Linear Algebra
Eigen Values and Eigen vectors

- **Eigen Values and Eigen Vectors:** Let A be a square matrix of order n. Then a nonzero vector in $\mathcal{R}^n$ is called an eigen vector of A if $AX = \lambda X$ for some $\lambda \in R$. Here λ is called an eigen value of A.

  Now $AX = \lambda X \rightarrow AX - \lambda X = 0$. Therefore $(A\lambda I)X = 0$. Thus X is an eigen vector of A iff it is a nonzero solution of the homogeneous system $(A - \lambda I)X = 0$. But we know that the system $(A\lambda I)X = 0$ has a nonzero solution iff $|A - \lambda I| = 0$.

  $|A-\lambda I| = 0$ is called characteristic equation of A. Its roots are eigen values of A and a nonzero solution of $(A - \lambda_i I)X = 0$ is an eigen vector corresponding the eigen value λ_i.

- **Characteristic Equation Of A:** The characteristic equation $|A - \lambda I| = 0$ can also be written as

  $$\lambda^n - \sigma_1 \lambda^{n-1} + \sigma_2 \lambda^{n-2} - \cdots + (-1)^n \sigma_n = 0$$

  where σ_i = sum of the determinants of order i containing i diagonal elements at a time.

  For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, ch. equation is $\lambda^2 - \sigma_1 \lambda + \sigma_2 = 0$ where $\sigma_1 = a_{11} + a_{22}$ and $\sigma_2 = |A|$.

  For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, ch. equation is $\lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3 = 0$

  where $\sigma_1 = a_{11} + a_{22} + a_{33}$ and

  $\sigma_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$ and $\sigma_3 = |A|$.

- **Algebraic and geometric multiplicity of an eigen value:** Let λ_i be an eigen value of a square matrix A of order n. The highest number $1 \leq k \leq n$ for which $(\lambda - \lambda_i)^k$ is a factor of the characteristic equation, is called the algebraic multiplicity of λ_i (i.e. order of λ_i as a root of the ch. equation.)

The geometric multiplicity of λ_i is the maximum number of linearly independent eigen vectors corresponding to λ_i.

The difference between algebraic and geometric multiplicity of an eigen value λ_i is known as defect of λ_i.

- **Properties of Eigen Values and Eigen Vectors:**

1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A (counted according to their multiplicities) then $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_n = |A|$ and
   $\lambda_1 + \lambda_2 + \ldots + \lambda_n =$ the trace of A. .
   Proof: Let $\lambda_1, \lambda_2 \ldots, \lambda_n$ be eigenvalues of A. Then
   $\lambda^n - \sigma_1 \lambda^{n-1} + \sigma_2 \lambda^{n-2} - \cdots + (-1)^n \sigma_n = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$
   Therefore $\lambda^n - \sigma_1 \lambda^{n-1} + \sigma_2 \lambda^{n-2} - \cdots + (-1)^n \sigma_n = \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots (-1)^n (\lambda_1 \times \lambda_2 \times \ldots \times \lambda_n)$
   Hence $\sigma_1 =$ Trace of $A = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ and
   $\sigma_n = |A| = \lambda_1 \times \lambda_2 \times \ldots \times \lambda_n$
   Hence the proof.

2. The matrix A is invertible if and only if 0 is not an eigenvalue of A.
   Proof: The matrix A is invertible iff $|A| \neq 0$.
   But $|A| = \lambda_1 \times \lambda_2 \times \ldots \times \lambda_n$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A.
   Hence the matrix A is invertile iff $\lambda_1 \times \lambda_2 \times \ldots \times \lambda_n \neq 0$.
   Hence the matrix A is invertile iff $\lambda_i \neq 0$ for all $i = 1, 2, \cdots n$.
   Hence the proof.

