Chern Classes

Deepak Badarinath

June 17 2020

$\mathbb{CP}^n$ Facts

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- $\mathbb{CP}^n := \frac{\mathbb{C}^{n+1}-\{0\}}{\sim}$, where $z \sim w$ iff $z = \lambda w$, where $\lambda \in \mathbb{C}^*$
- $\mathbb{CP}^n = S^{2n+1}/\sim$ where $z \sim w$, iff $z = e^{i\theta}w$, implies $\mathbb{CP}^n$ is compact.
- Let $U_i \subset \mathbb{CP}^n$, $U_i = \{[z_0 : z_1 :: ... : z_n] | z_i \neq 0\}$ $\phi_i : U_i \to \mathbb{C}^n$, $[z_0 : z_1 :: ... : z_n] \mapsto (\frac{z_0}{z_i}, \frac{z_1}{z_i}, ..., \frac{\hat{z_i}}{z_i}, .., \frac{z_n}{z_i})$
- ϕ_i give $\mathbb{CP}^n$, a complex manifold structure
- $\mathbb{CP}^n$ is thus orientable
- The subspace $[0 : z_1 : z_2 : ... : z_n] \cong \mathbb{CP}^{n-1} \subset \mathbb{CP}^n$ and $\mathbb{CP}^n/\mathbb{CP}^{n-1} \cong S^{2n}$
- $\mathbb{CP}^1 \cong S^2$

Cohomology groups of $\mathbb{CP}^n$

Theorem

The cohomology groups, with coefficients in $\mathbb{R}$ of $\mathbb{CP}^n$ vanish on odd dimensions and are one dimensional $\mathbb{R}$ vector spaces on all even dimensions $\leq 2n$

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$$H^{2k+1}(\mathbb{CP}^n) = 0 \forall k$$

$$H^{2i}(\mathbb{CP}^n) \cong \mathbb{R}, i = 0, 1, ...n$$

Computing the Cohomology groups of $\mathbb{CP}^n$(Sketch)

- Assume inductively that we know the cohomology groups of $\mathbb{CP}^m \;\; \forall \;\; 1 \leq m < n$ and it is given by $H^{2i}(\mathbb{CP}^m) \cong \mathbb{R} \; \forall i \leq m$ and $H^{2k+1}(\mathbb{CP}^m) = 0$
- $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n \to (\mathbb{CP}^n, \mathbb{CP}^{n-1})$
- Exicision and homotopy axiom implies that $H^i(\mathbb{CP}^n, \mathbb{CP}^{n-1}) \cong H^i(S^{2n}) \; \forall i \geq 1$
- Long exact sequence axiom (with coefficients in $\mathbb{R}$) gives us:

$$H^{2n}(\mathbb{CP}^{n-1}) = 0 \leftarrow H^{2n}(\mathbb{CP}^n) \leftarrow H^{2n}(S^{2n}) \cong \mathbb{R} \xleftarrow{\delta} H^{2n-1}(\mathbb{CP}^{n-1}) = 0$$

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$$0 \leftarrow H^k(\mathbb{CP}^{n-1}) \leftarrow H^k(\mathbb{CP}^n) \leftarrow 0$$

$\forall k < 2n-1$

- $H^{2i}(\mathbb{CP}^n) \cong \mathbb{R}, i = 0, 1, \ldots n$ and $H^{2k+1}(\mathbb{CP}^n) = 0 \forall$ k $\blacksquare$

Ring structure on DeRham Cohomology

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- Given a manifold M, let $\Omega^n(M)$ be the set of smooth n forms on M. Define a bilinear map-:

$$\Omega^n(M) \times \Omega^m(M) \to \Omega^{n+m}(M)$$

$$(\omega, \tau) \to \omega \wedge \tau$$

- This induces a map -:

$$H^n(M) \otimes H^m(M) \to H^{n+m}(M)$$

$$([\omega], [\tau]) \to [\omega \wedge \tau]$$

- $H^*(M) := \bigoplus_{i=0}^{\infty} H^i(M)$ is a graded commutative ring.

