

Foundation of Cryptography

Session 16

Date: 09 March 2021

Dr. V. K. Pachghare

Number Theory

- Euler Totient Function
- Extended Euclidean Algorithm
- Chinese Remainder Theorem

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$$4^{1023 \bmod \phi(25)} \bmod 25 \quad (\phi(25) = 20)$$

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$$4^{1023 \bmod \phi(25)} \bmod 25$$

$$(\phi(25) = 20)$$

$$= 4^3 \bmod 25$$

$$\text{Since } 1023 \bmod 20 = 3$$

$$= 4^3 \bmod 25$$

$$= 64 \bmod 25$$

14, 39, 64

out of these only 64 is divisible by 4.

The number which is power of 4 is selected and that number is the last two digits

So last two digits are 64

Factors of 100 are 5, 10, 20, 25 and 50

$4^{1023} \bmod 5$	$4^{1023} \bmod 10$	$4^{1023} \bmod 20$	$4^{1023} \bmod 50$
$4^{1023 \bmod \phi(5)} \bmod 5$	$4^{1023 \bmod \phi(10)} \bmod 10$	$4^{1023 \bmod \phi(20)} \bmod 20$	$4^{1023 \bmod \phi(50)} \bmod 50$
$4^{1023 \bmod 4} \bmod 5$	$4^{1023 \bmod 4} \bmod 10$	$4^{1023 \bmod 8} \bmod 20$	$4^{1023 \bmod 20} \bmod 50$
$4^3 \bmod 5$	$4^3 \bmod 10$	$4^7 \bmod 20$	$4^3 \bmod 50$
64 mod 5 = 4 mod 5	64 mod 10 = 4 mod 10	$4^7 = 4^3 * 4^3 * 4$ $4*4*4 \bmod 20$ = 64 mod 20 = 4 mod 20	64 mod 50 = 14 mod 50
4, 9, 14, 19, 24, 29, 34, 39, 44, 49, 54, 59, 64 , 69, 74, 79, 84, 89, 94, 99	4, 14, 24, 34, 44, 54, 64 , 74, 84, 94	4, 24, 44, 64 , 84	14, 64

The number which is power of 4 is the last two digits

What are the last two digits of $\underbrace{3^{3^{3^{\dots^3}}}}_{2017 \text{ times}} ?$

Solution:

We know that $\phi(100) = 40$;
 So, we need to compute $\underbrace{3^{3^{3^{\dots^3}}}}_{2017 \text{ times}}$

and raise 3 to that power.

$$\phi(40) = 16; \phi(16) = 8; \phi(8) = 4; \phi(4) = 2$$

In particular, $3^k = 3 \pmod{4}$ for any value of k .

