CVT – UNIT - V

LAPLACE TRANSFORM

PART - A (2 Marks)

1. State the conditions under which the Laplace transform of $f(t)$ exist.

i) $f(t)$ should be continuous or piecewise continuous in the given closed interval $[a,b]$, where $a>0$.

ii) $f(t)$ should be of exponential order

2. State and prove first shifting theorem.

Statement:

If $L[f(t)] = F(s)$

Then

$$\boxed{L[e^{-at} f(t)] = F(s+a)}$$

proof:

$$L[f(t)] = \int_0^\infty f(t) e^{-st} dt$$

$$L[e^{-at}f(t)] = \int_0^\infty e^{-at} f(t) e^{-st} dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t) dt$$

$$\boxed{L[e^{-at}f(t)] = F(s+a)}$$

3 state and prove Change of scale property.

Statement : $L f(t) = F(s)$, then $\boxed{L[f(at)] = \frac{1}{a} F\left[\frac{s}{a}\right]}$

Proof :

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$L[f(at)] = \int_0^\infty e^{-st} f(at) dt$$

Put $at = x$ | $t \to 0 \Rightarrow x \to 0$

$a dt = dx$ | $t \to \infty \Rightarrow x \to \infty$

$\Rightarrow \boxed{dt = \frac{dx}{a}}$

$$= \int_0^\infty e^{-sx/a} f(x) \frac{dx}{a}$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)x} f(x) dx$$

$$= \frac{1}{a} \int_0^\infty e^{-(s/a)t} f(t) dt$$ [$\because x$ is dummy variable]

$$\boxed{L[f(at)] = \frac{1}{a} F\left[\frac{s}{a}\right]}$$

4. Is the linearity property applicable to $L\left(\frac{1-\cos t}{t}\right)$

$$L\left[\frac{1-\cos t}{t}\right] = L\left[\frac{1}{t}\right] - L\left[\frac{\cos t}{t}\right]$$

$L\left[\frac{1}{t}\right]$ does not exist.

$$\lim_{t \to 0} \frac{1}{t} = \frac{1}{0} = \infty$$

$L\left[\frac{\cos t}{t}\right]$ does not exist.

$$\lim_{t \to 0} \frac{\cos t}{t} = \frac{1}{0} = \infty$$

$\therefore$ Linearity property is not applicable to $L\left[\frac{1-\cos t}{t}\right]$

5. Does $L\left(\frac{\cos at}{t}\right)$ exist?

$$L\left[\frac{\cos at}{t}\right] = \lim_{t \to 0} \frac{\cos at}{t}$$

$$= \frac{1}{0}$$

$$\boxed{L\left[\frac{\cos at}{t}\right] = \infty}$$

Therefore, $L\left[\frac{\cos at}{t}\right]$ does not exist.

(6) Find the $L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

$$\boxed{L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{t}\sin t}$$

(7) Find the inverse Laplace transform of $\log\left(\frac{s+1}{s-1}\right)$

Soln:

$$\boxed{L^{-1}[F(s)] = -\frac{1}{t} L^{-1}[F'(s)]}$$

$F(s) = \log\left(\frac{s+1}{s-1}\right)$

$$L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s+1}{s-1}\right)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}(\log(s+1) - \log(s-1))\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= -\frac{1}{t}\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s-1}\right)\right]$$

$$\boxed{\sinh t = \frac{e^{t}-e^{-t}}{2}}$$

$$= -\frac{1}{t}\left[e^{-t} - e^{t}\right]$$

$$= \frac{1}{t}\left[e^{t} - e^{-t}\right]$$

$$\boxed{L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{2}{t}\sinh t}$$

(8) Find the Laplace transform of Unit step function.

