Generation of Paths for Motion Planning for a Dubins Vehicle on Sphere*

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March 29, 2025

1 Derivation of Closed-Form Expressions for Paths on a Sphere

Consider the differential equations corresponding to the Sabban frame. As u_g is piecewise constant, the differential equations for each segment can be represented as

$$\begin{pmatrix}\mathbf{X}'(s) & \mathbf{T}'(s) & \mathbf{N}'(s)\end{pmatrix} = \begin{pmatrix}\mathbf{X}(s) & \mathbf{T}(s) & \mathbf{N}(s)\end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -u_g \\ 0 & u_g & 0 \end{pmatrix}}_{\Omega}. \tag{1.1}$$

The solution to the above differential equations can be obtained as

$$\begin{pmatrix}\mathbf{X}(s) & \mathbf{T}(s) & \mathbf{N}(s)\end{pmatrix} = \begin{pmatrix}\mathbf{X}(s_i) & \mathbf{T}(s_i) & \mathbf{N}(s_i)\end{pmatrix} \left(e^{\Omega^T \Delta s}\right)^T, \tag{1.2}$$

where $\Delta s = s - s_i$. The expression for $e^{\Omega \Delta s}$ can be obtained using the Euler-Rodriguez formula for the exponential of a skew-symmetric matrix. Moreover, since $s = \phi_G$ for a great circle turn, and $s = r\phi_L$ and $s = r\phi_R$ for the left and right tight turns, respectively, the solution to the differential equations can be written as

$$R_{after} = R_{before} R_{seg}, \tag{1.3}$$

where R_{after} and R_{before} denote the configurations after and before the segment, respectively. Moreover, R_{seg} represents the rotation matrix corresponding to a chosen segment, and is given by

$$R_{seg} = \begin{cases} R_G(\phi) = \begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, & u_g = 0 \\ R_L(r,\phi) = \begin{pmatrix} 1-(1-c\phi)r^2 & -rs\phi & (1-c\phi)r\sqrt{1-r^2} \\ rs\phi & c\phi & -s\phi\sqrt{1-r^2} \\ (1-c\phi)r\sqrt{1-r^2} & s\phi\sqrt{1-r^2} & c\phi+(1-c\phi)r^2 \end{pmatrix}, & u_g = U_{max} \\ R_R(r,\phi) = \begin{pmatrix} 1-(1-c\phi)r^2 & -rs\phi & -(1-c\phi)r\sqrt{1-r^2} \\ rs\phi & c\phi & s\phi\sqrt{1-r^2} \\ -(1-c\phi)r\sqrt{1-r^2} & -s\phi\sqrt{1-r^2} & c\phi+(1-c\phi)r^2 \end{pmatrix}, & u_g = -U_{max} \end{cases}. \tag{1.4}$$

For deriving the closed-form expressions for the paths, the following two vectors will be utilized:

$$\mathbf{u}_L := \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix}, \quad \mathbf{u}_R := \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix}.$$

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It should be noted that the vector $\mathbf{u}_L$ corresponds to the axial vector of the rotation matrix $R_L(r,\phi)$. Hence, $R_L(r,\phi)\mathbf{u}_L = \mathbf{u}_L$. Similarly, the vector $\mathbf{u}_R$ corresponds to the axial vector of the rotation matrix $R_R(r,\phi)$. Hence, $R_R(r,\phi)\mathbf{u}_R = \mathbf{u}_R$.

1.1 LGL path

The equation to be solved is given by

$$R_L(r,\phi_1)R_G(\phi_2)R_L(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{1.5}$$

Pre-multiplying Eq. (1.5) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_L$,

$$\mathbf{u}_L^T R_L(r,\phi_1)R_G(\phi_2)R_L(r,\phi_3)\mathbf{u}_L = \mathbf{u}_L^T R_G(\phi_2)\mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L.$$

$$\therefore \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} = \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix}.$$

Simplifying the above equation,

$$\begin{aligned} (1-r^2)c\phi_2 + r^2 &= (1-r^2)\alpha_{11} + r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31}) + r^2\alpha_{33}. \\ \implies c\phi_2 &= \frac{\alpha_{11} + r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31}) + r^2(\alpha_{33}-\alpha_{11}-1)}{1-r^2}, \end{aligned} \tag{1.6}$$

which yields at most two solutions for $\phi_2 \in [0, 2\pi)$ if the absolute value of the RHS is less than or equal to 1.

Consider pre-multiplying Eq. (1.5) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_L$, which yields

$$\mathbf{u}_R^T R_L(r,\phi_1)R_G(\phi_2)R_L(r,\phi_3)\mathbf{u}_L = \mathbf{u}_R^T R_L(r,\phi_1)R_G(\phi_2)\mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \tag{1.7}$$

The LHS of the above equation can be expanded as

$$\begin{aligned} &\mathbf{u}_R^T R_L(r,\phi_1)R_G(\phi_2)\mathbf{u}_L \\ &= \mathbf{u}_R^T R_L(r,\phi_1) \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} 1-(1-c\phi_1)r^2 & -rs\phi_1 & (1-c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & -s\phi_1\sqrt{1-r^2} \\ (1-c\phi_1)r\sqrt{1-r^2} & s\phi_1\sqrt{1-r^2} & c\phi_1 + (1-c\phi_1)r^2 \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2}c\phi_2 \\ \sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{1-r^2} + 2(1-c\phi_1)r^2\sqrt{1-r^2} & 2r\sqrt{1-r^2}s\phi_1 & 2(1-c\phi_1)r^3 + r(2c\phi_1 - 1) \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2}c\phi_2 \\ \sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\ &= -(1-r^2)c\phi_2 + 2(1-c\phi_1)r^2(1-r^2)c\phi_2 + 2r(1-r^2)s\phi_1 s\phi_2 + 2(1-c\phi_1)r^4 + 2r^2c\phi_1 - r^2 \\ &= \left((1-r^2)c\phi_2 + r^2\right)(-1+2r^2) + 2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2 s\phi_1. \end{aligned}$$

The RHS of Eq. (1.7) can be expanded as

$$
\begin{aligned}
&\mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L \\
&= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{1-r^2}\alpha_{11} + r\alpha_{31} & -\sqrt{1-r^2}\alpha_{12} + r\alpha_{32} & -\sqrt{1-r^2}\alpha_{13} + r\alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\
&= -(1-r^2)\alpha_{11} + r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13}) + r^2\alpha_{33} = -\alpha_{11} + r^2(\alpha_{11}+\alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13}).
\end{aligned}
$$

Substituting the obtained expressions in Eq. (1.7) and using the expression for $(1-r^2)c\phi_2+r^2$ from Eq. (1.6),

$$
\begin{aligned}
2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2 s\phi_1 &= -\alpha_{11} + r^2(\alpha_{11}+\alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13}) \\
&\quad - \left((1-r^2)c\phi_2 + r^2\right)(-1+2r^2) \\
&= -\alpha_{11} + r^2(\alpha_{11}+\alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13}) \\
&\quad - \left(\alpha_{11} + r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31}) + r^2(\alpha_{33}-\alpha_{11})\right)(-1+2r^2) \\
&= 2(\alpha_{33}-\alpha_{11})r^2(1-r^2) - 2\alpha_{13}r^3\sqrt{1-r^2} + 2\alpha_{31}r(1-r^2)^{\frac{3}{2}}. \\
\implies r(1-c\phi_2)c\phi_1 + s\phi_2 s\phi_1 = (\alpha_{33}-\alpha_{11})r &- \alpha_{13}r^2(1-r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1-r^2}.
\end{aligned}
$$

It should be noted that if $\phi_2 \neq 0$, then the above equation can be used to solve for ϕ_1. Diving both sides by $\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}$ and defining $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}, s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}$, the above equation can be rewritten as

$$
c(\phi_1-\beta) = \frac{(\alpha_{33}-\alpha_{11})r - \alpha_{13}r^2(1-r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1-r^2}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}. \tag{1.8}
$$

$$
\begin{aligned}
\therefore \phi_1 &= \left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r - \alpha_{13}r^2(1-r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1-r^2}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right) + \beta\right)\%2\pi \\
&= \left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r - \alpha_{13}r^2(1-r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1-r^2}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right) + \operatorname{atan2}\left(s\phi_2, r(1-c\phi_2)\right)\right)\%2\pi.
\end{aligned} \tag{1.9}
$$

Important remark: From the above equation, for each solution of ϕ_2, at most two solutions can be obtained for ϕ_1, since $\cos^{-1}$ can be replaced with $2\pi - \cos^{-1}$ to obtain another solution. The same argument applies for the rest of the report whenever we encounter a solution using $\cos^{-1}$. Furthermore, for all angles, a modulus function is utilized to ensure that it is between 0 to 2π.