3. If λ is an eigenvalue of a matrix A, $\lambda^{-1} = \frac{1}{\lambda}$ is an eigenvalue of a matrix A^{-1}(if exists).
   Proof: Let A be an invertible matrix. Let λ be an eigenvalue of A. Since A is invertible, $\lambda \neq 0$.
   Since λ is an eigenvalue of A, $\exists X \neq 0$ such that $AX = \lambda X$.
   Premultiplying by A^{-1}, $A^{-1}(AX) = A^{-1}(\lambda X)$
   Therefore $(A^{-1}A)X = \lambda(A^{-1}X)$
   Therefore $X = \lambda(A^{-1}X)$

Therefore $\frac{1}{\lambda}X = (A^{-1}X)$

Therefore $\frac{1}{\lambda}$ is an eigen value of A^{-1}

4. If λ is an eigenvalue of a matrix A, $k\lambda$ is an eigenvalue of a matrix kA, where k is a non-zero scalar.

   Proof: Let λ be an eigenvalue of A.

   Therefore $\exists X \neq 0$ such that $AX = \lambda X$.

   Premultiplying by k, $k(AX) = k(\lambda X)$

   Therefore $(kA)X = k\lambda X$

   Therefore $k\lambda$ is an eigen value of kA

5. If λ is an eigenvalue of a matrix A, λ^2 is an eigenvalue of a matrix A^2.

   Proof: Let λ be an eigenvalue of A.

   Therefore $\exists X \neq 0$ such that $AX = \lambda X$.

   Premultiplying by A, $A(AX) = A(\lambda X)$

   Therefore $(AA)X = \lambda(AX)$

   Therefore $(A^2)X = \lambda(\lambda X) = \lambda^2 X$

   Therefore λ^2 is an eigen value of A^2.

6. The matrices A and A^T have the same eigenvalues.

   Proof: We know that $|A| = |A^T|$

   Therefore $|A - \lambda I| = |(A - \lambda I)^T|$

   Therefore $|A - \lambda I| = |(A^T - \lambda I^T|$ since $(A + B)^T = A^T + B^T$

   Therefore $|A - \lambda I| = |(A^T - \lambda I|$

   Hence Ch. equations of A and A^T are same. Hence A and A^T have same eigenvalues.

7. The eigen vectors corresponding to two distinct eigenvalues of any $n \times n$ matrix A are linearly independent.

   Proof: Let λ_1, λ_2 be distinct eigenvalues of a matrix A. Let X_1, X_2 be the corresponding eigen vectors respectively.

   Therefore $AX_1 = \lambda_1 X_1$ and $AX_2 = \lambda_2 X_2$.

   Claim : X_1 and X_2 are linearly independent.

Let $X_1 = kX_2$.
Therefore $AX_1 = A(kX_2)$
Therefore $\lambda_1 X_1 = k(AX_2) = k\lambda_2 X_2$
Therefore $\lambda_1 k X_2 = k\lambda_2 X_2)$
Therefore $k(\lambda_1 - \lambda_2)X_2 = 0$
Since X_2 is an eigen vector, it can not be zero. Since λ_1, λ_2 are distinct eigenvalues, $(\lambda_1 - \lambda_2) \neq 0$.
Hence $k = 0$.
Thus $X_1 = kX_2$ implies that $k = 0$. Hence X_1 and X_2 are linearly independent.

- **Similar Matrices:** Two square matrices of same order A and B are said to be similar if there exists a non singular matrix C, such that $B = C^{-1}AC$.

- **Similar matrices have same eigen values.**

  Proof: Let A and B be similar matrices. Let λ be an eigen value of B and X be the corresponding eigen vector.

  Then $BX = \lambda X$. But $B = C^{-1}AC$ for some non singular matrix C.

  Therefore $C^{-1}ACX = \lambda X$

  Premultiplying by C, $CC^{-1}ACX = C\lambda X$

  Therefore $A(CX) = \lambda(CX)$.

  Since C is nonsingular matrix and X is a nonzero vector, CX is a nonzero vector.

  Hence λ is an eigen value of A and the corresponding eigen vector is CX.

- **Diagonalization:** Let A have eigen values $\lambda_1, \lambda_2, \cdots \lambda_n$. If no eigen value has a defect, then there are n linearly independent eigen vectors say $X_1, X_2, \cdots X_n$. Then they form a basis of $\mathcal{R}^n$. It is known as Eigen Basis. Construct a matrix C with $X_1, X_2, \cdots X_n$ as columns. Since they are linearly independent C is non singular and invertible.

  Then $C^{-1}AC$ is a diagonal matrix with diagonal elements as $\lambda_1, \lambda_2, \cdots \lambda_n$.

  Then matrix A is said to be diagonalizable matrix.