

# Generation of Paths for Motion Planning for a Dubins Vehicle on Sphere\*

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## Abstract

In this article, the candidate optimal paths for a Dubins vehicle on a sphere are analytically derived. In particular, the arc angles for segments in  $CGC$ ,  $CCC$ ,  $CCCC$ , and  $CCCCC$  paths, which have previously been shown to be optimal depending on the turning radius  $r$  of the vehicle in [1], are analytically derived. The derived expressions are used for the implementation provided in <https://github.com/DeepakPrakashKumar/Motion-planning-on-sphere>.

## 1 Derivation of Closed-Form Expressions for Paths on a Sphere

Consider the differential equations corresponding to the Sabban frame. As  $u_g$  is piecewise constant, the differential equations for each segment can be represented as

$$\begin{pmatrix} \mathbf{X}'(s) & \mathbf{T}'(s) & \mathbf{N}'(s) \end{pmatrix} = \begin{pmatrix} \mathbf{X}(s) & \mathbf{T}(s) & \mathbf{N}(s) \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -u_g \\ 0 & u_g & 0 \end{pmatrix}}_{\Omega}. \quad (1.1)$$

The solution to the above differential equations can be obtained as

$$\begin{pmatrix} \mathbf{X}(s) & \mathbf{T}(s) & \mathbf{N}(s) \end{pmatrix} = \begin{pmatrix} \mathbf{X}(s_i) & \mathbf{T}(s_i) & \mathbf{N}(s_i) \end{pmatrix} \left( e^{\Omega^T \Delta s} \right)^T, \quad (1.2)$$

where  $\Delta s = s - s_i$ . The expression for  $e^{\Omega \Delta s}$  can be obtained using the Euler-Rodriguez formula for the exponential of a skew-symmetric matrix. Moreover, since  $s = \phi_G$  for a great circle turn, and  $s = r\phi_L$  and  $s = r\phi_R$  for the left and right tight turns, respectively, the solution to the differential equations can be written as

$$R_{after} = R_{before} R_{seg}, \quad (1.3)$$

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where  $R_{after}$  and  $R_{before}$  denote the configurations after and before the segment, respectively. Moreover,  $R_{seg}$  represents the rotation matrix corresponding to a chosen segment, and is given by

$$R_{seg} = \begin{cases} R_G(\phi) = \begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, & u_g = 0 \\ R_L(r, \phi) = \begin{pmatrix} 1 - (1 - c\phi)r^2 & -rs\phi & (1 - c\phi)r\sqrt{1 - r^2} \\ rs\phi & c\phi & -s\phi\sqrt{1 - r^2} \\ (1 - c\phi)r\sqrt{1 - r^2} & s\phi\sqrt{1 - r^2} & c\phi + (1 - c\phi)r^2 \end{pmatrix}, & u_g = U_{max} \\ R_R(r, \phi) = \begin{pmatrix} 1 - (1 - c\phi)r^2 & -rs\phi & -(1 - c\phi)r\sqrt{1 - r^2} \\ rs\phi & c\phi & s\phi\sqrt{1 - r^2} \\ -(1 - c\phi)r\sqrt{1 - r^2} & -s\phi\sqrt{1 - r^2} & c\phi + (1 - c\phi)r^2 \end{pmatrix}, & u_g = -U_{max} \end{cases} \quad (1.4)$$

For deriving the closed-form expressions for the paths, the following two vectors will be utilized:

$$\mathbf{u}_L := \begin{pmatrix} \sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix}, \quad \mathbf{u}_R := \begin{pmatrix} -\sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix}.$$

It should be noted that the vector  $\mathbf{u}_L$  corresponds to the axial vector of the rotation matrix  $R_L(r, \phi)$ . Hence,  $R_L(r, \phi)\mathbf{u}_L = \mathbf{u}_L$ . Similarly, the vector  $\mathbf{u}_R$  corresponds to the axial vector of the rotation matrix  $R_R(r, \phi)$ . Hence,  $R_R(r, \phi)\mathbf{u}_R = \mathbf{u}_R$ .