Soln: USF

$$u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[u(t-a)] = \int_0^a e^{-st} f(t)\,dt + \int_a^\infty e^{-st} f(t)\,dt$$

$$= 0 + \int_a^\infty e^{-st} (1)\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= -\frac{1}{s}\left[e^{-\infty} - e^{-as}\right]$$

$$= -\frac{1}{s}\left[0 - e^{-as}\right]$$

$$\boxed{L[u(t-a)] = \frac{e^{-as}}{s}}$$

(9) State initial and Final value theorem

Initial value theorem:

statement: If $L[f(t)] = F(s)$

Then

$$\boxed{\lim_{t \to 0} f(t) = \lim_{s \to 0} sF(s)}$$

Final value theorem:

Statement: If $L[f(t)] = F(s)$

Then

$$\boxed{\lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)}$$

(10) state convolution theorem.

If $f(t)$ & $g(t)$ are function definition for $t \geq 0$, then

$$\boxed{L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]}$$

PART-(B)

(1) Find the Laplace transform of the triangular
(i) wave function $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a \le t \le 2a \end{cases}$

with $f(t) = f(t+2a)$.

Soln:

$$L[f(t)] = \frac{1}{1-e^{-as}}\left[\int_0^a e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{2a} e^{-st} f(t)\,dt\right]$$

$\boxed{\text{Given Period} = 2a}$

$$= \frac{1}{1+e^{-2as}}\left[\int_0^a e^{-st} f(t) + \int_0^{2a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a t\,e^{-st}\,dt + \int_0^{2a} (2a-t)\,e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left\{\left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right\}$$

$$= \frac{1}{1-e^{-2as}}\left[-\cancel{\frac{a e^{-sa}}{s}} - \frac{e^{-sa}}{s^2} + 0 + \frac{1}{s^2} - 0 + \frac{e^{-2sa}}{s^2} + \cancel{\frac{a e^{-sa}}{s}} - \frac{e^{-sa}}{s^2}\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[1 + e^{-2sa} - 2e^{-sa}\right]$$

$\boxed{(a-b)^2 = a^2+b^2-2ab}$

$$= \frac{1}{s^2[1-e^{-2as}]} \cdot (1-e^{-as})^2$$

$$L[f(t)] = \frac{(1-e^{-as})^2}{s^2[1-(e^{-as})^2]}$$

$$= \frac{(1-e^{-as})^{\cancel{2}}}{s^2(1-\cancel{e^{-as}})(1+e^{-as})}$$

$$= \frac{(1-e^{-as})}{s^2(1+e^{-as})}$$

$$\boxed{\tanh\left(\frac{\theta}{2}\right) = \frac{1+e^{-\theta}}{1-e^{\theta}}}$$

$$\boxed{L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)}$$

(P)

(ii) find $L[t^2 e^{-t}\cos t]$

Soln:

$$\boxed{L[t^2 f(t)] = \frac{d^2}{ds^2}\, L[f(t)]}$$

$$L[t^2\cos t] = \frac{d^2}{ds^2}\, L[\cos t]$$

$$\boxed{L(\cos t) = \frac{s}{s^2+1}}$$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)\cdot 1-(s)(2s)}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2+1-2s^2}{(s^2+1)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]$$

$$= \frac{(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)\cdot 2s}{(s^2+1)^4}$$

$$= \frac{2s(\cancel{s^2+1})\left[-(s^2+1)-2(1-s^2)\right]}{(s^2+1)^{\cancel{4}\,3}}$$

$$= \frac{2s}{(s^2+1)^3}\left[-s^2-1-2+2s^2\right]$$

$$\boxed{L\left[t^2\cos t\right] = \frac{2s}{(s^2+1)^3}(s^2-3)}$$

Rep. $s \to s+1$

$$L\left[e^{-t}t^2\cos t\right] = \frac{2(s+1)\left[(s+1)^2-3\right]}{\left[(s+1)^2+1\right]^3}$$

$$\boxed{L\left[e^{-t}t^2\cos t\right] = \frac{2(s+1)\left[(s+1)^2-3\right]}{\left[(s+1)^2+1\right]^3}}$$

(2) (i) Find the Laplace transform of the

$$f(t) = \begin{cases} \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega \end{cases}$$

