23MA1206 – Complex Variables and Transforms

UNIT – 1

VECTOR CALCULUS

PART – A

1. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Soln:

Formula.

$$\boxed{\text{Directional Derivatives} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}}$$

Given,

$$\phi = x^2yz + 4xz^2$$

$$\boxed{\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}}$$

$$|\vec{a}| = \sqrt{4+1+4} = \sqrt{9}$$

$$\boxed{|\vec{a}| = 3}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

$$\nabla\phi_{(1,-2,-1)} = \vec{i}(4+4) + \vec{j}(-1) + \vec{k}(-2-8)$$

$$\boxed{\nabla\phi = 8\vec{i} - \vec{j} - 10\vec{k}}$$

$$D.D = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k}) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{16+1+20}{3} = \frac{37}{3}$$

$$\boxed{\text{Directional Derivative} = 37/3}$$

2. Find a unit normal to the surface $xy = z^2$ at the point $(1, 1, -1)$.

Soln.

Formula

$$\boxed{\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}}$$

Given,

$$\phi = xy - z^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}.$$

$$= \vec{i}(y) + \vec{j}(x) + \vec{k}(-2z)$$

$$\nabla\phi_{(1,1,-1)} = \vec{i} + \vec{j} + 2\vec{k}$$

$$\boxed{\nabla\phi = \vec{i} + \vec{j} + 2\vec{k}}.$$

$$|\nabla\phi| = \sqrt{1+1+4} = \sqrt{6}$$

$$\boxed{|\nabla\phi| = \sqrt{6}}.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\boxed{\hat{n} = \frac{\vec{i} + \vec{j} + 2\vec{k}}{\sqrt{6}}}$$

3. Find $\nabla(r^n)$.

Soln.

$$\nabla(r^n) = \Sigma\, \vec{i}\, \frac{\partial}{\partial x}(r^n)$$

$$= \Sigma\, \vec{i}\, n r^{n-1}\left(\frac{\partial r}{\partial x}\right)$$

$$= \Sigma\, \vec{i}\, n . r^{n-1}\left(\frac{x}{r}\right)$$

$$= \Sigma \vec{i}\, n \cdot r^{n-2} (x)$$

$$= n \cdot r^{n-2} [\Sigma \vec{i} x]$$

$$= n \cdot r^{n-2} [x\vec{i} + y\vec{j} + z\vec{k}]$$

$$\boxed{\nabla \cdot (r^n) = n \cdot r^{n-2} \cdot \vec{r}}.$$

4. Find (i) $\nabla \cdot \vec{r}$ (ii) $\nabla \times \vec{r}$

Soln.

$$\boxed{\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}}$$

$$\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1$$

$$\boxed{\nabla \cdot \vec{r} = 3}$$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\boxed{\nabla \times \vec{r} = 0}$$

5. Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Soln.

Formula.

$$\boxed{\nabla\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

$$\nabla\cdot\vec{F} = \frac{\partial(z)}{\partial x} + \frac{\partial(x)}{\partial y} + \frac{\partial(y)}{\partial z}$$

$$= 0 + 0 + 0$$

$$\boxed{\nabla\cdot\vec{F} = 0}$$

$\therefore$ The given sum is solenoidal.

Hence proved.

6. Prove that the vector $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

Soln.

Formula.

$$\boxed{\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$\nabla\times\vec{F} = \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z)$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$\boxed{\nabla \times \vec{F} = \vec{0}}$$

$\therefore$ $\vec{F}$ is irrotational.

Hence proved.

7. Find 'a' such that $(3x-2y+z)\vec{i}+(4x+ay-z)\vec{j}+(x-y+2z)\vec{k}$ is solenoidal.

Soln.

Given,

$\vec{F}$ is solenoidal

$$\boxed{\nabla \cdot \vec{F} = 0}$$

$$\boxed{\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

$$\nabla \cdot \vec{F} = \frac{\partial(3x)}{\partial x} + \frac{\partial(ay)}{\partial y} + \frac{\partial(2z)}{\partial z}$$

$$\nabla \cdot \vec{F} = 3 + a + 2$$

$$3 + a + 2 = 0$$

$$5 + a = 0$$

$$\boxed{a = -5}$$

8. Find the constants a, b, c so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Soln.

$$\boxed{\nabla \times \vec{F} = 0}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$\vec{i}[c+1]-\vec{j}[4-a]+\vec{k}[b-2] = 0\vec{i}-0\vec{j}+0\vec{k}$$

$c+1=0$ | $4-a=0$ | $b-2=0$

$\boxed{c=-1}$ | $\boxed{a=4}$ | $\boxed{b=2}$

$\therefore \boxed{a=4, \quad b=2, \quad c=-1}$.

9. If $\vec{F} = x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$ then find div (curl $\vec{F}$).

Soln.

To find,

$$\boxed{\nabla\cdot(\nabla\times\vec{F})}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \vec{i}(0-0)-\vec{j}(0-0)+\vec{k}(0-0)$$

$$\boxed{\nabla\times\vec{F} = 0}$$

$$\nabla\cdot(\nabla\times\vec{F}) = \nabla\cdot 0$$

$$\boxed{\nabla\cdot(\nabla\times\vec{F}) = 0}$$

$\therefore \boxed{\text{div (curl } \vec{F}) = 0}$.

10. State the physical interpretation of the line integral $\int_A^B \vec{F} \cdot d\vec{r}$.

Physically $\int_A^B \vec{F} \cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$, in displacing a particle from A to B along the curve C.

(or)

Line Integral is the total work done by a vector field to move a point along a curve.

PART – B

1. Verify Green's theorem for
$\int_C (3x - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region given by $x=0$, $y=0$, $x+y=1$.

Soln.