Finally, consider pre-multiplying Eq. (1.5) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_R$, which yields

$$
\mathbf{u}_L^T R_L(r,\phi_1) R_G(\phi_2) R_L(r,\phi_3)\mathbf{u}_R = \mathbf{u}_L^T R_G(\phi_2) R_L(r,\phi_3)\mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \tag{1.10}
$$

The LHS of the above equation can be expanded as

$$
\begin{aligned}
&\mathbf{u}_L^T R_G(\phi_2) R_L(r,\phi_3)\mathbf{u}_R \\
&= \begin{pmatrix}\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix} c\phi_2 & -s\phi_2 & 0\\ s\phi_2 & c\phi_2 & 0\\ 0 & 0 & 1\end{pmatrix} R_L(r,\phi_3)\mathbf{u}_R \\
&= \begin{pmatrix}\sqrt{1-r^2}c\phi_2 & -\sqrt{1-r^2}s\phi_2 & r\end{pmatrix}\begin{pmatrix} 1-(1-c\phi_3)r^2 & -rs\phi_3 & (1-c\phi_3)r\sqrt{1-r^2}\\ rs\phi_3 & c\phi_3 & -s\phi_3\sqrt{1-r^2}\\ (1-c\phi_3)r\sqrt{1-r^2} & s\phi_3\sqrt{1-r^2} & c\phi_3+(1-c\phi_3)r^2\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}\\ 0\\ r\end{pmatrix} \\
&= \begin{pmatrix}\sqrt{1-r^2}c\phi_2 & -\sqrt{1-r^2}s\phi_2 & r\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}+2(1-c\phi_3)r^2\sqrt{1-r^2}\\ -2r\sqrt{1-r^2}s\phi_3\\ 2(1-c\phi_3)r^3+r(2c\phi_3-1)\end{pmatrix} \\
&= -(1-r^2)c\phi_2+2(1-c\phi_3)r^2(1-r^2)c\phi_2+2r(1-r^2)s\phi_3 s\phi_2+2(1-c\phi_3)r^4+2r^2c\phi_3-r^2 \\
&= \left((1-r^2)c\phi_2+r^2\right)(-1+2r^2)+2r^2(1-r^2)(1-c\phi_2)c\phi_3+2r(1-r^2)s\phi_2 s\phi_3.
\end{aligned}
$$

The RHS of Eq. (1.10) can be expanded as

$$
\begin{aligned}
&\mathbf{u}_L^T\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R \\
&= \begin{pmatrix}\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}\\ 0\\ r\end{pmatrix} \\
&= \begin{pmatrix}\sqrt{1-r^2}\alpha_{11}+r\alpha_{31} & \sqrt{1-r^2}\alpha_{12}+r\alpha_{32} & \sqrt{1-r^2}\alpha_{13}+r\alpha_{33}\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}\\ 0\\ r\end{pmatrix} \\
&= -(1-r^2)\alpha_{11}+r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31})+r^2\alpha_{33} = -\alpha_{11}+r^2(\alpha_{11}+\alpha_{33})+r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31}).
\end{aligned}
$$

Substituting the obtained expressions in Eq. (1.10) and using the expression for $(1-r^2)c\phi_2+r^2$ from Eq. (1.6),

$$
\begin{aligned}
2r^2(1-r^2)(1-c\phi_2)c\phi_3+2r(1-r^2)s\phi_2 s\phi_3 &= -\alpha_{11}+r^2(\alpha_{11}+\alpha_{33})+r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31}) \\
&\quad -\left((1-r^2)c\phi_2+r^2\right)(-1+2r^2) \\
&= -\alpha_{11}+r^2(\alpha_{11}+\alpha_{33})+r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31}) \\
&\quad -\left(\alpha_{11}+r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31})+r^2(\alpha_{33}-\alpha_{11})\right)(-1+2r^2) \\
&= 2(\alpha_{33}-\alpha_{11})r^2(1-r^2)+2\alpha_{13}r(1-r^2)^{\frac{3}{2}}-2\alpha_{31}r^3\sqrt{1-r^2}. \\
\implies r(1-c\phi_2)c\phi_3+s\phi_2 s\phi_3 &= (\alpha_{33}-\alpha_{11})r+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}.
\end{aligned}
$$

It should be noted that if $\phi_2 \neq 0$, then the above equation can be used to solve for ϕ_3. Diving both sides by $\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}$ and defining $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}$, $s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}$, the above equation can be rewritten as

$$c(\phi_3-\beta)=\frac{(\alpha_{33}-\alpha_{11})r+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}. \tag{1.11}$$

$$\begin{aligned}\therefore \phi_3 &= \left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right)+\beta\right)\%2\pi\\ &= \left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r+\alpha_{13}\sqrt{1-r^2}-\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right)+\tan^{-1}\left(\frac{s\phi_2}{r(1-c\phi_2)}\right)\right)\%2\pi.\end{aligned} \tag{1.12}$$

From the above equation, for each solution of ϕ_2, at most two solutions can be obtained for ϕ_3.

Remark: In the case that $\phi_2=0$, the LGL path reduces to an L segment. Hence, in this case, ϕ_3 can be set to zero without loss of generality, and ϕ_1 can be solved using the first and second entries in the second row in the net rotation matrix (refer to the expression for R_L). That is,

$$\phi_1=\big(\text{atan2}\left(\alpha_{21}, r\alpha_{22}\right)\big)\%2\pi. \tag{1.13}$$

1.2 RGR path

The equation to be solved is given by

$$R_R(r,\phi_1)R_G(\phi_2)R_R(r,\phi_3)=\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}. \tag{1.14}$$

Pre-multiplying Eq. (1.14) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_R$,

$$\mathbf{u}_R^T R_R(r,\phi_1)R_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_R=\mathbf{u}_R^T R_G(\phi_2)\mathbf{u}_R=\mathbf{u}_R^T\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R.$$

$$\therefore \begin{pmatrix}-\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix}c\phi_2 & -s\phi_2 & 0\\ s\phi_2 & c\phi_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}\\ 0\\ r\end{pmatrix}=\begin{pmatrix}-\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}\\ 0\\ r\end{pmatrix}.$$

Simplifying the above equation,

$$\begin{aligned}(1-r^2)c\phi_2+r^2&=(1-r^2)\alpha_{11}-r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31})+r^2\alpha_{33}.\\ \implies c\phi_2&=\frac{\alpha_{11}-r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31})+r^2(\alpha_{33}-\alpha_{11}-1)}{1-r^2},\end{aligned} \tag{1.15}$$

which yields at most two solutions for $\phi_2\in[0,2\pi)$ if the absolute value of the RHS is less than or equal to 1.

Consider pre-multiplying Eq. (1.14) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_R$, which yields

$$\mathbf{u}_L^T R_R(r,\phi_1)R_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_R=\mathbf{u}_L^T R_R(r,\phi_1)R_G(\phi_2)\mathbf{u}_R=\mathbf{u}_L^T\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R. \tag{1.16}$$

The LHS of the above equation can be expanded as

$$
\begin{aligned}
&\mathbf{u}_L^T R_R(r,\phi_1) R_G(\phi_2)\mathbf{u}_R \\
&= \mathbf{u}_L^T R_R(r,\phi_1)\begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r\end{pmatrix} \\
&= \begin{pmatrix}\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix} 1-(1-c\phi_1)r^2 & -rs\phi_1 & -(1-c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & s\phi_1\sqrt{1-r^2} \\ -(1-c\phi_1)r\sqrt{1-r^2} & -s\phi_1\sqrt{1-r^2} & c\phi_1+(1-c\phi_1)r^2\end{pmatrix}\begin{pmatrix} -\sqrt{1-r^2}c\phi_2 \\ -\sqrt{1-r^2}s\phi_2 \\ r\end{pmatrix} \\
&= \begin{pmatrix}\sqrt{1-r^2}-2(1-c\phi_1)r^2\sqrt{1-r^2} & -2r\sqrt{1-r^2}s\phi_1 & 2(1-c\phi_1)r^3+r(2c\phi_1-1)\end{pmatrix}\begin{pmatrix} -\sqrt{1-r^2}c\phi_2 \\ -\sqrt{1-r^2}s\phi_2 \\ r\end{pmatrix} \\
&= -(1-r^2)c\phi_2 + 2(1-c\phi_1)r^2(1-r^2)c\phi_2 + 2r(1-r^2)s\phi_1 s\phi_2 + 2(1-c\phi_1)r^4 + 2r^2 c\phi_1 - r^2 \\
&= \left((1-r^2)c\phi_2 + r^2\right)(-1+2r^2) + 2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2 s\phi_1.
\end{aligned}
$$