Soln:

$$L[f(t)] = \frac{1}{1-e^{-as}}\left[\int_0^a e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\frac{2\pi}{\omega}} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st} f(t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{-1}{(1-e^{-\frac{2\pi s}{\omega}})(s^2+\omega^2)}\left[e^{-\frac{s\pi}{\omega}}\left(s\sin\cancel{\omega}\frac{\pi}{\cancel{\omega}} + \omega\cos\cancel{\omega}\frac{\pi}{\cancel{\omega}}\right) - e^0(s\sin 0 + \omega\cos 0)\right]$$

$$= \frac{-1}{(1-e^{-\frac{2\pi s}{\omega}})(s^2+\omega^2)}\left[e^{-\frac{s\pi}{\omega}}(0+\omega(-1)) - (0+\omega)\right]$$

$\sin 0 = 0$ $\sin\pi = 0$
$\cos 0 = 1$ $\cos\pi = -1$

formula

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)}$$

$$L[f(t)] = \frac{-1(-\omega)}{(1-e^{\frac{-2\pi s}{\omega}})(s^2+\omega^2)} \left[e^{-\frac{s\pi}{\omega}} + 1 \right]$$

$$= \frac{\omega}{(s^2+\omega^2)} \frac{(1+e^{-s\pi/\omega})}{1-(e^{-\pi s/\omega})^2}$$

$$= \frac{\omega}{(s^2+\omega^2)} \cdot \frac{(1+\cancel{e^{-s\pi/\omega}})}{(1+\cancel{e^{-s\pi/\omega}})(1-e^{-s\pi/\omega})}$$

$$\boxed{\therefore L[f(t)] = \frac{\omega}{(s^2+\omega^2)(1-e^{-s\pi/\omega})}}$$

(2) (ii) find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using convolution theo.

Soln:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

Formula:

$$\boxed{f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du}$$

$$= \cos at * \frac{\sin at}{a}$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)\, du$$

$$= \frac{1}{a}\int_0^t \sin(at-au) \cos au\, du$$

Formula:

$$\boxed{\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]}$$

$$\boxed{\begin{aligned} A &= at - au \\ B &= au \\ A+B &= at \\ A-B &= at - 2au \end{aligned}}$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(at-2au)]\,du$$

$$= \frac{1}{2a}\left[\sin at\,(u) - \frac{\cos(at-2au)}{-2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \cancel{\frac{\cos at}{2a}} - \cancel{\frac{\cos at}{2a}}\right]$$

$$\boxed{L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t\sin at}{2a}}$$

(3) (i) Find the Laplace transform of the square wave function

$$f(t) = \begin{cases} E, & 0 \le t \le a/2 \\ -E, & a/2 \le t \le a \end{cases} \qquad \boxed{f(t) = f(t+a)}$$

Soln:

$$L[f(t)] = \frac{1}{1-e^{-as}}\left[\int_0^a e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[E\int_0^{a/2} e^{-st}\,dt - E\int_{a/2}^{a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{1-e^{-as}}\left[-\frac{e^{-as/2}}{s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{-as/2}}{2}\right]$$

$$L[f(t)] = \frac{E}{s(1-e^{-as})}\left[1+e^{-as}-2e^{-as/2}\right]$$

$$= \frac{E}{s(1-e^{-as})}\cdot\left(1-e^{-as/2}\right)^2$$

$$= \frac{E}{s}\,\frac{\left(1-e^{-as/2}\right)^2}{1-\left(e^{-as/2}\right)^2}$$

$$= \frac{E}{s}\cdot\frac{\left(1-e^{-as/2}\right)^{\cancel{2}}}{\left(1+e^{-as/2}\right)\cancel{\left(1-e^{-as/2}\right)}}$$

$$= \frac{E}{s}\cdot\frac{\left(1-e^{-as/2}\right)}{\left(1+e^{-as/2}\right)}$$

$$\boxed{L[f(t)] = \frac{E}{s}\cdot\tanh\left(\frac{as}{4}\right)}$$

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(3) (ii) Find $L^{-1}\left[\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theo.