Working backwards

$$3^{\phi(100)} \bmod 100$$

$$3^3$$

$$3^{3^{\phi(40)}} \bmod 40$$

$$3^{3^{\phi(100)}} \bmod 100$$

$$3^3$$

$$3^{(3^{\text{mod } 4} (16)) \text{mod } 16}$$

$$3^{(3^{\text{mod } 4} (40)) \text{mod } 40}$$

$$3^{\text{mod } 4} (100) \text{mod } 100$$

$$3^{3^3}$$

$$3^{3^{3^{3^{\left(3^{\phi(8)}\right)} \bmod 8}}}$$

$$3^{3^{3^{3^{\left(3^{\phi(16)}\right)} \bmod 16}}}$$

$$3^{3^{3^{3^{\left(3^{\phi(40)}\right)} \bmod 40}}}$$

$$3^{3^{\phi(100)} \bmod 100}$$

$$3^{3^{3^3}}$$

$$3^{3^{3^3}}(3^{3 \bmod 4})_{\bmod 4}$$

$$3^{3^3}(3^{3 \bmod 8})_{\bmod 8}$$

$$3^3(3^{3 \bmod 16})_{\bmod 16}$$

$$3(3^{3 \bmod 40})_{\bmod 40}$$

$$3^{3 \bmod 100} \bmod 100$$

$$3^{3^{3^{3^3}}}$$

$$3^{3^{3^3}}(3^{3 \bmod \phi(2)}) \bmod 2$$

$$3^{3^{3^3}}(3^{3 \bmod \phi(4)}) \bmod 4$$

$$3^{3^3}(3^{3 \bmod \phi(8)}) \bmod 8$$

$$3^3(3^{3 \bmod \phi(16)}) \bmod 16$$

$$3(3^{3 \bmod \phi(40)}) \bmod 40$$

$$3^{3 \bmod \phi(100)} \bmod 100$$

$$3^{3^{3^{3^{3^3}}}}$$

$$3^{3^{3^3}}(3^{3 \bmod 2})_{\bmod 2}$$

$$(3^{3 \bmod 2})_{\bmod 2} = 3^{3 \bmod 2} \bmod 2 = 1$$

$$3^{3^{3^3}}(3^{3 \bmod 4})_{\bmod 4}$$

$$3^3(3^{3 \bmod 8})_{\bmod 8}$$

$$3^{(3^{3 \bmod 16})_{\bmod 6}}$$

$$3^{(3^{3 \bmod 40})_{\bmod 40}}$$

$$3^{3 \bmod (100)} \bmod 100$$

$$3^{3^{3^{3^{3^3}}}}$$

$$3^{3^{3^3}}(3^{3 \bmod \phi(2)})_{\bmod 2}$$

$$(3^{3 \bmod \phi(2)})_{\bmod 2} = 3^{3 \bmod 1} \bmod 2 = 1$$

$$3^{3^{3^3}}(3^{3 \bmod \phi(4)})_{\bmod 4}$$

$$(3^{3 \bmod \phi(4)})_{\bmod 4} = 3^{3 \bmod 2} \bmod 4 = 3$$

$$3^{3^3}(3^{3 \bmod \phi(8)})_{\bmod 8}$$

$$3^{3^3}(3^{3 \bmod \phi(16)})_{\bmod 16}$$

$$3^{3^3}(3^{3 \bmod \phi(40)})_{\bmod 40}$$

$$3^{3 \bmod \phi(100)} \bmod 100$$

$$3^{3^{3^{3^{3^3}}}}$$

$$3^{3^{3^3}}(3^{\cancel{3}^{\cancel{3}}\text{mod}(2)})_{\text{mod}2}$$

$$(\cancel{3}^{\cancel{3}}\text{mod}(2))_{\text{mod}2} = 3^{\cancel{3}}\text{mod}2 = 1$$

$$3^{3^{3^3}}(3^{\cancel{3}^{\cancel{3}}\text{mod}(4)})_{\text{mod}4}$$

$$(\cancel{3}^{\cancel{3}}\text{mod}(4))_{\text{mod}4} = 3^{\cancel{3}}\text{mod}4 = 3$$

$$3^3(3^{\cancel{3}^{\cancel{3}}\text{mod}(8)})_{\text{mod}8}$$

$$(\cancel{3}^{\cancel{3}}\text{mod}(8))_{\text{mod}8} = 3^{\cancel{3}}\text{mod}8 = 3$$