The RHS of Eq. (1.16) can be expanded as

$$
\begin{aligned}
&\mathbf{u}_L^T\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R \\
&= \begin{pmatrix}\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r\end{pmatrix} \\
&= \begin{pmatrix}\sqrt{1-r^2}\alpha_{11}+r\alpha_{31} & \sqrt{1-r^2}\alpha_{12}+r\alpha_{32} & \sqrt{1-r^2}\alpha_{13}+r\alpha_{33}\end{pmatrix}\begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r\end{pmatrix} \\
&= -(1-r^2)\alpha_{11} + r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31}) + r^2\alpha_{33} = -\alpha_{11} + r^2(\alpha_{11}+\alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31}).
\end{aligned}
$$

Substituting the obtained expressions in Eq. (1.16) and using the expression for $(1-r^2)c\phi_2 + r^2$ from Eq. (1.15),

$$
\begin{aligned}
2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2 s\phi_1 &= -\alpha_{11} + r^2(\alpha_{11}+\alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31}) \\
&\quad - \left((1-r^2)c\phi_2 + r^2\right)(-1+2r^2) \\
&= -\alpha_{11} + r^2(\alpha_{11}+\alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13}-\alpha_{31}) \\
&\quad - \left(\alpha_{11} + r^2(\alpha_{33}-\alpha_{11}) - r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31})\right)(-1+2r^2) \\
&= 2(\alpha_{33}-\alpha_{11})r^2(1-r^2) + 2\alpha_{13}r^3\sqrt{1-r^2} - 2\alpha_{31}r(1-r^2)^{\frac{3}{2}}. \\
\implies r(1-c\phi_2)c\phi_1 + s\phi_2 s\phi_1 &= (\alpha_{33}-\alpha_{11})r + \alpha_{13}r^2(1-r^2)^{-\frac{1}{2}} - \alpha_{31}(1-r^2)^{\frac{1}{2}}.
\end{aligned}
$$

It should be noted that if $\phi_2 \neq 0$, then the above equation can be used to solve for ϕ_1. Diving both sides by $\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}$ and defining $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}$, $s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}$, the above equation can be rewritten as

$$c(\phi_1-\beta)=\frac{(\alpha_{33}-\alpha_{11})r+\alpha_{13}r^2(1-r^2)^{-\frac{1}{2}}-\alpha_{31}(1-r^2)^{\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}. \tag{1.17}$$

$$\begin{aligned}\therefore \phi_1&=\left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r+\alpha_{13}r^2(1-r^2)^{-\frac{1}{2}}-\alpha_{31}(1-r^2)^{\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right)+\beta\right)\%2\pi\\&=\left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r+\alpha_{13}r^2(1-r^2)^{-\frac{1}{2}}-\alpha_{31}(1-r^2)^{\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right)+\operatorname{atan2}\left(s\phi_2,r(1-c\phi_2)\right)\right)\%2\pi.\end{aligned} \tag{1.18}$$

From the above equation, for each solution of ϕ_2, at most two solutions can be obtained for ϕ_1.

Finally, consider pre-multiplying Eq. (1.14) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_L$, which yields

$$\mathbf{u}_R^TR_R(r,\phi_1)R_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_L=\mathbf{u}_R^TR_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_L=\mathbf{u}_R^T\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\mathbf{u}_L. \tag{1.19}$$

The LHS of the above equation can be expanded as

$$\begin{aligned}&\mathbf{u}_R^TR_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_L\\&=\begin{pmatrix}-\sqrt{1-r^2}&0&r\end{pmatrix}\begin{pmatrix}c\phi_2&-s\phi_2&0\\s\phi_2&c\phi_2&0\\0&0&1\end{pmatrix}R_R(r,\phi_3)\mathbf{u}_L\\&=\begin{pmatrix}-\sqrt{1-r^2}c\phi_2&\sqrt{1-r^2}s\phi_2&r\end{pmatrix}\begin{pmatrix}1-(1-c\phi_3)r^2&-rs\phi_3&-(1-c\phi_3)r\sqrt{1-r^2}\\rs\phi_3&c\phi_3&s\phi_3\sqrt{1-r^2}\\-(1-c\phi_3)r\sqrt{1-r^2}&-s\phi_3\sqrt{1-r^2}&c\phi_3+(1-c\phi_3)r^2\end{pmatrix}\begin{pmatrix}\sqrt{1-r^2}\\0\\r\end{pmatrix}\\&=\begin{pmatrix}-\sqrt{1-r^2}c\phi_2&\sqrt{1-r^2}s\phi_2&r\end{pmatrix}\begin{pmatrix}\sqrt{1-r^2}-2(1-c\phi_3)r^2\sqrt{1-r^2}\\2r\sqrt{1-r^2}s\phi_3\\2(1-c\phi_3)r^3+r(2c\phi_3-1)\end{pmatrix}\\&=-(1-r^2)c\phi_2+2(1-c\phi_3)r^2(1-r^2)c\phi_2+2r(1-r^2)s\phi_3s\phi_2+2(1-c\phi_3)r^4+2r^2c\phi_3-r^2\\&=\left((1-r^2)c\phi_2+r^2\right)(-1+2r^2)+2r^2(1-r^2)(1-c\phi_2)c\phi_3+2r(1-r^2)s\phi_2s\phi_3.\end{aligned}$$

The RHS of Eq. (1.19) can be expanded as

$$\begin{aligned}&\mathbf{u}_R^T\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\mathbf{u}_L\\&=\begin{pmatrix}-\sqrt{1-r^2}&0&r\end{pmatrix}\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\begin{pmatrix}\sqrt{1-r^2}\\0\\r\end{pmatrix}\\&=\begin{pmatrix}-\sqrt{1-r^2}\alpha_{11}+r\alpha_{31}&-\sqrt{1-r^2}\alpha_{12}+r\alpha_{32}&-\sqrt{1-r^2}\alpha_{13}+r\alpha_{33}\end{pmatrix}\begin{pmatrix}\sqrt{1-r^2}\\0\\r\end{pmatrix}\\&=-(1-r^2)\alpha_{11}+r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13})+r^2\alpha_{33}=-\alpha_{11}+r^2(\alpha_{11}+\alpha_{33})+r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13}).\end{aligned}$$

Substituting the obtained expressions in Eq. (1.19) and using the expression for $(1-r^2)c\phi_2+r^2$ from

Eq. (1.15),

$$\begin{aligned}2r^2(1-r^2)(1-c\phi_2)c\phi_3+2r(1-r^2)s\phi_2 s\phi_3 &= -\alpha_{11}+r^2(\alpha_{11}+\alpha_{33})+r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13})\\ &\quad -\left((1-r^2)c\phi_2+r^2\right)(-1+2r^2)\\ &= -\alpha_{11}+r^2(\alpha_{11}+\alpha_{33})+r\sqrt{1-r^2}(\alpha_{31}-\alpha_{13})\\ &\quad -\left(\alpha_{11}+r^2(\alpha_{33}-\alpha_{11})-r\sqrt{1-r^2}(\alpha_{13}+\alpha_{31})\right)(-1+2r^2)\\ &= 2(\alpha_{33}-\alpha_{11})r^2(1-r^2)-2\alpha_{13}r(1-r^2)^{\frac{3}{2}}+2\alpha_{31}r^3\sqrt{1-r^2}.\\ \implies r(1-c\phi_2)c\phi_3+s\phi_2 s\phi_3 &= (\alpha_{33}-\alpha_{11})r-\alpha_{13}(1-r^2)^{\frac{1}{2}}+\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}.\end{aligned}$$

It should be noted that if $\phi_2 \neq 0$, then the above equation can be used to solve for ϕ_3. Diving both sides by $\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}$ and defining $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}$, $s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}$, the above equation can be rewritten as

$$c(\phi_3-\beta) = \frac{(\alpha_{33}-\alpha_{11})r-\alpha_{13}(1-r^2)^{\frac{1}{2}}+\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}. \tag{1.20}$$

$$\begin{aligned}\therefore \phi_3 &= \left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r-\alpha_{13}(1-r^2)^{\frac{1}{2}}+\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right)+\beta\right)\%2\pi\\ &= \left(\cos^{-1}\left(\frac{(\alpha_{33}-\alpha_{11})r-\alpha_{13}(1-r^2)^{\frac{1}{2}}+\alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2+s^2\phi_2}}\right)+\operatorname{atan2}\left(s\phi_2, r(1-c\phi_2)\right)\right)\%2\pi.\end{aligned} \tag{1.21}$$

From the above equation, for each solution of ϕ_2, at most two solutions can be obtained for ϕ_3.

Remark: The case of $\phi_2 = 0$ can be handled similar to the LGL path, wherein ϕ_3 can be set to zero and the expression for ϕ_1 obtained is the same as that given in Eq. (1.13).

