(PCC) Robot Kinematics and Dynamics

Syllabus Contents:

- Basic concepts of linear algebra and feedback control, Rigid bodies and homogeneous transformations, Differential kinematics
- Robot modelling, Direct kinematics, Inverse kinematics problem, Inverse kinematics algorithms
- Trajectory planning, Kinematic solutions and trajectory planning
- Geometric Jacobian / Analytical Jacobian, Singularities and redundancy, Statics and manipulability

- INTRODUCTION TO ROBOT DYNAMICS: Forward Dynamics and Inverse Dynamics – Importance – Spatial description and transformations – Different types of dynamic formulation schemes – Lagrangian formulation for equation of motion for robots and manipulators. Properties of the dynamic model, Dynamic model of simple manipulator structures, Dynamic parameters identification, Operational space dynamics model,

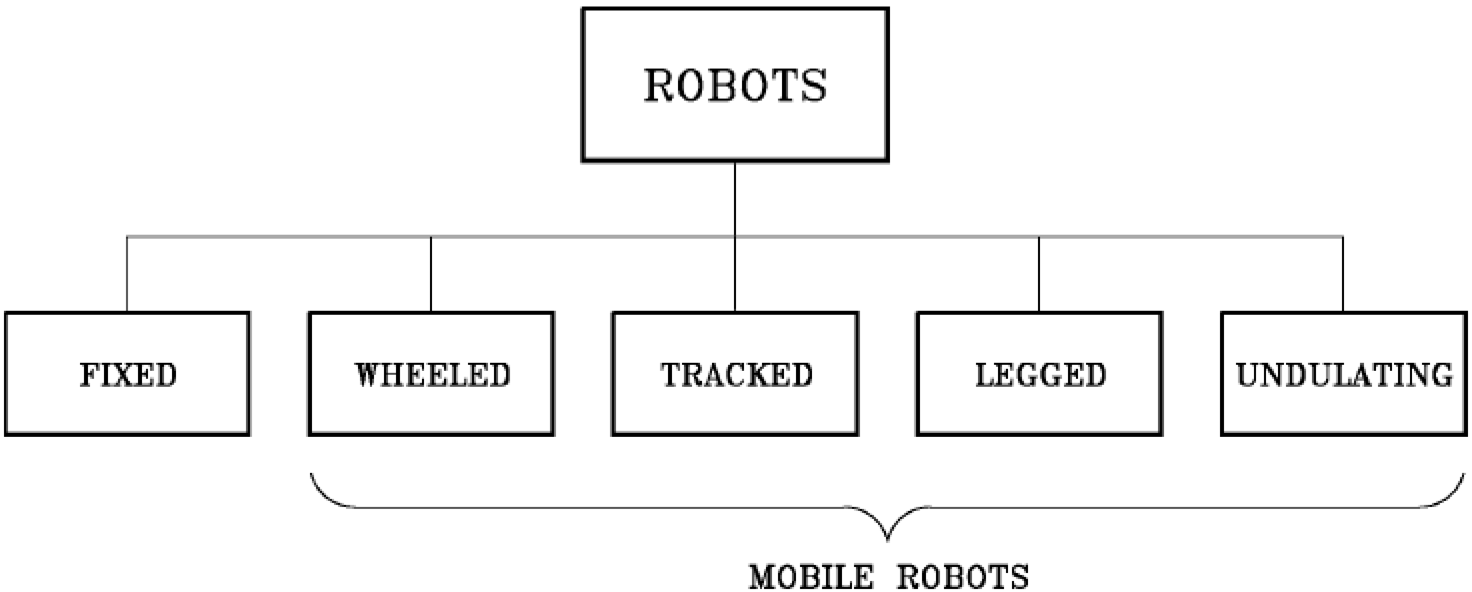
- DYNAMIC MODELING AND SIMULATION: Modeling of motion of robots and manipulators using Newton – Euler equations – State space representation of equation of motion and system properties – Importance of Simulation and its types –
- Numeric Integration solvers and their role in numeric simulation - Numeric simulation of robots and manipulators using MATLAB / Simulink module.

- INTRODUCTION TO ROBOT CONTROL: Introduction – Need and types of control schemes for robots – joint space control schemes with an example – task space control schemes with an example.

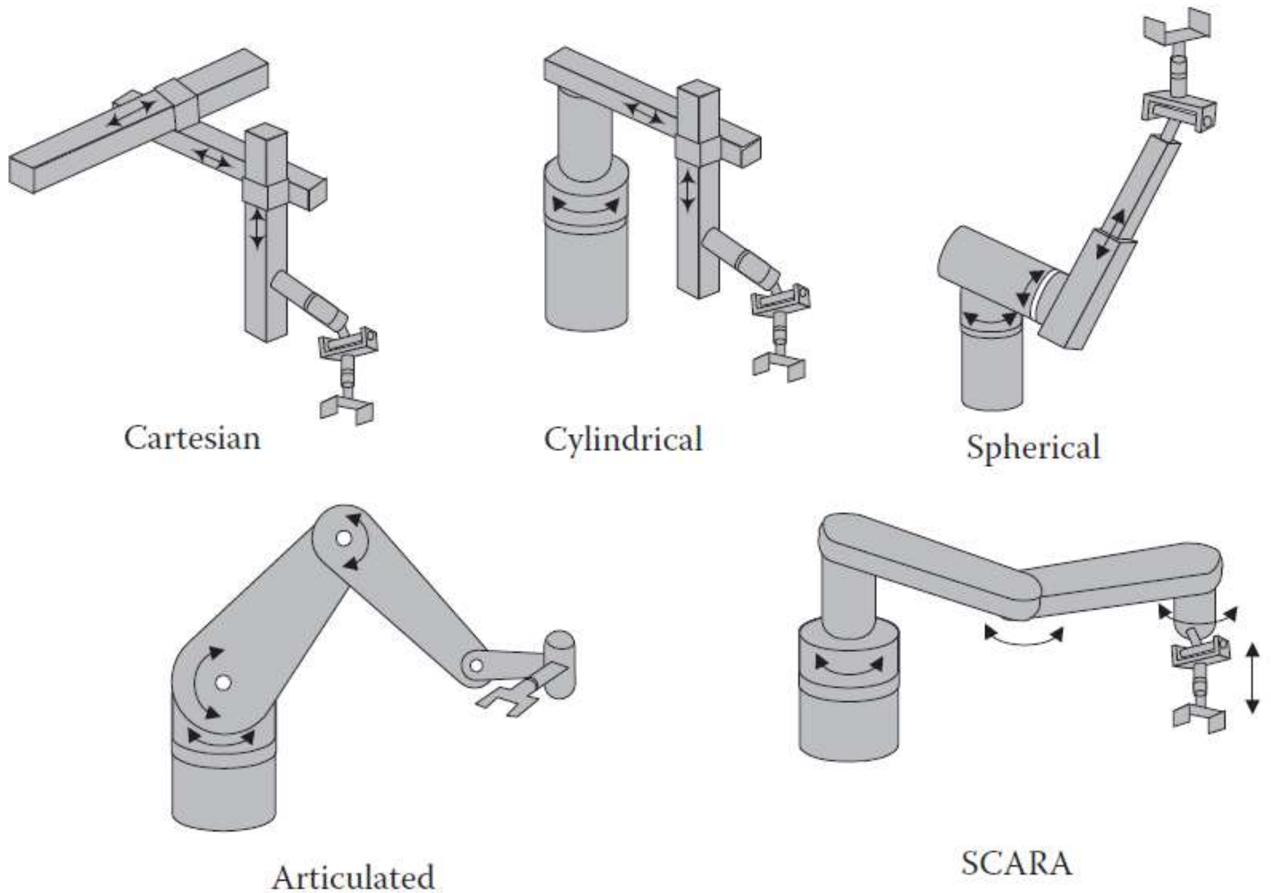
Basic Concepts

- *Robotics is concerned with the study of those machines that can replace human* beings in the execution of a task, as regards both physical activity and decision making.

Robot Classification



Robot Configurations



Work space

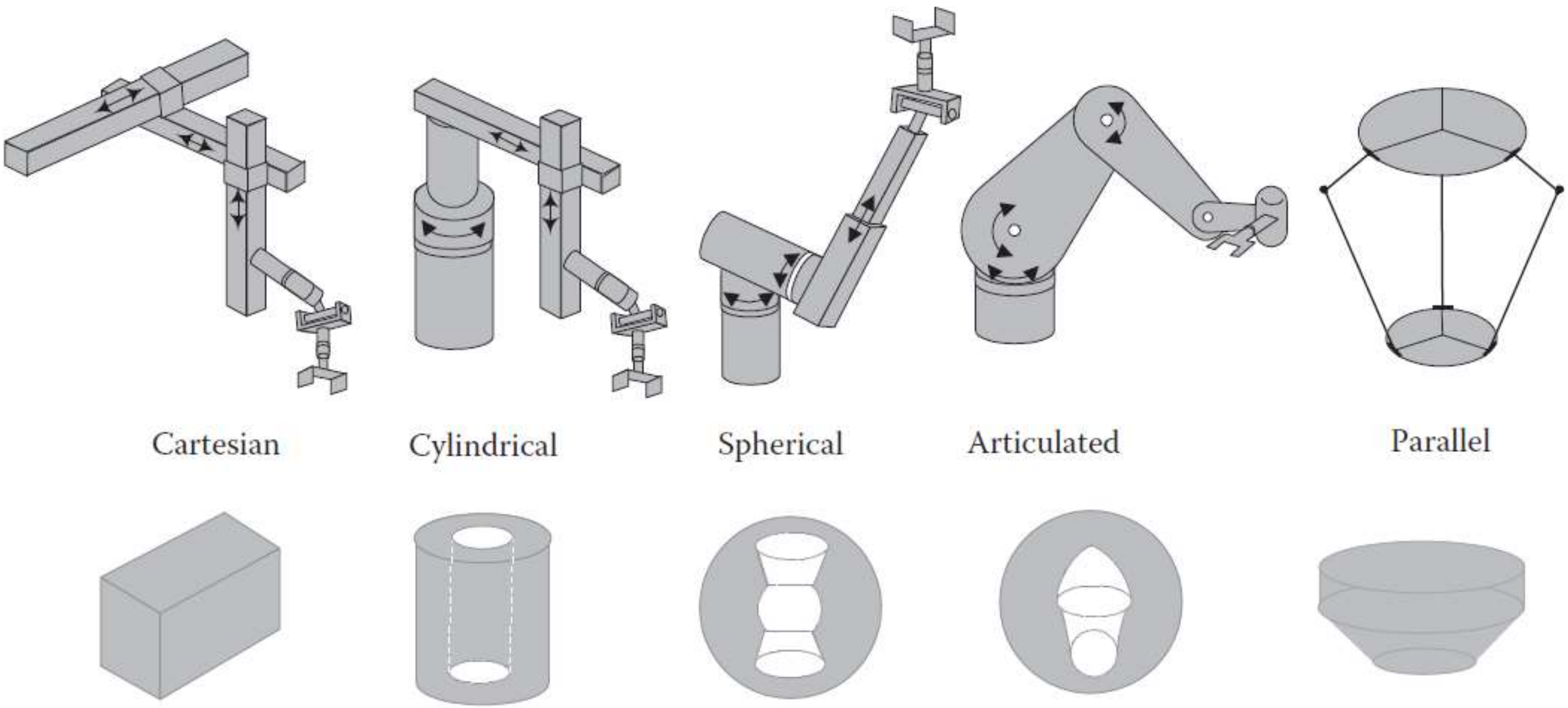
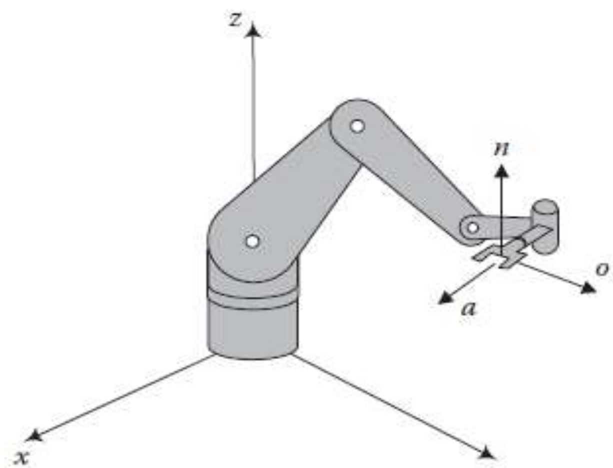
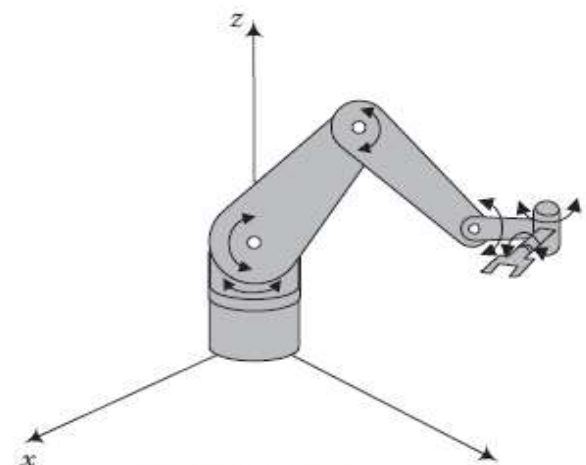


Figure Typical approximate workspaces for common robot configurations.