Formula.

$$\boxed{\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy}$$

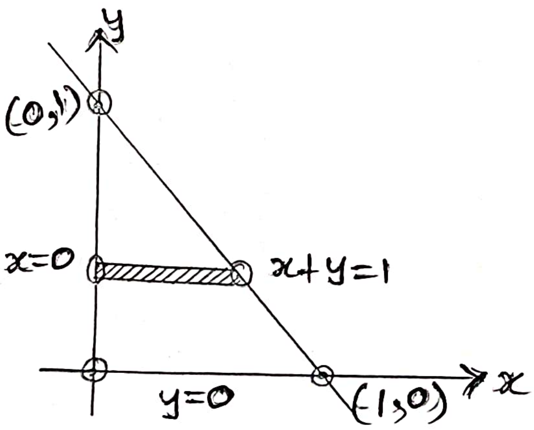
Given:

$u = 3x - 8y^2$	$v = 4y - 6xy$
$\frac{\partial u}{\partial y} = -16y$	$\frac{\partial v}{\partial x} = -6y$

Limits

$\boxed{x=0}$ to $\boxed{x=1-y}$

$\boxed{y=0}$ to $\boxed{y=1}$



To find RHS $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

RHS $= \iint_R (-6y + 16y)\,dx\,dy$

$= \iint_R 10y\,dx\,dy$

$= \int_0^1 \int_0^{1-y} 10y\,dx\,dy \qquad = \int_0^1 \left[10xy\right]_0^{1-y} dy$

$$= \int_0^1 (10(1-y)y - 0)\,dy \quad = \int_0^1 10(y-y^2)\,dy$$

$$= 10\int_0^1 (y-y^2)\,dy \quad = 10\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1$$

$$= 10\left[\frac{1}{2}-\frac{1}{3}\right] = 10\left[\frac{3-2}{6}\right] = 10\times\frac{1}{6}$$

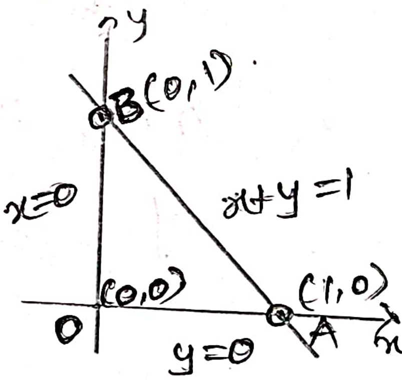
$$= \frac{10}{6} = \frac{5}{3}$$

$$\boxed{RHS = \frac{5}{3}}.$$

To find LHS.

$$\int_C (u\,dx + v\,dy) = \int_C (3x-8y^2)\,dx + (4y-6xy)\,dy$$

$$\boxed{\int_C = \int_{OA} + \int_{AB} + \int_{BO}} \quad -(1)$$



Along OA $[y=0,\ dy=0]$

$$\int_{OA} 3x\,dx = 3\int_0^1 x\,dx = 3\left[\frac{x^2}{2}\right]_0^1 = 3\left[\frac{1}{2}\right] = \frac{3}{2}.$$

$$\boxed{\int_{OA} = \frac{3}{2}} \quad -(2)$$

Along AB $[x+y=1,\ y=1-x,\ dy=-dx]$

$$\int_{AB} \left[3x - 8(1-x)^2\right]dx + \left[4(1-x) - 6x[1-x]\right](-dx)$$

$$\int_{AB} [3x - 8(1+x^2-2x) - (4+4x+6x-6x^2)]\,dx$$

$$= \int_1^0 (3x - 8 - 8x^2 + 16x - 4 + 4x + 6x - 6x^2)\,dx.$$

$$= \int_1^0 (-14x^2 + 29x - 12)\,dx = \left[\frac{-14x^3}{3} + 29\frac{x^2}{2} - 12x\right]_1^0$$

$$= \frac{14}{3} - \frac{29}{2} + 12$$

$$= \frac{28 - 87 + 72}{6}$$

$$\therefore \boxed{\int_{AB} = \frac{13}{6}} \quad — (3)$$

<u>Along</u> <u>BO</u> $[x=0,\ dx=0]$

$$\int_{BO} 4y\,dy = 4\int_1^0 y\,dy = \cancel{4}^{2}\left[\frac{y^2}{\cancel{2}}\right]_1^0$$

$$\boxed{\int_{BO} = -2} \quad — (4)$$

Sub (2), (3), (4) in (1)

$$\int_C = \int_{OA} + \int_{AB} + \int_{BO}$$

$$= \frac{3}{2} + \frac{13}{6} - 2$$

$$\int_C = \frac{9 + 13 - 12}{6} = \frac{10}{6} = \frac{5}{3}$$

$$\boxed{\int_C = LHS = \frac{5}{3}}$$

$$\boxed{LHS = RHS = \frac{5}{3}}$$

$\therefore$ Green's theorem is verified.

2. Verify the Green's theorem in the xy plane for $\int_C (xy + y^2)\,dx + x^2 dy$, where c is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Soln,

Formula.

$$\boxed{\int_C (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy}$$

$$\int_C (xy + y^2)\,dx + x^2 dy$$

Given:

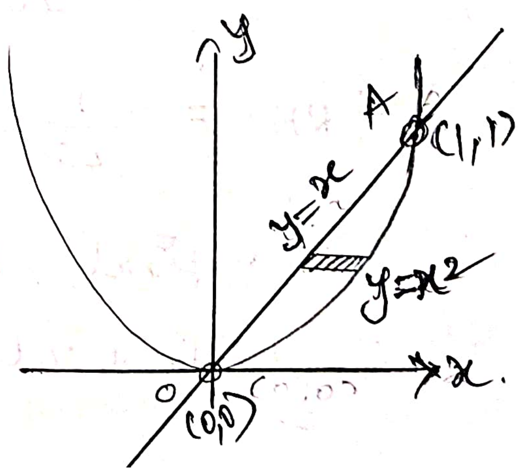
$u = xy + y^2$	$v = x^2$
$\frac{\partial u}{\partial y} = x + 2y$	$\frac{\partial v}{\partial x} = 2x$