1.3 LGR path

The equation to be solved is given by

$$R_L(r,\phi_1)R_G(\phi_2)R_R(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}. \tag{1.22}$$

Pre-multiplying Eq. (1.22) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_R$,

$$\mathbf{u}_L^T R_L(r,\phi_1)R_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_R = \mathbf{u}_L^T R_G(\phi_2)\mathbf{u}_R = \mathbf{u}_L^T\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R.$$

$$\therefore \begin{pmatrix}\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix}c\phi_2 & -s\phi_2 & 0\\ s\phi_2 & c\phi_2 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}\\ 0\\ r\end{pmatrix} = \begin{pmatrix}\sqrt{1-r^2} & 0 & r\end{pmatrix}\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\begin{pmatrix}-\sqrt{1-r^2}\\ 0\\ r\end{pmatrix}.$$

Simplifying the above equation,

$$\begin{aligned}-\left(1-r^2\right)c\phi_2+r^2 &= -\left(1-r^2\right)\alpha_{11}+r\sqrt{1-r^2}\left(\alpha_{13}-\alpha_{31}\right)+r^2\alpha_{33}.\\ \implies c\phi_2 &= \frac{\left(1-r^2\right)\alpha_{11}+r\sqrt{1-r^2}\left(\alpha_{31}-\alpha_{13}\right)+r^2(1-\alpha_{33})}{(1-r^2)},\end{aligned} \tag{1.23}$$

which yields at most two solutions for $\phi_2 \in [0, 2\pi)$ if the absolute value of the RHS is less than or equal to 1.

Consider pre-multiplying Eq. (1.22) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_R$, which yields

$$\mathbf{u}_R^T R_L(r,\phi_1) R_G(\phi_2) R_R(r,\phi_3)\mathbf{u}_R = \mathbf{u}_R^T R_L(r,\phi_1) R_G(\phi_2)\mathbf{u}_R = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \tag{1.24}$$

The LHS of the above equation can be expanded as

$$\begin{aligned}
&\mathbf{u}_R^T R_L(r,\phi_1) R_G(\phi_2)\mathbf{u}_R \\
&= \mathbf{u}_R^T R_L(r,\phi_1) \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} 1-(1-c\phi_1)r^2 & -rs\phi_1 & (1-c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & -s\phi_1\sqrt{1-r^2} \\ (1-c\phi_1)r\sqrt{1-r^2} & s\phi_1\sqrt{1-r^2} & c\phi_1+(1-c\phi_1)r^2 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2}c\phi_2 \\ -\sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\
&= (1-2r^2)\left(\left(1-r^2\right)c\phi_2 - r^2\right) + 2r\sqrt{1-r^2}\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)c\phi_1 - 2r\left(1-r^2\right)s\phi_2 s\phi_1.
\end{aligned}$$

The RHS of Eq. (1.24) can be expanded as

$$\begin{aligned}
\mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R &= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\
&= \left(1-r^2\right)\alpha_{11} - r\sqrt{1-r^2}\alpha_{31} - r\sqrt{1-r^2}\alpha_{13} + r^2\alpha_{33}.
\end{aligned}$$

Substituting the obtained expressions in Eq. (1.24) and using the expression for $\left(1-r^2\right)c\phi_2 - r^2$ from Eq. (1.23),

$$\begin{aligned}
&2r\sqrt{1-r^2}\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)c\phi_1 - 2r\left(1-r^2\right)s\phi_2 s\phi_1 \\
&= \left(1-r^2\right)\alpha_{11} - r\sqrt{1-r^2}\alpha_{31} - r\sqrt{1-r^2}\alpha_{13} + r^2\alpha_{33} \\
&\quad - (1-2r^2)\left(\left(1-r^2\right)\alpha_{11} + r\sqrt{1-r^2}\alpha_{31} - r\sqrt{1-r^2}\alpha_{13} - r^2\alpha_{33}\right) \\
&= 2r^2\left(1-r^2\right)\alpha_{11} - 2r(1-r^2)\sqrt{1-r^2}\alpha_{31} - 2r^3\sqrt{1-r^2}\alpha_{13} + 2r^2(1-r^2)\alpha_{33}. \\
\implies &\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)c\phi_1 - \sqrt{1-r^2}s\phi_2 s\phi_1 \\
&= r\sqrt{1-r^2}\alpha_{11} - \left(1-r^2\right)\alpha_{31} - r^2\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}.
\end{aligned} \tag{1.25}$$

It should be noted that if $\phi_2 = \pi$, the coefficient of $s\phi_1$ and $c\phi_1$ reduces to 0. This special case will be treated separately.

When $\phi_2 \neq \pi$, dividing both sides of Eq. (1.25) by $\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}$ and defining

$$c\beta := \frac{r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}}, \quad s\beta := \frac{\sqrt{1-r^2}s\phi_2}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}},$$

Eq. (1.25) can be rewritten as

$$c(\phi_1+\beta)=\frac{r\sqrt{1-r^2}\alpha_{11}-\left(1-r^2\right)\alpha_{31}-r^2\alpha_{13}+r\sqrt{1-r^2}\alpha_{33}}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2+r\sqrt{1-r^2}\right)^2+(1-r^2)s^2\phi_2}}. \tag{1.26}$$

$$\therefore \phi_1=\left(\cos^{-1}\left(\frac{r\sqrt{1-r^2}\alpha_{11}-\left(1-r^2\right)\alpha_{31}-r^2\alpha_{13}+r\sqrt{1-r^2}\alpha_{33}}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2+r\sqrt{1-r^2}\right)^2+(1-r^2)s^2\phi_2}}\right)-\beta\right)\%2\pi, \tag{1.27}$$

where $\beta=\operatorname{atan2}\left(s\phi_2, r\left(c\phi_2+1\right)\right)$. From the above equation, for each solution of ϕ_2, at most two solutions can be obtained for ϕ_1.

Consider pre-multiplying Eq. (1.22) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_L$, which yields

$$\mathbf{u}_L^T R_L(r,\phi_1)R_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_L=\mathbf{u}_L^T R_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_L=\mathbf{u}_L^T\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\mathbf{u}_L. \tag{1.28}$$

The LHS of the above equation can be expanded as

$$\begin{aligned}
&\mathbf{u}_L^T R_G(\phi_2)R_R(r,\phi_3)\mathbf{u}_L\\
&=\begin{pmatrix}\sqrt{1-r^2}&0&r\end{pmatrix}\begin{pmatrix}c\phi_2&-s\phi_2&0\\s\phi_2&c\phi_2&0\\0&0&1\end{pmatrix}R_R(r,\phi_3)\mathbf{u}_L\\
&=\begin{pmatrix}\sqrt{1-r^2}c\phi_2&-\sqrt{1-r^2}s\phi_2&r\end{pmatrix}\begin{pmatrix}1-(1-c\phi_3)r^2&-rs\phi_3&-(1-c\phi_3)r\sqrt{1-r^2}\\rs\phi_3&c\phi_3&s\phi_3\sqrt{1-r^2}\\-(1-c\phi_3)r\sqrt{1-r^2}&-s\phi_3\sqrt{1-r^2}&c\phi_3+(1-c\phi_3)r^2\end{pmatrix}\begin{pmatrix}\sqrt{1-r^2}\\0\\r\end{pmatrix}\\
&=(1-2r^2)\left(\left(1-r^2\right)c\phi_2-r^2\right)+2r\sqrt{1-r^2}\left(r\sqrt{1-r^2}c\phi_2+r\sqrt{1-r^2}\right)c\phi_3-2r\left(1-r^2\right)s\phi_2 s\phi_3.
\end{aligned}$$

The RHS of Eq. (1.28) can be expanded as

$$\begin{aligned}
\mathbf{u}_L^T\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\mathbf{u}_L&=\begin{pmatrix}\sqrt{1-r^2}&0&r\end{pmatrix}\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\begin{pmatrix}\sqrt{1-r^2}\\0\\r\end{pmatrix}\\
&=\left(1-r^2\right)\alpha_{11}+r\sqrt{1-r^2}\alpha_{31}+r\sqrt{1-r^2}\alpha_{13}+r^2\alpha_{33}.
\end{aligned}$$

Substituting the obtained expressions in Eq. (1.28) and using the expression for $\left(1-r^2\right)c\phi_2-r^2$ from Eq. (1.23),

$$\begin{aligned}
&2r\sqrt{1-r^2}\left(r\sqrt{1-r^2}c\phi_2+r\sqrt{1-r^2}\right)c\phi_3-2r\left(1-r^2\right)s\phi_2 s\phi_3\\
&=\left(1-r^2\right)\alpha_{11}+r\sqrt{1-r^2}\alpha_{31}+r\sqrt{1-r^2}\alpha_{13}+r^2\alpha_{33}\\
&\quad-(1-2r^2)\left(\left(1-r^2\right)\alpha_{11}+r\sqrt{1-r^2}\alpha_{31}-r\sqrt{1-r^2}\alpha_{13}-r^2\alpha_{33}\right)\\
&=2r^2\left(1-r^2\right)\alpha_{11}+2r^3\sqrt{1-r^2}\alpha_{31}+2r(1-r^2)\sqrt{1-r^2}\alpha_{13}+2r^2(1-r^2)\alpha_{33}.\\
\implies&\left(r\sqrt{1-r^2}c\phi_2+r\sqrt{1-r^2}\right)c\phi_3-\sqrt{1-r^2}s\phi_2 s\phi_3\\
&=r\sqrt{1-r^2}\alpha_{11}+r^2\alpha_{31}+\left(1-r^2\right)\alpha_{13}+r\sqrt{1-r^2}\alpha_{33}.
\end{aligned} \tag{1.29}$$