## 1.1 LGL path

The equation to be solved is given by

$$R_L(r, \phi_1)R_G(\phi_2)R_L(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.5)$$

Pre-multiplying Eq. (1.5) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_L$ ,

$$\begin{aligned} \mathbf{u}_L^T R_L(r, \phi_1)R_G(\phi_2)R_L(r, \phi_3)\mathbf{u}_L &= \mathbf{u}_L^T R_G(\phi_2)\mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \\ \therefore \begin{pmatrix} \sqrt{1 - r^2} & 0 & r \end{pmatrix} \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix} &= \begin{pmatrix} \sqrt{1 - r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix}. \end{aligned}$$

Simplifying the above equation,

$$\begin{aligned} (1 - r^2)c\phi_2 + r^2 &= (1 - r^2)\alpha_{11} + r\sqrt{1 - r^2}(\alpha_{13} + \alpha_{31}) + r^2\alpha_{33}. \\ \implies c\phi_2 &= \frac{\alpha_{11} + r\sqrt{1 - r^2}(\alpha_{13} + \alpha_{31}) + r^2(\alpha_{33} - \alpha_{11} - 1)}{1 - r^2}, \end{aligned} \quad (1.6)$$

which yields at most two solutions for  $\phi_2 \in [0, 2\pi)$  if the absolute value of the RHS is less than or equal to 1.

Consider pre-multiplying Eq. (1.5) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_L$ , which yields

$$\mathbf{u}_R^T R_L(r, \phi_1)R_G(\phi_2)R_L(r, \phi_3)\mathbf{u}_L = \mathbf{u}_R^T R_L(r, \phi_1)R_G(\phi_2)\mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \quad (1.7)$$

The LHS of the above equation can be expanded as

$$\begin{aligned}
& \mathbf{u}_R^T R_L(r, \phi_1) R_G(\phi_2) \mathbf{u}_L \\
&= \mathbf{u}_R^T R_L(r, \phi_1) \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} 1 - (1-c\phi_1)r^2 & -rs\phi_1 & (1-c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & -s\phi_1\sqrt{1-r^2} \\ (1-c\phi_1)r\sqrt{1-r^2} & s\phi_1\sqrt{1-r^2} & c\phi_1 + (1-c\phi_1)r^2 \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2}c\phi_2 \\ \sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{1-r^2} + 2(1-c\phi_1)r^2\sqrt{1-r^2} & 2r\sqrt{1-r^2}s\phi_1 & 2(1-c\phi_1)r^3 + r(2c\phi_1-1) \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2}c\phi_2 \\ \sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\
&= -(1-r^2)c\phi_2 + 2(1-c\phi_1)r^2(1-r^2)c\phi_2 + 2r(1-r^2)s\phi_1s\phi_2 + 2(1-c\phi_1)r^4 + 2r^2c\phi_1 - r^2 \\
&= \left( (1-r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) + 2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2s\phi_1.
\end{aligned}$$

The RHS of Eq. (1.7) can be expanded as

$$\begin{aligned}
& \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L \\
&= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{1-r^2}\alpha_{11} + r\alpha_{31} & -\sqrt{1-r^2}\alpha_{12} + r\alpha_{32} & -\sqrt{1-r^2}\alpha_{13} + r\alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\
&= -(1-r^2)\alpha_{11} + r\sqrt{1-r^2}(\alpha_{31} - \alpha_{13}) + r^2\alpha_{33} = -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31} - \alpha_{13}).
\end{aligned}$$

Substituting the obtained expressions in Eq. (1.7) and using the expression for  $(1-r^2)c\phi_2 + r^2$  from Eq. (1.6),

$$\begin{aligned}
2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2s\phi_1 &= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31} - \alpha_{13}) \\
&\quad - \left( (1-r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) \\
&= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31} - \alpha_{13}) \\
&\quad - \left( \alpha_{11} + r\sqrt{1-r^2}(\alpha_{13} + \alpha_{31}) + r^2(\alpha_{33} - \alpha_{11}) \right) (-1 + 2r^2) \\
&= 2(\alpha_{33} - \alpha_{11})r^2(1-r^2) - 2\alpha_{13}r^3\sqrt{1-r^2} + 2\alpha_{31}r(1-r^2)^{\frac{3}{2}}. \\
\implies r(1-c\phi_2)c\phi_1 + s\phi_2s\phi_1 &= (\alpha_{33} - \alpha_{11})r - \alpha_{13}r^2(1-r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1-r^2}.
\end{aligned}$$

It should be noted that if  $\phi_2 \neq 0$ , then the above equation can be used to solve for  $\phi_1$ . Diving both sides by  $\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}$  and defining  $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ ,  $s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ , the above equation can

be rewritten as

$$c(\phi_1 - \beta) = \frac{(\alpha_{33} - \alpha_{11})r - \alpha_{13}r^2(1 - r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1 - r^2}}{\sqrt{r^2(1 - c\phi_2)^2 + s^2\phi_2}}. \quad (1.8)$$

$$\begin{aligned} \therefore \phi_1 &= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r - \alpha_{13}r^2(1 - r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1 - r^2}}{\sqrt{r^2(1 - c\phi_2)^2 + s^2\phi_2}} \right) + \beta \right) \% 2\pi \\ &= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r - \alpha_{13}r^2(1 - r^2)^{-\frac{1}{2}} + \alpha_{31}\sqrt{1 - r^2}}{\sqrt{r^2(1 - c\phi_2)^2 + s^2\phi_2}} \right) + \text{atan2}(s\phi_2, r(1 - c\phi_2)) \right) \% 2\pi. \end{aligned} \quad (1.9)$$