Soln:

$$L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\cdot\frac{1}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+b^2)}\right]$$

$=$

$$= \cos at * \frac{\sin bt}{b}$$

$$= \frac{1}{b}\int_0^t \cos au \, \sin b(t-u)\, du$$

$$= \frac{1}{b}\int_0^t \sin(bt-bu) \cos au \, du$$

$$\boxed{\begin{aligned} A &= bt-bu \\ B &= au \\ A+B &= (a-b)u+bt \\ A-B &= (a+b)u+bt \end{aligned}}$$

$$= \frac{1}{2b}\int_0^t \Big(\sin[(a-b)u+bt] + \sin[(a+b)u+bt]\Big)\, du$$

$$= \frac{1}{2b}\left[-\frac{\cos[(a-b)u+bt]}{(a-b)} - \frac{\cos[(a+b)u+bt]}{-(a+b)}\right]_0^t$$

$$= \frac{1}{2b}\left[-\frac{\cos[(a-b)u+bt]}{a-b} + \frac{\cos[(a+b)u+bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[-\frac{\cos(at \cancel{-bt} \cancel{+bt})}{a-b} + \frac{\cos(at \cancel{\pm bt} \cancel{+bt})}{a+b} + \frac{\cos bt}{a-b} - \frac{\cos(-bt)}{a+b}\right]$$

$$= \frac{1}{2b}\left[-\frac{\cos at}{a-b} + \frac{\cos at}{a+b} + \frac{\cos bt}{a-b} - \frac{\cos bt}{a+b}\right]$$

$$= \frac{1}{2b}\left[\cos at\left(\frac{-1}{a-b} + \frac{1}{a+b}\right) + \cos bt\left(\frac{1}{a-b} - \frac{1}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left[\cos at\left(\frac{\cancel{a}-b+\cancel{a}-b}{a^2-b^2}\right) - \cos bt\left(\frac{\cancel{a}+b-\cancel{a}+b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{2b(a^2-b^2)}\left[\cos at\,(-2b) - \cos bt\,(2b)\right]$$

$$= \frac{(-\cancel{2b})(\cos at - \cos bt)}{2\cancel{b}(a^2-b^2)}$$

$$\boxed{L^{-1}\left[\frac{s}{(s^2+a^2)(s^2+b^2)}\right] = \frac{\cos at - \cos bt}{b^2-a^2}}$$

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(4) (i) Find the Laplace Transform of the function $f(t) = \begin{cases} K, & 0 \le t \le a \\ -K, & a \le t \le 2a \end{cases}$

& $f(t+2a) = f(t)$.

Soln:

$$L[f(t)] = \frac{1}{1-e^{-as}}\left[\int_0^a e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^{2a} e^{-st} f(t)\,dt\right]$$

$\boxed{\text{Given Period} = 2a}$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[K\int_0^a e^{-st}\,dt - K\int_a^{2a} e^{-st}\,dt\right]$$

$$= \frac{K}{(1-e^{-2as})}\left[\left(\frac{e^{-st}}{-s}\right)_0^a - \left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[-e^{-as} + 1 + e^{-2as} - e^{-as}\right]$$

$$= \frac{K}{s(1-e^{-2as})}\left[1 + e^{-2as} - 2e^{-as}\right]$$

$$= \frac{K}{s(1-e^{-2as})}(1-e^{-as})^2$$

$$= \frac{K(1-e^{-as})^2}{s\left[1-(e^{-as})^2\right]} = \frac{K}{s}\cdot\frac{(1-e^{-as})^{\cancel{2}}}{\cancel{(1-e^{-as})}(1+e^{-as})}$$

$$= \frac{K}{s}\cdot\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\boxed{L[f(t)] = \frac{K}{s}\tanh\left(\frac{as}{2}\right)}$$

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(4) (ii) Find $L\left[\frac{\cos at - \cos bt}{t}\right]$