Similar to the derivation of the closed-form expression for ϕ_1 for the LGR path, the above equation can be used to solve for ϕ_3 for any case apart from $\phi_2 = \pi$. In this case, the coefficient of both $c\phi_3$ and $s\phi_3$ is zero. This case is considered separately. For any other case, dividing both sides of Eq. (1.29) by $\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}$, Eq. (1.29) can be rewritten as

$$c(\phi_3 + \beta) = \frac{r\sqrt{1-r^2}\alpha_{11} + r^2\alpha_{31} + \left(1-r^2\right)\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}}. \tag{1.30}$$

$$\therefore \phi_3 = \left(\cos^{-1}\left(\frac{r\sqrt{1-r^2}\alpha_{11} + r^2\alpha_{31} + \left(1-r^2\right)\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}}\right) - \beta\right) \%2\pi, \tag{1.31}$$

where $\beta = \text{atan2}\left(s\phi_2, r\left(c\phi_2 + 1\right)\right)$. From the above equation, for each solution of ϕ_2, at most two solutions can be obtained for ϕ_3.

1.3.1 Case with $\phi_2 = \pi$

In this case, it was observed that ϕ_1 and ϕ_3 cannot be solved by pre- and post-multiplying by vectors. Hence, this case is addressed by solving the matrix equation given in Eq. (1.22) directly, which is simplified as

$$R_L(r, \phi_1) R_G(\pi) R_R(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{1.32}$$

Expanding the LHS,

$$\begin{aligned}
&R_L(r, \phi_1) R_G(\pi) R_R(r, \phi_3) \\
&= R_L(r, \phi_1) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1-(1-c\phi_3)r^2 & -rs\phi_3 & -(1-c\phi_3)r\sqrt{1-r^2} \\ rs\phi_3 & c\phi_3 & s\phi_3\sqrt{1-r^2} \\ -(1-c\phi_3)r\sqrt{1-r^2} & -s\phi_3\sqrt{1-r^2} & c\phi_3 + (1-c\phi_3)r^2 \end{pmatrix} \\
&= \begin{pmatrix} 1-(1-c\phi_1)r^2 & -rs\phi_1 & (1-c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & -s\phi_1\sqrt{1-r^2} \\ (1-c\phi_1)r\sqrt{1-r^2} & s\phi_1\sqrt{1-r^2} & c\phi_1 + (1-c\phi_1)r^2 \end{pmatrix} \begin{pmatrix} -1+(1-c\phi_3)r^2 & rs\phi_3 & (1-c\phi_3)r\sqrt{1-r^2} \\ -rs\phi_3 & -c\phi_3 & -s\phi_3\sqrt{1-r^2} \\ -(1-c\phi_3)r\sqrt{1-r^2} & -s\phi_3\sqrt{1-r^2} & c\phi_3 + (1-c\phi_3)r^2 \end{pmatrix} \\
&= \begin{pmatrix} -1+r^2-r^2c(\phi_1+\phi_3) & rs(\phi_1+\phi_3) & r\sqrt{1-r^2}(1-c(\phi_1+\phi_3)) \\ -rs(\phi_1+\phi_3) & -c(\phi_1+\phi_3) & -\sqrt{1-r^2}s(\phi_1+\phi_3) \\ -r\sqrt{1-r^2}(1-c(\phi_1+\phi_3)) & -\sqrt{1-r^2}s(\phi_1+\phi_3) & r^2+(1-r^2)c(\phi_1+\phi_3) \end{pmatrix}.
\end{aligned}$$

Therefore,

$$\tan\left(\phi_1 + \phi_3\right) = \frac{\alpha_{12}}{-r\alpha_{22}}. \tag{1.33}$$

Noting that infinitely many solutions exist, and the left and right turns have the same cost, ϕ_3 can be set to zero without loss of generality, and ϕ_1 can be solved from the above equation.

1.4 RGL path

An RGL path can be constructed using the derived LGR path. To this end, assuming without loss of generality that the initial configuration corresponds to the identity matrix, we reflect the final configuration about the XY plane, construct the LGR path, and use the obtained parameters for the RGL path by reflecting it about the XY plane. To this end, the final configuration is modified to be

$$R_{f,mod} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & -\alpha_{13} \\ \alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{1.34}$$

An LGR path is constructed to the modified final configuration. The obtained values for ϕ_1, ϕ_2, and ϕ_3 are the parameters of the RGL path to attain the initially provided final configuration (before modification).

1.5 LRL path

Consider an LRL path, wherein the angle of the first L segment is ϕ_1, the angle of the middle R is equal to ϕ_2, and the angle of the final L segment is ϕ_3. The equation to be solved is given by

$$R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{1.35}$$

Pre-multiplying Eq. (1.35) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_L$,

$$\mathbf{u}_L^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_L = \mathbf{u}_L^T R_R(r, \phi_2) \mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \tag{1.36}$$

Both sides of the above equation can be expanded to obtain

$$\left(1 - 2r^2\right)^2 + 4r^2 \left(1 - r^2\right) \cos \phi_2 = \left(1 - r^2\right) \alpha_{11} + r\sqrt{1 - r^2} \left(\alpha_{13} + \alpha_{31}\right) + r^2 \alpha_{33}.$$

The above equation can be rearranged to obtain

$$\cos \phi_2 = \frac{\left(1 - r^2\right) \alpha_{11} + r\sqrt{1 - r^2} \left(\alpha_{13} + \alpha_{31}\right) + r^2 \alpha_{33} - \left(1 - 2r^2\right)^2}{4r^2 \left(1 - r^2\right)}.$$

At most one solution for $\phi_2 \in (\pi, 2\pi)$ can be obtained from the above equation.

Now, it is desired to obtain the solution for ϕ_1. Pre-multiplying Eq. (1.35) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_L$, the equation obtained is given by

$$\mathbf{u}_R^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_L = \mathbf{u}_R^T R_L(r, \phi_1) R_R(r, \phi_2) \mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \tag{1.37}$$

Expanding both sides of the above equation, the equation obtained is given by

$$\begin{aligned} &4r^2 \left(1 - r^2\right) \sin(\phi_1) \sin(\phi_2) + 4 \left(2r^2 - 1\right) \left(1 - r^2\right) r^2 \left(1 - \cos\left(\phi_2\right)\right) \cos(\phi_1) \\ &+ 8r^6 - 12r^4 + 6r^2 - 1 - 4 \left(2r^6 - 3r^4 + r^2\right) \cos(\phi_2) = \left(r^2 - 1\right) \alpha_{11} + r\sqrt{1 - r^2} \left(\alpha_{31} - \alpha_{13}\right) + r^2 \alpha_{33}. \end{aligned}$$

The above equation can be rewritten as

$$\begin{aligned} &\left(2r^2 - 1\right) \left(1 - \cos \phi_2\right) \cos \phi_1 + \sin \phi_2 \sin \phi_1 \\ &= \frac{\left(r^2 - 1\right) \alpha_{11} + r\sqrt{1 - r^2} \left(\alpha_{31} - \alpha_{13}\right) + r^2 \alpha_{33} - \left(8r^6 - 12r^4 + 6r^2 - 1 - 4 \left(2r^6 - 3r^4 + r^2\right) \cos(\phi_2)\right)}{4r^2 \left(1 - r^2\right)}. \end{aligned}$$

Noting that $\sin\phi_2 \neq 0$, the coefficient of $\sin\phi_1 \neq 0$. Therefore, at most two solutions can be obtained for ϕ_1 from the above equation for each ϕ_2.