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- $H^*(M) := \bigoplus_{i=0}^{\infty} H^i(M)$ is a graded commutative ring.
- If f a smooth map from $M \to N$, then $f^* : H^*(N) \to H^*(M)$ is a map of graded commutative rings.
- In particular, if $M \cong N$, then $H^*(M) \cong H^*(N)$ as graded commutative rings

Recap

Recap

Thom Class and Direct Sum-Proposition 6.19

Let E_1, E_2 be two oriented vector bundles over M with canonical projections $\pi_i : E_1 \oplus E_2 \to E_i$ and let the Thom class of E be given by $\Phi(E)$,then

$$\Phi(E_1 \oplus E_2) = \pi_1^* \Phi(E_1) \wedge \pi_2^* \Phi(E_2)$$

Proof Idea: $\pi_1^* \Phi(E_1) \wedge \pi_2^* \Phi(E_2)$ is a class in $H_{cv}^{m+n}(E_1 \oplus E_2)$ whose restriction to each fiber, generates the compact cohomology of the fiber

Poincare Dual and Thom class- Proposition 6.24a)

The Poincare dual of a closed oriented submanifold S in an oriented Riemannian manifold M and the Thom class of the normal bundle of S can be represented by the same forms(in cohomology).

$$[\omega_S] = \Phi(N_S)$$

Transversal Intersection

Transversal Intersection

Definition

Let S, P be submanifolds of a manifold M, then S and P are said to intersect transversally, if $\forall x \in S \cap T$, $T_xS + T_xP = T_xM$

- Let S and P be closed oriented submanifolds of an oriented Riemannian manifold M.
- Assume S and P intersect transversally in M, then we can show that, $codim(S \cap P) = codim(S) + codim(P)$, where $codim(S) := dim(M) - dim(S)$
- Thus the normal bundle of $S \cap P$ in M is the direct sum of the normal bundles of S and P, ie $N_{P \cap S} = N_S \oplus N_P$
- $$\Phi(N_{P \cap S}) = \Phi(N_S \oplus N_P) = \Phi(N_S) \wedge \Phi(N_P)$$

$$\omega_{S \cap P} = \omega_S \wedge \omega_P$$

Geometric interpretation of wedge product

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Theorem-Geometric interpretation of wedge product

Let M be a closed oriented manifold and S, P be closed oriented submanifolds of M which meet transversally, let $[\omega_S], [\omega_P]$ be the cohomology classes of the Poincare dual of S, P. Then the Poincare dual of $S \cap P$ is $[\omega_S] \wedge [\omega_P]$

Cohomology Ring of $\mathbb{CP}^n$

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Theorem

The cohomology ring (with coefficients in $\mathbb{R}$) of $\mathbb{CP}^n$ is isomorphic to $\mathbb{R}[X]/(X^{n+1})$

Computing the Cohomology ring of $\mathbb{CP}^n$

Computing the Cohomology ring of $\mathbb{CP}^n$

- Let $M := \mathbb{CP}^n$ and $[\omega]$ be a generator of $H^2_{dR}(M)$
- We show that $\omega^n = \omega \wedge \omega ... \wedge \omega \neq [0] \in H^{2n}_{dR}(M)$
- Let $N_i = [z_0 : z_1 : ... : z_{i-1} : 0 : z_{i+1} : .. : z_n] \cong \mathbb{CP}^{n-1} \subset M$ $\forall i \in \{1, 2, 3, ...n\}$
- Let $[\omega_i] \in H^2(M)$ be the Poincare dual of N_i, since $H^2(M) \cong \mathbb{R}$, $\exists \lambda_i \in \mathbb{R}$ s.t $[\omega_i] = \lambda_i \omega$
- Note that N_i meets $\bigcap_{j \neq i} N_j$ transversally $\forall i$ and $\bigcap_i N_i = [1 : 0 : 0...0] := x$
- We know that the Poincare dual of any point $\{x\}$ is $dV_M \neq 0$
- By previous theorem, Poincare dual of $\{x\}$ $=dV_M = (\Pi_i \lambda_i)\omega^n \neq 0$ ■