<u>Soln:</u>

$$\boxed{L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds}$$

<u>W.K.T</u> $F(s) = L[f(t)] = L[\cos at - \cos bt]$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty L\left[\cos at - \cos bt\right] ds$$

$$= \int_s^\infty \left[L(\cos at) - L(\cos bt)\right] ds$$

$$\boxed{\int \frac{2s}{s^2+a^2} = \log(s^2+a^2)}$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$\boxed{\frac{1}{\infty} = 0}$$

$$\boxed{\log 1 = 0}$$

$$= \frac{1}{2}\left[\log \frac{\cancel{s^2}(1+a^2/s^2)}{\cancel{s^2}(1+b^2/s^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{(s^2+a^2)/\cancel{s^2}}{(s^2+b^2)/\cancel{s^2}}\right)\right]$$

$$= -\frac{1}{2}\log\left[\frac{s^2+a^2}{s^2+b^2}\right]$$

$$\boxed{\therefore L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log\left[\frac{s^2+b^2}{s^2+a^2}\right]}$$

$$\textcircled{5}$$ Find $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ using convolution theo.

Soln:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{s}{s^2+b^2}\right]$$

$$= \cos at * \cos bt$$

$$= \int_0^t \cos au \cos b(t-u)\,du$$

$$= \int_0^t \cos au \cos(bt-bu)\,du$$

$$\boxed{\begin{aligned} &A = au \\ &B = bt-bu \\ &A+B = (a-b)u+bt \\ &A-B = (a+b)u-bt \end{aligned}}$$

$$= \frac{1}{2}\int_0^t \left(\cos[(a-b)u+bt] + \cos[(a+b)-bt]\right)du$$

$$\boxed{\begin{aligned} &\cos A\cos B = \\ &\tfrac{1}{2}[\cos(A+B) + \\ &\quad \cos(A-B)] \end{aligned}}$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{(a+b)}\right]_0^t$$

$$\boxed{\begin{aligned} &\sin(-\theta) = -\sin\theta \\ &\cos(-\theta) = \cos\theta \end{aligned}}$$

$$= \frac{1}{2}\left[\frac{\sin(at-\cancel{bt}+\cancel{bt})}{a-b} + \frac{\sin(at+\cancel{bt}-\cancel{bt})}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin(-bt)}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b}+\frac{1}{a+b}\right)+\sin bt\left(-\frac{1}{a-b}+\frac{1}{a+b}\right)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{a+\cancel{b}+a-\cancel{b}}{a^2-b^2}\right)+\sin bt\left(\frac{-\cancel{a}-b+\cancel{a}-b}{a^2-b^2}\right)\right]$$

$$= \frac{1}{2(a^2-b^2)}\left[2a\sin at-2b\sin bt\right]$$

$$= \frac{\cancel{2}}{\cancel{2}(a^2-b^2)}(a\sin at-b\sin bt)$$

$$\boxed{\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]=\frac{a\sin at-b\sin bt}{a^2-b^2}}$$

(6) (i) verify Initial value & Final value theorem for $f(t) = 1+e^{-t}(\sin t+\cos t)$

Soln: (i) Initial value theorem

$$\boxed{\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)}$$

$$F(s) = L[f(t)] = L[1+e^{-t}(\sin t+\cos t)]$$

$$= L(1)+L(e^{-t}\sin t)+L(e^{-t}\cos t)$$

$$= \frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}$$

$$\boxed{F(s) = \frac{1}{s}+\frac{s+2}{(s+1)^2+1}}$$

LHS (IVT) $\lim_{t \to 0} f(t)$

$$\Rightarrow \lim_{t \to 0} f(t) = \lim_{t \to 0} \left[1 + e^{-t}(\sin t + \cos t)\right]$$

$$= 1 + e^{0}(\sin 0 + \cos 0)$$

$$= 1 + (0 + 1) = 1 + 1$$

$$\boxed{LHS = \lim_{t \to 0} f(t) = 2}$$

RHS (IVT) $\lim_{s \to \infty} sF(s)$

$$\Rightarrow \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \lim_{s \to \infty} \left[1 + \frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= \lim_{s \to \infty} \left[1 + \frac{\cancel{s}\cdot\cancel{s}(1+2/s)}{\cancel{s^2}[(1+1/s)^2 + 1/s^2]}\right]$$