Now, it is desired to obtain the solutions for ϕ_3. To this end, pre-multiplying Eq. (1.35) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_R$, the equation obtained is given by

$$\mathbf{u}_L^T R_L(r,\phi_1) R_R(r,\phi_2) R_L(r,\phi_3) \mathbf{u}_R = \mathbf{u}_L^T R_R(r,\phi_2) R_L(r,\phi_3) \mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \tag{1.38}$$

Both sides of the above equation can be expanded to obtain

$$\begin{aligned} &4r^2\left(1-r^2\right)\sin(\phi_3)\sin(\phi_2) + 4\left(2r^2-1\right)\left(1-r^2\right) r^2\left(1-\cos\left(\phi_2\right)\right)\cos(\phi_3) \\ &+ 8r^6 - 12r^4 + 6r^2 - 1 - 4\left(2r^6 - 3r^4 + r^2\right)\cos(\phi_2) = \left(r^2-1\right)\alpha_{11} + r\sqrt{1-r^2}\left(\alpha_{13}-\alpha_{31}\right) + r^2\alpha_{33}. \end{aligned}$$

The above equation can be rewritten as

$$\begin{aligned} &\left(2r^2-1\right)\left(1-\cos\phi_2\right)\cos\phi_3 + \sin\phi_2\sin\phi_3 \\ &= \frac{\left(r^2-1\right)\alpha_{11} + r\sqrt{1-r^2}\left(\alpha_{13}-\alpha_{31}\right) + r^2\alpha_{33} - \left(8r^6 - 12r^4 + 6r^2 - 1 - 4\left(2r^6 - 3r^4 + r^2\right)\cos(\phi_2)\right)}{4r^2\left(1-r^2\right)}. \end{aligned}$$

Noting that $\sin\phi_2 \neq 0$, the coefficient of $\sin\phi_3 \neq 0$. Therefore, at most two solutions can be obtained for ϕ_1 from the above equation for each ϕ_2.

1.6 RLR path

An *RLR* path is constructed by swapping the initial and final configurations, and swapping the tangent vector and tangent-normal directions. That is, the initial configuration and final configuration are modified to be

$$R_i = \begin{pmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ \alpha_{21} & -\alpha_{22} & -\alpha_{23} \\ \alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix},$$
$$R_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An *LRL* path is then constructed. The obtained parameters ϕ_1, ϕ_2, and ϕ_3 are the angles of the final R segment, the middle L segment, and first R segment of the *RLR* path.

1.7 $LR_\pi L$ path

Consider an $LR_\pi L$ path, wherein the angle of the first L segment is ϕ_1, the angle of the middle R segment is π, and the angle of the final L segment is ϕ_3. The equation to be solved is given by

$$R_L(r,\phi_1) R_R(r,\pi) R_L(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{1.39}$$

Pre-multiplying Eq. (1.39) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_L$, the equation obtained is given by

$$\begin{aligned} &\mathbf{u}_R^T R_L(r,\phi_1) R_R(r,\pi) R_L(r,\phi_3) \mathbf{u}_L \\ &= \mathbf{u}_R^T R_L(r,\phi_1) R_R(r,\pi) \mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_1 is given by

$$-\left(2r^2-1\right)\left(-8r^4+8\left(r^2-1\right)r^2\cos\phi_1+8r^2-1\right)=\alpha_{11}\left(r^2-1\right)+r\left(\left(\alpha_{31}-\alpha_{13}\right)\sqrt{1-r^2}+\alpha_{33}r\right).$$

In the case of $r \neq \frac{1}{\sqrt{2}}$, which is a special case that will be handled separately, the above equation can be rewritten as

$$\cos\phi_1=\frac{1}{8\left(r^2-1\right)r^2}\left(1-8r^2+8r^4+\frac{\alpha_{11}\left(r^2-1\right)+r\left(\left(\alpha_{31}-\alpha_{13}\right)\sqrt{1-r^2}+\alpha_{33}r\right)}{1-2r^2}\right). \tag{1.40}$$

The above equation yields at most two solutions for ϕ_1.

The equation for ϕ_3 can be obtained by pre-multiplying Eq. (1.39) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_R$ as

$$\begin{aligned}&\mathbf{u}_L^T R_L(r,\phi_1)R_R(r,\pi)R_L(r,\phi_3)\mathbf{u}_R\\&=\mathbf{u}_L^T R_R(r,\pi)R_L(r,\phi_3)\mathbf{u}_L=\mathbf{u}_L^T\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}\mathbf{u}_R.\end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_3 is given by

$$-\left(2r^2-1\right)\left(-8r^4+8\left(r^2-1\right)r^2\cos\phi_3+8r^2-1\right)=\alpha_{11}\left(r^2-1\right)+r\left(\left(\alpha_{13}-\alpha_{31}\right)\sqrt{1-r^2}+\alpha_{33}r\right).$$

In the case of $r \neq \frac{1}{\sqrt{2}}$, which is a special case that will be handled separately, the above equation can be rewritten as

$$\cos\phi_3=\frac{1}{8\left(r^2-1\right)r^2}\left(1-8r^2+8r^4+\frac{\alpha_{11}\left(r^2-1\right)+r\left(\left(\alpha_{13}-\alpha_{31}\right)\sqrt{1-r^2}+\alpha_{33}r\right)}{1-2r^2}\right). \tag{1.41}$$

The above equation yields at most two solutions for ϕ_3.

1.7.1 Special case of $r = \frac{1}{\sqrt{2}}$

In this case, the net rotation matrix corresponding to the $LR_\pi L$ path reduces to

$$R_L\left(\frac{1}{\sqrt{2}},\phi_1\right)R_R\left(\frac{1}{\sqrt{2}},\pi\right)R_L\left(\frac{1}{\sqrt{2}},\phi_3\right)=\begin{pmatrix}\frac{1}{2}(\cos\left(\phi_1-\phi_3\right)-1) & \frac{\sin(\phi_1-\phi_3)}{\sqrt{2}} & -\cos^2\left(\frac{\phi_1-\phi_3}{2}\right)\\ \frac{\sin(\phi_1-\phi_3)}{\sqrt{2}} & -\cos(\phi_1-\phi_3) & -\frac{\sin(\phi_1-\phi_3)}{\sqrt{2}}\\ -\cos^2\left(\frac{\phi_1-\phi_3}{2}\right) & -\frac{\sin(\phi_1-\phi_3)}{\sqrt{2}} & \frac{1}{2}(\cos\left(\phi_1-\phi_3\right)-1).\end{pmatrix}$$

Noting that this matrix must equal the RHS matrix, $\phi_1-\phi_3$ can be obtained as

$$\phi_1-\phi_3=\operatorname{atan2}\left(\sqrt{2}\alpha_{21},-\alpha_{22}\right),$$

which yields an angle in $(-\pi,\pi]$. If the obtained angle is negative, then ϕ_1 is set to zero and ϕ_3 is computed. If the obtained angle is positive, ϕ_3 is set to zero, and ϕ_1 is computed.

1.8 $RL_\pi R$ path

The construction of an $RL_\pi R$ path using the construction of the $LR_\pi L$ follows similar to the construction of an RLR path using the LRL path construction. In particular, the final configuration is reflected about the XY plane, the $LR_\pi L$ path is constructed, the parameters of which correspond to the $RL_\pi R$ path prior to reflection.

1.9 LRLR path

Consider an $LRLR$ path, wherein the angle of the first L segment is ϕ_1, the angle of the middle R and L segments are equal to ϕ_2, and the angle of the final R segment is ϕ_3. The equation to be solved is given by

$$R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_3) = \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}. \tag{1.42}$$

Pre-multiplying Eq. (1.42) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_R$,

$$\begin{aligned}&\mathbf{u}_L^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_3)\mathbf{u}_R\\ &= \mathbf{u}_L^T R_R(r,\phi_2)R_L(r,\phi_2)\mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R.\end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_2 is given by

$$\begin{aligned}&-1+10r^2-16r^4+8r^6-8\left(r^2-3r^4+2r^6\right)\cos\phi_2+8r^4\left(r^2-1\right)\cos^2\phi_2\\ &=\alpha_{11}\left(r^2-1\right)+r\left(\alpha_{13}\sqrt{1-r^2}-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right).\end{aligned}$$

Noting that the above equation is a quadratic equation in terms of $\cos\phi_2$, at most two real solutions can be obtained from $\cos\phi_2$. Furthermore, noting that for every value of $\cos\phi_2$, two solutions exist, wherein one solution lies in $[0,\pi]$, and another solution lies in $[\pi, 2\pi)$, at most one solution for ϕ_2 is selected since $\phi_2 \in (\pi, 2\pi)$ for optimality. Therefore, from the obtained equation, at most two solutions for $\phi_2 \in (\pi, 2\pi)$ can be obtained.