**Important remark:** From the above equation, for each solution of  $\phi_2$ , at most two solutions can be obtained for  $\phi_1$ , since  $\cos^{-1}$  can be replaced with  $2\pi - \cos^{-1}$  to obtain another solution. The same argument applies for the rest of the report whenever we encounter a solution using  $\cos^{-1}$ . Furthermore, for all angles, a modulus function is utilized to ensure that it is between 0 to  $2\pi$ .

Finally, consider pre-multiplying Eq. (1.5) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_R$ , which yields

$$\mathbf{u}_L^T R_L(r, \phi_1) R_G(\phi_2) R_L(r, \phi_3) \mathbf{u}_R = \mathbf{u}_L^T R_G(\phi_2) R_L(r, \phi_3) \mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \quad (1.10)$$

The LHS of the above equation can be expanded as

$$\begin{aligned} &\mathbf{u}_L^T R_G(\phi_2) R_L(r, \phi_3) \mathbf{u}_R \\ &= \begin{pmatrix} \sqrt{1 - r^2} & 0 & r \end{pmatrix} \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_L(r, \phi_3) \mathbf{u}_R \\ &= \begin{pmatrix} \sqrt{1 - r^2}c\phi_2 & -\sqrt{1 - r^2}s\phi_2 & r \end{pmatrix} \begin{pmatrix} 1 - (1 - c\phi_3)r^2 & -rs\phi_3 & (1 - c\phi_3)r\sqrt{1 - r^2} \\ rs\phi_3 & c\phi_3 & -s\phi_3\sqrt{1 - r^2} \\ (1 - c\phi_3)r\sqrt{1 - r^2} & s\phi_3\sqrt{1 - r^2} & c\phi_3 + (1 - c\phi_3)r^2 \end{pmatrix} \begin{pmatrix} -\sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1 - r^2}c\phi_2 & -\sqrt{1 - r^2}s\phi_2 & r \end{pmatrix} \begin{pmatrix} -\sqrt{1 - r^2} + 2(1 - c\phi_3)r^2\sqrt{1 - r^2} \\ -2r\sqrt{1 - r^2}s\phi_3 \\ 2(1 - c\phi_3)r^3 + r(2c\phi_3 - 1) \end{pmatrix} \\ &= -(1 - r^2)c\phi_2 + 2(1 - c\phi_3)r^2(1 - r^2)c\phi_2 + 2r(1 - r^2)s\phi_3s\phi_2 + 2(1 - c\phi_3)r^4 + 2r^2c\phi_3 - r^2 \\ &= \left( (1 - r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) + 2r^2(1 - r^2)(1 - c\phi_2)c\phi_3 + 2r(1 - r^2)s\phi_2s\phi_3. \end{aligned}$$

The RHS of Eq. (1.10) can be expanded as

$$\begin{aligned} &\mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R \\ &= \begin{pmatrix} \sqrt{1 - r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1 - r^2}\alpha_{11} + r\alpha_{31} & \sqrt{1 - r^2}\alpha_{12} + r\alpha_{32} & \sqrt{1 - r^2}\alpha_{13} + r\alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix} \\ &= -(1 - r^2)\alpha_{11} + r\sqrt{1 - r^2}(\alpha_{13} - \alpha_{31}) + r^2\alpha_{33} = -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1 - r^2}(\alpha_{13} - \alpha_{31}). \end{aligned}$$

Substituting the obtained expressions in Eq. (1.10) and using the expression for  $(1-r^2)c\phi_2+r^2$  from Eq. (1.6),

$$\begin{aligned}
2r^2(1-r^2)(1-c\phi_2)c\phi_3 + 2r(1-r^2)s\phi_2s\phi_3 &= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13} - \alpha_{31}) \\
&\quad - \left( (1-r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) \\
&= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13} - \alpha_{31}) \\
&\quad - \left( \alpha_{11} + r\sqrt{1-r^2}(\alpha_{13} + \alpha_{31}) + r^2(\alpha_{33} - \alpha_{11}) \right) (-1 + 2r^2) \\
&= 2(\alpha_{33} - \alpha_{11})r^2(1-r^2) + 2\alpha_{13}r(1-r^2)^{\frac{3}{2}} - 2\alpha_{31}r^3\sqrt{1-r^2}. \\
\implies r(1-c\phi_2)c\phi_3 + s\phi_2s\phi_3 &= (\alpha_{33} - \alpha_{11})r + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}.
\end{aligned}$$