$\boxed{\frac{1}{\infty} = 0}$

$$= \lim_{s \to \infty} \left[1 + \frac{(1+2/s)}{(1+1/s)^2 + 1/s^2}\right]$$

$$= 1 + \frac{(1+0)}{1+0} = 1 + 1$$

$$\boxed{RHS = \lim_{s \to \infty} sF(s) = 2}$$

$\therefore$ LHS = RHS = 2

$$\boxed{\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)}$$ IVT verified

(ii) Final value theo

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

LHS (FVT) $\lim_{t\to\infty} f(t)$

$$\Rightarrow \lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right]$$

$$= 1+e^{-\infty} = 1+0 = 1$$

$$\boxed{LHS = \lim_{t\to\infty} f(t) = 1}$$

RHS (FVT) $\lim_{s\to 0} sF(s)$

$$\Rightarrow \lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1+\frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= 1+0 = 1$$

$$\boxed{\therefore\ RHS = \lim_{t\to 0} sF(s) = 1}$$

$$\boxed{LHS = RHS = 1}$$

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

FVT verified

$\therefore$ Initial value & Final value theo verified.

(6) (iii) find $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

Soln:

$$\boxed{L\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds}$$

$$F(s) = L[f(t)] = L\left[e^{-at}-e^{-bt}\right]$$

$$\therefore F(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

$$\Rightarrow L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \int_s^\infty \left(\frac{1}{s+a}-\frac{1}{s+b}\right) ds$$

$$= \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\frac{(s+a)}{(s+b)}\right]_s^\infty$$

$$= \left[\log\frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right]_s^\infty$$

$$= \log(1) - \log\left(\frac{1+a/s}{1+b/s}\right)$$

$$= 0 - \left[\log\frac{(s+a)/\cancel{s}}{(s+b)/\cancel{s}}\right]$$

$$= -\log\left[\frac{(s+a)}{s+b}\right]$$

$$\boxed{L\left[\frac{e^{-at}-e^{-bt}}{t}\right] = \log\left[\frac{s+b}{s+a}\right]}$$

7) Solve $y'' - 3y' + 2y = 1$, given that $y(0) = 1$, $y'(0) = 1$, by using Lap. Transform

Soln:

Given: $y''(t) - 3y'(t) + 2y(t) = 1$

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(1)$

$s^2 L[y(t)] - s\,y(0) - y'(0) - 3[sL[y(t)] - y(0)]$

$+ 2L[y(t)] = \frac{1}{s}$

$L[y(t)](s^2 - 3s + 2) - s - 1 + 3 = \frac{1}{s}$

$L[y(t)](s^2 - 3s + 2) - s + 2 = \frac{1}{s}$

$L[y(t)](s-1)(s-2) = \frac{1}{s} + s - 2$

$\Rightarrow L[y(t)] = \frac{1}{s(s-1)(s-2)} + \frac{\cancel{s-2}}{(s-1)\cancel{(s-2)}}$

$\Rightarrow L[y(t)] = \frac{1}{s(s-1)(s-2)} + \frac{1}{s-1} \longrightarrow (1)$

consider $\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$\Rightarrow 1 = A(s-1)(s-2) + B\,s(s-2) + C\,s(s-1)$

put $\boxed{s=1} \Rightarrow 1 = 0 + B(-1) + 0 \quad \therefore \boxed{B = -1}$

$\boxed{s=2} \Rightarrow 1 = 0 + 0 + C(2)(1) \quad \therefore \boxed{C = 1/2}$

$\boxed{s=0} \Rightarrow 1 = A(-1)(-2) + 0 + 0$

$\Rightarrow \boxed{A = 1/2}$

Sub (A), (B), (C) in Equ: (1)

$$L[y(t)] = \frac{1/2}{s} - \frac{1}{s-1} + \frac{1/2}{s-2} + \frac{1}{s-1}$$

$$\therefore y(t) = 1/2\, L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s-1}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\left(\frac{1}{s-1}\right)$$

$$= \frac{1}{2}(1) - \cancel{e^{t}} + \frac{1}{2} e^{2t} + \cancel{e^{t}}$$

$$\boxed{y(t) = \frac{1}{2} + \frac{e^{2t}}{2}}$$

(8) Solve $y'' - 3y' + 2y = e^{3t}$, given that $y(0) = 1$, $y'(0) = 0$, using Laplace Transform.