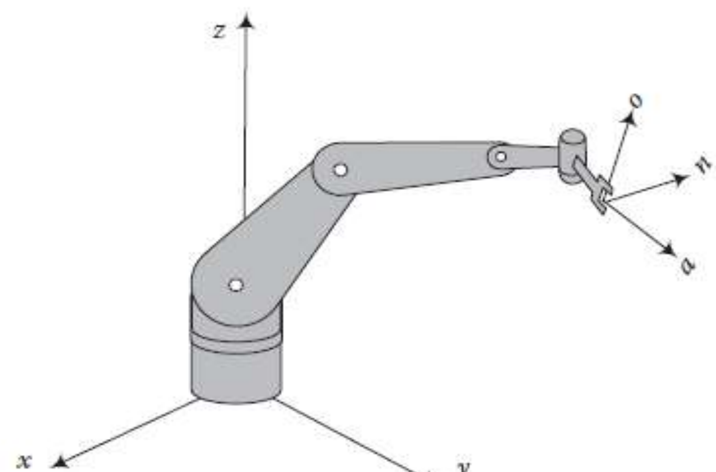
A robot’s World, Joint, and Tool reference frames



World Reference Frame



Joint Reference Frame



Tool Reference Frame

Components of Robotic System

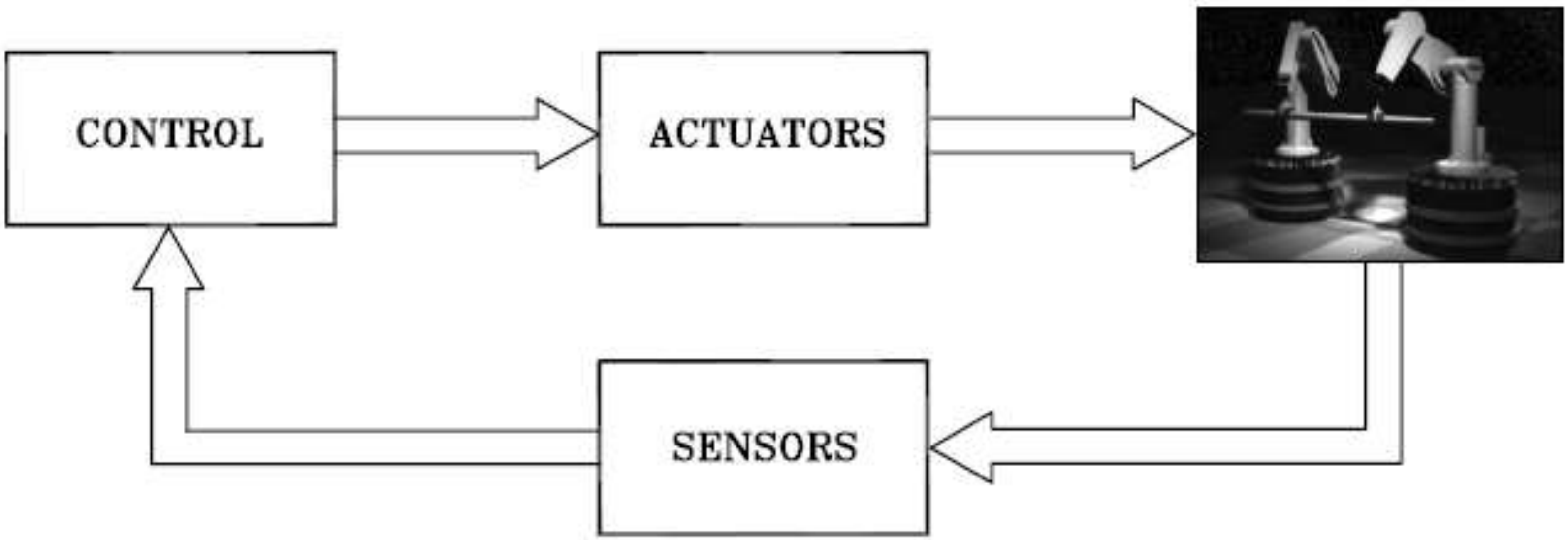


Fig. 1.1. Components of a robotic system

Robot manipulator

- The mechanical structure of a *robot manipulator consists of a sequence of rigid* bodies (*links) interconnected by means of articulations (joints);*
- *A manipulator* is characterized by an *arm that ensures mobility, a wrist that confers dexterity,* and an *end-effector that performs the task required of the robot.*
- The fundamental structure of a manipulator is the serial or *open kinematic chain. From a topological viewpoint, a kinematic chain is termed open when* there is only one sequence of links connecting the two ends of the chain.
- Alternatively, a manipulator contains a *closed kinematic chain when a sequence* of links forms a loop.

Manipulator

- A *manipulator can be schematically represented from a mechanical viewpoint* as a kinematic chain of rigid bodies (*links) connected by means of revolute* or prismatic *joints.*
- *One end of the chain is constrained to a base, while an end-effector is mounted to the other end.*
- *The resulting motion of the structure* is obtained by composition of the elementary motions of each link with respect to the previous one.
- Therefore, in order to manipulate an object in space, it is necessary to describe the end-effector position and orientation.
- The end-effector position and orientation (*pose) to be expressed as a function* of the joint variables of the mechanical structure with respect to a reference frame.

Rigid body

- A **rigid body**, sometimes called a link, represents a solid **body** that cannot deform.
- The distance between any two points on a single **rigid body** remains constant.

Rigid bodies

- A *rigid body is completely described in space by its position and orientation* (in brief *pose) with respect to a reference frame.*

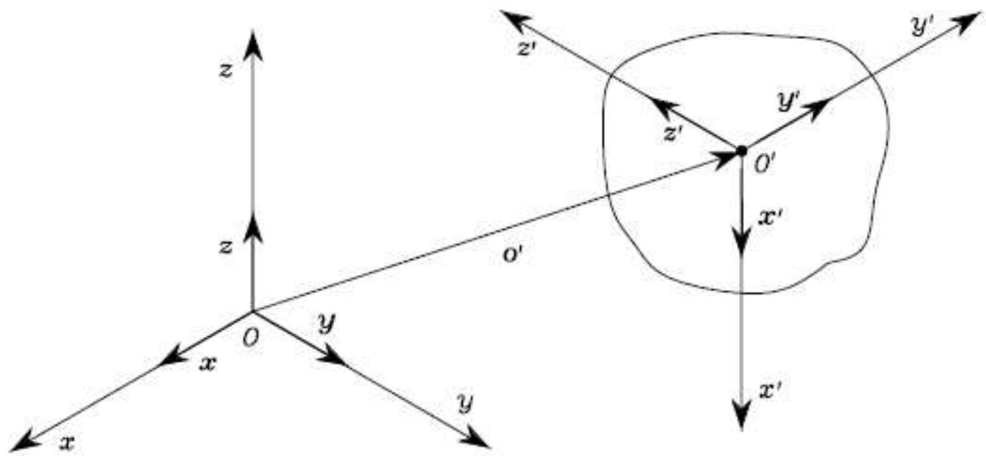


Fig. 2.1. Position and orientation of a rigid body

where o'_x, o'_y, o'_z denote the components of the vector $\boldsymbol{o}' \in \mathbb{R}^3$ along the frame axes; the position of O' can be compactly written as the (3×1) vector

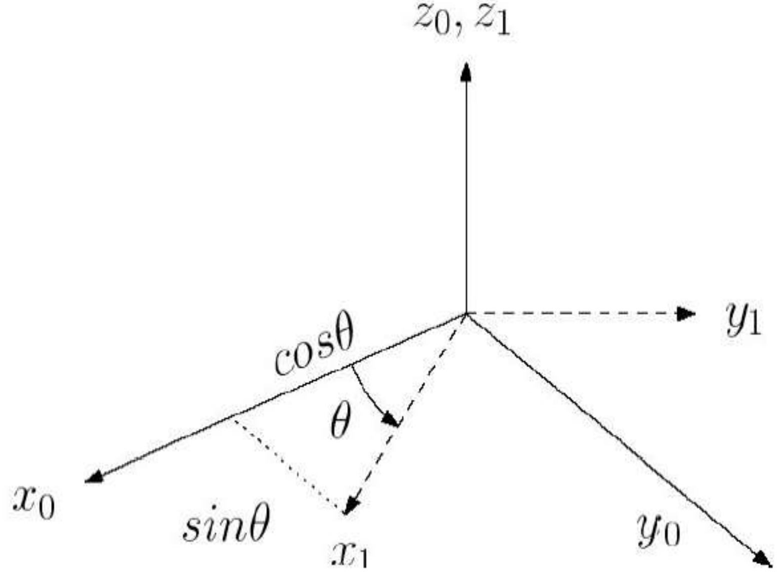
$$\boldsymbol{o}' = \begin{bmatrix} o'_x \\ o'_y \\ o'_z \end{bmatrix}. \qquad (2.1)$$

Rotation in 3D

Representation of Rotations in 3D:

We need to project frame $\{1\}$ into frame $\{0\}$:

$$R_1^0 = [\ x_1^0 |\ y_1^0 |\ z_1^0\ ] = \begin{bmatrix} \hat{x}_1.\hat{x}_0 & \hat{y}_1.\hat{x}_0 & \hat{z}_1.\hat{x}_0 \\ \hat{x}_1.\hat{y}_0 & \hat{y}_1.\hat{y}_0 & \hat{z}_1.\hat{y}_0 \\ \hat{x}_1.\hat{z}_0 & \hat{y}_1.\hat{z}_0 & \hat{z}_1.\hat{z}_0 \end{bmatrix}$$



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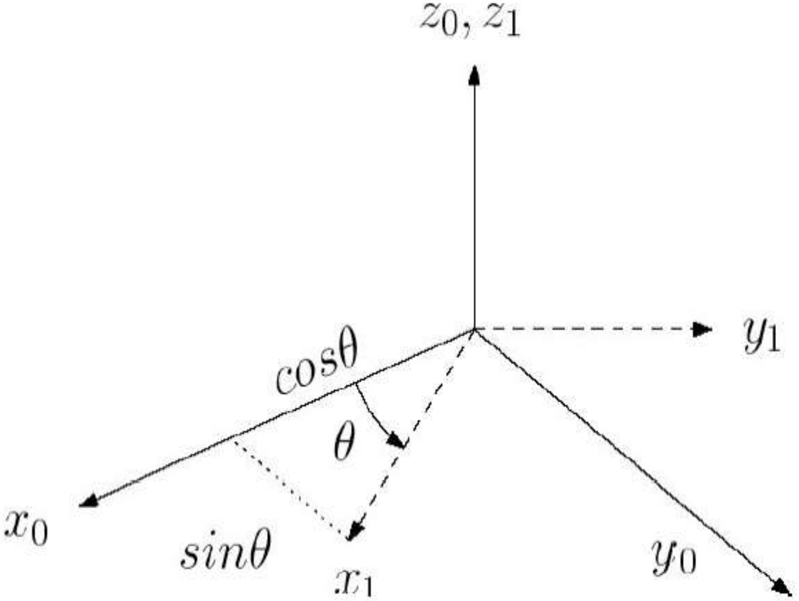
$$(R_1^0) = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_{z,\theta}$$

$R_{z,\theta}$ is the basic rotation matrix around z-axis.

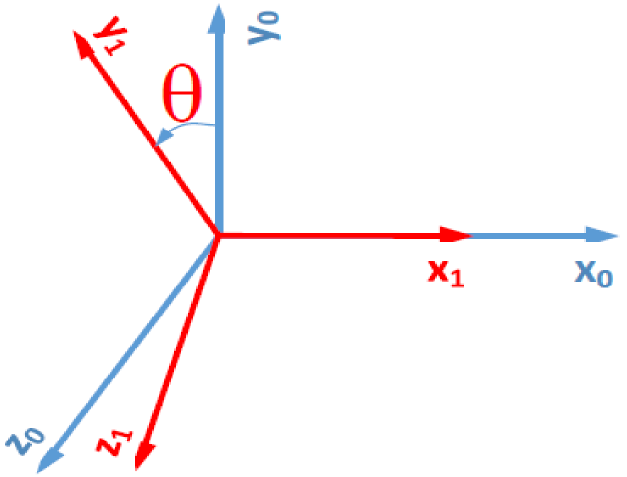


Rotation in 3D

Representation of Rotations in 3D:

Basic Rotation Matrices:

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$



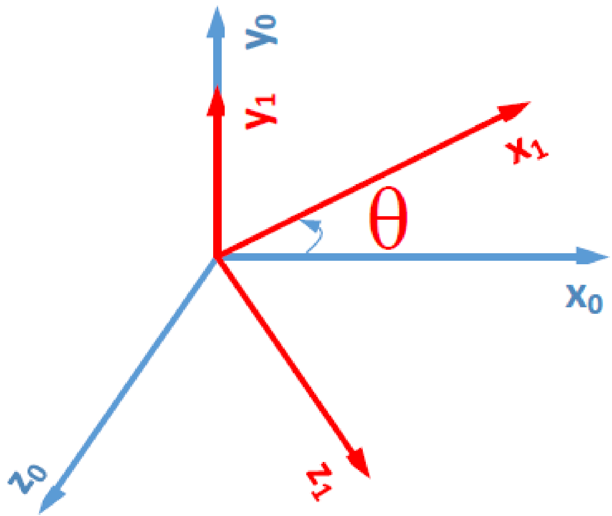
$R_{x,\theta}$

Rotation in 3D

Representation of Rotations in 3D:

Basic Rotation Matrices:

$$R_{y,\theta} = \begin{bmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{bmatrix}$$



$R_{y,\theta}$

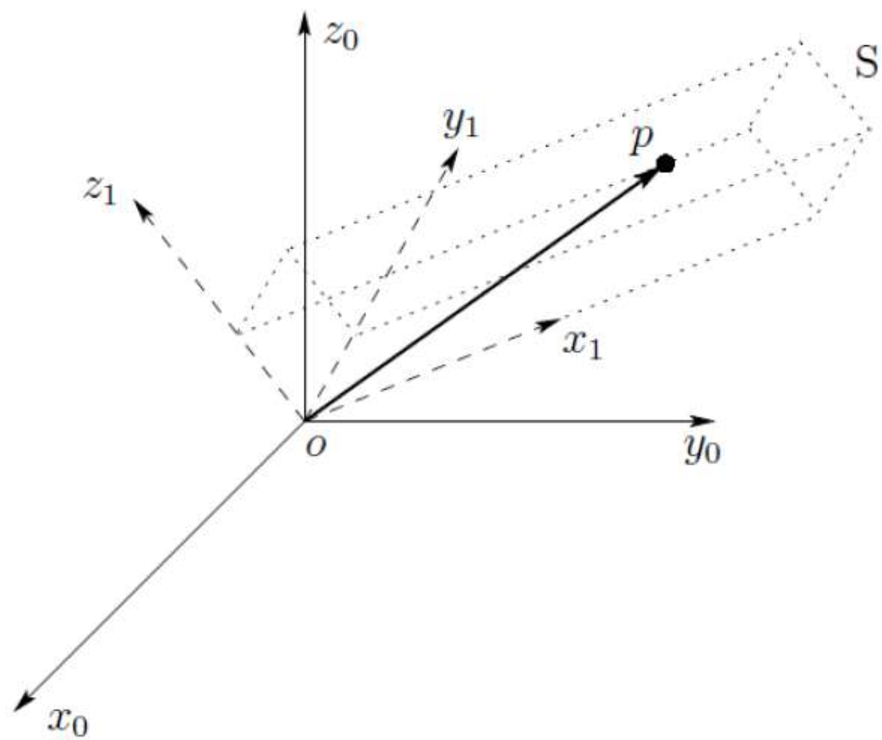
Transformations

Rotational Transformations:

- The $\{0\}$-frame is our fixed frame, the $\{1\}$-frame is fixed to a rigid body.
- What will happen with points of body (let say p) if we rotate the body, i.e. the $\{1\}$-frame?
- The coordinates of point p in the 1-frame are constant p^1, but in the 0-frame they are changed.

- The coordinates of the point p in 0-frame

is: $$p^0 = R_1^0\, p^1$$



Rotational Transformation

Composition of Rotations:

[1] **Rotation about Current Frame:**
Suppose that we have 3 frames:

$$\{0\} = (o_0, x_0, y_0, z_0)$$

$$\{1\} = (o_1, x_1, y_1, z_1)$$

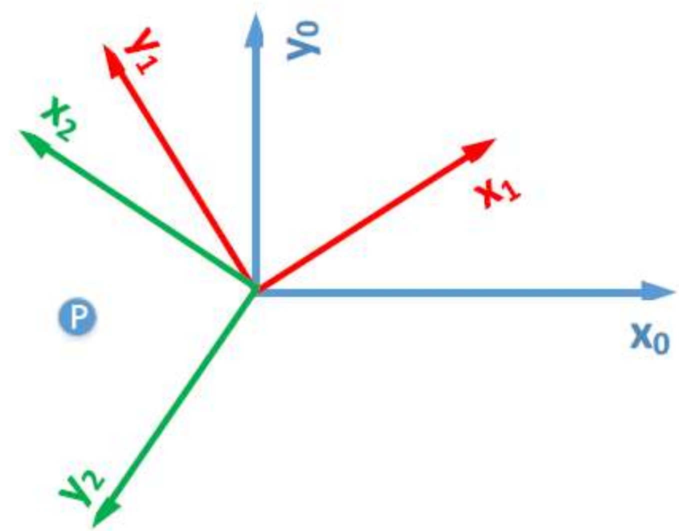
$$\{2\} = (o_2, x_2, y_2, z_2)$$

Any point p can be represented in any of the three coordinates:

$$p^0 = R_1^0 \, p^1$$

$$p^1 = R_2^1 \, p^2$$

$$p^0 = R_2^0 \, p^2$$



Rotational Transformation

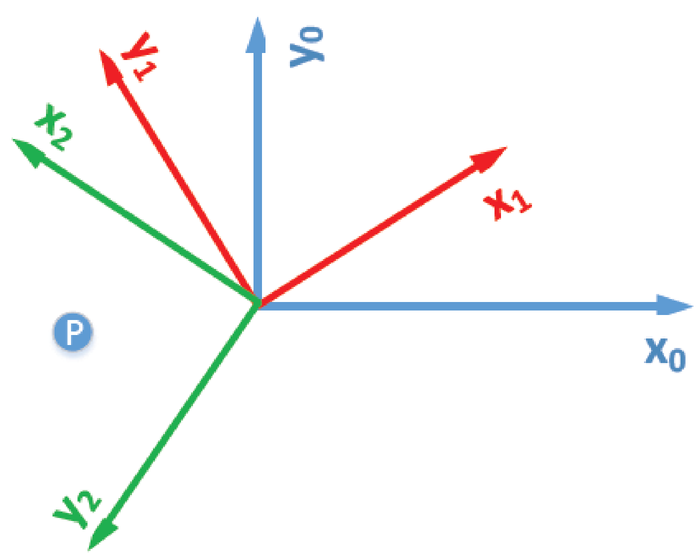
Composition of Rotations:

[1] **Rotation about Current Frame:**

$$p^0 = R_1^0 \; p^1$$

$$p^1 = R_2^1 \; p^2$$

$$p^0 = R_2^0 \; p^2$$



We can write:

$$p^0 = R_1^0 \; p^1 = R_1^0 \; R_2^1 \; p^2$$

$$p^0 = R_1^0 \; R_2^1 \; p^2$$

$$p^0 = R_2^0 \; p^2$$

$$\boxed{R_2^0 = R_1^0 \; R_2^1}$$ Law of composite rotation

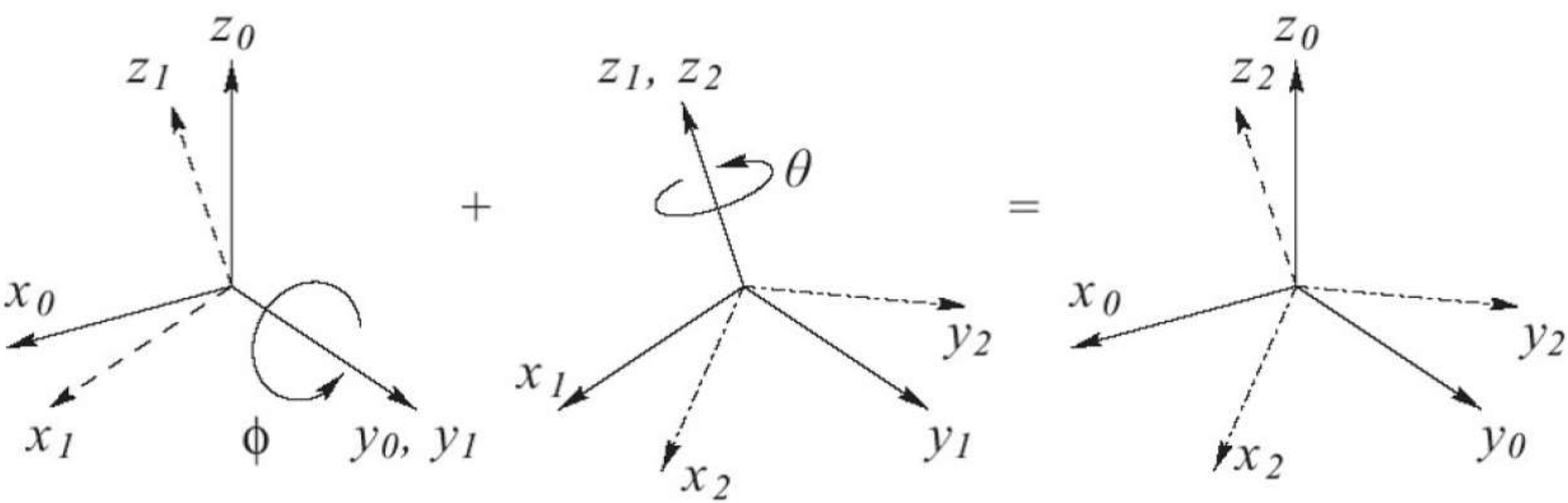
Rotational Transformation

Composition of Rotations:

[1] **Rotation about Current Frame:**

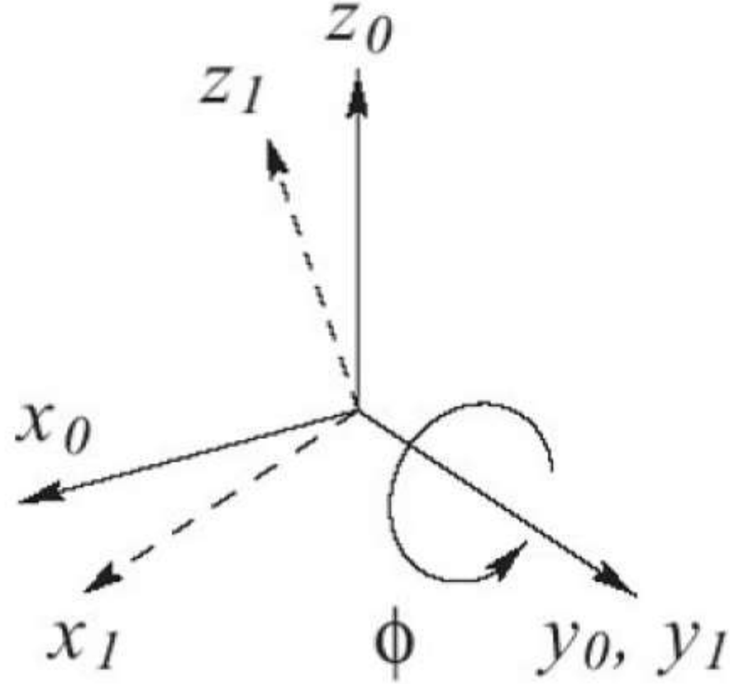
Example: Suppose we rotate:

1. first the frame by angle ϕ around **current** y-axis,
2. then rotate by angle θ around the **current** z-axis. Find the combined rotation ?



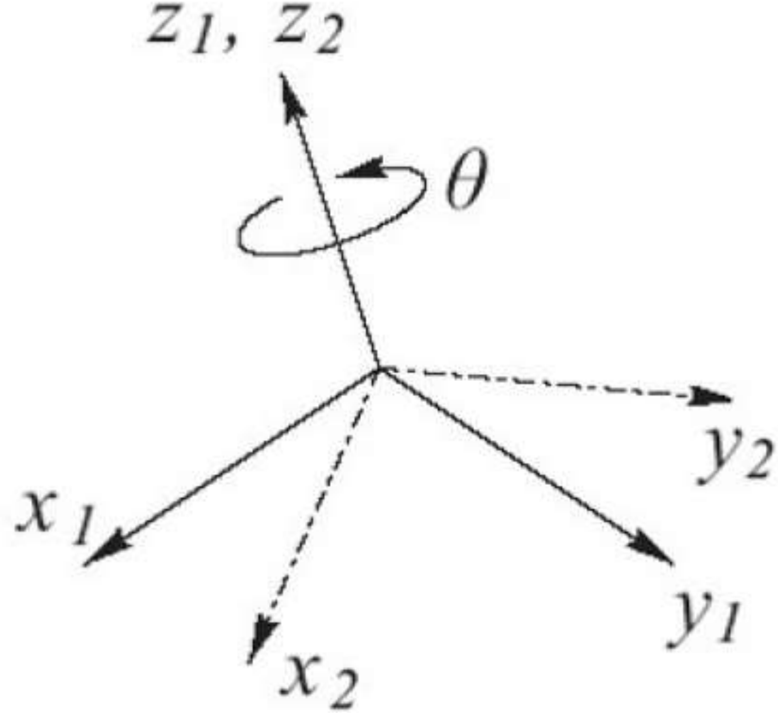
Rotational Transformation

$$R_{y,\phi} = \begin{bmatrix} cos(\phi) & 0 & sin(\phi) \\ 0 & 1 & 0 \\ -sin(\phi) & 0 & cos(\phi) \end{bmatrix}$$



Rotational Transformation

$$R_{z,\theta} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

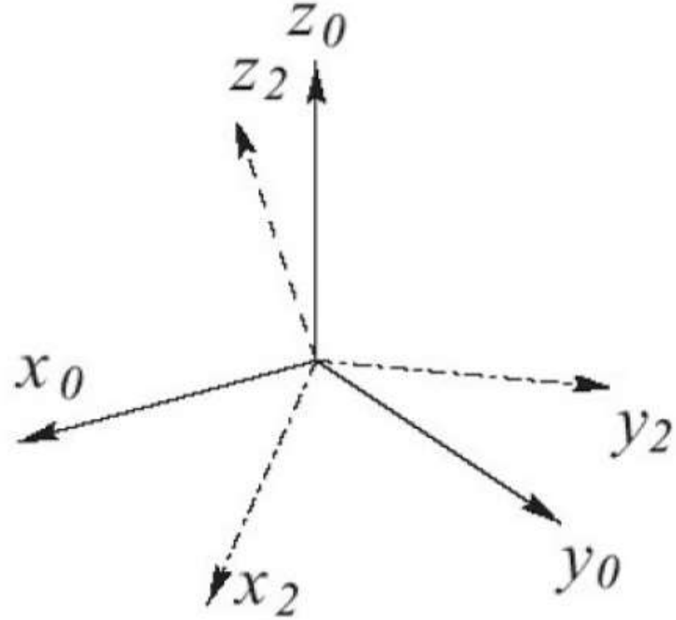


Rotational Transformation

Composition of Rotations:

[1] **Rotation about Current Frame:**

$$R_2^0 = R_{y,\phi}\ R_{z,\theta}$$



$$R_2^0 = \begin{bmatrix} cos(\phi) & 0 & sin(\phi) \\ 0 & 1 & 0 \\ -sin(\phi) & 0 & cos(\phi) \end{bmatrix} \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2^0 = \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}$$

Note: $s_\phi = sin(\phi)$ and $c_\theta = cos(\theta)$

Rotational Transformation

[1] **Rotation about Current Frame:**

Important Observation: Rotations do not commute.

$$R_{y,\phi}\,R_{z,\theta} \neq R_{z,\theta}\,R_{y,\phi}$$

So that the **order of rotations** is important!