Cohomology ring of $\mathbb{CP}^n$

Cohomology ring of $\mathbb{CP}^n$

- Define $\Gamma : H^*_{dR}(\mathbb{CP}^n) \to \mathbb{R}[X]/(X^{n+1})$, $[\omega] \to X$, then Γ is a well defined ring isomorphism
- Hence $H^*_{dR}(\mathbb{CP}^n) \cong \mathbb{R}[X]/(X^{n+1})$
- Define the Poincare series of a manifold M to be $P_t(M) = \Sigma_{n\in\mathbb{Z}} dim(H^n_{dR}(M))t^n$

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- Hence $H^*_{dR}(\mathbb{CP}^n) \cong \mathbb{R}[X]/(X^{n+1})$
- Define the Poincare series of a manifold M to be $P_t(M) = \Sigma_{n \in \mathbb{Z}} dim(H^n_{dR}(M))t^n$
- The Poincare series of $\mathbb{CP}^n = 1 + t^2 + ... + t^{2n} = \frac{1-t^{2n}}{1-t^2}$

Complex Vector Bundles

Complex Vector Bundles

Definition

A Complex Vector Bundle of complex dimension n over a manifold M is a fiber bundle over M with fibres $\mathbb{C}^n$ and structure group $GL_n(\mathbb{C})$
A complex line bundle is a $\mathbb{C}$ vector bundle of rank 1

Remarks

- Can reduce structure group of a complex vector bundle to $U(n)$, similar to how we reduced the structure group to $O(n)$ in the real case
- $\mathbb{C}$-vector bundles of dim n are $\mathbb{R}$-vector bundles of dim $2n$
- Since $U(1) \cong SO(2)$, we have a bijection between $\{\mathbb{C} \text{ line bundles}\} \cong \{\mathbb{R} \text{ Oriented 2 dim vector bundles}\}$

First Chern Class and some properties

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Definition

The First Chern Class of a complex line bundle(L) over M, is defined as the *Euler Class* of it's underlying real 2 dimensional bundle($L_{\mathbb{R}}$)

$$c_1(L) := e(L_{\mathbb{R}})$$

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Definition

The First Chern Class of a complex line bundle(L) over M, is defined as the *Euler Class* of it's underlying real 2 dimensional bundle($L_{\mathbb{R}}$)

$$c_1(L) := e(L_{\mathbb{R}})$$

- If L, L' are $\mathbb{C}$ line bundles with transition maps $\{g_{\alpha\beta}, g'_{\alpha\beta}\}$ and $g_{\alpha\beta}, g'_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$
- $L \otimes L'$ is the $\mathbb{C}$ line bundle with transition maps $\{g_{\alpha\beta}.g'_{\alpha\beta}\}$
- Recall the formula for the Euler class for 2 dim vector bundles $e(E) = -(2\pi i)^{-1} \sum_\gamma d(\rho_\alpha d(log(g_{\gamma\alpha})) \text{ on } U_\alpha$
- $c_1(L \otimes L') = c_1(L) + c_1(L')$

Properties of first chern class ctd

Properties of first chern class ctd

- Let L^* be the dual of the $\mathbb{C}$ line bundle L, then we have $L \otimes L^* = Hom(L, L)$
- $Hom(L, L)$ has a nowhere vanishing section, namely the identity map
- Thus $0 = e(L \otimes L^*) = c_1(L \otimes L^*) = c_1(L) + c_1(L^*)$
- $c_1(L) = -c_1(L^*)$
- If $f : M \to N$ and E a rank 2 vector bundle over N, then $c_1(f^{-1}(E)) = f^*(c_1(E))$ (Naturality of Euler Class)

Examples of complex vector bundles

Examples of complex vector bundles

Let V be a complex vector space of dim n and let $P(V) := \{1 \; dim \; subspaces \; of \; V\}$, P(V) is the projectivization of V
On $P(V)$ there are several natural vector bundles.