Limits

$\boxed{x = y}$ to $\boxed{x = \sqrt{y}}$

$\boxed{y = 0}$ to $\boxed{y = 1}$

To find RHS.



$$= \iint_R (2x - x - 2y)\,dx\,dy$$

$$= \iint_R (x - 2y)\,dx\,dy$$

$$= \int_0^1 \int_y^{\sqrt{y}} (x - 2y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\left(\frac{y}{2} - 2y^{1/2}y\right) - \left(\frac{y^2}{2} - 2y^2\right)\right] dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{3/2} - \frac{y^2}{2} + 2y^2\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{3/2} + \frac{3y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{4} - \frac{2y^{5/2}}{5/2} + \frac{\cancel{3}}{2} \times \frac{y^3}{\cancel{3}}\right]_0^1$$

$$= \frac{1}{4} - 2 \cdot \frac{2}{5} + \frac{1}{2} = \frac{3}{4} - \frac{4}{5}$$

$$= \frac{15 - 16}{20} = -1/20$$

$$\boxed{RHS = -\frac{1}{20}}$$

<u>To find LHS</u>

$$\int_C (u\,dx + v\,dy) = \int_C (xy + y^2)\,dx + x^2\,dy$$

$$\boxed{\int_C = \int_{OA} + \int_{AO}} \quad \text{---} ①$$

Along $\underline{OA}$ $[y = x^2, \; dy = 2x\,dx]$

$$\int_{OA} = \int_{OA} (x^3 + x^4)\,dx + x^2(2x\,dx)$$

$$= \int_0^1 (3x^3 + x^4)\,dx = \left[3 \cdot \frac{x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{15+4}{20} = \frac{19}{20}$$

$$\boxed{\int_{OA} = \frac{19}{20}} \quad \text{---} ②$$

Along $\underline{AO}$ $[y = x, \; dy = dx]$

$$\int_{AO} = \int_{AO} (2x^2)\,dx + x^2\,dx$$

$$= \int_1^0 3x^2\,dx = \cancel{3}\left[\frac{x^3}{\cancel{3}}\right]_1^0 = -1$$

$$\boxed{\int_{AO} = -1} \quad \text{---} ③$$

Sub ② & ③ in ①

$$\int_C = \int_{OA} + \int_{AO}$$

$$= \frac{19}{20} - 1$$

$$= \frac{19-20}{20}$$

$$\boxed{LHS = -1/20}$$

$$\boxed{LHS = RHS = -1/20}$$

$\therefore$ Green's theorem is verified.

3. Evaluate $\int_c (y - \sin x)dx + \cos x\, dy$ by using Green's theorem where c is the triangle formed by the lines $y=0$, $x = \pi/2$ and $y = \frac{2x}{\pi}$.

Soln.

Formula.

$$\boxed{\int_c (u\,dx + v\,dy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy}$$

$$\int_c (y - \sin x)dx + \cos x\, dy$$

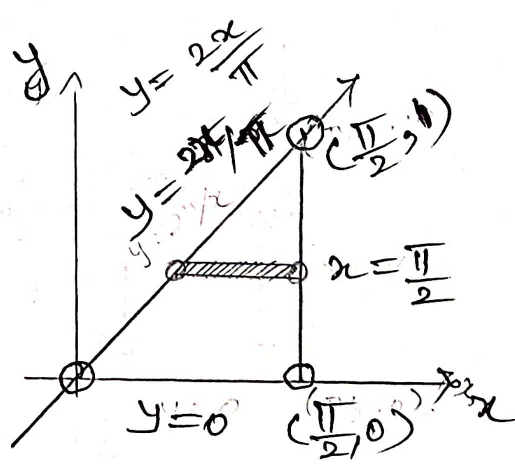
Given:

$u = y - \sin x$	$v = \cos x$
$\frac{\partial u}{\partial y} = 1$	$\frac{\partial v}{\partial x} = -\sin x$

Limits

$\boxed{x = \frac{\pi y}{2}}$ to $\boxed{x = \frac{\pi}{2}}$

$\boxed{y=0}$ to $\boxed{y=1}$



Using Green's theorem.

$$\int_c (y - \sin x)dx + \cos x\, dy$$

$$= \iint_R (-\sin x - 1)\, dx\, dy$$

$$= \int_0^1 \int_{\frac{\pi y}{2}}^{\pi/2} (-\sin x - 1)\, dx\, dy$$

$$= \int_0^1 \left[\cos x - x\right]_{\frac{\pi y}{2}}^{\pi/2} dy = \int_0^1 \left[\cos\frac{\pi}{2} - \frac{\pi}{2}\right] - \left[\cos\frac{\pi y}{2} - \frac{\pi y}{2}\right] dy$$

$$= \int_0^1 \left[-\frac{\pi}{2} - \cos\frac{\pi y}{2} + \frac{\pi y}{2}\right] dy = \left[-\frac{\pi y}{2} - \frac{\sin \pi/2\, y}{\pi/2} + \frac{\pi y^2}{4}\right]_0^1$$

$$= -\frac{\pi}{2} - \frac{2}{\pi}\sin\frac{\pi}{2} + \frac{\pi}{4}.$$

$$= -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4} = -\frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

$$\boxed{= -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]}.$$

4. Verify Gauss divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln.

Formula.