Rule of composite rotation around the current (new) frame:

For successive rotations about the current reference frame we use the **post-multiplication** to find the total rotation matrix.

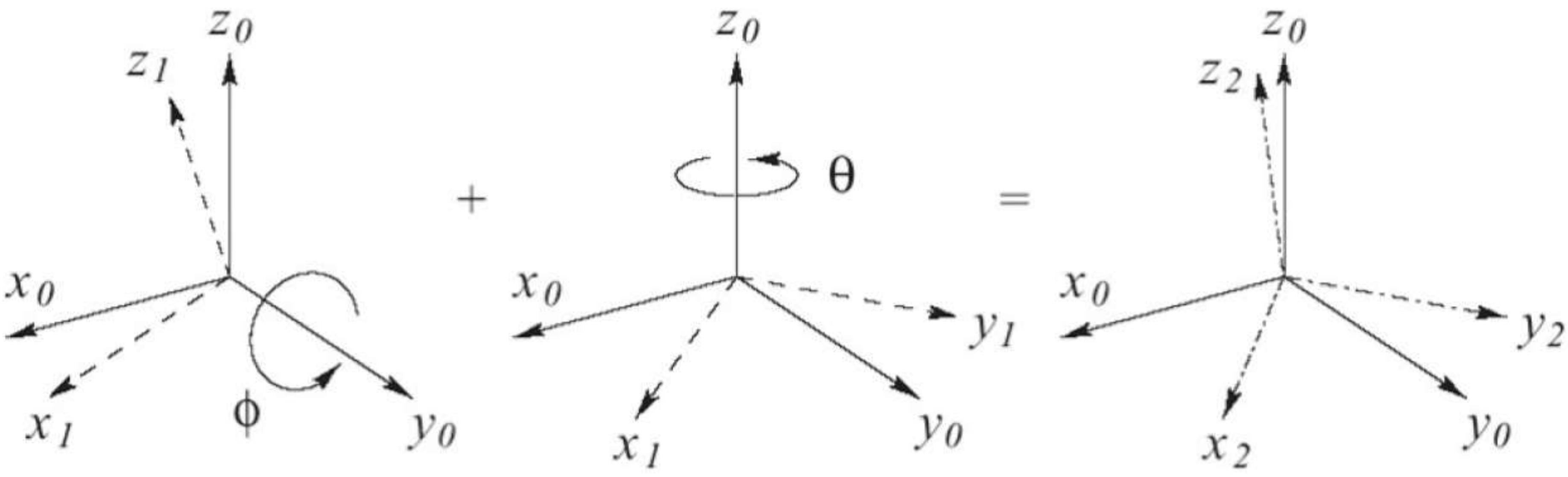
Rotational Transformation

Composition of Rotations:

[2] **Rotation with respect to Fixed Frame:**

Example: Suppose we rotate:

1. the first rotation is by angle ϕ around y_0-axis.
2. then, a rotation by angle θ around z_0-axis (**not z_1-axis**). What is the total rotation ?



Rotational Transformation

Composition of Rotations:

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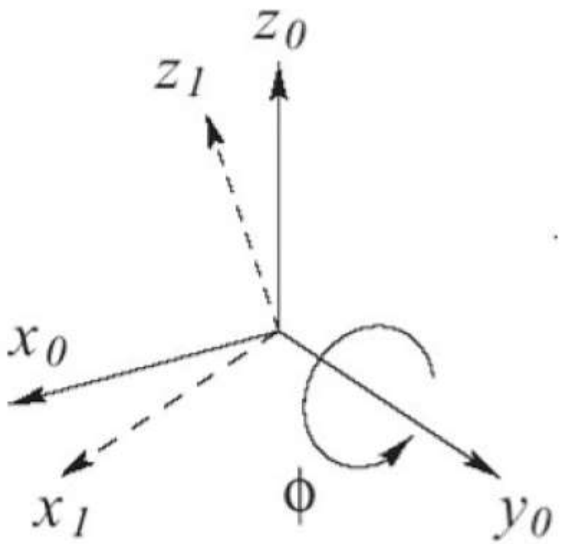
$$R_{y0,\phi} = \begin{bmatrix} cos(\phi) & 0 & sin(\phi) \\ 0 & 1 & 0 \\ -sin(\phi) & 0 & cos(\phi) \end{bmatrix}$$



Rotational Transformation

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Example: Suppose we rotate:

1. the first rotation is by angle ϕ around y_0-axis.
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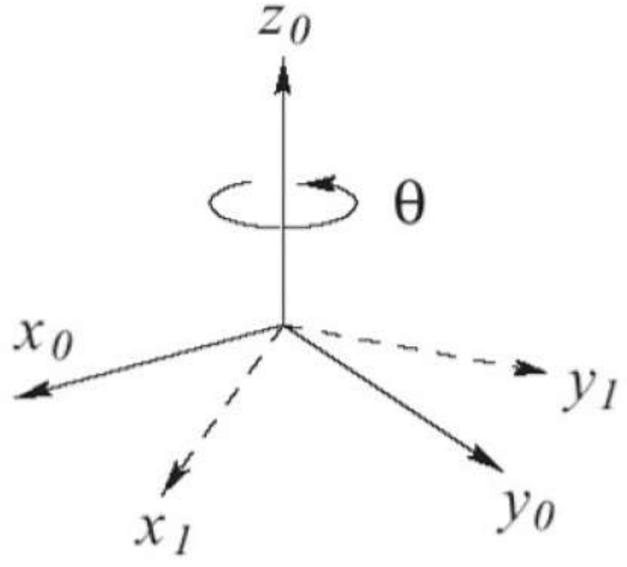
$$R_{z_0,\theta} = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



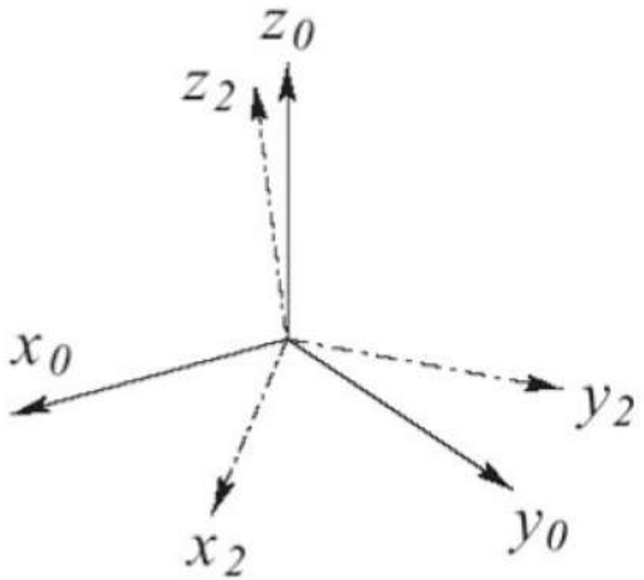
Rotational Transformation

Composition of Rotations:

[2] **Rotation with respect to Fixed Frame:**

$$R_2^0 = R_{z,\theta}\, R_{y,\phi} \qquad \text{note the order!}$$

$$R_2^0 = \begin{bmatrix} cos(\theta) & -sin(\theta) & 0 \\ sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\phi) & 0 & sin(\phi) \\ 0 & 1 & 0 \\ -sin(\phi) & 0 & cos(\phi) \end{bmatrix}$$



Rotational Transformation

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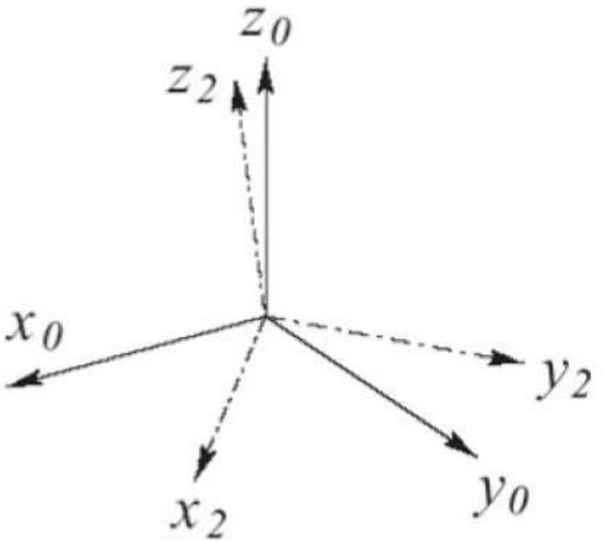


Rule of composite rotation around the fixed (original) frame:

For successive rotations about the fixed reference frame we use the **pre-multiplication** to find the total rotation matrix.

Example:

Find the rotation R defined by the following basic rotations:

1. A rotation of θ about the current axis x;
2. A rotation of ϕ about the current axis z;
3. A rotation of α about the fixed axis z;
4. A rotation of β about the current axis y;
5. A rotation of δ about the fixed axis x.

$$\delta = 15^o,\ \alpha = 30^o,\ \theta = 45^o,\ \phi = 60^o,\ \beta = 90^o$$

Composition of Rotations:

Around fixed frame ?
Pre-multiply

Around current frame ?
Post-multiply

Example:

Find the rotation R defined by the following basic rotations:

1. A rotation of θ about the current axis x;
2. A rotation of ϕ about the current axis z;
3. A rotation of α about the fixed axis z;
4. A rotation of β about the current axis y;
5. A rotation of δ about the fixed axis x.

Solution:

$$R = R_{x,\delta}\, R_{z,\alpha}\, R_{x,\theta}\, R_{z,\phi}\, R_{y,\beta}$$

$\delta = 15^o,\ \alpha = 30^o,\ \theta = 45^o,\ \phi = 60^o,\ \beta = 90^o$

$R = ?$

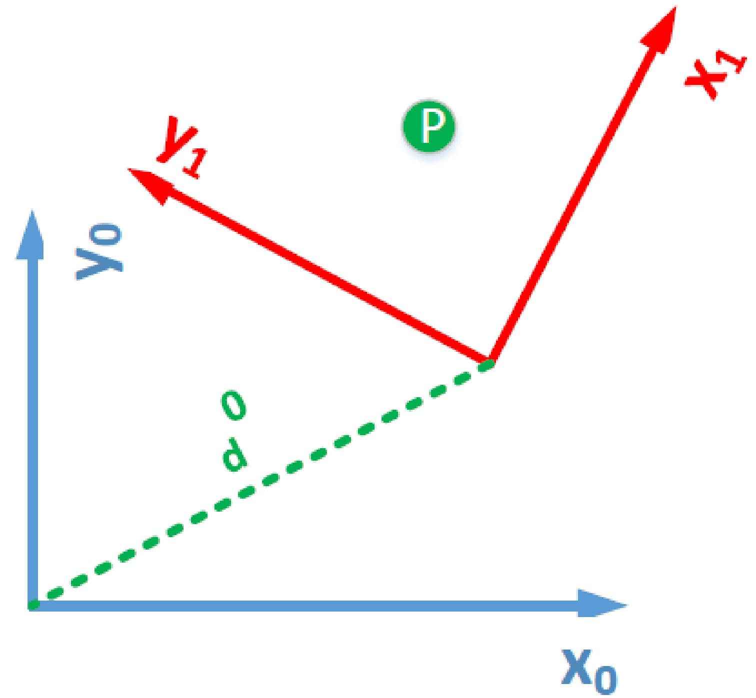
Homogeneous Transformation

Homogeneous Transformation:

Rigid Motions:

- A rigid motion is an ordered pair (R, d) of rotation R and translation d.

$$p^0 = R_1^0\, p^1 + d_1^0$$



Homogeneous Transformation

Homogeneous Transformation:

Rigid Motions:

- A rigid motion is an ordered pair (R, d) of rotation R and translation d.