Consider pre-multiplying Eq. (1.42) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_R$. The equation obtained is given by

$$\begin{aligned}&\mathbf{u}_R^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_3)\mathbf{u}_R\\ &= \mathbf{u}_R^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)\mathbf{u}_R = \mathbf{u}_R^T \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R.\end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_1 is given by

$$\begin{aligned}&\left(2r^2-1\right)\left(12r^6-20r^4+10r^2+4\left(r^2-1\right)r^4\cos\left(2\phi_2\right)-8\left(2r^6-3r^4+r^2\right)\cos\left(\phi_2\right)-1\right)\\ &-4r^2\left(r^2-1\right)\left(2r^2\cos(\phi_2)-2r^2+1\right)\left(\cos(\phi_1)\left(\left(2r^2-1\right)\cos(\phi_2)-2r^2+2\right)-\sin(\phi_1)\sin(\phi_2)\right)\\ &=\alpha_{11}\left(1-r^2\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right).\end{aligned}$$

The above equation can be rewritten as

$$A\cos\phi_1+B\sin\phi_1+C=\alpha_{11}\left(1-r^2\right)+r\left(\alpha_{13}\left(-\sqrt{1-r^2}\right)-\alpha_{31}\sqrt{1-r^2}+\alpha_{33}r\right), \tag{1.43}$$

where

$$\begin{aligned}A&=4r^2\left(1-r^2\right)\left(2r^2\cos(\phi_2)-2r^2+1\right)\left(\left(2r^2-1\right)\cos(\phi_2)-2r^2+2\right),\\ B&=4r^2\left(1-r^2\right)\left(2r^2\cos(\phi_2)-2r^2+1\right)\left(-\sin\phi_2\right),\\ C&=\left(2r^2-1\right)\left(12r^6-20r^4+10r^2+4\left(r^2-1\right)r^4\cos\left(2\phi_2\right)-8\left(2r^6-3r^4+r^2\right)\cos\left(\phi_2\right)-1\right).\end{aligned}$$

In the case that $2r^2\cos\phi_2 - 2r^2 + 1 = 0$, ϕ_1 cannot be uniquely solved from the above equation. This special case will be considered separately.

Suppose $2r^2\cos\phi_2 - 2r^2 + 1 \neq 0$. It is desired to determine if $A^2 + B^2 \neq 0$ to obtain a finite number of solutions from the above equation. It should be noted that if $A^2 + B^2 = 0$, then $A = 0, B = 0$. However, for $B = 0$, it is necessary that $\sin\phi_2 = 0$. However, for an $LRLR$ path to be optimal, $\phi_2 \in (\pi, 2\pi)$. Hence, since $B \neq 0$, $A^2 + B^2 \neq 0$. Therefore, at most two solutions can be obtained for ϕ_1 from Eq. (1.43).

Now, it is desired to obtain the expression for ϕ_3. Pre-multiplying Eq. (1.42) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_L$, the equation obtained is given by

$$\begin{aligned}&\mathbf{u}_L^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_3)\mathbf{u}_L\\ &= \mathbf{u}_L^T R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_3)\mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_L.\end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_3 is given by

$$A\cos\phi_3 + B\sin\phi_3 + C = \left(1 - r^2\right)\alpha_{11} + r\sqrt{1-r^2}\,(\alpha_{13} + \alpha_{31}) + r^2\alpha_{33},$$

where the expressions for A, B, and C are the same as obtained for the equation for ϕ_1. At most two solutions can be obtained when $2r^2\cos\phi_2 - 2r^2 + 1 \neq 0$ for each ϕ_2.

1.9.1 Special case when $\cos\phi_2 = 1 - \frac{1}{2r^2}$

In this case, ϕ_1 and ϕ_3 cannot be uniquely solved. Noting that using $\cos\phi_2 = 1 - \frac{1}{2r^2}$, a solution for $\phi_2 \in (\pi, 2\pi)$ exists for $r \in \left(\frac{1}{2}, 1\right)$ it is desired to determine whether the path exists or not. Since $\phi_2 \in (\pi, 2\pi)$, the corresponding value of $\sin\phi_2$ can be obtained as $\sin\phi_2 = -\sqrt{1-\cos^2\phi_2} = \frac{-\sqrt{4r^2-1}}{2r^2}$. Using these values of $\sin\phi_2$ and $\cos\phi_2$, the net rotation matrix in Eq. (1.42) reduces to

$$R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_3) = \begin{pmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\\ -\gamma_{12} & \gamma_{22} & \gamma_{23}\\ -\gamma_{13} & \gamma_{23} & \gamma_{33}\end{pmatrix},$$

where

$$\begin{aligned}
\gamma_{11} &= \frac{1}{2}\left(\sqrt{4r^2-1}\sin(\phi_1+\phi_3) + \left(1-2r^2\right)\cos(\phi_1+\phi_3) + 2r^2 - 2\right),\\
\gamma_{12} &= \frac{\left(2r^2-1\right)\sin(\phi_1+\phi_3) + \sqrt{4r^2-1}\cos(\phi_1+\phi_3)}{2r},\\
\gamma_{13} &= \frac{\sqrt{1-r^2}\left(\sqrt{4r^2-1}\sin(\phi_1+\phi_3) + \left(1-2r^2\right)\cos(\phi_1+\phi_3) + 2r^2\right)}{2r},\\
\gamma_{22} &= \frac{\sqrt{4r^2-1}\sin(\phi_1+\phi_3) + \left(1-2r^2\right)\cos(\phi_1+\phi_3)}{2r^2},\\
\gamma_{23} &= -\frac{\sqrt{1-r^2}\left(\left(2r^2-1\right)\sin(\phi_1+\phi_3) + \sqrt{4r^2-1}\cos(\phi_1+\phi_3)\right)}{2r^2},\\
\gamma_{33} &= \frac{2r^4 + \left(r^2-1\right)\sqrt{4r^2-1}\sin(\phi_1+\phi_3) + \left(-2r^4+3r^2-1\right)\cos(\phi_1+\phi_3)}{2r^2}.
\end{aligned}$$

Clearly, either no solution or infinitely many solutions can be obtained for ϕ_1 and ϕ_3, since unique solutions cannot be obtained for ϕ_1 and ϕ_3.

If infinitely many solutions are obtained, then, ϕ_3 can be set to zero since the left and right turns have the same radius and cost. Then, using γ_{12} and γ_{22},

$$\begin{pmatrix}\frac{\sqrt{4r^2-1}}{2r} & \frac{2r^2-1}{2r}\\ -\frac{2r^2-1}{2r^2} & \frac{\sqrt{4r^2-1}}{2r^2}\end{pmatrix}\begin{pmatrix}\cos(\phi_1)\\ \sin(\phi_1)\end{pmatrix} = \begin{pmatrix}\alpha_{12}\\ \alpha_{22}\end{pmatrix}.$$

The determinant of the matrix in the left-hand side is r^2, which is non-zero. Hence, the expression for $\cos\phi_1$ and $\sin\phi_1$ can be obtained as

$$\begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1) \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} \frac{\sqrt{4r^2-1}}{2r^2} & -\frac{2r^2-1}{2r} \\ \frac{2r^2-1}{2r^2} & \frac{\sqrt{4r^2-1}}{2r} \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix}.$$

It should be noted that the obtained expressions for $\cos\phi_1$ and $\sin\phi_1$ must be verified to satisfy $\cos^2\phi_1 + \sin^2\phi_1 = 1$.

1.10 RLRL path

The construction of an $RLRL$ path can be performed using the construction of the $LRLR$ path by first reflecting the final configuration about the XY plane (assuming the initial configuration is the identity matrix without loss of generality). The $LRLR$ path can then be constructed to attain the modified final configuration. The obtained solutions for ϕ_1, ϕ_2, and ϕ_3 will correspond to the arc angles of the first R segment, intermediary L and R segments, and final L segment in the $RLRL$ path connecting to the initially provided final configuration. To this end, the final configuration is modified as given in Eq. (1.34), using which the $LRLR$ path is constructed.

1.11 LRLRL path

Consider an $LRLRL$ path, wherein the angle of the first L segment is ϕ_1, the angle of the middle R, L, and R segments are equal to ϕ_2, and the angle of the final L segment is ϕ_3. The equation to be solved is given by

$$R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)R_L(r,\phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \tag{1.44}$$

Pre-multiplying Eq. (1.44) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_L$,

$$\begin{aligned} &\mathbf{u}_L^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)R_L(r,\phi_3)\mathbf{u}_L \\ &= \mathbf{u}_L^T R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)\mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_2 is given by

$$\begin{aligned} &16r^8 - 48r^6 + 48r^4 - 16r^2 + 1 - 16r^2\left(1-r^2\right)^2\left(3r^2-1\right)\cos\phi_2 + 16r^4\left(2-5r^2+3r^4\right)\cos^2\phi_2 \\ &+ 16r^6\left(1-r^2\right)\cos^3\phi_2 = \alpha_{11}\left(1-r^2\right) + r\left(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r\right). \end{aligned}$$

At most three real solutions can be obtained for $\cos\phi_2$. The solutions that lie in $[-1,1]$ are selected; for each solution of $\cos\phi_2$, at most one solution lies in $(\pi, 2\pi)$. It is now desired to obtain the corresponding solutions for ϕ_1 and ϕ_3.