Soln: Given: $y''(t) - 3y'(t) + 2y(t) = e^{3t}$

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{3t})$$

$$s^2 L[y(t)] - s y(0) - y'(0) - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s-3}$$

$$L[y(t)](s^2 - 3s + 2) - s - 0 + 3 = \frac{1}{s-3}$$

$$L[y(t)](s^2 - 3s + 2) = \frac{1}{s-3} + s - 3$$

$$= \frac{1 + (s-3)^2}{s-3}$$

$$= \frac{1 + s^2 + 9 - 6s}{s-3}$$

$$= \frac{s^2 - 6s + 10}{s-3}$$

$$L[y(t)](s-1)(s-2) = \frac{s^2-6s+10}{s-3}$$

$$L[y(t)] = \frac{s^2-6s+10}{(s-1)(s-2)(s-3)}$$

$$L[y(t)] = \frac{s^2-6s+10}{(s-1)(s-2)(s-3)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} \longrightarrow (1)$$

$$\Rightarrow s^2-6s+10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

put $\boxed{s=1}$ $\Rightarrow 1-6+10 = A(-1)(-2)$

$\Rightarrow 5 = 2A \quad \therefore \boxed{A = 5/2}$

put $\boxed{s=2}$ $\Rightarrow 4-12+10 = B(-1)(1)$

$\Rightarrow 2 = -B \quad \therefore \boxed{B = -2}$

put $\boxed{s=3}$ $\Rightarrow 9-8+10 = C(2)(1)$

$\Rightarrow 1 = 2C \quad \therefore \boxed{C = 1/2}$

Sub A, B, C values in Equ (1)

$$L[y(t)] = \frac{5/2}{s-1} + \frac{(-2)}{s-2} + \frac{1/2}{s-3}$$

$$\therefore y(t) = \frac{5}{2}L^{-1}\left(\frac{1}{s-1}\right) - 2L^{-1}\left(\frac{1}{s-2}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s-3}\right)$$

$$\boxed{\therefore y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}}$$

(9) Solve $(D^2+9)y = \cos 2t$, given that $y(0)=1$, $y(\frac{\pi}{2}) = -1$

Soln: Given: $(D^2+9)y = \cos 2t$

$y''(t) + 9y(t) = \cos 2t$

$\boxed{\begin{array}{l} D^2y = y'' \\ Dy = y' \end{array}}$

$L[y''(t)] + 9L[y(t)] = L(\cos 2t)$

$s^2 L[y(t)] - s y(0) - y'(0) + 9L[y(t)] = \frac{s}{s^2+4}$

$L[y(t)](s^2+9) - s(1) - c = \frac{s}{s^2+4}$

$L[y(t)](s^2+9) = \frac{s}{s^2+4} + s + c$

$L[y(t)] = \frac{s}{(s^2+4)(s^2+9)} + \frac{s}{s^2+9} + \frac{c}{s^2+9} \longrightarrow (1)$

Consider $\frac{1}{(s^2+4)(s^2+9)} = \frac{A}{(s^2+4)} + \frac{B}{(s^2+9)} \longrightarrow (2)$

$1 = A(s^2+9) + B(s^2+4)$

put $\boxed{s^2=-9} \Rightarrow 1 = A(-9+9) + B(-9+4)$

$1 = 0 + B(-5)$

$\therefore \boxed{B = -1/5}$

put $\boxed{s^2=-4} \Rightarrow 1 = 0 + A(-4+9)$

$1 = 5A \quad \therefore \boxed{A = 1/5}$

Sub (A) & (B) values in (2)

$$\therefore \frac{1}{(s^2+4)(s^2+9)} = \frac{1/5}{s^2+4} - \frac{1/5}{s^2+9}$$

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{1}{5}\left(\frac{1}{s^2+4}\right) - \frac{1}{5}\left(\frac{1}{s^2+9}\right)$$