$$p^0 = R_1^0 \, p^1 + d_1^0$$

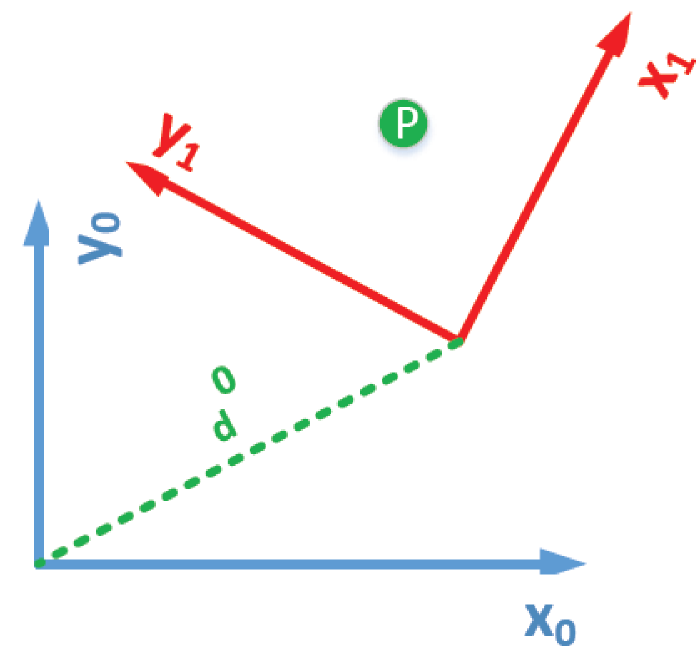
- If there are 3 frames corresponding to 2 rigid motions:

$$p^1 = R_2^1 \, p^2 + d_2^1$$

$$p^0 = R_1^0 \, p^1 + d_1^0$$

Then the overall motion is:

$$\boxed{p^0 = R_1^0 \, R_2^1 \, p^2 + R_1^0 \, d_2^1 + d_1^0}$$



Homogeneous Transformation

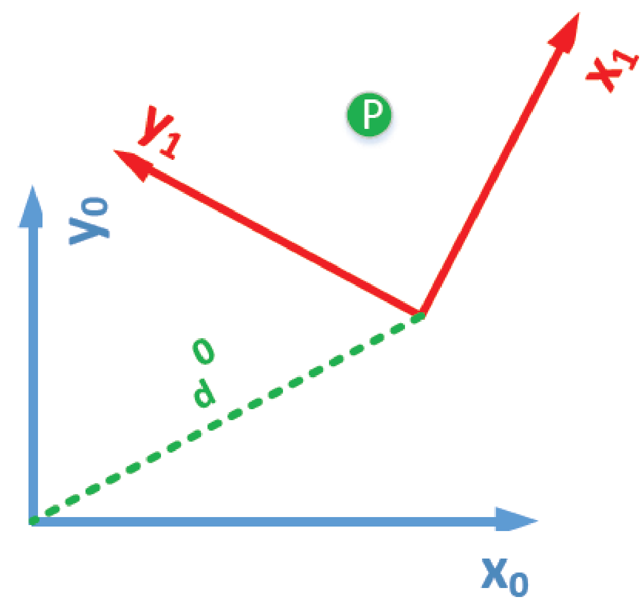
Homogeneous Transformation:

- Homogeneous Transformation is a convenient way to write the formula in a $4 * 4$ matrix:

$$\boxed{p^0 = R_1^0\ R_2^1\ p^2 + R_1^0\ d_2^1 + d_1^0}$$

- Given a rigid motion (R, d), the $4 * 4$-matrix T:

$$T = \begin{bmatrix} \mathbf{R}_{3*3} & \mathbf{d}_{3*1} \\ 0_{3*1} & 1 \end{bmatrix}$$

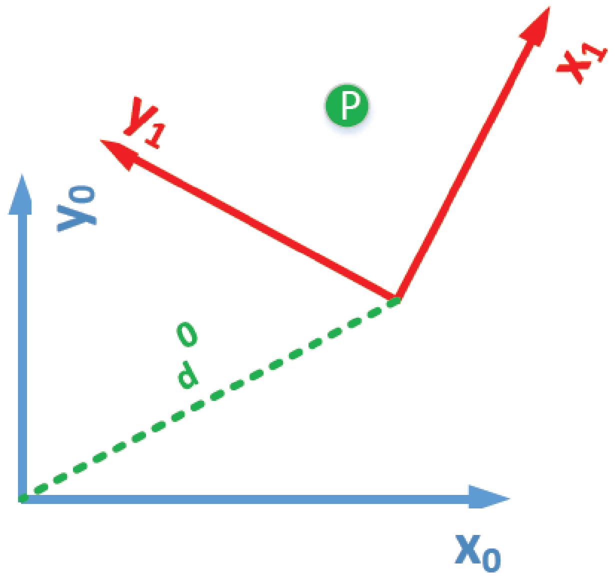


Homogeneous Transformation

Homogeneous Transformation:

- To use HTs in computing coordinates of point p, we need to extend the vectors of a point by one coordinate:

$$P^0 = T_1^0 \; P^1 = \begin{bmatrix} \mathbf{R}_{3*3} & \mathbf{d}_{3*1} \\ 0_{3*1} & 1 \end{bmatrix} \begin{bmatrix} p_x^1 \\ p_y^1 \\ p_z^1 \\ \mathbf{1} \end{bmatrix}$$



For composite homogeneous transformation, the rule for **pre** and **post** multiply is valid as rotation.

Homogeneous Transformation:

Basic Homogeneous Transformation:

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad Rot_{x,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation

Example

Find homogeneous transformation matrix T that represents a rotation by angle α about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle θ about the current z-axis

Homogeneous Transformation

Example

Find homogeneous transformation matrix T that represents a rotation by angle α about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle θ about the current z-axis, is given by:

$$T = Rot_{x,\alpha}\, Trans_{x,b}\, Trans_{z,d}\, Rot_{z,\theta}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation

Parameterization of Rotations:

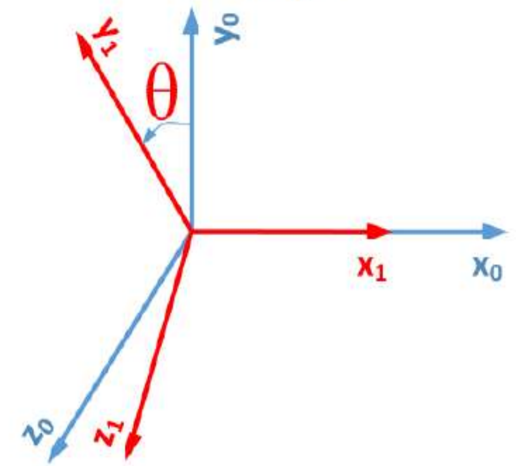
- A general rotation matrix R consists of nine elements r_{ij}.
- These nine elements are not independent quantities due to these constraints:
  1. The columns of a rotation matrix are unit vectors:

$$\sum_i r_{ij}^2 = 1, \qquad j \in \{1, 2, 3\}$$

  2. Columns of a rotation matrix are **mutually orthogonal**:

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \qquad i \neq j$$

Example



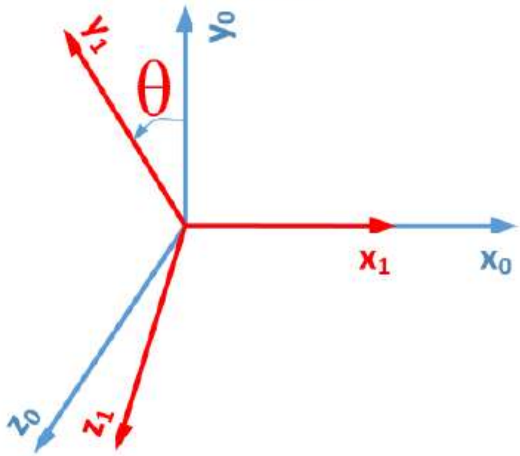
$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

These constraints define **six independent equations** with nine unknowns, so **there are three free variables** required to define a general rotation.

Homogeneous Transformation

Parameterization of Rotations:

Example



- A general rotation matrix R consists of nine elements r_{ij}.
- These nine elements are not independent quantities due to these constraints:
  1. The columns of a rotation matrix are unit vectors:

$$\sum_i r_{ij}^2 = 1, \qquad j \in \{1,2,3\}$$

  2. Columns of a rotation matrix are **mutually orthogonal**:

$$r_{1i}r_{1j} + r_{2i}r_{2j} + r_{3i}r_{3j} = 0, \qquad i \neq j$$

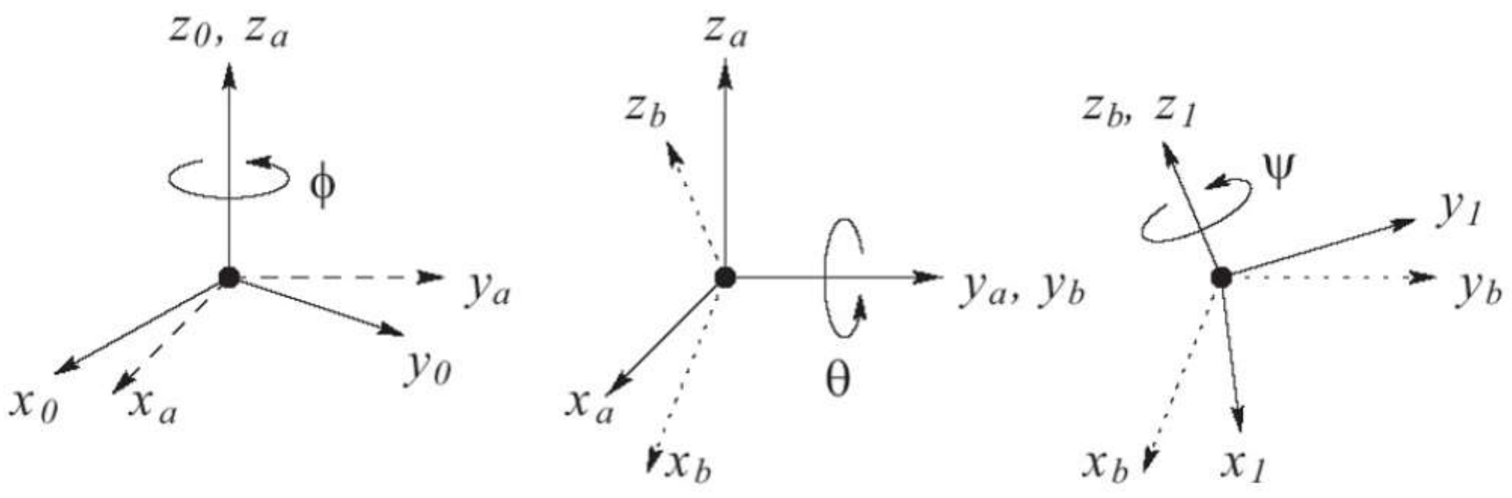
$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) \\ 0 & sin(\theta) & cos(\theta) \end{bmatrix}$$

Two ways are discussed to represent any arbitary rotations by three variable: **Euler Angles** and **Roll-Pith-Yaw** parametrization

Homogeneous Transformation

Parameterization of Rotations: ZYZ-Euler Angles

- It is a common method of specifying a rotation matrix in terms of three independent quantities called Euler Angles $\{\phi, \theta, \psi\}$.
- Any arbitrary rotation could be represented by three successive rotations of:
  1. Rotation by ϕ about the z-axis,
  2. Followed by rotation by θ about the **current** y-axis.
  3. then followed by ψ about the **current** z-axis.



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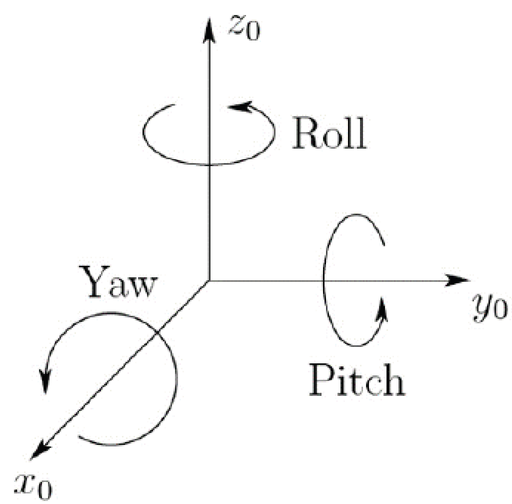
$$R_{ZYZ} = R_{z,\phi} R_{y,\theta} R_{z,\psi} = \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta c_\psi - s_\phi s_\psi & -c_\phi c_\theta s_\psi - s_\phi c_\psi & c_\phi s_\theta \\ s_\phi c_\theta c_\psi + c_\phi s_\psi & -s_\phi c_\theta s_\psi + c_\phi c_\psi & s_\phi s_\theta \\ -s_\theta c_\psi & s_\theta s_\psi & c_\theta \end{bmatrix}$$

Homogeneous Transformation

Parameterization of Rotations: Roll. Pitch and Yaw Angles

- A rotation matrix R could be represented as a product of three successive rotations about the **fixed coordinates**.
- These rotations define the three angles: roll, pitch, yaw, $\{\phi, \theta, \psi\}$:
  1. Rotation by ϕ about the x_0-axis,
  2. Followed by rotation by θ about the **fixed** y_0-axis.
  3. then followed by ψ about the **fixed** z_0-axis.



