = 2y - y^2, x = 0$ about x axis. (Ans: $8\pi/3$)
6. $y = x^2, y = 0, x = 1$ about $x = -1$. (Ans : $7\pi/6$)

5. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4, y = x, x = 0$ about
   (a) x axis using washer method (Ans: 16π)
   (b) the line $y = 8$ using the washer method (Ans: 48π)

6. Find the length of the curve
   (a) $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt, -\pi/4 \le y \le \pi/4$ (Ans : 2)
   (b) $y = x^{3/2}$ from $x = 0$ to $x = 4$.
   (c) $x = (y^2/4) + (1/8y^2)$ from $y = 1$ to $y = 2$. (Ans: 123/32)

7. Find the length of astroid $x^{2/3} + y^{2/3} = 1$ by finding the length of half the first quadrant portion, $y = (1 - x^{2/3})^{3/2}, \sqrt{2}/4 \le x \le 1$, and multiplying by 8. (Ans : 6)

8. Derive the formula for circumference of the circle $x^2 + y^2 = r^2$ by finding the length of quarter circle and multiplying by 4.

9. Evaluate the following improper integrals :
   (a) $\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\, ds$ (Ans : $2 + \pi/2$)
   (b) $\int_0^\infty \frac{16 \tan^{-1} x}{1+x^2}\, dx$ (Ans : $2\pi^2$)

10. Test the convergence of the following integrals :
    (a) $\int_\pi^\infty \frac{1+\sin x}{x^2}\, dx$ (Ans : Convergent)
    (b) $\int_2^\infty \frac{x}{\sqrt{x^4-1}}\, dx$ (Ans : Divergent)

11. Prove the following recurrence relations and state the values of n for which they are valid. Note : m,n are nonnegative integers.
    (a) If $I_n = \int (\ln x)^n dx$, then prove that $I_n = x(\ln x)^n - nI_{n-1}$,.

12. Prove the following :
    (a) Show that $\beta(p,q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int_0^1 \frac{y^{p-1} + y^{q-1}}{(1+y)^{p+q}} dy$. Hence evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}} dx$ (Ans : 1/5005)
    (b) Prove that $\int_0^1 \left(1 - x^{1/n}\right)^m dx = \frac{m!n!}{(m+n)!}$, m, n are positive integers.
    (c) Prove that $\Gamma(n) = \int_0^1 [\ln(1/y)]^{n-1} dy \ n > 0$.
    Hence (a) Prove that $\int_0^1 x^{a-1} [\ln(1/x)]^{n-1} dx = \frac{\Gamma(n)}{a^n}$. (b) Evaluate $\int_0^1 x^5 [\ln(1/x)]^3 dx$ (Ans : 1/216).

(d) Given that $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}, 0 < n < 1,$

a) Prove $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$. b) Evaluate $\int_0^\infty \frac{1}{1+y^4}dy$ (Ans : $\frac{\sqrt{2}\pi}{4}$).

13. Compute the following.

1. $\frac{\Gamma(6)}{2\Gamma(3)}$ (Ans : 30) 2. $\frac{\Gamma(5/2)}{\Gamma(1/2)}$ (Ans : 3/4)

3. $\frac{\Gamma(3)\Gamma(2.5)}{\Gamma(5.5)}$ (Ans : 16/315) (4) $\Gamma(-5/2)$ (Ans : $\frac{-8\sqrt{\pi}}{15}$).

14. Evaluate the following integrals :

1. $\int_0^\pi x\cos^6 x dx$ (Ans : $\frac{5\pi^2}{32}$). 2. $\int_0^\infty \frac{x^2}{(1+x^6)^{7/2}}dx$ (Ans : $\frac{8}{45}$).

3. $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\cos^2 x dx$ (Ans : $\frac{\pi}{16}$). 4. $\int_0^\infty x^4 e^{-x^4}dx$ (Ans : $\frac{1}{16}\Gamma(\frac{1}{4})$).

5. $\int_0^1 \sqrt[3]{x\ln(1/x)}dx$ (Ans : $(\frac{3}{4})^{\frac{4}{3}}\Gamma(\frac{4}{3})$). 6. $\int_0^\infty \frac{x^c}{c^x}dx$ (Ans : $\frac{1}{(\ln c)^{c+1}}\Gamma(c+1)$).

7. $\int_0^\infty e^{-x^2}dx$ (Ans : $\frac{\sqrt{\pi}}{2}$). 8. $\int_0^\infty \sqrt{x}e^{-x^3}dx$ (Ans : $\frac{\sqrt{\pi}}{3}$).