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dV}$$

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$\nabla\cdot\vec{F} = 4z - 2y + y$$

$$\boxed{\nabla\cdot\vec{F} = 4z - y}$$

<u>To find</u> <u>RHS</u>.

$$= \iiint_V (4z-y)\,dv = \iiint_V (4z-y)\,dx\,dy\,dz.$$

$$= \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz.$$

$$= \int_0^1\int_0^1 \left[4xz - xy\right]_0^1 dy\,dz = \int_0^1\int_0^1 (4z-y)\,dy\,dz$$

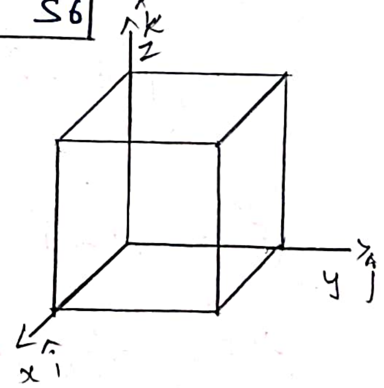
$$= \int_0^1 \left[4yz - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= \left[\frac{4z^2}{2} - \frac{z}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

$$\boxed{RHS = \frac{3}{2}}$$

<u>To find</u> <u>LHS</u>.

$$\boxed{\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}} \quad \text{—} (1)$$



To find S_1 & S_2 [x-axis / ds = dydz]

$$\iint_{S_1} + \iint_{S_2} = \int_0^1 \int_0^1 4z\,dy\,dz$$

$$= \int_0^1 4z[y]_0^1\,dz$$

$$= \int_0^1 4z\,dz$$

$$= 4\left[\frac{z^2}{2}\right]_0^1$$

$$= 2$$

	S_1	S_2
$\hat{n}$	$\vec{i}$	$-\vec{i}$
$\vec{F}\cdot\hat{n}$	$4xz$	$-4xz$
Eqn	$x=1$	$x=0$
$\iint \vec{F}\cdot\hat{n}\,ds$	$\int_0^1\int_0^1 4z\,dy\,dz$	0

$$\boxed{\iint_{S_1} + \iint_{S_2} = 2} \quad \text{— (2)}$$

To find S_3 & S_4 [y-axis / ds = dxdz]

$$\iint_{S_3} + \iint_{S_4} = \int_0^1 \int_0^1 (-1)\,dx\,dz$$

$$= \int_0^1 [-x]_0^1\,dz$$

$$= \int_0^1 -dz$$

$$= [-z]_0^1$$

$$= -1$$

	S_3	S_4
$\hat{n}$	$\vec{j}$	$-\vec{j}$
$\vec{F}\cdot\hat{n}$	$-y^2$	y^2
Eqn	$y=1$	$y=0$
$\iint \vec{F}\cdot\hat{n}\,ds$	$\int_0^1\int_0^1 -dx\,dz$	0

$$\boxed{\iint_{S_3} + \iint_{S_4} = -1} \quad \text{— (3)}$$

To find S_5 & S_6 [z-axis / ds = dxdy]

$$\iint_{S_5} + \iint_{S_6} = \int_0^1 \int_0^1 y\,dx\,dy$$

$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx$

$= \frac{1}{2}\int_0^1 dx$

$= \frac{1}{2}[x]_0^1$

$= \frac{1}{2}$

	S5	S6
$\hat{n}$	$\vec{k}$	$-\vec{k}$
$\bar{F}\cdot\hat{n}$	yz	$-yz$
Eqn	$z=1$	$z=0$
$\iint \bar{F}\cdot\hat{n}\,ds$	$\int_0^1\int_0^1 y\,dx\,dy$	0

$$\boxed{\iint_{S5} + \iint_{S6} = \frac{1}{2}} \quad -④$$

Sub ②, ③, ④ in ①

$\iint_S = 2 - 1 + \frac{1}{2}$

$\iint_S = 1 + \frac{1}{2}$

$\iint_S = \frac{3}{2}$

$$\boxed{LHS = \frac{3}{2}}$$

$\therefore$ $\boxed{RHS = LHS = \frac{3}{2}}$

$\therefore$ Gauss divergence theorem is verified.

5. Verify Gauss divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$ taken over the rectangular parallelopiped bounded by $x=0, x=a, y=0, y=b, z=0, z=c$.

A

Soln.

Formula.

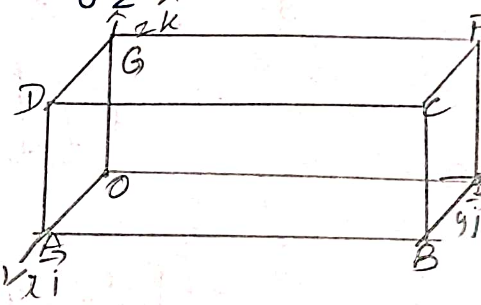
$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV}$$

Given:

$$\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy).$$

$$\nabla\cdot\vec{F} = 2x+2y+2z$$



$$\boxed{\nabla\cdot\vec{F} = 2(x+y+z)}$$

To find RHS.

$$\iiint_V 2(x+y+z)dV = 2\int_0^c\int_0^b\int_0^a (x+y+z)\,dx\,dy\,dz.$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2}+yx+xz\right]_0^a dy\,dz.$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2}+ay+az\right] dy\,dz.$$

$$= 2\int_0^c \left[\frac{a^2y}{2}+\frac{ay^2}{2}+ayz\right]_0^b dz$$

$$= 2\int_0^c \frac{a^2b}{2}+\frac{ab^2}{2}+abz\;dz.$$

$$= 2\left[\frac{a^2bz}{2}+\frac{ab^2z}{2}+\frac{abz^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2}+\frac{ab^2c}{2}+\frac{abc^2}{2}\right]$$

$= \frac{\cancel{2}}{\cancel{2}} abc\,[a+b+c]$

$\boxed{RHS = abc\,[a+b+c]}$

To find LHS.