Pre-multiplying Eq. (1.44) by $\mathbf{u}_R^T$ and post-multiplying by $\mathbf{u}_L$,

$$\begin{aligned} &\mathbf{u}_R^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)R_L(r,\phi_3)\mathbf{u}_L \\ &= \mathbf{u}_R^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)\mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_1 is given by

$$A\cos\phi_1 + B\sin\phi_1 + C = \left(r^2-1\right)\alpha_{11} + r\sqrt{1-r^2}\left(\alpha_{31}-\alpha_{13}\right) + r^2\alpha_{33},$$

where

$$
\begin{aligned}
A &= \gamma \tan\left(\frac{\phi_2}{2}\right)\left(-4r^4 + 2\left(2r^2-1\right)r^2\cos(\phi_2) + 6r^2 - 1\right),\\
B &= \gamma\left(1 - 2r^2 + 2r^2\cos\phi_2\right),\\
C &= \left(1-2r^2\right)\Big[4r^8\cos(3\phi_2) - 40r^8 - 4r^6\cos(3\phi_2) + 88r^6 - 64r^4 + 16r^2 - 8\left(3r^4 - 5r^2 + 2\right)r^4\cos(2\phi_2)\\
&\qquad + 4\left(15r^6 - 31r^4 + 20r^2 - 4\right)r^2\cos(\phi_2) - 1\Big],\\
\gamma &= 8r^2\left(1-r^2\right)\left(1 - r^2 + r^2\cos\phi_2\right)\sin\phi_2.
\end{aligned}
$$

It is claimed that A^2+B^2 can be zero only when $\gamma = 0$. When $\gamma = 0$, the corresponding value of ϕ_2 must correspond to $\cos\phi_2 = 1 - \frac{1}{r^2}$. This is a case that will be considered separately.

Consider $\gamma \neq 0$. Then, $B = 0$ only when $\cos\phi_2 = 1 - \frac{1}{2r^2}$, which has a solution in $(\pi, 2\pi)$ when $r \in \left(\frac{1}{2}, 1\right]$. Since $\phi_2 \in (\pi, 2\pi)$, the corresponding expression for $\sin\phi_2 = -\sqrt{1-\cos^2\phi_2} = \frac{-\sqrt{4r^2-1}}{2r^2}$. Since $\tan\left(\frac{\phi_2}{2}\right) \neq 0$, A can be zero only if $-4r^4 + 2\left(2r^2-1\right)r^2\cos\phi_2 + 6r^2 - 1$ reduces to zero. Substituting the expression for $\cos\phi_2$ in the considered expression, we get

$$
-4r^4 + 2\left(2r^2-1\right)r^2\cos\phi_2 + 6r^2 - 1 = 2r^2,
$$

which is non-zero. Hence, $A^2+B^2 \neq 0$ if $\gamma \neq 0$. Therefore, at most two solutions can be obtained for ϕ_1 for each solution of ϕ_2.

Remark: For the implementation, the expressions for A and B can be alternately written as

$$
\begin{aligned}
A &= 16r^2\left(r^2-1\right)\sin^2\left(\frac{\phi_2}{2}\right)\left(-6r^6 + 11r^4 - 7r^2 + \left(r^4 - 2r^6\right)\cos(2\phi_2) + \left(8r^4 - 12r^2 + 3\right)r^2\cos(\phi_2) + 1\right),\\
B &= 8r^2\left(1-r^2\right)\sin(\phi_2)\left(r^4\cos(2\phi_2) + 3r^4 - 3r^2 + \left(3r^2 - 4r^4\right)\cos(\phi_2) + 1\right).
\end{aligned}
$$

It is now desired to obtain the expression for ϕ_3. Pre-multiplying Eq. (1.44) by $\mathbf{u}_L^T$ and post-multiplying by $\mathbf{u}_R$,

$$
\begin{aligned}
&\mathbf{u}_L^T R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)R_L(r,\phi_3)\mathbf{u}_R\\
&= \mathbf{u}_L^T R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)R_L(r,\phi_3)\mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13}\\ \alpha_{21} & \alpha_{22} & \alpha_{23}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}\end{pmatrix}\mathbf{u}_R.
\end{aligned}
$$

Expanding both sides of the above equation, the equation obtained in terms of ϕ_3 is given by

$$
A\cos\phi_1 + B\sin\phi_1 + C = \left(r^2-1\right)\alpha_{11} + r\sqrt{1-r^2}\,(\alpha_{13} - \alpha_{31}) + r^2\alpha_{33},
$$

where the expressions for A, B, and C are the same as that obtained for the equation for ϕ_1. Hence, at most two solutions can be obtained for ϕ_3 for each ϕ_2 if $\gamma \neq 0$.

1.11.1 Case with $\cos\phi_2 = 1 - \frac{1}{r^2}$

In this case, the net rotation matrix corresponding to the $LRLRL$ path, given by the LHS of Eq. (1.44), reduces to

$$
R_L(r,\phi_1)R_R(r,\phi_2)R_L(r,\phi_2)R_R(r,\phi_2)R_L(r,\phi_3) = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13}\\ -\delta_{12} & \delta_{22} & \delta_{23}\\ \delta_{13} & -\delta_{23} & \delta_{33}\end{pmatrix},
$$

where $\sin\phi_2 = -\sqrt{1-\cos^2\phi_2} = -\frac{\sqrt{2r^2-1}}{r^2}$ was used. Here,

$$\begin{aligned}
\delta_{11} &= -2\sin\left(\frac{\phi_1+\phi_3}{2}\right)\left(\left(r^2-1\right)\sin\left(\frac{\phi_1+\phi_3}{2}\right)+\sqrt{2r^2-1}\cos\left(\frac{\phi_1+\phi_3}{2}\right)\right),\\
\delta_{12} &= -\frac{\left(r^2-1\right)\sin(\phi_1+\phi_3)+\sqrt{2r^2-1}\cos(\phi_1+\phi_3)}{r},\\
\delta_{13} &= \frac{\sqrt{1-r^2}\left(\sqrt{2r^2-1}\sin(\phi_1+\phi_3)-\left(r^2-1\right)\cos(\phi_1+\phi_3)+r^2\right)}{r},\\
\delta_{22} &= \frac{\left(r^2-1\right)\cos(\phi_1+\phi_3)-\sqrt{2r^2-1}\sin(\phi_1+\phi_3)}{r^2},\\
\delta_{23} &= -\frac{\sqrt{1-r^2}\left(\left(r^2-1\right)\sin(\phi_1+\phi_3)+\sqrt{2r^2-1}\cos(\phi_1+\phi_3)\right)}{r^2},\\
\delta_{33} &= \frac{r^4+\left(r^2-1\right)\sqrt{2r^2-1}\sin(\phi_1+\phi_3)-\left(r^2-1\right)^2\cos(\phi_1+\phi_3)}{r^2}.
\end{aligned}$$

Noting that the net rotation matrix must equal the desired final configuration, given by

$$R_f = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix},$$

the expression for $\phi_1+\phi_3$ can be obtained by equation δ_{12} to δ_{22} to α_{12} and α_{22}, respectively, to obtain

$$\begin{pmatrix} \sqrt{2r^2-1} & \left(r^2-1\right) \\ \left(r^2-1\right) & -\sqrt{2r^2-1} \end{pmatrix}\begin{pmatrix} \cos\left(\phi_1+\phi_3\right) \\ \sin\left(\phi_1+\phi_3\right) \end{pmatrix} = \begin{pmatrix} -r\alpha_{12} \\ r^2\alpha_{22} \end{pmatrix}.$$

The determinant of the matrix on the left hand side is $-r^4$, which is non-zero. Hence, the matrix on the left hand side can be inverted to obtain

$$\begin{aligned}
\begin{pmatrix} \cos\left(\phi_1+\phi_3\right) \\ \sin\left(\phi_1+\phi_3\right) \end{pmatrix} &= \frac{-1}{r^4}\begin{pmatrix} -\sqrt{2r^2-1} & -\left(r^2-1\right) \\ -\left(r^2-1\right) & \sqrt{2r^2-1} \end{pmatrix}\begin{pmatrix} -r\alpha_{12} \\ r^2\alpha_{22} \end{pmatrix}\\
&= \frac{-1}{r^4}\begin{pmatrix} \sqrt{2r^2-1}r\alpha_{12} - r^2\left(r^2-1\right)\alpha_{22} \\ r\left(r^2-1\right)\alpha_{12}+r^2\sqrt{2r^2-1}\alpha_{22} \end{pmatrix}.
\end{aligned}$$

Setting $\phi_3 = 0$ without loss of generality, a solution for ϕ_1 can be obtained from the above equation. It should be noted that the obtained expressions for $\cos\phi_1$ and $\sin\phi_1$ must be verified to satisfy $\cos^2\phi_1+\sin^2\phi_1 = 1$.

1.12 RLRLR path

The *RLRLR* path can be constructed by utilizing the construction of an *LRLRL* path. To this end, the final configuration is reflected about the XY plane, and the *LRLRL* path is constructed to the reflected final configuration. The modified final configuration is given in Eq. (1.34). The obtained solutions for ϕ_1, ϕ_2, and ϕ_3 correspond to the parameters for the *RLRLR* path.