It should be noted that if  $\phi_2 \neq 0$ , then the above equation can be used to solve for  $\phi_3$ . Diving both sides by  $\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}$  and defining  $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ ,  $s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ , the above equation can be rewritten as

$$c(\phi_3 - \beta) = \frac{(\alpha_{33} - \alpha_{11})r + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}. \quad (1.11)$$

$$\begin{aligned}
\therefore \phi_3 &= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}} \right) + \beta \right) \% 2\pi \\
&= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r + \alpha_{13}\sqrt{1-r^2} - \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}} \right) + \tan^{-1} \left( \frac{s\phi_2}{r(1-c\phi_2)} \right) \right) \% 2\pi.
\end{aligned} \quad (1.12)$$

From the above equation, for each solution of  $\phi_2$ , at most two solutions can be obtained for  $\phi_3$ .

**Remark:** In the case that  $\phi_2 = 0$ , the *LGL* path reduces to an *L* segment. Hence, in this case,  $\phi_3$  can be set to zero without loss of generality, and  $\phi_1$  can be solved using the first and second entries in the second row in the net rotation matrix (refer to the expression for  $R_L$ ). That is,

$$\phi_1 = (\text{atan2}(\alpha_{21}, r\alpha_{22})) \% 2\pi. \quad (1.13)$$

## 1.2 RGR path

The equation to be solved is given by

$$R_R(r, \phi_1)R_G(\phi_2)R_R(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.14)$$

Pre-multiplying Eq. (1.14) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_R$ ,

$$\begin{aligned}
\mathbf{u}_R^T R_R(r, \phi_1)R_G(\phi_2)R_R(r, \phi_3)\mathbf{u}_R &= \mathbf{u}_R^T R_G(\phi_2)\mathbf{u}_R = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \\
\therefore \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} &= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix}.
\end{aligned}$$

Simplifying the above equation,

$$\begin{aligned}
(1-r^2)c\phi_2 + r^2 &= (1-r^2)\alpha_{11} - r\sqrt{1-r^2}(\alpha_{13} + \alpha_{31}) + r^2\alpha_{33}. \\
\implies c\phi_2 &= \frac{\alpha_{11} - r\sqrt{1-r^2}(\alpha_{13} + \alpha_{31}) + r^2(\alpha_{33} - \alpha_{11} - 1)}{1-r^2},
\end{aligned} \quad (1.15)$$

which yields at most two solutions for  $\phi_2 \in [0, 2\pi)$  if the absolute value of the RHS is less than or equal to 1. Consider pre-multiplying Eq. (1.14) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_R$ , which yields

$$\mathbf{u}_L^T R_R(r, \phi_1) R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_R = \mathbf{u}_L^T R_R(r, \phi_1) R_G(\phi_2) \mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \quad (1.16)$$

The LHS of the above equation can be expanded as

$$\begin{aligned} & \mathbf{u}_L^T R_R(r, \phi_1) R_G(\phi_2) \mathbf{u}_R \\ &= \mathbf{u}_L^T R_R(r, \phi_1) \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} 1 - (1-c\phi_1)r^2 & -rs\phi_1 & -(1-c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & s\phi_1\sqrt{1-r^2} \\ -(1-c\phi_1)r\sqrt{1-r^2} & -s\phi_1\sqrt{1-r^2} & c\phi_1 + (1-c\phi_1)r^2 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2}c\phi_2 \\ -\sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1-r^2} - 2(1-c\phi_1)r^2\sqrt{1-r^2} & -2r\sqrt{1-r^2}s\phi_1 & 2(1-c\phi_1)r^3 + r(2c\phi_1 - 1) \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2}c\phi_2 \\ -\sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\ &= -(1-r^2)c\phi_2 + 2(1-c\phi_1)r^2(1-r^2)c\phi_2 + 2r(1-r^2)s\phi_1s\phi_2 + 2(1-c\phi_1)r^4 + 2r^2c\phi_1 - r^2 \\ &= \left( (1-r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) + 2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2s\phi_1. \end{aligned}$$