$$3^{3^3}(3^{\cancel{3}^{\cancel{3}}\text{mod}(16)})_{\text{mod}16}$$

$$3^{3^3}(3^{\cancel{3}^{\cancel{3}}\text{mod}(40)})_{\text{mod}40}$$

$$3^{\cancel{3}^{\cancel{3}}\text{mod}(100)}\text{mod}100$$

$$3^{3^{3^{3^{3^3}}}}$$

$$3^{3^{3^3(3^{\phi(2)})} \bmod 2}$$

$$(3^{\phi(2)}) \bmod 2 = 3^1 \bmod 2 = 1$$

$$3^{3^{3^3(3^{\phi(4)})} \bmod 4}$$

$$(3^{\phi(4)}) \bmod 4 = 3^2 \bmod 4 = 1$$

$$3^{3^3(3^{\phi(8)}) \bmod 8}$$

$$(3^{\phi(8)}) \bmod 8 = 3^4 \bmod 8 = 1$$

$$3^{(3^{\phi(16)}) \bmod 16}$$

$$(3^{\phi(16)}) \bmod 16 = 3^8 \bmod 16 = 1$$

$$3^{(3^{\phi(40)}) \bmod 40}$$

$$3^{\phi(100)} \bmod 100$$

$$3^{3^{3^{3^{3^3}}}}$$

$$3^{3^{3^{3^3}}(3^{3 \bmod \phi(2)}) \bmod a}$$

$$(3^{\text{mod}(2)})_{\text{mod } 2} = 3^{\text{mod } 1} \text{ mod } 2 = 1$$

$$3^{3^{3^3(3^{3 \bmod \phi(4)}) \bmod 4}}$$

$$(3^{\text{mod}(4)})_{\text{mod } 4} = 3^{3 \text{ mod } 4} \text{ mod } 4 = 3$$

$$3^{3^{(3^{3 \bmod \phi(8)}) \bmod 8}}$$

$$(3^{\text{mod}(8)})_{\text{mod } 8} = 3^{\text{mod } 4} \text{ mod } 8 = 3$$

$$3^{(3^{3 \bmod 16}) \bmod 16}$$

$$(3^{\text{mod}(16)})_{\text{mod } 6} = 3^{\text{mod } 8} \text{ mod } 6 = 11$$

$$[3^4 \bmod 40 = 1] \Rightarrow [3^1 = (3^4)^2 3^3 \bmod 40 = 1] \Rightarrow [3^3 \bmod 40 = 27]$$

$$3^{(3 \bmod 40) \bmod 40} = 3^{(3 \bmod 4)} = 3^3 \bmod 40 = 3^7 \bmod 40 = 3^{11} \bmod 40 = 27$$

$$3 \bmod (100) \bmod 100$$

333333333

$$3^{3^{3^3(3^{\phi(2)}) \bmod 2}} \quad (3^{\phi(2)}) \bmod 2 = 3^{\phi(2)} \bmod 2 = 1$$

$$3^{3^{3^3(3^{\phi(4)}) \bmod 4}} \quad (3^{\phi(4)}) \bmod 4 = 3^{\phi(4)} \bmod 4 = 3$$

$$3^{3^3(3^{\phi(8)}) \bmod 8} \quad (3^{\phi(8)}) \bmod 8 = 3^{\phi(8)} \bmod 8 = 3$$

$$3^{3^{(3^{\phi(16)}) \bmod 16}} \quad (3^{\phi(16)}) \bmod 16 = 3^{\phi(16)} \bmod 16 = 11$$

$$[3^4 \bmod 40 = 1] \Rightarrow [3^{11} = (3^4)^2 3^3 \bmod 40 = 1] \Rightarrow [3^3 \bmod 40 = 27]$$

$$3^{3^{3^{3^3(3^{\phi(40)}) \bmod 40}}} \quad (3^{\phi(40)}) \bmod 40 = 3^{\phi(40)} \bmod 40 = 3^{11} \bmod 40 = 27$$

$$3^{\phi(100)} \bmod 100 = 3^{27 \bmod 40} \bmod 100$$

$$= 3^{27} \bmod 100 =$$

$$3^{3^{3^{3^3}}}$$

$$\begin{aligned}
& 27^1 \text{mod} \cancel{100} \\
& = 27^1 \text{mod} \cancel{40} \quad \text{as } \cancel{100} = 40 \\
& = 3 \times 3^2 \text{mod} \cancel{00} = 3 \times (3^3)^2 \text{mod} \cancel{00} \\
& = 3 \times (3 \times (3)^2)^2 \text{mod} \cancel{00} \\
& = 87 \text{mod} \cancel{00}
\end{aligned}$$