Unit 3 Differential Calculus

1. **Function :**

   A function f from a set D to a set Y is a rule that assigns a unique single element $f(x) \in Y$ to each element $x \in D$. The set D of all possible input values is called **domain** and the set of all values of $f(x)$ as x varies throughout D is called the**range** of the function. In calculus, D and Y are subsets of set of real numbers.

2. **Limit :**

   Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

   $$\lim_{x \to x_0} f(x) = L,$$

   if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x

   $$0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \epsilon$$

3. **Continuity :**

   Interior Point : A function $y = f(x)$ is continuous at an interior point c of its domain if

   $$\lim_{x \to c} f(x) = f(c).$$

   End Point : A function $y = f(x)$ is continuous at a left end point a or is continuous at a right end point b of its domain if

   $$\lim_{x \to a^+} f(x) = f(a) \qquad \textit{or} \qquad \lim_{x \to b^-} f(x) = f(b), \qquad \text{respectively.}$$

4. **Theorem - Continuity Test :** A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

   1. $f(c)$ exists (c lies in the domain of f).
   2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
   3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

5. **Derivative :** The derivative of a function f at a point x_0, denoted $f'(x_0)$, is

   $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   provided this limit exists.

6. **Differentiable Function :**

   A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval [a,b] if it is differentiable on the interior of (a,b) and if the limits

   $$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \qquad \lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h}$$

   exist at the end points.

7. **Theorem :**

   If f has a derivative at $x = c$, then f is continuous at $x = c$.

8. **Remark** Converse of above theorem is not true. The function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

9. **Absolute Extreme Values**

   Let f be a function with domain D. Then f has an absolute maximum value on D at a point c if

   $$f(x) \leq f(c) \qquad \text{for all } x \in D$$

   and an absolute minimum value on D at c if

   $$f(x) \geq f(c) \qquad \text{for all } x \in D$$

10. **The Extreme Value Theorem :**

    If f is continuous on a closed interval $[a, b]$, then f attains both absolute extreme values in $[a, b]$. That is, there are numbers $x_1, x_2 \in [a, b]$ with $f(x_1) = m;\ \ f(x_2) = M;\ \ m \leq f(x) \leq M$ for all$x \in [a, b]$.

11. **Remark** The requirements in theorem are key ingredients. Without them, the conclusion of the theorem need not hold. The function $y = x$ on $(-\infty, \infty)$ is continuous but has domain neither finite nor closed and it has neither absolute maximum nor absolute minimum. This shows that the domain must be closed and finite, only continuity is not sufficient. The function

    $$y = \begin{cases} 1/2 & x = 0 \\ x & 0 < x < 1 \\ 1/2 & x = 1 \end{cases}$$

    is defined on closed and finite interval $[0, 1]$ but not continuous at $x = 0$ and $x = 1$ and has neither absolute maximum nor absolute minimum.

12. **Local Extreme Values :**

    A function f has local maximum value at a point c within its domain D if $f(x) \leq f(c)$ for all x in some open interval containing c in D.

    A function f has local minimum value at a point c within its domain D if $f(x) \geq f(c)$ for all x in some open interval containing c in D.

    If the domain of f is the closed interval $[a, b]$, then f has local maximum or minimum at the end points a and b considering the half open intervals containing a or b in D.

    Note that the absolute extreme values of a function f are also local extreme values.

13. **Theorem - The First Derivative Test for Local Extreme Values :**

    If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

    Proof: Let us assume that f has local maximum at an interior point $c \in D$ and $f'(c)$ exists. We want to prove that $f'(c) = 0$.

    Since f has local maximum at $x = c$, $f(x) \leq f(c)$ for all x in some open interval I containing c in D.

Therefore $f(x) - f(c) \leq 0$; for all $x \in I$ $\quad (i)$

Since $f'(c)$ exists, the limits $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c}$ and $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c}$ exist and

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c) \qquad (ii)$$

But when $x \to c^+$,

$$x - c \geq 0; \qquad \text{and hence } \frac{f(x) - f(c)}{x - c} \leq 0$$

and when $x \to c^-$,

$$x - c \leq 0; \qquad \text{and hence } \frac{f(x) - f(c)}{x - c} \geq 0$$

Therefore, $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} \leq 0$ and $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{x - c} \geq 0 \qquad (iii)$

But from (ii), these two limits are equal, hence this is possible only if they both are zero. That is

$$\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = f'(c) = 0 \qquad (iv)$$

Hence, it is proved that $f'(c) = 0$. Similarly we can prove that if f has local minimum at $c \in D$ then also $f'(c) = 0$.

Hence the proof.