- Product bundle: $\hat{V} := V \times P(V)$
- Universal Subbundle: S which is the subbundle of $\hat{V}$ defined as,
  $S := \{(l, v) \in P(V) \times V | v \in l\}$
  Note that the fibre of S over a point $l \in P(V)$ is the l considered as a complex line in V
- Universal Quotient Bundle: Q which is defined by the exact sequence of bundles,
  $0 \to S \to \hat{V} \to Q \to 0$
  This short exact sequence is called the Tautological Exact Sequence over $P(V)$
  S^* is called the Hyperplane Bundle

Euler class of the universal subbundle over $\mathbb{CP}^n$

Euler class of the universal subbundle over $\mathbb{CP}^n$

- Let $e(S_{\mathbb{CP}^n}) :=$ Euler class of $S_{\mathbb{CP}^n}$. We claim that $[e(S_{CP^n})] \in H^2(\mathbb{CP}^n)$ generates the cohomology ring of $H^*(\mathbb{CP}^n)$.
- Enough to show that $[e(S_{\mathbb{CP}^n})] \neq 0$
- Let $N := [z_0 : z_1 : 0 : 0, .. : 0]$, $N \hookrightarrow \mathbb{CP}^n$ and $N \cong \mathbb{CP}^1$
- Note that $\iota^{-1} S_{\mathbb{CP}^n} = S_N$ where ι is the inclusion map
- By naturality of the Euler class, we get that $\iota^*(e(S_{\mathbb{CP}^n})) = e(S_N)$
- $N \cong \mathbb{CP}^1$ and also this will induce a bundle iso $S_N \cong S_{\mathbb{CP}^1}$
- Enough to show that $e(S_{\mathbb{CP}^1})$ is not zero

Euler class of the universal subbundle over $\mathbb{CP}^1$(Sketch)

Euler class of the universal subbundle over $\mathbb{CP}^1$(Sketch)

- Since the dimension of $\mathbb{CP}^1$ is equal to the (real)dimension of the line bundle $S_{\mathbb{CP}^1}$, we can use the Hopf index theorem to calculate the Euler number of $S_{\mathbb{CP}^1}$
- We define a section to the sphere bundle of $S_{\mathbb{CP}^1}$ with a singularity.
- We show that the local degree around the singularity is $\neq 0$, then we are done by the Index theorem
- Let $U_i := \{[z_0 : z_1] : z_i \neq 0\}, i \in \{0, 1\}$
- Define a section with a singularity at $[0 : 1]$ by $s : U_0 \to S|_{U_0}$, $[1 : z] \to ([1 : z], (1, z))$
- Find index of the singularity at $[0 : 1]$ using the coordinate chart on U_1, $[z_0 : z_1] \xrightarrow{\phi_1} \frac{z_0}{z_1}$
- $s|_{U_1 \cap U_0}$ in local coordinates on U_1 is the map $\tilde{s} : \mathbb{C} - 0 \to \mathbb{C} - 0 \times \mathbb{C} - 0$, $z \to (z, \frac{1}{z})$
- Local degree of the map $S^1 \to S^1, z \to \frac{1}{z}$ is -1
- Euler class of the univ subbundle over $\mathbb{CP}^1$ is not 0 $\blacksquare$

Remark

Remark

- After making a choice of basis for any $\mathbb{C}$ vector space(V) of dimension n, we have $V \cong \mathbb{C}^n$,$P(V) \cong \mathbb{CP}^n$ and a bundle isomorphism of their universal subbundles, $S_{P(V)} \cong S_{\mathbb{CP}^n}$
- By naturality of the Euler Class, we then see that $H^*(P(V))$ is generated by $e(S_{P(V)}))$