S_1 & S_2 [x-axis/ ds=dydz]

	S_1	S_2
$\hat{n}$	$\vec{i}$	$-\vec{i}$
$\bar{F}\cdot\hat{n}$	(x^2-yz)	$-(x^2-yz)$
Eqn	$x=a$	$x=0$
$\int \bar{F}\cdot\hat{n}\,ds$	$\int_0^c\int_0^b (a^2-yz)\,dydz$	$\int_0^c\int_0^b -(-yz)\,dydz$

$$\iint_{S_1} + \iint_{S_2} = \int_0^c \int_0^b (a^2-yz)\,dydz$$

$$+ \int_0^c \int_0^b -(-yz)\,dydz.$$

$$= \int_0^c \left[a^2y - \frac{y^2z}{2}\right]_0^b dz +$$

$$\int_0^c \left[\frac{y^2z}{2}\right]_0^b dz$$

$$= \int_0^c \left[a^2b - \frac{b^2z}{2}\right]dz + \int_0^c \left[\frac{b^2z}{2}\right]dz$$

$$= \left[a^2bz - \frac{b^2z^2}{2}\right]_0^c + \left[\frac{b^2z^2}{2}\right]_0^c$$

$$= a^2bc - \frac{b^2c^2}{2} + \frac{b^2c^2}{2}$$

$$\boxed{\iint_{S_1} + \iint_{S_2} = a^2bc} \quad —(1)$$

To find

S_3 & S_4 [y-axis/ ds= dxdz]

$$\iint_{S_3} + \iint_{S_4} = \int_0^c \int_0^a (b^2 - zx)\,dx\,dz + \int_0^c \int_0^a (zx)\,dx\,dz.$$

	S_3	S_4
$\hat{n}$	$\vec{j}$	$-\vec{j}$
$\bar{F}\cdot\hat{n}$	$y^2 - zx$	$-y^2 + zx$
Eqn	$y = b$	$y = 0$
$\int \bar{F}\cdot\hat{n}\,ds$	$\int_0^c\int_0^a b^2 - zx\,dx\,dz$	$\int_0^c\int_0^a zx\,dx\,dz$

$$= \int_0^c \left[b^2 x - \frac{zx^2}{2} \right]_0^a dz + \int_0^c \left[\frac{zx^2}{2} \right]_0^a dz$$

$$= \int_0^c \left[b^2 a - \frac{za^2}{2} \right] dz + \int_0^c \left[\frac{za^2}{2} \right] dz.$$

$$= \left[b^2 a z - \frac{z^2 a^2}{2} \right]_0^c + \left[\frac{z^2 a^2}{2} \right]_0^c$$

$$= b^2 a c - \frac{c^2 a^2}{2} + \frac{c^2 a^2}{2}$$

$$\boxed{\iint_{S_3} + \iint_{S_4} = b^2 a c} \quad \text{—} (2)$$

To find

S_5 & S_6 [z-axis / $ds = dx\,dy$]

	S_5	S_6
$\hat{n}$	$\vec{k}$	$-\vec{k}$
$\bar{F}\cdot\hat{n}$	$(z^2 - xy)$	$-(z^2 + xy)$
Eqn	$z = c$	$z = 0$
$\iint \bar{F}\cdot\hat{n}\,ds$	$\int_0^b\int_0^a (c^2 - xy)\,dx\,dy$	$\int_0^b\int_0^a xy\,dx\,dy$

$$\iint_{S_5} + \iint_{S_6} = \int_0^b \int_0^a (c^2 - xy)\,dx\,dy + \int_0^b \int_0^a xy\,dx\,dy$$

$$= \int_0^b \left[c^2 x - \frac{x^2 y}{2} \right]_0^a dy + \int_0^b \left[\frac{x^2 y}{2} \right]_0^a dy$$

$$= \int_0^b \left[c^2 a - \frac{a^2 y}{2}\right] dy + \int_0^b \left[\frac{a^2 y}{2}\right] dy$$

$$= \left[c^2 ay - \frac{a^2 y^2}{2}\right]_0^b + \left[\frac{a^2 y^2}{2}\right]_0^b$$

$$= c^2 ab - \frac{a^2 b^2}{2} + \frac{a^2 b^2}{2}$$

$$\boxed{\iint_{S_5} + \iint_{S_6} = c^2 ab.} \quad \text{—} \ (3)$$

Sub (1), (2), (3) in (A).

$$\iint_S = a^2 bc + b^2 ac + c^2 ab$$

$$\iint_S = abc[a+b+c]$$

$$\boxed{LHS = abc\,[a+b+c]}$$

$$\boxed{RHS = LHS}$$

$\therefore$ Gauss divergence theorem is verified.

---

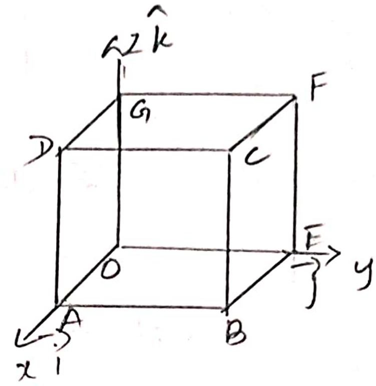
6. Verify Gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ taken over the cube bounded by the planes $x=0, x=a, y=0, y=b, z=0, z=c$.

<u>Soln.</u>

Soln:

Formula.