III^{ly}

$$\boxed{\frac{s}{(s^2+4)(s^2+9)} = \frac{1}{5}\left(\frac{s}{s^2+4}\right) - \frac{1}{5}\left(\frac{s}{s^2+9}\right)} \rightarrow (3)$$

Sub (3) in Equ (1)

$$L[y(t)] = \frac{1}{5}\left(\frac{s}{s^2+4}\right) - \frac{1}{5}\left(\frac{s}{s^2+9}\right) + \frac{s}{s^2+9} + \frac{c}{s^2+9}$$

$$= \frac{1}{5}\left(\frac{s}{s^2+4}\right) + \frac{4}{5}\left(\frac{s}{s^2+9}\right) + \frac{c}{s^2+9}$$

$$y(t) = \frac{1}{5}L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{4}{5}L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$+ c\,L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$\boxed{y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{c}{3}\sin 3t} \rightarrow (4)$$

Given: $\boxed{y(\pi/2) = -1}$

$$y\left(\frac{\pi}{2}\right) = \frac{1}{5}\cos \cancel{2}\frac{\pi}{\cancel{2}} + \frac{4}{5}\cos 3\frac{\pi}{2} + \frac{c}{3}\sin 3\frac{\pi}{2}$$

$$= -1$$

$$\Rightarrow \frac{1}{5}(-1) + 0 + \frac{c}{3}(-1) = -1$$

$$-\frac{1}{5} - \frac{c}{3} = -1$$

$$\Rightarrow -\frac{c}{3} = -1 + \frac{1}{5} = -\frac{4}{5}$$

$$\therefore \boxed{c = \frac{12}{5}}$$

$\cos\pi = -1$

$\cos\frac{3\pi}{2} = 0$

$\sin\frac{3\pi}{2} = -1$

Sub c value in Equ.

$$\boxed{y(t) = \frac{\cos 2t}{5} + \frac{4}{5}\cos 3t + \frac{12}{5}\sin 3t}$$

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(10) Solve $\frac{d^2y}{dt^2} + \frac{dy}{dt} = t^2 + 2t$, given that $y = 4$, $y' = -2$, when $t = 0$, using Laplace Transform.

Soln:

Given: $y''(t) + y'(t) = t^2 + 2t$

$$L[y''(t)] + L[y'(t)] = L(t^2) + 2L(t)$$

$$s^2 L[y(t)] - s\,y(0) - y'(0) + s L[y(t)] - y(0) = \frac{2}{s^3} + 2\cdot\frac{1}{s^2}$$

Given: $y(0) = 4$, $y'(0) = -2$

$L[y(t)](s^2+s) - 4s + 2 - 4 = \frac{2}{s^3} + \frac{2}{s^2}$

$L[y(t)]\,s(s+1) = \frac{2+2s}{s^3} + 4s + 2$

$L[y(t)] = \frac{2(\cancel{s+1})}{s^3 \cdot s(\cancel{s+1})} + \frac{4\cancel{s}}{\cancel{s}(s+1)} + \frac{2}{s(s+1)}$

$\therefore L[y(t)] = \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s(s+1)} \longrightarrow (1)$

Consider

$\frac{2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$

$\Rightarrow 2 = A(s+1) + B(s)$

put $\boxed{s=0} \Rightarrow 2 = A(1) \quad \therefore \boxed{A=2}$

$\boxed{s=-1} \Rightarrow 2 = B(-1) \quad \therefore \boxed{B=-2}$

Sub (A) & (B) values in Equ: (1)

$L[y(t)] = \frac{2}{s^4} + \frac{4}{s+1} + \frac{2}{s} - \frac{2}{s+1}$

$= \frac{2}{s^4} + \frac{2}{s+1} + \frac{2}{s}$

$y(t) = 2L^{-1}\left(\frac{1}{s^4}\right) + 2L^{-1}\left(\frac{1}{s+1}\right) + 2L^{-1}\left(\frac{1}{s}\right)$

$$\boxed{y(t) = \cancel{2}\cdot\frac{t^3}{\cancel{6}\,3} + 2e^{-t} + 2}$$