Chapter 3

Forest Fires

Power law distributions are found in many different phenomena in fields as diverse as biology, demography, computer science, economics, information theory, language, and astronomy [1]. Also, several natural disasters present power-law statistics, such as the extension of forest fires [2] and the intensity of earthquakes [3]. Power-law frequency-size distributions are often associated with self-organized critical behaviour, which can be found in a number of different models: the sandpile model [4], the slider-block model [5, 6], and the forest fire model [7]. Many other fundamental models studied in statistical physics, such as the Ising model, have critical points, where correlations diverge while phenomena become complex and interesting in many ways. All of these models generate results where power-law statistics appear. In this chater, we will focus on a forest fire model and study a simple model of forests grown under exposure to periodic forest fires. We analyse the size distribution of fires which appears to follow a power law. All the necessary details regarding the numerical modelling for this exercise are explained here, but to read more about the forest fire model and its consistency with the observational data, please refer to "Forest fires: An example of self-organized critical behavior" [2].

In this exercise, we will consider a proposed solution that is called *self-organized criticality* (SOC). The idea is that something about the system drives it towards a critical point, regardless of the initial conditions. In the forest fire model [8] which we will study here, trees grow and fires burn down connected clusters of trees. Thus, when there are few trees, fires have little effect and the forest grows. However, when the density increases the fires tend to have more effect, burning down large swats of forest. There is thus a balancing act between the growing and burning effects, and the model is held by its own dynamics close to a critical point.

The model consist of a square $N \times N$–lattice on which the trees grow. Each site either has a tree or is empty. At each time step each empty site has a probability p to turn into an occupied one. Each time step also has a probability f of a lightning strike, occurring at a random site. The strike ignites the tree, if any, at that site. The tree ignites its von Neumann neighbors[1] (using periodic boundary conditions), they ignite their neighbors, and so on. Finally, all ignited trees are removed from the lattice. In other words, the lightning strike starts a forest fire which burns down a connected cluster of trees (see Fig. 3.1). The fire can spread one step each time step or burn down the whole cluster during a single time step (corresponding to taking $p \to 0$ with p/f finite, separating the timescales for growth and burning completely). Use the latter!

The number of trees burnt down in a single strike is the size of the fire. We will study the distribution of these sizes.

Rank-frequency plots, Plotting power law distributions can be problematic as the tail gets very noisy and binning introduces artifacts. A solution is the rank-frequency plot. It effectively plots the complementary cumulative distribution (cCDF)[2] of the data, allowing easy comparison to the cCDF of the hypothetical distribution. To do such a plot given some data, we sort the data, giving a series

[1]The nearest von Neumann neighbours of a cell are the four cells to its immediate right, left, top, and bottom.

[2]If a quantity has probability distribution $P(x)$, its complementary cumulative distribution is $cCDF(y) = \int_y^\infty P(x)dx$, that is, the probability of observing a value greater than or equal to y.

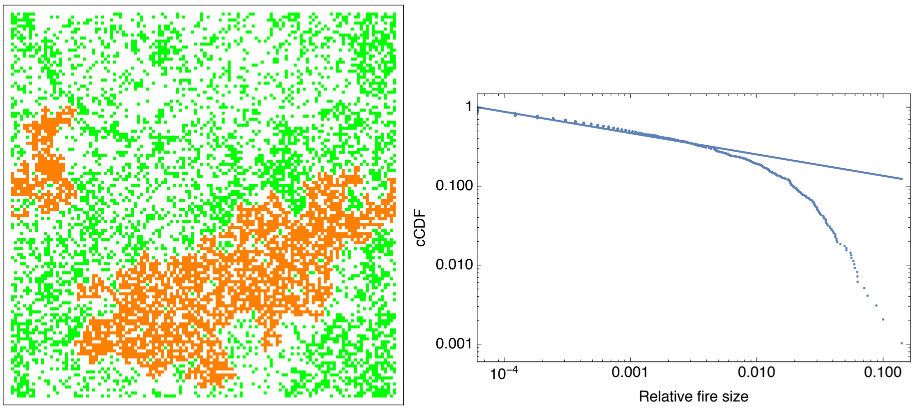

Figure 3.1: (Left) A snapshot of a forest where a cluster of 2605 trees is burning down, yielding a relative fire size of 0.16. Green patches are occupied by trees, orange are burning. (Right) Rank-frequency plot of the fire sizes on a 128×128 lattice with a power law fitted to the bulk of the distribution.