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv}$$



Given:

$$\boxed{\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}}$$

$$\boxed{\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}}.$$

$$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\nabla\cdot\vec{F} = 2x + 2y + 2z$$

$$\boxed{\nabla\cdot\vec{F} = 2(x+y+z)}$$

To find RHS.

$$\iiint_V 2(x+y+z)\,dv = \iiint_V 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b\int_0^a (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{x^2}{2} + xy + xz\right]_0^a dy\,dz$$

$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2} + ay + az\right] dy\,dz$$

$$= 2\int_0^c \left[\frac{a^2y}{2} + \frac{ay^2}{2} + ayz\right]_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right) dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_0^c.$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= \cancel{2}\, abc\left[\frac{a+b+c}{\cancel{2}}\right]$$

$$\boxed{RHS = abc\,[a+b+c]}$$

To find LHS:

$$\boxed{\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}} \quad — ①$$

To find S_1 & S_2

[x - axis, $ds = dy\,dz$]

	S_1	S_2
$\hat{n}$	$\vec{i}$	$-\vec{i}$
$\bar{F}\cdot\hat{n}$	x^2	$-x^2$
Eqn	$x=a$	$x=0$
$\iint \bar{F}\cdot\hat{n}\,ds$	$\int_0^c\int_0^b a^2\,dy\,dz$	0

$$\iint_{S_1} + \iint_{S_2} = \int_0^c\int_0^b a^2\,dy\,dz$$

$$= \int_0^c \left[a^2 y\right]_0^b dz$$

$$= \int_0^c \left[a^2 b\right] dz$$

$$= \left[a^2 b z\right]_0^c = a^2bc$$

$$\boxed{\iint_{S_1} + \iint_{S_2} = a^2bc} \quad — ②$$

To find S_3 & S_4

[y - axis, $ds = dz\,dx$]

$$\iint_{S_3} + \iint_{S_4} = \int_0^c\int_0^a b^2\,dx\,dz$$

	S_3	S_4
$\hat{n}$	$\vec{j}$	$-\vec{j}$
$\bar{F}\cdot\hat{n}$	y^2	$-y^2$
Eqn	$y=b$	$y=0$
$\iint \bar{F}\cdot\hat{n}\,ds$	$\int_0^c\int_0^a b^2\,dx\,dz$	0

$$= \int_0^c [b^2 x]_0^a\,dz$$

$$= \int_0^c [b^2 a]\,dz$$

$$= [b^2 a z]_0^c$$

$$= b^2 a c$$

$$\boxed{\iint_{S_3} + \iint_{S_4} = ab^2c} \quad \text{— (3)}$$

To find S_5 & S_6

[Z-axis / $ds = dx\,dy$]

	S_5	S_6
$\hat{n}$	$\vec{k}$	$-\vec{k}$
$\bar{F}\cdot\hat{n}$	z^2	$-z^2$
Eqn	$z=c$	0
$\iint \bar{F}\cdot\hat{n}\,ds$	$\int_0^b\int_0^a c^2\,dx\,dy$	0

$$\iint_{S_5} + \iint_{S_6} = \int_0^b\int_0^a c^2\,dx\,dy$$

$$= \int_0^b [c^2 x]_0^a\,dy$$

$$= \int_0^b [c^2 a]\,dy$$

$$= \int_0^b ac^2\,dy = [ac^2 y]_0^b$$

$$\boxed{\iint_{S_5} + \iint_{S_6} = c^2 b a} \quad \text{— (4)}$$

Sub (2), (3), (4) in (1)

$$\iint_S = a^2bc + ab^2c + abc^2$$

$$\boxed{\iint_S = abc[a+b+c] = LHS.}$$

$$\boxed{RHS = LHS = abc(a+b+c)}$$

$\therefore$ Gauss divergence theorem is verified.

7. Verify stoke's theorem, $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangular region bounded by the lines $x = \pm a$, $y = 0$, $y = b$.

Soln.

Formula.

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds}$$

Given:

$$\boxed{\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$\boxed{d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}}$$

$$\boxed{\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy}.$$

To find RHS.

$$\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds.$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\,(-2y-2y)$$

$$\boxed{\nabla \times \vec{F} = -4y\vec{k}}$$

$$\boxed{\hat{n} = \vec{k}}$$

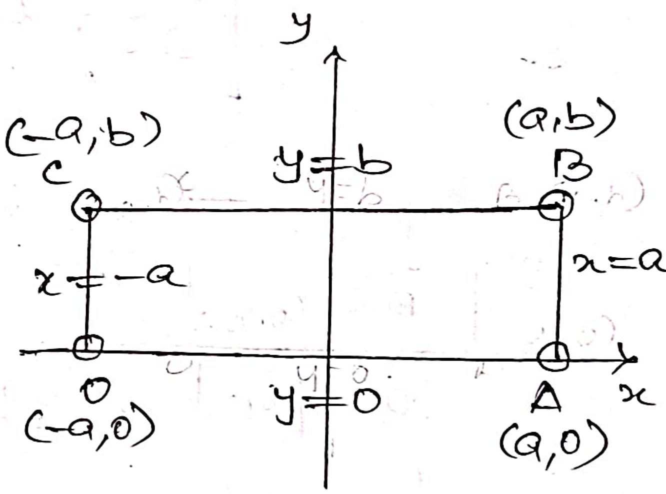
$$\boxed{\nabla \times \vec{F} \cdot \hat{n} = -4y}$$

$$\underline{\underline{RHS}} = \int_0^b \int_{-a}^{a} -4y\,dx\,dy = \int_0^b -4y\,[x]_{-a}^{a}\,dy$$

$$= \int_0^b -4y\,[a+a]\,dy = \int_0^b -8ay\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -8a\frac{b^2}{2}$$

$$\boxed{RHS = -4ab^2}$$



__To find__ __LHS.__

$$\boxed{\int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}} \quad \text{— (A)}$$