Projectivization of a Complex Vector Bundle

Projectivization of a Complex Vector Bundle

- Let $\rho : E \to M$ a $\mathbb{C}$ vector bundle, transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C})$
- $E_p :=$ fibre over p, $PGL(n, \mathbb{C}) := GL(n, \mathbb{C})/\{scalar\ \ matrices\}$
- Projectivization of E is defined as the fibre bundle $\pi : P(E) \to M$ whose fibres are $P(E_p)$ and transition maps are $\overline{g_{\alpha\beta}} : U_\alpha \cap U_\beta \to PGL(n, \mathbb{C})$ induced by $g_{\alpha\beta}$
- A point in $P(E)$ corresponds to a line l_p in E_p

Tautological Exact Sequence

We construct $\mathbb{C}$ vector bundles over $P(E)$

Tautological Exact Sequence

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- Pullback bundle: $\pi^{-1}E$, a bundle over $P(E)$ obtained by pulling back the vector bundle E over M via π
  Fibre of pullback bundle over $l_p \in P(E)$ is E_p
  When restricted to $\pi^{-1}(p)$, it becomes the trivial bundle,

  $$\pi^{-1}E|_{P(E)_p} = P(E)_p \times E_p$$

  as $\rho : E_p \to p$ is a trivial bundle
- Universal Subbundle: S over P(E) is defined as,

  $$S = \{(l_p, v) \in \pi^{-1}E | v \in l_p\}$$

  It's fibre at l_p is all the points in l_p
- Universal Quotient Bundle: $Q := Coker(S \to \pi^{-1}E)$

$$0 \to S \to \pi^{-1}E \to Q \to 0$$

$$\begin{tikzcd}
\pi^{-1}E \arrow[d] & \\
P(E) \arrow[dr, "\pi"'] & E \arrow[d, "\rho"] \\
 & M
\end{tikzcd}$$

Defining Chern Classes

Defining Chern Classes

- Define $x := c_1(S^*)$ then x is a cohomology class in $H^2(P(E))$
- Let $\tilde{S}$ be the universal subbundle of the projectivized vector space $P(E_p)$ and S the universal subbundle of $P(E)$
- Notice that S restricted to $P(E_p)$ is $\tilde{S}$
- Using naturality of the Euler class, to the inclusion map of $P(E_p) \to P(E)$ we get
  $c_1(\tilde{S})$ is the restriction of $-x$ to $P(E_p)$
- But we have seen that $\{1, -c_1(\tilde{S}), ..., (-c_1(\tilde{S}))^{n-1}\}$ generates the cohomology of $P(E_p)$
- Thus, the cohomology classes $1, x, ..., x^{n-1}$ on $P(E)$ restricted to each fiber $P(E_p)$ freely generate the cohomology of that fibre

Defining Chern Classes ctd

Defining Chern Classes ctd

- By the Leray-Hirsch Thm, $H^*(P(E))$ is a free module over $H^*(M)$ with basis $\{1, x, ..., x^{n-1}\}$
- So, x^n can be written as a unique linear combination of $1, x, ..., x^{n-1}$ with coefficients in $H^*(M)$
- These coefficients are by definition the Chern classes of the complex vector bundle E:

$$x^n + c_1(E)x^{n-1} + ... + c_n(E) = 0 \qquad c_i(E) \in H^{2i}(M)$$

  In this equation by $c_i(E)$ we really mean $\pi^* c_i(E)$
- The ith Chern class of E is defined as $c_i(E)$
- The total Chern class $c(E) := 1 + c_1(E) + ... + c_n(E) \in H^*(M)$
- The polynomial $x^n + c_1(E)x^{n-1} + ... + c_n(E)$ is called the Chern polynomial of E