14. **Critical Point :**

An interior point of the domain of a function f where $f' = 0$ or f' is undefined is a critical point of f.

We can identify critical points from the graph of a function easily. The points where tangent does not exist and the points where tangent is parallel to x- axis (horizontal tangent) are the critical points of that function. In fact, from the graph of a function, we can conclude whether there is a local maximum or minimum at the critical points. The function f has local maximum at a critical point if f is increasing at the left of that point and decreasing at the right of that point. Similarly, the function f has local minimum at a critical point if f is decreasing at the left of that point and increasing at the right of that point.

Without using graphs, we can identify the critical points by calculating derivative. To derive the algebraic conditions for increasing - decreasing functions we need following theorems.

15. **Rolle's Theorem :**

Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one c in (a, b) at which $f'(c) = 0$.

Proof: Being continuous on a closed interval, f has absolute maximum and minimum on $[a, b]$ by extreme value theorem. These can occur only at

(i) interior points where $f' = 0$

(ii) interior points where f' does not exist.

(iii) end points of the function's domain.

Since f is differentiable on (a, b), the second case is not possible.

If either the absolute maximum or minimum occurs at an interior point c, then $f'(c) = 0$ by first derivative test of extreme values. The existence of c proves Rolle's Theorem.

If both the absolute extreme values occur at end points only, then let f have minimum at a and maximum at b. Then $f(a) \leq f(x) \leq f(b)$ for all $x \in (a, b)$. But $f(a) = f(b)$. Hence $f(a) = f(x) = f(b)$ for all $x \in (a, b)$. Thus f is constant function and $f(x) = 0$ for all $x \in (a, b)$. This proves the Rolle's Theorem.

16. **Remark :** All the conditions in Rolle's theorem are essential. If any one of them is not true, there does not exist a c in the domain of function such that $f'(c) = 0$. (i.e. the graph may not have a horizontal tangent.)

17. **Lagrange's Mean Value Theorem (LMVT) :**

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Proof : Let

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Since f is continuous on $[a, b]$ and differentiable on (a, b), h is also continuous on $[a, b]$ and differentiable on (a, b). Further

$$h(a) = f(a) - f(a) - \frac{f(b) - f(a)}{b - a}(a - a) = 0$$

$$h(b) = f(b) - f(a) - \frac{f(b) - f(a)}{b - a}(b - a) = f(b) - f(a) - f(b) + f(a) = 0$$

Hence h satisfies all the conditions of Rolle's Theorem. Therefore there exists $c \in (a, b)$ such that $h'(c) = 0$.

But

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$$

.

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \Rightarrow h'(c) = \frac{f(b) - f(a)}{b - a}$$

Hence the proof.

18. **Remark :** The number $\frac{f(b)-f(a)}{b-a}$ can be treated as average change in f over $[a, b]$ and $f'(c)$ is instantaneous change. Then the mean value theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

19. **Cauchy's Mean Value Theorem (CMVT) :**

    Suppose $f(x)$ and $g(x)$ are continuous on a closed interval $[a, b]$ and differentiable on (a, b) and also $g'(x) \neq 0$ on (a, b). Then there is at least one point $c \in (a, b)$ such that
    $$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

    Proof : First we show that $g(a) \neq g(b)$.

    If $g(a) = g(b)$, then by applying Rolle's Theorem to $g(x)$ on $[a, b]$, we get $g'(c) = 0$ for some $c \in (a, b)$. This is not true since it is given that $g'(x) \neq 0$ on (a, b).

    Therefore $g(a) \neq g(b)$.

    Now consider
    $$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)]$$

    This function is continuous on $[a, b]$, differentiable on (a, b). Further $F(a) = F(b) = 0$.

    By Rolle's Theorem, there exists $c \in (a, b)$ such that $F'(c) = 0$

    But $F'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$

    Therefore $F'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$

    Hence $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ since $g'(c) \neq 0$.

20. **Taylor's Theorem:** If f and its first n derivatives $f', f'', \cdots f^{(n)}$ are continuous on $[a, b]$ and $f^{(n)}$ is differentiable on (a, b), then there exists a point $c \in (a, b)$ such that
    $$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2!} + \cdots + f^{(n)}(a)\frac{(b-a)^n}{n!} + f^{(n+1)}(c)\frac{(b-a)^{n+1}}{(n+1)!}$$

21. **L'H'ôpital's Rule :**

    Suppose that functions f and g are differentiable on an open interval I containing a, $f(a) = g(a) = 0$, $g'(x) \neq 0$ on I. If $x \neq a$, then

    $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ assuming that the limit at right exists.

