Elements of Probability Theory:

A simple definition of a random experiment requires that the experiment can be repeated any number of times under the same set of conditions and it's outcome is ~~ktoo~~ known only after the completion of the experiment

- Simple event: A possible outcome of a random experiment is called a simple event (or elementary event) & denoted by ω_i.
- Sample space: The set of all possible outcomes $\{\omega_1, \omega_2, \ldots, \omega_k\}$ is called the sample space & is denoted as Ω, i.e., $\Omega = \{\omega_1, \omega_2, \ldots, \omega_k\}$.
- Events: Subsets of Ω are called events and are denoted by capital letters such as A, B, C.

  The set of all simple events that are contained in the event A is denoted by Ω_A.
- The ~~te~~ event $\bar{A}$ refers to the non-occuring of A and is called a composite or complementary event.
- Ω is an event. Since it contains all possible outcomes we say that Ω will always occur & we call it a sure event or certain event.
- ~~ot~~ On other hand, if we consider the null set $\phi = \{\}$ as an event, then this event can never occur and we call it an impossible event.

∴ The sure event therefore is the set of all elementary events and the impossible event is the set with no elementary events.

e.g.: (i) Rolling of dice.
(ii) Rolling of 2 dice.

If an event A is defined as upper faces of both the dice contain the same number of dots, then the sample space is $\Omega_A = \{(1,1), (2,2), \cdots, (6,6)\}$.

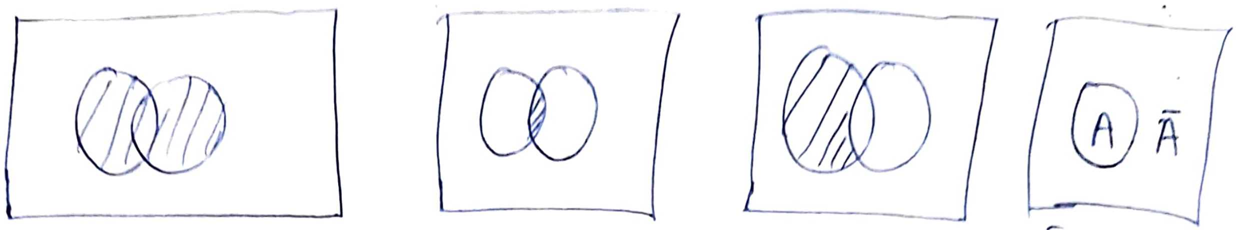
If another event B is defined as "the sum of numbers on the upper faces is 6", then the sample space is $\Omega_B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$.

A sure event is get either an even number or an odd number and impossible event would be the sum of the two dice is greater than 13.

It is possible to view events as sets of simple events. This helps to determine how diffrent events relate to each other. A popular technique to visualize this approach is to use Venn diagrams.

In Venn diagrams, two or more sets are visualized by circles. Overlapping circles imply that both events have one or more identical simple events. Separated circles means that none of the simple events of A are contained in the sample space of B.

: The union of events ... theorem



$A \cup B$ $\quad$ $A \cap B$ $\quad$ $A \setminus B$ $\quad$ $\bar{A} = \Omega \setminus A$

$A \cup B$: The union of events $A \cup B$ is the set of all simple events of A and B which occurs if at least one of the simple event A <u>or</u> B occurs.

$A \cap B$: The intersection of events $A \cap B$ is the set of all simple events A and B which occurs when a simple event occurs that belongs to A <u>&</u> B.

$A \setminus B$: The event $A \setminus B$ contains all simple events of A which are not contained in B. The event "A but not B" or "A minus B" occurs, if A occurs but B does not occur. Also <u>$A \setminus B = A \cap \bar{B}$</u>.

$\bar{A}$ = The events $\bar{A}$ contains all simple events of Ω, which are not contained in A. The complementary event of A (which is Not-A or $\bar{A}$ occurs whenever A does not occur).

$A \subseteq B$: A is a subset of B. This means that all simple events of A are also part of the sample space of B.

<u>Notation:</u>

$A+B$ for $A \cup B$
AB for $A \cap B$
$A-B$ for $A \setminus B$

Defn: Two events A & B are disjoint if $A \cap B = \phi$ i.e, if both events cannot occur simultaneously.

e.g: The events A & $\bar{A}$ are disjoint.