__To find__ __OA__ $[y=0,\ dy=0]$

$$\int_{OA} x^2\,dx = \int_{-a}^{a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} + \frac{a^3}{3}$$

$$\boxed{\int_{OA} = \frac{2a^3}{3}} \quad \text{— (1)}$$

__To find__ __AB__ $[x=a,\ dx=0]$

$$\int_{AB} -2ay\,dy = -2a\int_0^b y\,dy$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b = -\cancel{2}a\,\frac{b^2}{\cancel{2}}$$

$$\boxed{\int_{AB} = -ab^2} \quad -(2)$$

To find BC $[y=b,\ dy=0]$

$$\int_{BC}(x^2+b^2)\,dx = \int_{+a}^{-a}\left(\frac{x^3}{3}+xb^2\right)dx$$

$$= \left[\frac{x^3}{3}+xb^2\right]_a^{-a} = \left[-\frac{a^3}{3}-ab^2\right]-\left[\frac{a^3}{3}+ab^2\right]$$

$$\boxed{\int_{BC} = \frac{-2a^3}{3} - 2ab^2} \quad -(3)$$

To find CO $[x=-a,\ dx=0]$

$$\int_{CO} 2ay\,dy = 2a\int_b^0 y\,dy = 2a\left[\frac{y^2}{2}\right]_b^0$$

$$= -\cancel{2}a\,\frac{b^2}{\cancel{2}}$$

$$\boxed{\int_{CO} = -ab^2} \quad -(4)$$

Sub (1), (2), (3), (4) in (A).

$$\int_C = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2$$

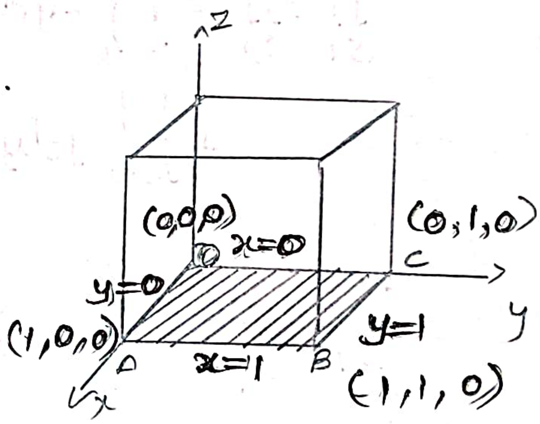
$\boxed{\int_C = -4ab^2} \Rightarrow \boxed{LHS = -4ab^2}$

$\boxed{LHS = RHS = -4ab^2}$

$\therefore$ Stoke's theorem is verified.

---

8. Verify stoke's theorem for $\vec{F} = (y-z)\vec{i} + (yz)\vec{j} - xz\vec{k}$ where the surface bounded by the planes $x=0, x=1, y=0, y=1, z=0, z=1$ not included in the XOY plane (above XOY plane).



Soln.

Given: $\boxed{\vec{F} = (y-z)\vec{i} + (yz)\vec{j} - xz\vec{k}}$

$\boxed{d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}}$

$\boxed{\vec{F}\cdot d\vec{r} = (y-z)dx + (yz)dy - (xz)dz}$.

To find RHS.

$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y-z) & yz & -xz \end{vmatrix}$$

$$= \vec{i}(0-y) - \vec{j}(-z+1) + \vec{k}(0-1)$$

$$\boxed{\nabla \times \vec{F} = -y\vec{i} + (z-1)\vec{j} - \vec{k}}.$$

To find RHS.

$$\boxed{\iint_S = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5}} \quad -(A)$$

To find S_1 & S_2

[x-axis/ ds = dydz].

$$\iint_{S1} + \iint_{S2} = \int_0^1\int_0^1 -y\,dy\,dz + \int_0^1\int_0^1 y\,dy\,dz$$

$$\boxed{\iint_{S_1} + \iint_{S_2} = 0} \quad -(1)$$

	S_1	S_2
$\hat{n}$	$\vec{i}$	$-\vec{i}$
$(\nabla\times F)\cdot\hat{n}$	$-y$	y
Eqn	$x=1$	$x=0$
$\iint(\nabla\times F)\cdot\hat{n}\,ds$	$\int_0^1\int_0^1 -y\,dy\,dz$	$\int_0^1\int_0^1 y\,dy\,dz$

To find S_3 & S_4

[y-axis / ds = dxdz]

$$\iint_{S3} + \iint_{S4} = \int_0^1\int_0^1 (z-1)\,dx\,dz + \int_0^1\int_0^1 -(z-1)\,dx\,dz$$

$$\boxed{\iint_{S_3} + \iint_{S4} = 0}. \quad -(2)$$

	S_3	S_4
$\hat{n}$	$\vec{j}$	$-\vec{j}$
$(\nabla\times F)\cdot\hat{n}$	$(z-1)$	$-(z-1)$
Eqn	$y=1$	$y=0$
$\iint(\nabla\times F)\cdot\hat{n}\,ds$	$\int_0^1\int_0^1 (z-1)\,dx\,dz$	$\int_0^1\int_0^1 -(z-1)\,dx\,dz$

To find S_5 [z-axis / $ds = dx\,dy$]

	S_5
$\hat{n}$	$\vec{k}$
$(\nabla\times\vec{F})\cdot\hat{n}$	-1
Eqn	$z=1$
$\iint ds$	$\int_0^1\int_0^1 -dx\,dy$

$$\iint_{S_5} = \int_0^1\int_0^1 -dx\,dy$$

$$= \int_0^1 [-x]_0^1\, dy$$

$$= \int_0^1 -1\, dy$$

$$= [-y]_0^1$$

$$\boxed{\iint_{S5} = -1} \quad -(3)$$

Sub (1), (2), (3) in (A)

$$\iint_S = -1$$

$$\boxed{RHS = -1}$$

To find LHS.

$$\boxed{\int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}} \quad -(B)$$

To find OA [$y=0,\ z=0,\ dy=0,\ dz=0$]

$$\vec{F}\cdot d\vec{r} = (y-z)dx + yz\,dy - xz\,dz$$

$$\boxed{\int_{OA} = 0} \quad -(4)$$

To find AB [$x=1,\ z=0,\ dx=0,\ dz=0$]

$$\boxed{\int_{AB} = 0} \quad -(5)$$

To find BC $[y=1, z=0, dy=0, dz=0]$

$$\int_{BC} = \int_1^0 dx = [x]_1^0 = -1$$

$$\boxed{\int_{BC} = -1} \quad -(6)$$

To find CO $[x=0, z=0, dx=0, dz=0]$

$$\boxed{\int_{CO} = 0} \quad -(7)$$

Sub (4), (5), (6), (7) in (B)

$$\int_C = 0 + 0 - 1 + 0$$

$$\boxed{\int_C = -1}$$

$$\boxed{LHS = -1}$$

$$\boxed{RHS = LHS = -1}$$

$\therefore$ Stoke's theorem is verified.