$\{y_i\}_{i=1}^n$ with $y_i \geq y_{i+1}$, and assign to each value its (normalized) rank, giving a new data set $\{(y_i, i/n)\}_{i=1}^n$. This is the empirical cCDF, ready to be plotted in a log-log graph. See Fig. 3.1 for an example.

Generating power law data There are several methods for generating random numbers with a given distribution $P(x)$. The most general is rejection sampling, but if one knows the CDF and can invert it, the more direct method of generating uniform numbers $r_i \sim \text{Unif}[0,1]$ and picking $X_i = CDF^{-1}(r_i)$ is much more efficient, giving data with the desired distribution $X_i \sim P(X_i)$. This approach is known as inverse transform sampling. For a power law distribution $P(x) \propto (x/x_{min})^{-\tau}$, the formula for the cCDF becomes $X_i = x_{min}(1 - r_i)^{-\frac{1}{\tau-1}}$.

Exercises

1. Implement the basic model and visualize it for $N = 128$, see Fig. 3.1. What are reasonable values for the parameters p and f? Note the shapes of the clusters removed by a lightning strike. **To demonstrate (6 points)**: Figures similar to the left plot in Fig. 3.1 for different parameters. Describe and

explain the characteristic behaviour for each choice of parameters. Restrict yourself to four figures.

2. Study the size distribution of the fires for $N = 128$. For each fire do the following: 1) record its size; 2) compute the density of the forest just before the fire; 3) generate a random forest (grown without fires) with similar density; and 4) start a fire in it (pick random site with three) and record its size. Generate and compare the rank-frequency plots of the distributions of fire sizes for the two types of forests. **To demonstrate (7 points)**: The distributions shown in a single plot. Discuss how and why they differ.

3. It is generally claimed that the self-organized fire distribution follows a power law with exponent $\tau = 1.15$ [9], $P(s) \propto s^{-1.15}$, in the limit of large systems. Try to evaluate this claim. Normally we should use a maximum likelihood estimator (MLE) to estimate the exponent [10], but in this case it works badly due to the cut-off of the distribution. An alternative would be to model the cut-off as an exponential and derive the MLE for the resulting distribution, but that is outside of the scope of this exercise. Instead, to get a rough approximation, use the heretical method of drawing straight lines in your plot and calculate the slope (which is $1-\tau$ in a rank-frequency plot). Check that this works acceptably by generating a synthetic data set (see above) from the power law and compare. **To demonstrate (7 points)**: 1) A rank-frequency plot with the fire data and the linear fit used to find the exponent τ; 2) A rank-frequency plot comparing the fire data and the synthetic power law data. 3) A discussion of your results. For example: Is the result sensitive to the choice of parameters p and f? What factors might impact the result?

4. The value for the exponent depends on the size of the lattice N, so do a finite size analysis. Repeat the above exercise 3 for the different values of $N \in \{8, 16, 32, 64, 128, 256, 512\}$. Plot the resulting exponents as a function of $1/N$ and extrapolate (using some suitable function) to zero (i.e. $N \to \infty$) to obtain the limiting exponent in the case of an infinite lattice. **To demonstrate (5 points)**: The resulting plot with the extrapolation and a discussion of your result.

For most models, studies such as the one in this homework is enough (provided you did it on reasonably large lattices); increasing system size merely leads to more significant digits in the critical parameters. This model is different though. Grassberger [11] did extensive simulations up to $N = 65536$ (by representing each tree with a single bit, you can fit such a lattice into a 1 GB memory) and found new behavior for extremely large systems, with e.g. $\tau = 1.19 \pm 0.01$, noting that nothing prevents still new behavior at even larger scales. Looking at the density of trees, he also found that the previous result of $\rho \approx 0.408$ is an underestimate: the density continues to increase for ever larger systems, albeit very slowly. As the density cannot exceed the percolation density $\rho_c = 0.592...$, Grassberger conjectured that it will end up there. A simple extrapolation of his data gives that this point is reached somewhere around $N = 10^{20}\dots$

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