22. **Monotonic Function :**

    A function f is said to be increasing on an interval I, if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ for all$x_1, x_2 \in I$. A function f is said to be decreasing on an interval I, if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ for all$x_1, x_2 \in I$. A function that is increasing or decreasing on an interval is said to be monotonic on that interval.

23. **First Derivative Test for Increasing / Decreasing Functions :**

    Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$. If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof : Let f be continuous on $[a, b]$ and differentiable on (a, b). Let x_1 and x_2 be any two points on $[a, b]$ with $x_1 < x_2$. Then the LMVT applied to f on $[x_1, x_2]$ says that there exists a $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Since $x_2 - x_1 > 0$, the signs of $f(x_2) - f(x_1)$ and $f'(c)$ are same. Hence if $f'(x) > 0$ for all $x \in (a, b)$, $f(x_2) - f(x_1) > 0 \Rightarrow f$ is increasing on $[a, b]$. if $f'(x) < 0$ for all $x \in (a, b)$, $f(x_2) - f(x_1) < 0 \Rightarrow f$ is decreasing on $[a, b]$.

Hence the proof.

24. **Remark :** If the derivative f' is continuous but never zero on the interval (a, b), then by the Intermediate Value Theorem applies to f', the derivative must be everywhere positive or everywhere negative on (a, b). Hence we can determine the sign of f' on (a, b) simply by evaluating the derivative at a single point x in (a, b).

25. **First Derivative Test for Local Extrema :**

    Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across this interval from left to right,

    (1) if f' changes sign from negative to positive at c, then f has a local minimum at c.

    (2) if f' changes sign from positive to negative at c, then f has a local maximum at c.

    (3) if f' does not change sign at c, then f has no local extreme at c.

    Proof : (1) Since the sign of f' changes from negative to positive at c, there are numbers a and b such that $a < c < b$, $f' < 0$ on (a, c), and $f' > 0$ on (c, b). That is f is decreasing on (a, c) and increasing on (c, b). For all $x \in (a, c), x < c \Rightarrow f(x) > f(c)$ (f decreasing). For all $x \in (c, b), x > c \Rightarrow f(x) > f(c)$ (f increasing). Thus $f(x) > f(c)$ for all $x \in (a, b)$. Hence f has local minimum at c.

    We can prove the parts (2) and (3) similarly.

26. **Concavity :**

    The graph of a differentiable function $y = f(x)$ is

    (a) concave up on an open interval I if f' is increasing on I.

    (b) concave down on an open interval I if f' is decreasing on I.

27. **Second Derivative test for concavity :**

    Let f be twice differentiable on an interval I.

    1. If $f'' > 0$ on I, the graph of f over I is concave up.

    2. If $f'' < 0$ on I, the graph of f over I is concave down.

28. **Point of Inflection :**

    A point where the graph of a function has a tangent line and where the concavity changes is a point of inflection.

29. **Remark :** At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

30. **Second Derivative Test for Local Extrema :**

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has local maximum at $x = c$.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has local minimum at $x = c$.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum or local minimum or neither.

Proof : (1) Since f'' is continuous, if $f''(c) < 0$, $f''(x) < 0$ on open interval I containing c. Therefore, f' is decreasing over I. But $f'(c) = 0$. This implies f' changes sign from positive to negative at c. So f has local maximum at c by first derivative test.

(2) Similar to (1)

(3) Consider the functions $f_1 = x^4$, $f_2 = -x^4$, $f_3 = x^3$. For all the functions, second derivative is continuous and first and second derivatives are zero at $x = 0$. But f_1 has local minimum at $x = 0$, f_2 has local maximum at $x = 0$, f_3 has neither maximum nor minimum at $x = 0$.

The above analysis does not give us only the points where we get local extreme values but it helps in deciding the shape of the graph of the function. f' gives us critical points, intervals in which function is increasing/ decreasing and f'' gives us concavity.

Algorithms with Illustrative Examples

1. **Procedure of Graphing** $y = f(x)$ **:**

   (a) Identify the domain of f any symmetries the curve may have. Find the x and y intercepts.

   (b) Find the derivatives f' and f''.

   (c) Find the critical points of f, if any and identify the function's behavior at each one.

   (d) Find where the curve is increasing and where it is decreasing.

   (e) Find the points of inflection, if any and determine the concavity of the curve.

   (f) Identify any asymptotes that may exist.

   (g) Plot key points, such as the intercepts, critical points, points of inflection and sketch the curve accordingly.

   (h) From the graph, conclude about absolute extreme values if any, local extreme values at end points in case of closed domain.

2. **Determine the absolute extreme values for the given function :**

   Note that if the function is continuous and the domain is finite closed, it has absolute extreme values.