Homogeneous Transformation

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  3. then followed by ψ about the **fixed** z_0-axis.

$$R_{XYZ} = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix}$$

$$= \begin{bmatrix} c_\phi c_\theta & -s_\phi c_\psi + c_\phi s_\theta s_\psi & s_\phi s_\psi + c_\phi s_\theta c_\psi \\ s_\phi c_\theta & c_\phi c_\psi + s_\phi s_\theta s_\psi & -c_\phi s_\psi + s_\phi s_\theta c_\psi \\ -s_\theta & c_\theta s_\psi & c_\theta c_\psi \end{bmatrix}$$

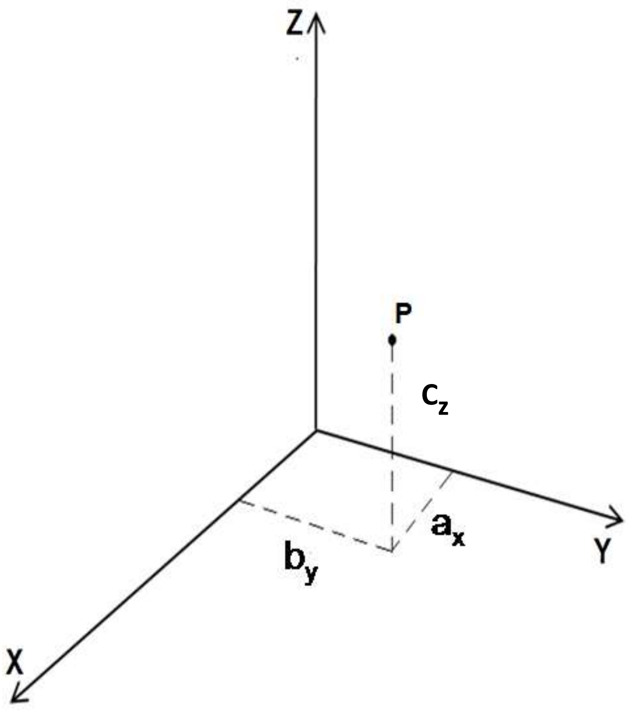
Robot Kinematics

A Point in Space:

- A point ‘P’ in space can be represented by its 3 coordinates relative to a reference frame.

$$P = a_x \hat{\imath} + b_y \hat{\jmath} + c_z \hat{k}$$

- Where $a_{x,}$ b_y and c_z are the 3 coordinates of point represented in the reference frame.



A Vector in Space:

- A vector can be represented by 3 coordinates of its tail and head.

Robot Kinematics

- If the vector start at point A and ends at point B, then it can be represented by,

$$\overline{P}_{AB} = (B_x - A_x)\hat{\imath} + (B_y - A_y)\hat{\jmath} + (B_z - A_z)\hat{k}$$

- If the vector starts at the origin, then

$$\overline{P} = a_x\hat{\imath} + b_y\hat{\jmath} + c_z\hat{k}$$

- Where, a_x, b_y, c_z are 3 components of vector in the reference frame in matrix form,

$$\overline{P} = \begin{bmatrix} a_x \\ b_y \\ c_z \end{bmatrix}$$

- This representation can be modified to also include a scale factor 'w' such that
- If x,y,z are divided by w they will yield a_x, b_y, c_z

Robot Kinematics

$$\overline{P} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \quad \text{Where,} \quad a_x = \frac{x}{w},\ b_y = \frac{y}{w},\ c_z = \frac{z}{w}$$

- Variable ‘w’ may be any member, and as it changes, it can change the overall size of the vector.
- This is similar to zooming a picture in computer graphics.

If $w > 1$, all vector components enlarge.

If $w < 1$,all vector components becomes smaller.

If $w = 1$, size all vector components remain unchanged.

If $w = 0$, then a_x, b_y and c_z will be infinity i.e. x,y,z will represent a vector where length is infinite, but has direction.

i.e. directional vector can be represented by a scale factor $w = 0$, where length is not of importance, but direction is represent by 3 components of the vector.

Robot Kinematics

Ex. A vector is described as $\overline{P} = 3\hat{\imath} + 5\hat{\jmath} + 2\hat{k}$

- Express the vector in matrix form with a scale factor of 2
- Express the vector in matrix form if it were to describe a direction as unit vector

The vector can be described in matrix form with a scale factor of 2 as well as 0 for direction as,

$$\overline{P} = \begin{bmatrix} 6 \\ 10 \\ 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \overline{P} = \begin{bmatrix} 3 \\ 5 \\ 2 \\ 0 \end{bmatrix}$$

- However, to make it into unit vector, we normalize the length such that new length will equal unity.

Robot Kinematics

$$i.e. \ \frac{each\ component\ of\ the\ vector\ (P_x,\ P_y,\ P_z)}{Squre\ root\ of\ sum\ of\ squares\ of\ 3\ componenets\ (\sqrt{P_x{}^2 + \ P_y{}^2 + P_z{}^2}\ )}$$

$$A = \sqrt{P_x{}^2 + \ P_y{}^2 + P_z{}^2} = \sqrt{3^2 + 5^2 + 2^2} = \sqrt{38} = 6.16$$

$$P_x = \frac{3}{6.16} = 0.487, \qquad P_y = \frac{5}{6.16} = 0.811, \quad P_z = \frac{2}{6.16} = 0.324$$

$$\overline{P} = \begin{bmatrix} 0.487 \\ 0.811 \\ 0.324 \\ 0 \end{bmatrix}$$

Robot Kinematics

Representation of a Frame at the Origin of Fixed Frame:

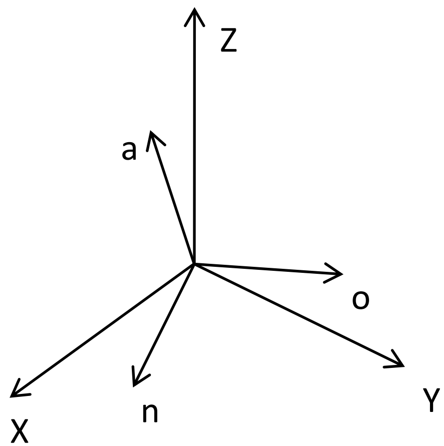
- A frame centered at the origin of a reference frame is represented by 3 vectors, mutually perpendicular to each other called unit vectors

$\bar{n}$ **,ō, ā**

$\bar{n}$ for normal vector

ō for orientation vector

ā for approach vector



- Each unit vector is represented by

  its 3 components in reference frame.

Thus, frame F in matrix is represented as :-

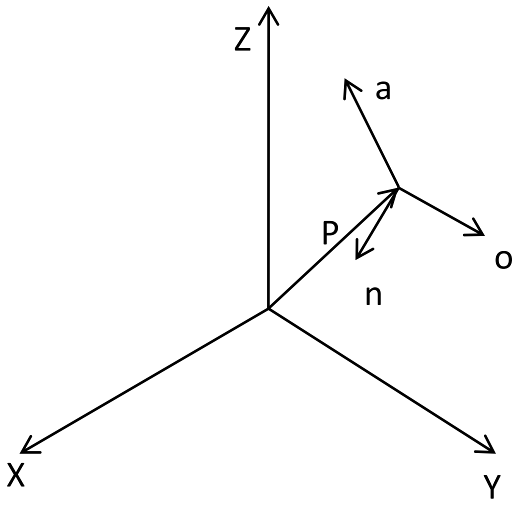
$$F = \begin{bmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$

Robot Kinematics

Representation of a Frame in A Fixed Reference Frame:

- If a frame is not at the origin, then the location of the origin of the frame relative to the reference frame is expressed by 3 vectors describing the directional unit vector as well a forth vector describing its location.

$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- The first three vectors are directional vectors with w = 0, representing the directions of three unit vectors of the frame $\overline{n}$, **ō**, **ā**

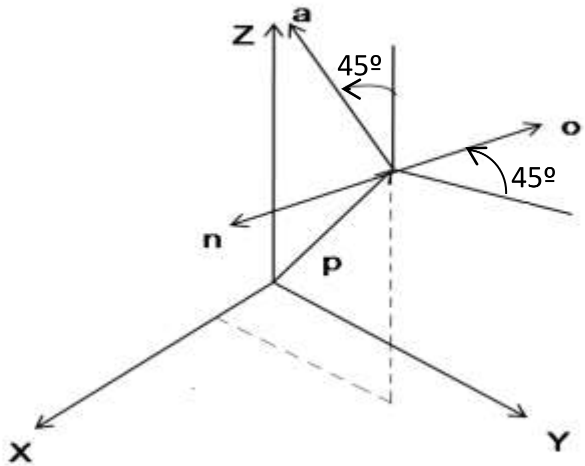
Robot Kinematics

- While, forth vector with w = 1 represents location of the origin of frame relative to the reference frame unlike unit vectors, length of vector P in important, so we use scale factor w = 1.

Example 2:

The frame is located at 3,5,7 units, with its n-axis is parallel to x axis, its o axis at 45º relative to y axis, it’s a- axis at 45º relative to z- axis.

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.707 & -0.707 & 5 \\ 0 & 0.707 & 0.707 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

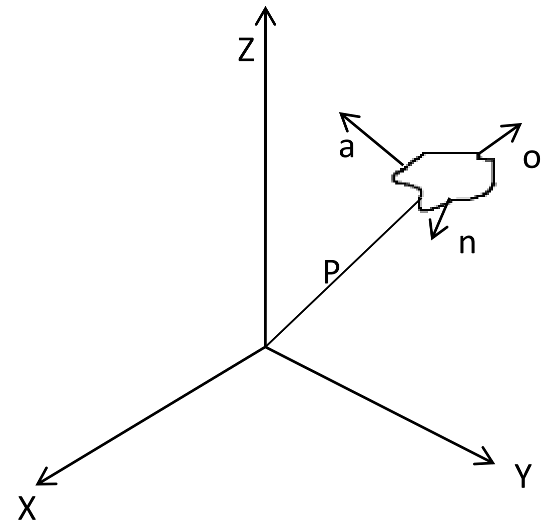


Robot Kinematics

REPRESENTATION OF A RIGID BODY:

- An object can be represented in space by attaching a frame to it and representing the frame in space.
- A frame in space can be represented by a matrix, where the origin of frame, as well as 3 vectors representing the orientation relative to the reference frame

$$F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- A point in space has 3 d.o.f. (T_x, T_y, T_z)
- A Rigid Body in space has 6 d.o.f. (T_x, T_y, T_z, R_x, R_y, R_z)

Robot Kinematics

- However, in above matrix, 12 pieces of information are given for orientation, 3 for position. (This excludes the scale factors on the last row of the matrix because they do not add to this information).
- There must be some constraints present in this representation to reduce the information from 12 to 6 pieces i.e. 6 constraints.

These 6 Constraints are:

- 3 unit vectors $\bar{n}$, ō, ā are mutually perpendicular, their dot product must be zero

$$\bar{n} \bullet \bar{o} = 0, \quad \bar{n}\ \bar{a} = 0, \quad \bar{a} \bullet \bar{o} = 0$$

- Each unit vectors length must be equal to unity, i.e. magnitude of the length of vector must be 1.

$$|n| = 1, \quad |o| = 1, \quad |a| = 1$$

Robot Kinematics

Example 3:

For the following frame, find the values of missing elements and complete the matrix representation of the frame

$$F = \begin{bmatrix} ? & 0 & ? & 5 \\ 0.707 & ? & ? & 3 \\ ? & ? & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robot Kinematics

- Values 5,3,2 represent position of the origin of the frame and do not affect constraint equations.
- Only three values of directional vectors are given:

$$O_y = 0,\ n_y = 0.707,\ a_z = 0.$$

$n_x\, o_x + n_y\, o_y + n_z\, o_z = 0$ **or**	$n_x\,(o) + 0.707\, o_y + n_z\, o_z = 0$ **or**	$0.707\, o_y + n_z\, o_z = 0$
$n_x\, a_x + n_y\, a_y + n_z\, a_z = 0$	$n_x\,(a_x) + 0.707\,(a_y) + n_z\,(0) = 0$	$n_x\, a_x + 0.707\, a_y = 0$
$a_x\, o_x + a_y\, o_y + a_z\, o_z = 0$	$a_x\,(o) + a_y\,(o_y) + 0\,(a_z) = 0$	$a_y\, o_y = 0$
$n_x^2 + n_y^2 + n_z^2 = 1$	$n_x^2 + 0.707^2 + n_z^2 = 1$	$n_x^2 + n_z^2 = 0.5$
$o_x^2 + o_y^2 + o_z^2 = 1$	$o^2 + o_y^2 + o_z^2 = 1$	$o_y^2 + o_z^2 = 1$
$a_x^2 + a_y^2 + a_Z^2 = 1$	$a_x^2 + a_y^2 + 0^2 = 1$	$a_x^2 + a_y^2 = 1$

Robot Kinematics

- Solving these six equations,

  $n_x = \pm 0.707$, $o_y = 0$, o_z +1, $a_x = \pm 0.707$, $a_y = 0.707$

  n_x and a_x must have same sign, i.e. it is possible to have 2 sets of mutually perpendicular vector in opposite direction, i.e. it is possible to have multiple solutions.

  $$F = \begin{bmatrix} 0.707 & 0 & 0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{Or} \quad F = \begin{bmatrix} -0.707 & 0 & -0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

  Values represent by three vector and not arbitrary but bounded by these equations. Do not arbitrary use any desired values in matrix.