The RHS of Eq. (1.16) can be expanded as

$$\begin{aligned} & \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R \\ &= \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{1-r^2}\alpha_{11} + r\alpha_{31} & \sqrt{1-r^2}\alpha_{12} + r\alpha_{32} & \sqrt{1-r^2}\alpha_{13} + r\alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= -(1-r^2)\alpha_{11} + r\sqrt{1-r^2}(\alpha_{13} - \alpha_{31}) + r^2\alpha_{33} = -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13} - \alpha_{31}). \end{aligned}$$

Substituting the obtained expressions in Eq. (1.16) and using the expression for  $(1-r^2)c\phi_2 + r^2$  from Eq. (1.15),

$$\begin{aligned} 2r^2(1-r^2)(1-c\phi_2)c\phi_1 + 2r(1-r^2)s\phi_2s\phi_1 &= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13} - \alpha_{31}) \\ &\quad - \left( (1-r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) \\ &= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{13} - \alpha_{31}) \\ &\quad - \left( \alpha_{11} + r^2(\alpha_{33} - \alpha_{11}) - r\sqrt{1-r^2}(\alpha_{13} + \alpha_{31}) \right) (-1 + 2r^2) \\ &= 2(\alpha_{33} - \alpha_{11})r^2(1-r^2) + 2\alpha_{13}r^3\sqrt{1-r^2} - 2\alpha_{31}r(1-r^2)^{\frac{3}{2}}. \\ \implies r(1-c\phi_2)c\phi_1 + s\phi_2s\phi_1 &= (\alpha_{33} - \alpha_{11})r + \alpha_{13}r^2(1-r^2)^{-\frac{1}{2}} - \alpha_{31}(1-r^2)^{\frac{1}{2}}. \end{aligned}$$

It should be noted that if  $\phi_2 \neq 0$ , then the above equation can be used to solve for  $\phi_1$ . Diving both sides by  $\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}$  and defining  $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ ,  $s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ , the above equation can

be rewritten as

$$c(\phi_1 - \beta) = \frac{(\alpha_{33} - \alpha_{11})r + \alpha_{13}r^2(1 - r^2)^{-\frac{1}{2}} - \alpha_{31}(1 - r^2)^{\frac{1}{2}}}{\sqrt{r^2(1 - c\phi_2)^2 + s^2\phi_2}}. \quad (1.17)$$

$$\begin{aligned} \therefore \phi_1 &= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r + \alpha_{13}r^2(1 - r^2)^{-\frac{1}{2}} - \alpha_{31}(1 - r^2)^{\frac{1}{2}}}{\sqrt{r^2(1 - c\phi_2)^2 + s^2\phi_2}} \right) + \beta \right) \% 2\pi \\ &= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r + \alpha_{13}r^2(1 - r^2)^{-\frac{1}{2}} - \alpha_{31}(1 - r^2)^{\frac{1}{2}}}{\sqrt{r^2(1 - c\phi_2)^2 + s^2\phi_2}} \right) + \text{atan2}(s\phi_2, r(1 - c\phi_2)) \right) \% 2\pi. \end{aligned} \quad (1.18)$$

From the above equation, for each solution of  $\phi_2$ , at most two solutions can be obtained for  $\phi_1$ .

Finally, consider pre-multiplying Eq. (1.14) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_L$ , which yields

$$\mathbf{u}_R^T R_R(r, \phi_1) R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_L = \mathbf{u}_R^T R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \quad (1.19)$$

The LHS of the above equation can be expanded as

$$\begin{aligned} &\mathbf{u}_R^T R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_L \\ &= \begin{pmatrix} -\sqrt{1 - r^2} & 0 & r \end{pmatrix} \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_R(r, \phi_3) \mathbf{u}_L \\ &= \begin{pmatrix} -\sqrt{1 - r^2}c\phi_2 & \sqrt{1 - r^2}s\phi_2 & r \end{pmatrix} \begin{pmatrix} 1 - (1 - c\phi_3)r^2 & -rs\phi_3 & -(1 - c\phi_3)r\sqrt{1 - r^2} \\ rs\phi_3 & c\phi_3 & s\phi_3\sqrt{1 - r^2} \\ -(1 - c\phi_3)r\sqrt{1 - r^2} & -s\phi_3\sqrt{1 - r^2} & c\phi_3 + (1 - c\phi_3)r^2 \end{pmatrix} \begin{pmatrix} \sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{1 - r^2}c\phi_2 & \sqrt{1 - r^2}s\phi_2 & r \end{pmatrix} \begin{pmatrix} \sqrt{1 - r^2} - 2(1 - c\phi_3)r^2\sqrt{1 - r^2} \\ 2r\sqrt{1 - r^2}s\phi_3 \\ 2(1 - c\phi_3)r^3 + r(2c\phi_3 - 1) \end{pmatrix} \\ &= -(1 - r^2)c\phi_2 + 2(1 - c\phi_3)r^2(1 - r^2)c\phi_2 + 2r(1 - r^2)s\phi_3s\phi_2 + 2(1 - c\phi_3)r^4 + 2r^2c\phi_3 - r^2 \\ &= \left( (1 - r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) + 2r^2(1 - r^2)(1 - c\phi_2)c\phi_3 + 2r(1 - r^2)s\phi_2s\phi_3. \end{aligned}$$