Defn: The events $A_1, A_2, \cdots, A_m$ are said to be mutually or pairwise disjoint if $A_i \cap A_j = \phi$ whenever $i \neq j = 1, 2, \cdots m$.

exa: Rolling of a dice:

If $A = \{1, 3, 5\}$ & $B = \{2, 4, 6\}$ are the sets of odd & even numbers, respectively, then the events A and B are disjoint.

Defn: The events $A_1, A_2, \cdots, A_m$ form a complete decomposition of Ω if and only if

$$A_1 \cup A_2 \cup \cdots \cup A_m = \Omega$$

and

$$A_i \cap A_j = \phi \quad \text{for all } i \neq j$$

exa: $A_1 = \{1\}$, $A_2 = \{2\}$, $\cdots$, $A_6 = \{6\}$ form a complete decomposition.

3/9 1: ① Rolling of a die:

If a die is rolled once, then the possible outcomes are the number of dots on the upper surface: $1, 2, \cdots, 6$.

$\therefore$ The sample space is set of simple events

$\omega_1 = 1, \quad \cdots \quad, \omega_6 = 6$. & $\Omega = \{\omega_1, \omega_2, \cdots, \omega_6\}$.

- Any subset of Ω can be used to define an event.
- If $A = \{\omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$ & B is the set of even numbers, then $B = \{\omega_2, \omega_4, \omega_6\}$ & thus $B \subseteq A$.
- If $A = \{\omega_2, \omega_4, \omega_6\}$ is the set of even numbers & $B = \{\omega_3, \omega_6\}$ is set of all numbers which are divisible by 3, then $A \cup B = \{\omega_2, \omega_3, \omega_4, \omega_6\}$ is the collection of simple events for which the number is either even or divisible by 3 or both.

* Relative frequency and Laplace probability.

There is a close connection between the relative frequency and the probability of an event. A random experiment is described by it's possible outcomes, for example getting a number between 1 & 6 when rolling a die. Suppose an experiment has m possible outcomes (events) $A_1, A_2, \cdots, A_m$ and the experiment is repeated n times. ~~Now we can count how many~~

We can count how many times each of the possible outcome has occured.

- We can calculate the absolute frequency $n_i = n(A_i)$ which is equal to the number of times an event A_i, $i = 1, 2, \ldots, m$ occurs.

- The relative frequency $f_i = f(A_i)$ of a random event A_i, with n repetitions of the experiment, is calculated as

$$f_i = f(A_i) = \frac{n_i}{n}.$$

exa: Suppose a fair coin is tossed $n = 30$ times and we observe the number of heads $n(A_1) = 10$ times & number of tails $n(A_2) = 12$ times.

(Fair coin ⇒ probabilities of head & tail are equal).

Then, the relative frequencies in the experiment are

$$f(A_1) = \frac{10}{30} = 0.33 \quad \& \quad f(A_2) = \frac{20}{30} = 0.66.$$

When the coin is tossed a large number of times & n tends to infinity, then both $f(A_1)$ & $f(A_2)$ will have a limiting value 0.5 which is simply the probability of getting a head or tail in tossing a fair coin.

Laplace experiment: An experiment is a Laplace experiment if the number of possible simple event is finite and all the outcomes are equally probable.

- The probability of an arbitrary event A is then defined as follows:

The proportion $P(A) = \frac{|A|}{|\Omega|} = \frac{\text{Number of favourable simple events for A}}{\text{Total number of possible simple events}}$

is called the Laplace probability, where $|A|$ is the cardinal number of A, i.e., the number of simple events contained in the set A and $|\Omega|$ is the cardinal number of Ω, i.e, the number of simple events contained in the set Ω.

exa: Rolling two dice: Suppose we throw two dice simultaneously and an event is defined as the number of dots observed on the upper surface of both the dice; then, there are 36 simple events defined as (no. of dots on first die, no. of dots on second die).

Event A: the sum of the dots on the two dice is at least 4 and at most 6.

The probability of the event A is therefore $\frac{12}{36} = \frac{1}{3}$.