Robot Kinematics

Homogeneous transformations:

- In robots, both for kinematics (forward and inverse) and for differential motions, we desire square matrices.
- To represent both orientation and position in the same matrix. We add scale factors to make it a 4×4 Matrix that is square matrix.
- Matrices of this type are called homogeneous matrices.

Robot Kinematics

- Homogeneous or square matrices have the following advantages:
  - Easier to calculate inverse of square Matrix than rectangular matrix.
  - Multiplication of square matrices becomes easier since dimensions of matrix always match.

    ($m \times m$ and $m \times m$ dimensions)

Robot Kinematics

- When a frame (a vector, an object or a moving frame) moves in space relative to a fixed frame, we can represent this motion (change in location and orientation) in a form similar to frame representation.
- Transformation maybe:-
  - Pure translational
  - Pure rotation about an axis
  - Combination of translations or rotations

Robot Kinematics

Pure Translation:

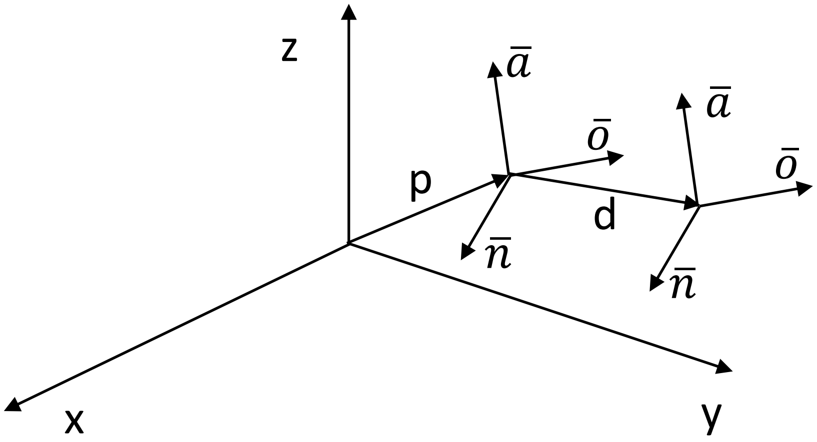
- If a frame moves in space without any change in orientation, the transformation is pure translational.
- In this case, directional unit vectors $\bar{n}$, ō, ā remain in same direction and do not change.

$$T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Where d_x,d_y,d_z are the 3 components of pure translational vector d relative to x, y, and z axes of the reference frame.

Robot Kinematics

- First three columns represent <u>no rotational movement (equivalent to unity)</u> while values in last column represent the translation.



Robot Kinematics

- The new location of the frame relative to the fixed reference frame can be found by adding the vector representing the translation to the vector representing the original location of the origin of the frame.

$$F_{new} = T_{trans}(dx\ dy\ dz) \times F_{old}$$

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{new} = \begin{bmatrix} n_x & o_x & a_x & p_x + dx \\ n_y & o_y & a_y & p_y + dy \\ n_z & o_z & a_z & p_z + dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robot Kinematics

Example:

A frame F has been moved 9 units along x axis and 5 units along z- axis of the reference frame. Find new location of frame.

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Robot Kinematics

Solution:

$$F_{new} = T_{rans}(9,0,5) \times F_{old}$$

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{new} = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 14 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

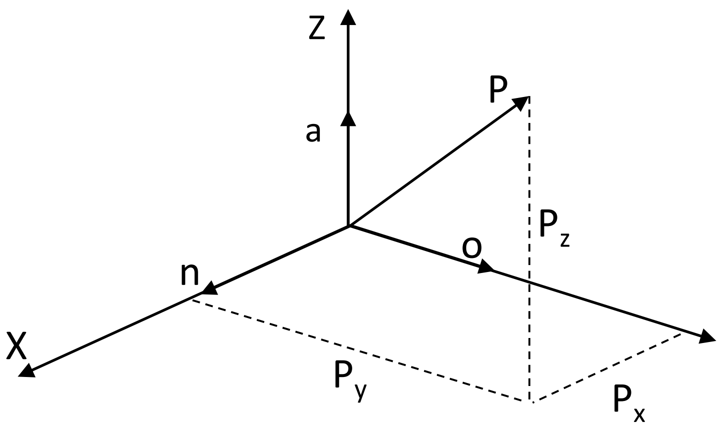
Robot Kinematics

Pure rotation about an axis:

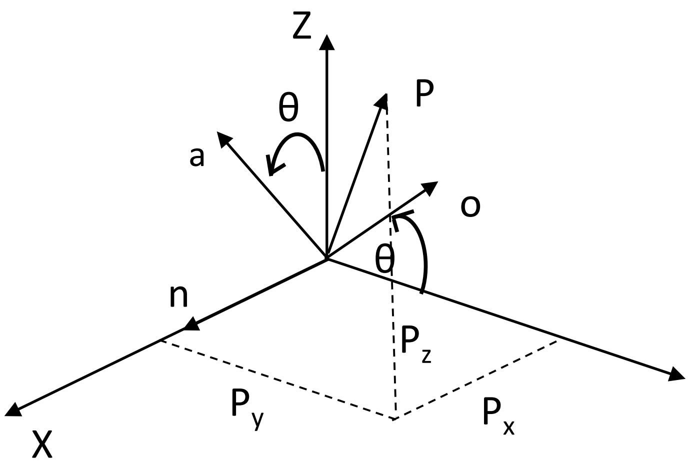
- Assume that frame $\bar{n}, \bar{o}, \bar{a}$ located at the origin of the reference frame $\bar{x}, \bar{y}, \bar{z}$ will rotate through an angle 'ϑ' about the *X*-Axis of the reference frame.
- Assume that attach to the rotating frame $\bar{n}, \bar{o}, \bar{a}$ is a point *P* with coordinates P_x, P_y, P_z relative to reference frame and P_n, P_o and P_a relative to the moving frame.
- As the frame rotates about *X*-Axis point be attached to the frame will also rotate with it.
- Before rotation coordinates of point 'P' in both reference and $\bar{n}, \bar{o}, \bar{a}$ and $\bar{x}, \bar{y}, \bar{z}$ are the same. (Both frames are at the same location in (same-origin) and parallel to each other).
-

Robot Kinematics

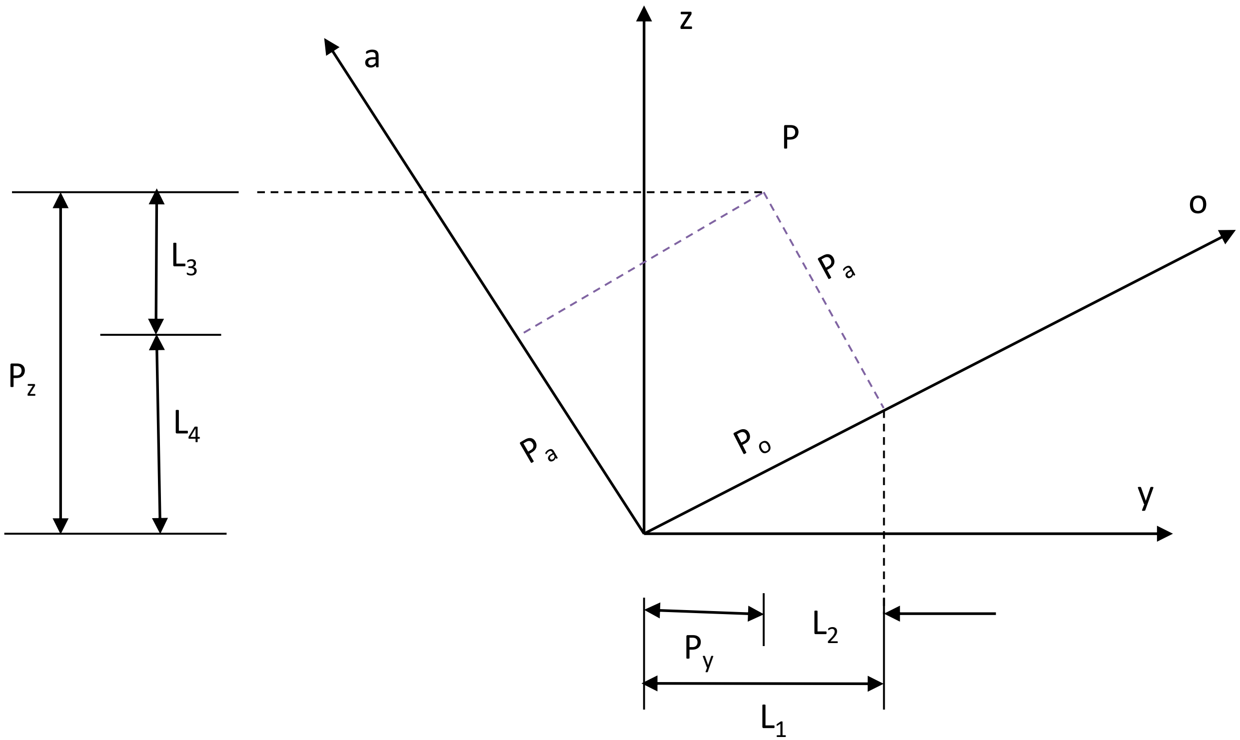
- After rotation P_n, P_o and P_a co-ordinates of the point remains same in its own face $\bar{n}$, $\bar{o}$, $\bar{a}$. But P_x, P_y, P_z co-ordinates will be different in $\bar{x}$, $\bar{y}$, $\bar{z}$ frame.
- Find the new co-ordinates of the point relative to the fix reference frame after the moving frame has rotated.



Before Rotation



Robot Kinematics



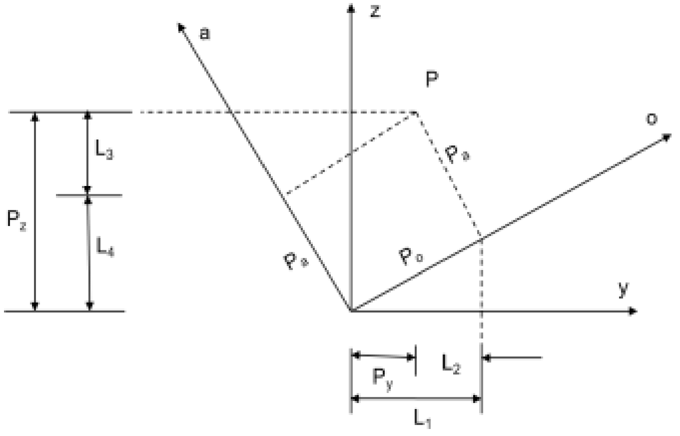
Robot Kinematics

- P_x does not change as the frame rotates about the x axis but the values of P_y and P_z do change.

$$P_x = P_n$$

$$P_y = L_1 - L_2 = P_o \cos\theta - P_a \sin\theta$$

$$P_z = L_3 + L_4 = P_o \sin\theta + P_a \cos\theta$$



In matrix form-

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} P_n \\ P_o \\ P_a \end{bmatrix}$$

- Indicated coordinates along X axis have not change

Robot Kinematics

- The coordinates of point (or vector) P in the rotated frame must be pre-multiplied by rotation matrix, to get coordinates in reference frame.
- This rotation is about x-axis of the reference frame and is denoted as

$$P_{xyz} = R_{ot}(x,\theta) \times P_{noa}$$

$$R_{ot}(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

Similarly,

$$R_{ot}(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \qquad R_{ot}(z,\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Robot Kinematics

$$^{U}P = {}^{U}T_{R} + {}^{R}P$$

i.e. ${}^{U}T_{R}$ = Transformation of frame relative to frame U for universe

P_{noa} as ${}^{R}P$- P relative to frame R

P_{xyz} as ${}^{U}P$- P relative to frame U

Cancelling R gives coordinates of point P relative to U.

Robot Kinematics

- **Example:**

A point P (2, 3, 4) is attached to a rotating frame the frame rotates 90°about X-Axis of the reference frame. Find out the coordinates of the point relative to the reference frame after the rotation.

$$P_{xyz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} P_n \\ P_o \\ P_a \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

Combined Transformation

Combined transformations consist of a number of successive translations and rotations about:

1. the fixed reference frame axis or
2. the moving current frame axis.

Assume a frame *n̄, ō, ā* assume a frame an a subjected to following three transformations relative to the fixed reference frame x,y,z.

i. Rotation of α degrees about x axis.
ii. Followed by translation of (L_1,L_2,L_3) relative to x,y,z axis
iii. Followed by a rotation of β degrees about y axis.

Point P_{noa} is attached to the rotating frame at the origin of the reference frame.

$$P_{1,xyz} = Rot(x,\alpha) \times P_{noa}$$

Where $P_{1,xyz}$ is the coordinate of the point after the first transformation relative to the reference frame .

Combined Transformation

The coordinates of the point relative to the reference frame at the conclusion of the second transformation will be

$P_{2,xyz}$ =Trans(L_1,L_2,L_3) X $P_{1,xyz}$ = Trans(L_1,L_2,L_3) X Rot(x,α) X P_{noa}

similarly after third transformation, the coordinates of the point relative to reference frame will be

P_{xyz} = $P_{3,xyz}$ = Rot(y,β) X $P_{2,xyz}$

= Rot(y,β) X Trans(L_1,L_2,L_3) X Rot(x,α) X P_{noa}

The coordinates of the point relative to the reference frame at the conclusion of each transformation is found by pre-multiplying the coordinates of the point by each transformation matrix.