The RHS of Eq. (1.19) can be expanded as

$$\begin{aligned} &\mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L \\ &= \begin{pmatrix} -\sqrt{1 - r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{1 - r^2}\alpha_{11} + r\alpha_{31} & -\sqrt{1 - r^2}\alpha_{12} + r\alpha_{32} & -\sqrt{1 - r^2}\alpha_{13} + r\alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1 - r^2} \\ 0 \\ r \end{pmatrix} \\ &= -(1 - r^2)\alpha_{11} + r\sqrt{1 - r^2}(\alpha_{31} - \alpha_{13}) + r^2\alpha_{33} = -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1 - r^2}(\alpha_{31} - \alpha_{13}). \end{aligned}$$

Substituting the obtained expressions in Eq. (1.19) and using the expression for  $(1 - r^2)c\phi_2 + r^2$  from

Eq. (1.15),

$$\begin{aligned}
2r^2(1-r^2)(1-c\phi_2)c\phi_3 + 2r(1-r^2)s\phi_2s\phi_3 &= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31} - \alpha_{13}) \\
&\quad - \left( (1-r^2)c\phi_2 + r^2 \right) (-1 + 2r^2) \\
&= -\alpha_{11} + r^2(\alpha_{11} + \alpha_{33}) + r\sqrt{1-r^2}(\alpha_{31} - \alpha_{13}) \\
&\quad - \left( \alpha_{11} + r^2(\alpha_{33} - \alpha_{11}) - r\sqrt{1-r^2}(\alpha_{13} + \alpha_{31}) \right) (-1 + 2r^2) \\
&= 2(\alpha_{33} - \alpha_{11})r^2(1-r^2) - 2\alpha_{13}r(1-r^2)^{\frac{3}{2}} + 2\alpha_{31}r^3\sqrt{1-r^2}. \\
\implies r(1-c\phi_2)c\phi_3 + s\phi_2s\phi_3 &= (\alpha_{33} - \alpha_{11})r - \alpha_{13}(1-r^2)^{\frac{1}{2}} + \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}.
\end{aligned}$$

It should be noted that if  $\phi_2 \neq 0$ , then the above equation can be used to solve for  $\phi_3$ . Diving both sides by  $\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}$  and defining  $c\beta := \frac{r(1-c\phi_2)}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ ,  $s\beta := \frac{s\phi_2}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}$ , the above equation can be rewritten as

$$c(\phi_3 - \beta) = \frac{(\alpha_{33} - \alpha_{11})r - \alpha_{13}(1-r^2)^{\frac{1}{2}} + \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}}. \quad (1.20)$$

$$\begin{aligned}
\therefore \phi_3 &= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r - \alpha_{13}(1-r^2)^{\frac{1}{2}} + \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}} \right) + \beta \right) \% 2\pi \\
&= \left( \cos^{-1} \left( \frac{(\alpha_{33} - \alpha_{11})r - \alpha_{13}(1-r^2)^{\frac{1}{2}} + \alpha_{31}r^2(1-r^2)^{-\frac{1}{2}}}{\sqrt{r^2(1-c\phi_2)^2 + s^2\phi_2}} \right) + \text{atan2}(s\phi_2, r(1-c\phi_2)) \right) \% 2\pi.
\end{aligned} \quad (1.21)$$

From the above equation, for each solution of  $\phi_2$ , at most two solutions can be obtained for  $\phi_3$ .

**Remark:** The case of  $\phi_2 = 0$  can be handled similar to the *LGL* path, wherein  $\phi_3$  can be set to zero and the expression for  $\phi_1$  obtained is the same as that given in Eq. (1.13).