9\. i) Prove that $\vec{F} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$ is irrotational vector and find the scalar potential such that $\vec{F} = \nabla\phi$.

Soln,

Irrotational vector.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (6xy+z^3) & (3x^2-z) & (3xz^2-y) \end{vmatrix}$$

$$\nabla \times \vec{F} = \vec{i}\,(-1+1) - \vec{j}\,(3z^2-3z^2) + \vec{k}\,(6x-6x)$$

$\boxed{\nabla \times \vec{F} = 0}$ $\therefore$ F is Irrotational vector

Scalar Potential.

$$\nabla\phi = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$$

$$\int \frac{\partial\phi}{\partial x} = \int (6xy+z^3)\,dx$$

$$\phi = \frac{6x^2y}{2} + xz^3$$

$$\boxed{\phi = 3x^2y + xz^3}$$

$$\int \frac{\partial\phi}{\partial y} = \int (3x^2-z)\,dy$$

$$\phi = 3x^2y - zy$$

$$\boxed{\phi = 3x^2y - zy}$$

$$\int \frac{\partial\phi}{\partial z} = \int (3xz^2-y)\,dz$$

$$\phi = \frac{3xz^3}{3} - yz$$

$$\boxed{\phi = xz^3 - yz}$$

$\therefore$ Scalar Potential $\boxed{\phi = 3x^2y + xz^3 - yz}$.

(ii) Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at $(2, -1, 2)$

Soln.

Formula.

$$\boxed{\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}}$$

Given:

$$\phi_1 = x^2 + y^2 - z - 3$$

$$\phi_2 = x^2 + y^2 + z^2 - 9$$

$$\nabla\phi_1 = \vec{i}\,\frac{\partial\phi_1}{\partial x} + \vec{j}\,\frac{\partial\phi_1}{\partial y} + \vec{k}\,\frac{\partial\phi_1}{\partial z}$$

$$\nabla\phi_1 = \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\boxed{\nabla\phi_1 (2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}}$$

$$\nabla\phi_2 = \vec{i}\,\frac{\partial\phi_2}{\partial x} + \vec{j}\,\frac{\partial\phi_2}{\partial y} + \vec{k}\,\frac{\partial\phi_2}{\partial z}.$$

$$\nabla\phi_2 = \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(2z)$$

$$\boxed{\nabla\phi_2 (2,-1,2) = 4\vec{i} - 2\vec{j} + 4\vec{k}.}$$

$$|\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\boxed{|\nabla\phi_1| = \sqrt{21}}$$

$$|\nabla\phi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\boxed{|\nabla\phi_2| = 6.}$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$\cos\theta = \frac{(4\vec{i}-2\vec{j}-\vec{k})\cdot(4\vec{i}-2\vec{j}+4\vec{k})}{(\sqrt{21})\,6}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}}$$

$$\cos\theta = \frac{16}{6\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]}$$

---

10. Determine a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

Soln. Given:

$$\phi_1 = ax^2 - (a+2)x - byz$$

$$\phi_1 = ax^2 - ax - 2x - byz$$

$$\phi_2 = 4x^2y + z^3 - 4$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$\nabla\phi_1 = \vec{i}(2ax - a - 2) + \vec{j}(-bz) + \vec{k}(-by)$

$\nabla\phi_2 = \vec{i}\dfrac{\partial\phi_2}{\partial x} + \vec{j}\dfrac{\partial\phi_2}{\partial y} + \vec{k}\dfrac{\partial\phi_2}{\partial z}$.

$\nabla\phi_2 = \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(3z^2)$

$\nabla\phi_1(1,-1,2) = \vec{i}(2a - a - 2) + \vec{j}(-2b) + \vec{k}(b)$

$\boxed{\nabla\phi_1(1,-1,2) = \vec{i}(a-2) + \vec{j}(-2b) + b\vec{k}}$

$\nabla\phi_2(1,-1,2) = \vec{i}(-8) + \vec{j}(4) + \vec{k}(12)$

$\boxed{\nabla\phi_2(1,-1,2) = -8\vec{i} + 4\vec{j} + 12\vec{k}}$

$\nabla\phi_1 \cdot \nabla\phi_2 = \left((a-2)\vec{i} - 2b\vec{j} + b\vec{k}\right) \cdot \left(-8\vec{i} + 4\vec{j} + 12\vec{k}\right)$

$\nabla\phi_1 \cdot \nabla\phi_2 = -8(a-2) - 8b + 12b$

$\nabla\phi_1 \cdot \nabla\phi_2 = -8a + 16 - 8b + 12b$

$\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$

$-8a + 16 - 8b + 12b = 0$

$-8a + 16 + 4b = 0$

$-8a + 4b = -16$

$(\div -4)$ $\boxed{2a - b = 4}$ $\longrightarrow$ (1)

$\phi_1(1,-1,2) = a + 2b - a - 2$

$\boxed{2b = 2}$ $\Rightarrow$ $\boxed{b = 1}$ $\longrightarrow$ (2)

$\boxed{b=1}$

sub ② in ①

$$2a-1=4$$

$$2a=5$$

$$a=5/2.$$

$\therefore$ $\boxed{a=5/2}$ $\boxed{b=1.}$