Observations

Observations

- Cohomology ring of $P(E)$ is given by-:

$$H^*(P(E)) \cong H^*(M)[x]/(x^n + c_1(E)x^{n-1} + ... + c_n(E))$$

  where $x = c_1(S^*)$ and n is rank of E
- $H^*(P(E)) \cong H^*(M) \otimes H^*(\mathbb{C}P^{n-1})$ (By Leray Hirsch)
- This implies that the Poincare Series of $P(E)$ is given by-:

$$P_t(P(E)) = P_t(M)\frac{1 - t^{2n}}{1 - t^2}$$

First Chern class of a line bundle

First Chern class of a line bundle

- We have 2 definitions of the first chern class of a line bundle L, as the Euler class of $L_{\mathbb{R}}$ and as a coefficient in the polynomial from the previous slide
- These two definitions can be seen to agree. (Bott and Tu Pg 271)

Properties of Chern Classes

Properties of Chern Classes

- Naturality: If $f : Y \to X$ and E is a complex vector bundle over X, then $c(f^{-1}(E)) = f^*c(E)$

$$\begin{tikzcd} f^{-1}E \arrow[d] & E \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

  Proof: Let S_E be the universal subbundle over $P(E)$.
  $f^{-1}PE = P(f^{-1}E)$ and $f^{-1}S_E^* = S_{f^{-1}E}^*$,
  if $x_E = c_1(S_E^*)$, then

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$$\begin{array}{ccc} f^{-1}E & & E \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

Proof: Let S_E be the universal subbundle over $P(E)$.
$f^{-1}PE = P(f^{-1}E)$ and $f^{-1}S_E^* = S_{f^{-1}E}^*$,
if $x_E = c_1(S_E^*)$, then
$x_{f^{-1}E} = c_1(S_{f^{-1}E}^*) = c_1(f^{-1}S_E^*) = f^* x_E$

Proof of Naturality Ctd

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- Applying f^* to

$$x_E^n + c_1(E)x_E^{n-1} + ... + c_n(E) = 0$$

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we get

$$x_{f^{-1}E}^n + f^*c_1(E)x_{f^{-1}E}^{n-1} + ... + f^*c_n(E) = 0$$

Hence,

$$c_i(f^{-1}E) = f^*c_i(E)$$

■

Naturality Consequences

Naturality Consequences

- 1) If E and F are isomorphic vector bundles over X, then $c(E) = c(F)$
- 2) If E is a trivial complex vector bundle over M then $c_i(E) = 0$ for all i (E would be the pullback of a vector bundle over a point)
- Therefore, Chern classes measure "twisting" of a vector bundle.

Properties of Chern Classes

Properties of Chern Classes

- Let V be a complex vector space. If S^* is the hyperplane bundle over $P(V)$, then $c_1(S^*)$ generates the algebra $H^*(P(V))$
- Whitney Product Formula: $c(E \oplus F) = c(E)c(F)$ (Proof in next talk)
- If E is a rank n complex vector bundle, then $c_i(E) = 0$ for $i > n$

Properties of Chern Classes

Properties of Chern Classes

- If E has a nonvanishing section then the top Chern class $c_n(E)$ is 0
- Proof: We get a section s_1 of $P(E)$ as follows, $\forall$ p in X(the base manifold) define $s_1(p)$ as the line connecting the origin to $s(p)$ in E_p

$$\begin{tikzcd} P(E) \arrow[d, two heads, "\pi"] \\ X \arrow[u, bend left, "s_1"] \end{tikzcd}$$

- $s_1^{-1}S_E$ is a line bundle on X with a non vanishing section. Hence it is trivial
- From naturality of Chern classes, we get $s_1^* c_1(S_E) = 0$
- Implies $s_1^* x = 0$
- Applying s_1^* to, $x^n + c_1(E)x^{n-1} + ... + c_n(E) = 0$,we get $s_1^* c_n = 0$
- By our abuse of notation we really mean, $s_1^* \pi^* c_n(E) = 0$ then $c_n(E) = 0$ ∎

Final Property of Chern class

Final Property of Chern class

- The top Chern class of a complex vector bundle E is the Euler class of it's realization
  $c_n(E) = e(E_{\mathbb{R}})$, where $n = \mathrm{rank(E)}$
- Proof in later talks