   (a) Identify the domain of the function.

   (b) Find f'.

   (c) Find critical points of f.

   (d) Find the values of f at critical points and end points. Whichever maximum is absolute maximum and whichever minimum is absolute minimum.

   If the domain is not closed plot the graph using first algorithm and then determine the absolute and local extreme values if any.

3. **Determine local extreme values / Determine the intervals where the function is increasing/ decreasing** Do the appropriate steps in Algorithm 1

4. **Test whether the given function satisfies hypothesis of LMVT. If yes, find the value of $'c'$**

   (a) Check whether the function is continuous at all the points in given interval.

   $e^x, \sin, \cos,$, polynomials are continuous functions at all real numbers. The addition, subtraction, multiplication, composition of two continuous functions is always continuous. The rational function is continuous except at those points where denominator is zero. The function is not defined at the points where negative sign under radical occurs. We have to check whether these points are in the given interval or not. If the definition of function differs in left and right of the some point, then we have to check the continuity at that point.

   (b) Check whether the function is differentiable in given open interval or not,

   Find $f'(x)$. The function is not differentiable at the points where denominator is zero or negative sign occurs in radical sign in f'. Check whether these points are in the interval or not.

   If both the conditions are satisfied, then conclude that the function satisfies hypothesis of LMVT.

   (c) Solve the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$ for c. Check whether $c \in (a, b)$. If yes that is the required value.

5. **Prove the Inequality -**

   (a) Compare the inequality with $f'(b)(b-a) \leq f(b) - f(a) \leq f'(a)(b-a)$ or $f'(a)(b-a) \leq f(b) - f(a) \leq f'(b)(b-a)$

   (b) Identify f and a, b such that the hypothesis of LMVT is satisfied for f on $[a, b]$

   (c) Now by LMVT, there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. But write the appropriate inequality between $f'(a), f'(b), f'(c)$. Put the value of $f'(c)$ as $\frac{f(b)-f(a)}{b-a}$ and simplify to get required inequality.

6. **Show that the given function has exactly one zero in given interval**

   (a) Check the signs of $f(a)$ and $f(b)$ where (a, b) is given interval. If they both have opposite signs, f has one root in the interval (a, b).

(b) Let us assume that the given function has two distinct zeros x_1, x_2 in given interval (a, b). Let $x_1 < x_2$

(c) Observe that the function is continuous on $[a, b]$, differentiable on (a, b). Hence the function is continuous on $[x_1, x_2]$ and differntiable on (x_1, x_2). Further $f(x_1) = f(x_2) = 0$. Hence f satisfies the hypothesis of Rolle's Theorem.

(d) Therefore there exists a $c \in (x_1, x_2)$, such that $f'(c) = 0$.

(e) Solve above equation to get value of c which indicates a contradiction to $c \in (x_1, x_2)$

(f) Hence the assumption is wrong. There can not exist two distinct roots. Therefore there exists esactly one root in the given interval

7. bf Sketch the graph of a function satisfying some conditions:

   (a) Identify the points mentioned in the conditions, local maximum, local minimum on the graph first. At the left and right of these points apply the remaining conditions carefully.

   (b) Note that $f' > 0$ implies f is increasing and $f' < 0$ implies f is decreasing. $f' = 0$ at some point means f has a tangent parallel to x axis at that point. Note that $f'' > 0$ implies f is concave up and $f'' < 0$ implies f is concave down.

8. **Theoretical Problems :**

   (a) If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

   (b) If f' is zero at each point x of an open interval (a, b), then prove that $f(x) = c$ for all $x \in (a, b)$, where c is a constant.

   (c) If $f(x) = g'(x)$ for each point $x \in (a, b)$, then prove that $f(x) = g(x) + c$ for all $x \in (a, b)$ for some constant c.

   (d) Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b).

   (e) Show that a cubic polynomial can have at most three real zeros.

   (f) A marathoner ran the 26.2 m marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph, assuming the initial and final speeds are zero.

   (g) (D. Andrica) If f is continuous on $[a, b]$ and differentiable on (a, b) and $f(x) \neq 0 \ \forall x \in (a, b)$, then prove that $\exists c \in (a, b)$ such that $\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}$
   Proof : Let $g(x) = f(x)(x - a)(x - b)$. $g(a) = g(b) = 0$. $g'(x) = f'(x)(x - a)(x - b) + f(x)(x - b) + f(x)(x - a)$ $g'(c) = 0 \rightarrow f'(c)(c - a)(c - b) = f(c)(b - c) + f(c)(a - c)$ $f'(c)/f(c) = 1/(a - c) + 1/(b - c)$