### 1.3 LGR path

The equation to be solved is given by

$$R_L(r, \phi_1)R_G(\phi_2)R_R(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.22)$$

Pre-multiplying Eq. (1.22) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_R$ ,

$$\begin{aligned}
\mathbf{u}_L^T R_L(r, \phi_1) R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_R &= \mathbf{u}_L^T R_G(\phi_2) \mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \\
\therefore \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} &= \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix}.
\end{aligned}$$

Simplifying the above equation,

$$\begin{aligned}
-\left(1-r^2\right)c\phi_2+r^2 &= -\left(1-r^2\right)\alpha_{11}+r\sqrt{1-r^2}\left(\alpha_{13}-\alpha_{31}\right)+r^2\alpha_{33}. \\
\implies c\phi_2 &= \frac{\left(1-r^2\right)\alpha_{11}+r\sqrt{1-r^2}\left(\alpha_{31}-\alpha_{13}\right)+r^2\left(1-\alpha_{33}\right)}{\left(1-r^2\right)},
\end{aligned} \quad (1.23)$$



which yields at most two solutions for  $\phi_2 \in [0, 2\pi)$  if the absolute value of the RHS is less than or equal to 1. Consider pre-multiplying Eq. (1.22) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_R$ , which yields

$$\mathbf{u}_R^T R_L(r, \phi_1) R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_R = \mathbf{u}_R^T R_L(r, \phi_1) R_G(\phi_2) \mathbf{u}_R = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \quad (1.24)$$

The LHS of the above equation can be expanded as

$$\begin{aligned} & \mathbf{u}_R^T R_L(r, \phi_1) R_G(\phi_2) \mathbf{u}_R \\ &= \mathbf{u}_R^T R_L(r, \phi_1) \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} 1 - (1-c\phi_1)r^2 & -rs\phi_1 & (1-c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & -s\phi_1\sqrt{1-r^2} \\ (1-c\phi_1)r\sqrt{1-r^2} & s\phi_1\sqrt{1-r^2} & c\phi_1 + (1-c\phi_1)r^2 \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2}c\phi_2 \\ -\sqrt{1-r^2}s\phi_2 \\ r \end{pmatrix} \\ &= (1-2r^2) \left( (1-r^2)c\phi_2 - r^2 \right) + 2r\sqrt{1-r^2} \left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right) c\phi_1 - 2r(1-r^2)s\phi_2s\phi_1. \end{aligned}$$

The RHS of Eq. (1.24) can be expanded as

$$\begin{aligned} \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R &= \begin{pmatrix} -\sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} -\sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= (1-r^2)\alpha_{11} - r\sqrt{1-r^2}\alpha_{31} - r\sqrt{1-r^2}\alpha_{13} + r^2\alpha_{33}. \end{aligned}$$

Substituting the obtained expressions in Eq. (1.24) and using the expression for  $(1-r^2)c\phi_2 - r^2$  from Eq. (1.23),

$$\begin{aligned} & 2r\sqrt{1-r^2} \left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right) c\phi_1 - 2r(1-r^2)s\phi_2s\phi_1 \\ &= (1-r^2)\alpha_{11} - r\sqrt{1-r^2}\alpha_{31} - r\sqrt{1-r^2}\alpha_{13} + r^2\alpha_{33} \\ &\quad - (1-2r^2) \left( (1-r^2)\alpha_{11} + r\sqrt{1-r^2}\alpha_{31} - r\sqrt{1-r^2}\alpha_{13} - r^2\alpha_{33} \right) \\ &= 2r^2(1-r^2)\alpha_{11} - 2r(1-r^2)\sqrt{1-r^2}\alpha_{31} - 2r^3\sqrt{1-r^2}\alpha_{13} + 2r^2(1-r^2)\alpha_{33}. \\ &\Rightarrow \left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right) c\phi_1 - \sqrt{1-r^2}s\phi_2s\phi_1 \\ &= r\sqrt{1-r^2}\alpha_{11} - (1-r^2)\alpha_{31} - r^2\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}. \end{aligned} \quad (1.25)$$

It should be noted that if  $\phi_2 = \pi$ , the coefficient of  $s\phi_1$  and  $c\phi_1$  reduces to 0. This special case will be treated separately.

When  $\phi_2 \neq \pi$ , dividing both sides of Eq. (1.25) by  $\sqrt{\left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right)^2 + (1-r^2)s^2\phi_2}$  and defining

$$c\beta := \frac{r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}}{\sqrt{\left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right)^2 + (1-r^2)s^2\phi_2}}, \quad s\beta := \frac{\sqrt{1-r^2}s\phi_2}{\sqrt{\left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right)^2 + (1-r^2)s^2\phi_2}},$$

Eq. (1.25) can be rewritten as

$$c(\phi_1 + \beta) = \frac{r\sqrt{1-r^2}\alpha_{11} - (1-r^2)\alpha_{31} - r^2\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}}{\sqrt{(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2})^2 + (1-r^2)s^2\phi_2}}. \quad (1.26)$$

$$\therefore \phi_1 = \left( \cos^{-1} \left( \frac{r\sqrt{1-r^2}\alpha_{11} - (1-r^2)\alpha_{31} - r^2\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}}{\sqrt{(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2})^2 + (1-r^2)s^2\phi_2}} \right) - \beta \right) \% 2\pi, \quad (1.27)$$

where  $\beta = \text{atan2}(s\phi_2, r(c\phi_2 + 1))$ . From the above equation, for each solution of  $\phi_2$ , at most two solutions can be obtained for  $\phi_1$ .