If we assume that the experiment is repeated large number of times (Mathematically, this would mean that n tends to infinity) & the experiment conditions remain the same (at least approximately) over all the repetitions, then the relative frequency $f(A)$ converges to a limiting value of A.

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This limiting value is interpreted as the probability of A and denoted by $P(A)$, i.e.,

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n},$$ where $n(A)$ denotes the number of times an event A occurs out of n times.

* The Axiomatic Def^n of Probability:

An important foundation for modern probability theory was established by A. N. Kolmogorov in 1933 when he proposed the following axioms of probability. holds.

Axiom 1: Every random event A has a probability in closed interval $[0,1]$, i.e., $0 \leq P(A) \leq 1$.

Axiom 2: The sure event has probability 1, i.e.,

$$P(\Omega) = 1.$$

om 3: If A_1 and A_2 are disjoint events, then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2)$$

- Axiom 3 also holds for three or more disjoint events and is called the theorem of additivity of disjoint events.

e.g: A_1, A_2, A_3 are disjoint events, then

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3).$$

exa: Rolling ~~of~~ a die: (event: the number of points observed on the upper surface of die)
There are six simple events, i.e, the natural numbers 1, 2, ..., 6. These events are disjoint & they have equal probability of occuring: $P(1) = P(2) = \cdots = P(6) = \frac{1}{6}$.
The probability of getting odd number is

$$P(\text{odd number}) = P(1) + P(3) + P(5) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

* Corollary:
We know that $A \cup \bar{A} = \Omega$ (sure event). Since A and $\bar{A}$ are disjoint, then by Axiom 3 we have

$$P(A \cup \bar{A}) = P(A) + P(\bar{A}) = 1.$$

- **Corollary 1**:- The probability of the complement event of A (i.e. $\bar{A}$) is $P(\bar{A}) = 1 - P(A)$.

Exa: Suppose a box of 30 chocolates contains chocolates of 6 diffrent flavours with 5 chocolates of each flavour. Suppose an event A is defined as $A = \{\text{marzipan flavour}\}$. The probability of finding a marzipan chocolate (without looking into the box) is

$$P(\text{marzipan}) = \frac{5}{30}.$$

The probability of the complementary event $\bar{A}$ i.e. the probability of not finding a marzipan chocolate is therefore

$$P(\text{no marzipan flavour}) = 1 - P(\text{marzipan flavour})$$
$$= \frac{25}{30}.$$

Corollary 2: The probability of occurrence of an impossible event ϕ is zero

$$P(\phi) = P(\bar{\Omega}) = 1 - P(\Omega) = 0.$$

Corollary 3: Let A_1 & A_2 be not necessarily disjoint event. The probability of occurrence of A_1 or A_2 is $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$. —— (*)

The rule ⊛ is known as the additive theorem of probability.

We use word "or" in statistical sense: either A_1 is occurring, A_2 is occurring, or both of them.

⇒ We have to add the probabilities $P(A_1)$ & $P(A_2)$ but need to make sure that the simple events which are contained in both sets are not counted twice, thus we subtract $P(A_1 \cap A_2)$.

exa: There are 16 actors acting in play. Two actors one of whom is male, are portraying evil characters. In total, there are 8 females actors.

Let an event A describe wether the actor is male and another event B describe whether the character is evil. Suppose we want to know the probability of a random chosen actor being male or evil.

P(actor is male or evil) = P(actor is male) + P(actor is evil) − P(actor is male & evil).

$$= \frac{8}{16} + \frac{2}{16} - \frac{1}{16} = \frac{9}{16}.$$

Corollary 4: If $A \subseteq B$, then $P(A) \leq P(B)$.

Proof: We use the representation $B = A \cup (\bar{A} \cap B)$, where A & $\bar{A} \cap B$ are the disjoint events.

By Axiom 3 & Axiom 1, we get

$$P(B) = P(A) + P(\bar{A} \cap B) \geq P(A).$$

Rules:

① $0 \leq P(A) \leq 1$

② $P(\Omega) = 1$

③ $P(A_1 \cup A_2) = P(A_1) + P(A_2)$, if A_1 and A_2 are disjoint

④ $P(\phi) = 0$

⑤ $P(\bar{A}) = 1 - P(A)$

⑥ $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

⑦ $P(A) \leq P(B)$, if $A \subseteq B$.