Thus, the order of matrices written is the opposite of the order of transformation performed.

Combined Transformation

Ex.

A point P(7,3,2) is attached to the frame $\bar{n}, \bar{o}, \bar{a}$ and is subjected to the following transformations.

i. Rotation of 90^o about the z-axis
ii. Followed by a rotation of 90^o about the y-axis
iii. Followed by a translation of (4,-3,7)

Find the coordinates of the point relative to the reference frame at the conclusions of the transformations.

Solution:

P_{xyz} = Trans(4,-3,7) X Rot(y,90^o) X Rot(z, 90^o) X P_{noa}

Combined Transformation

Solution:

P_{xyz} = Trans(4,-3,7) X Rot(y,90^o) X Rot(z, 90^o) X P_{noa}

$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{X} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{X} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{X} \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

Combined Transformation

Ex.

A point P(7,3,2) is attached to the frame $\bar{n}, \bar{o}, \bar{a}$ and is subjected to the following transformations but in different order i.e.

i. Rotation of 90^o about the z- axis

ii. Followed by a translation of (4,-3,7)

iii. Followed by a rotation of 90^o about the y axis, then

Find the coordinates of the point relative to the reference frame at the conclusions of the transformations.

Solution:

$P_{xyz} = Rot(Y,90^o) \times Trans(4,-3,7) \times Rot(Z, 90^o) \times P_{noa}$

Combined Transformation

Solution:

P_{xyz} = Rot(y,90^o) X Trans(4,-3,7) X Rot(z, 90^o) X P_{noa}

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} X \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 9 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

Combined Transformation

Ex. A moving frame noa is obtained from frame xyz by rotation of 90^o about the z axis followed by a rotation of 90^o about the x axis. Then, noa locates a point P_{noa} at n=10, o=20 and a=30. Determine its coordinates with respect to reference frame xyz.

Solution:

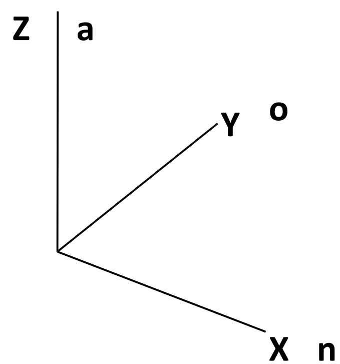
P_{xyz} = Rot(x,90^o) X Rot(z, 90^o) X P_{noa}

Rot(x,z)= Rot(x,90^o) X Rot(z, 90^o)

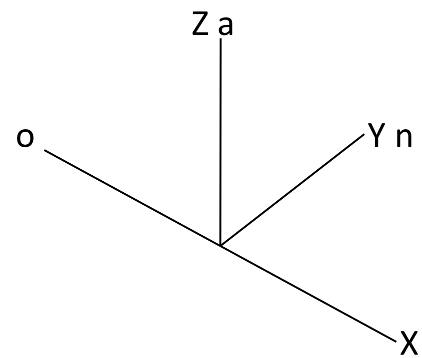
$$\mathrm{Rot}(x, 90^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90 & -\sin 90 \\ 0 & \sin 90 & \cos 90 \end{bmatrix} \qquad \mathrm{Rot}(z, 90^\circ) = \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

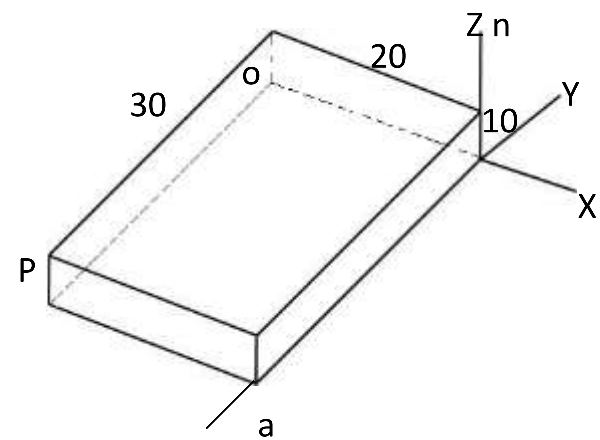
Combined Transformation



Initial position



After Rot (z, 90˚)
(The first rotation)



After Rot(x , 90˚)
(The second rotation)
Final position

Combined Transformation

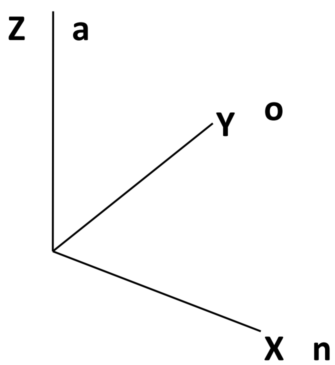
$$\text{Rot}(x,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \text{X} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Rot}(x,z) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

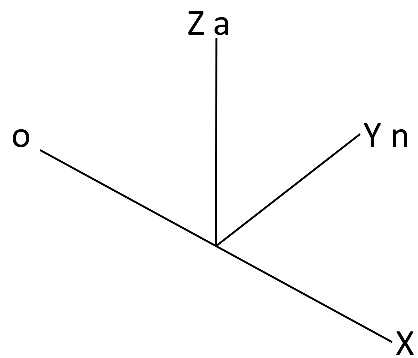
$$\mathbf{P}_{\mathbf{xyz}} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \text{X} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$\mathbf{P}_{\mathbf{xyz}} = \begin{bmatrix} -20 \\ -30 \\ 10 \end{bmatrix}$$

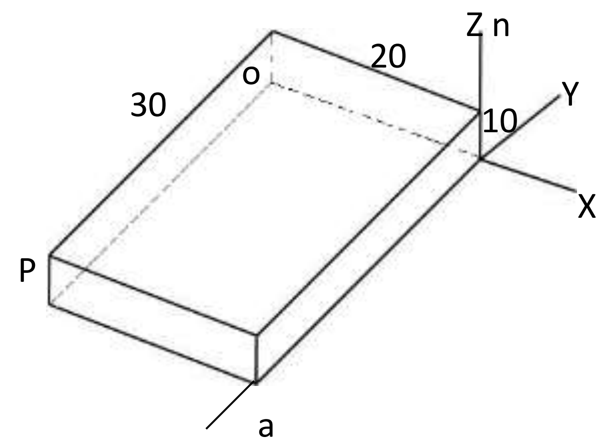
Combined Transformation



Initial position



After Rot (z, 90˚)
(The first rotation)



After Rot(x , 90˚)
(The second rotation)
Final position

Combined Transformation

Ex.A moving frame noa is obtained from frame xyz by rotation of 90^o about the z axis followed by a rotation of 90^o about the n-axis. Then, noa locates a point P_{noa} at n=10, o=20 and a=30. Determine its coordinates with respect to reference frame xyz.

Solution:

P_{xyz} = Rot(z,90^o) X Rot(n, 90^o) X P_{noa}

Rot(z,n)= Rot(z,90^o) X Rot(n, 90^o)

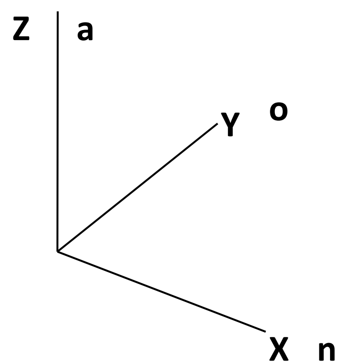
$$\text{Rot}(z, 90^\circ) = \begin{bmatrix} \cos 90 & -\sin 90 & 0 \\ \sin 90 & \cos 90 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

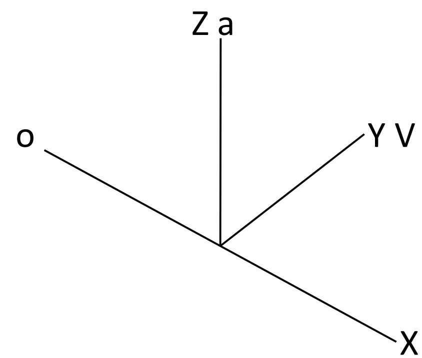
$$\text{Rot}(n, 90^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90 & -\sin 90 \\ 0 & \sin 90 & \cos 90 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

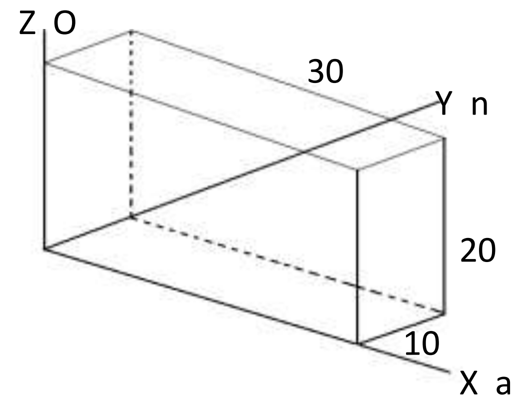
Combined Transformation



Initial position



After Rot(z, 90˚)
(The first rotation)



After Rot(x , 90˚)
(The second rotation)
Final position

Combined Transformation

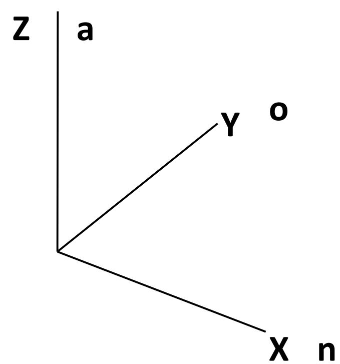
$$\mathrm{Rot}(z, n) = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{X} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathrm{Rot}(z, n) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

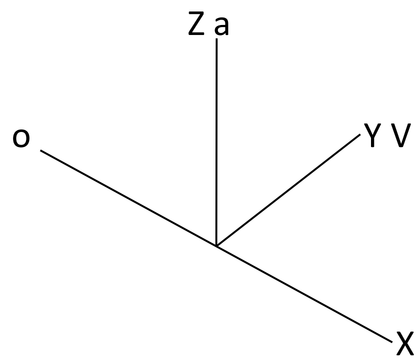
$$\mathbf{P}_{\mathbf{xyz}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathrm{X} \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}$$

$$\mathbf{P}_{\mathbf{xyz}} = \begin{bmatrix} 30 \\ 10 \\ 20 \end{bmatrix}$$

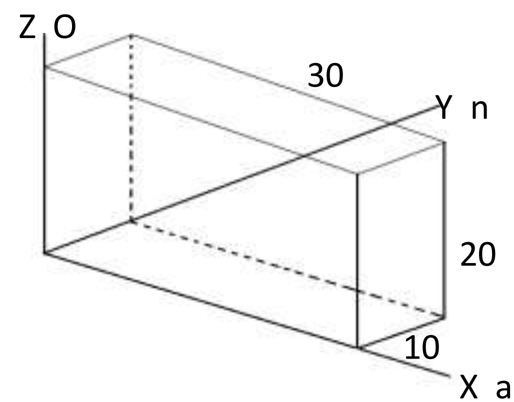
Combined Transformation



Initial position



After Rot(z, 90˚)
(The first rotation)



After Rot(x , 90˚)
(The second rotation)
Final position

Combined Transformation

Transformation relative to rotating frame:

It is possible to make a transformation relative to the axis of a moving or current frame.

i.e. A rotation of 90^o be made relative to $\bar{n}$ axis of the moving or current frame and not the x- axis of the reference frame.

To do this, transformation matrix is post-multiplied, Since the position of a point or an object attached to a moving frame is always measured relative to that moving frame, the position matrix describing the point or object is also always post multiplied.

Combined Transformation

Ex.8

The same point P(7,3,2) is subjected to transformations relative to the current moving frame, find the coordinates of the point relative to the reference frame after the transformations are completed.

i. Rotation of 90^o about the $\bar{a}$ axis
ii. Then a Followed by a translation of (4,-3,7) along *$\bar{n}$, $\bar{o}$, $\bar{a}$*
iii. Followed by a rotation of 90^o about the $\bar{o}$ axis.

Solution:

$P_{xyz} = Rot(\bar{a}, 90^o) \text{ X } Trans(4,-3,7) \text{ X } Rot(\bar{o}, 90^o) \text{ X } P_{noa}$

Combined Transformation

Solution:

P_{xyz} = Rot($\bar{a}$,90^o) X Trans(4,-3,7) X Rot($\bar{o}$, 90^o) X P_{noa}

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{X} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{X} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{X} \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

Combined Transformation

Ex.9

A frame B, was rotated about the $\bar{n}$ axis 90^o,it was then translated about the current $\bar{a}$ axis 3mm before being rotated about the z-axis 90^o. Finally it was translated about the current $\bar{o}$ axis 5mm

i. Write a equation describing the motion

ii. Find the final location of a point P(1,5,4) attached to the frame relative to the reference frame.

Solution:

$$^{U}T_{B} = Rot(z, 90^o) \times Rot(\bar{n}, 90^o) \times Trans(0,0,3) \times Trans(0,5,0)$$

Combined Transformation

Solution:

$^{U}T_{B}$= Rot(z,90°) X Rot($\bar{n}$,90°) X Trans(0,0,3) X Trans(0,5,0)

$^{U}P = {}^{U}T_{B}$ X ^{B}P

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{X} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{X} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{X} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathrm{X} \begin{bmatrix} 1 \\ 5 \\ 4 \\ 1 \end{bmatrix}$$

$$P_{xyz} = \begin{bmatrix} 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$