Consider pre-multiplying Eq. (1.22) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_L$ , which yields

$$\mathbf{u}_L^T R_L(r, \phi_1) R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_L = \mathbf{u}_L^T R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \quad (1.28)$$

The LHS of the above equation can be expanded as

$$\begin{aligned} & \mathbf{u}_L^T R_G(\phi_2) R_R(r, \phi_3) \mathbf{u}_L \\ &= \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} c\phi_2 & -s\phi_2 & 0 \\ s\phi_2 & c\phi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} R_R(r, \phi_3) \mathbf{u}_L \\ &= \begin{pmatrix} \sqrt{1-r^2}c\phi_2 & -\sqrt{1-r^2}s\phi_2 & r \end{pmatrix} \begin{pmatrix} 1 - (1-c\phi_3)r^2 & -rs\phi_3 & -(1-c\phi_3)r\sqrt{1-r^2} \\ rs\phi_3 & c\phi_3 & s\phi_3\sqrt{1-r^2} \\ -(1-c\phi_3)r\sqrt{1-r^2} & -s\phi_3\sqrt{1-r^2} & c\phi_3 + (1-c\phi_3)r^2 \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= (1-2r^2) \left( (1-r^2)c\phi_2 - r^2 \right) + 2r\sqrt{1-r^2} \left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right) c\phi_3 - 2r(1-r^2)s\phi_2s\phi_3. \end{aligned}$$

The RHS of Eq. (1.28) can be expanded as

$$\begin{aligned} \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L &= \begin{pmatrix} \sqrt{1-r^2} & 0 & r \end{pmatrix} \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} \sqrt{1-r^2} \\ 0 \\ r \end{pmatrix} \\ &= (1-r^2)\alpha_{11} + r\sqrt{1-r^2}\alpha_{31} + r\sqrt{1-r^2}\alpha_{13} + r^2\alpha_{33}. \end{aligned}$$

Substituting the obtained expressions in Eq. (1.28) and using the expression for  $(1-r^2)c\phi_2 - r^2$  from Eq. (1.23),

$$\begin{aligned} & 2r\sqrt{1-r^2} \left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right) c\phi_3 - 2r(1-r^2)s\phi_2s\phi_3 \\ &= (1-r^2)\alpha_{11} + r\sqrt{1-r^2}\alpha_{31} + r\sqrt{1-r^2}\alpha_{13} + r^2\alpha_{33} \\ &\quad - (1-2r^2) \left( (1-r^2)\alpha_{11} + r\sqrt{1-r^2}\alpha_{31} - r\sqrt{1-r^2}\alpha_{13} - r^2\alpha_{33} \right) \\ &= 2r^2(1-r^2)\alpha_{11} + 2r^3\sqrt{1-r^2}\alpha_{31} + 2r(1-r^2)\sqrt{1-r^2}\alpha_{13} + 2r^2(1-r^2)\alpha_{33}. \\ &\implies \left( r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2} \right) c\phi_3 - \sqrt{1-r^2}s\phi_2s\phi_3 \\ &= r\sqrt{1-r^2}\alpha_{11} + r^2\alpha_{31} + (1-r^2)\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}. \end{aligned} \quad (1.29)$$

Similar to the derivation of the closed-form expression for  $\phi_1$  for the *LGR* path, the above equation can be used to solve for  $\phi_3$  for any case apart from  $\phi_2 = \pi$ . In this case, the coefficient of both  $c\phi_3$  and  $s\phi_3$  is zero. This case is considered separately. For any other case, dividing both sides of Eq. (1.29) by  $\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}$ , Eq. (1.29) can be rewritten as

$$c(\phi_3 + \beta) = \frac{r\sqrt{1-r^2}\alpha_{11} + r^2\alpha_{31} + (1-r^2)\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}}. \quad (1.30)$$

$$\therefore \phi_3 = \left( \cos^{-1} \left( \frac{r\sqrt{1-r^2}\alpha_{11} + r^2\alpha_{31} + (1-r^2)\alpha_{13} + r\sqrt{1-r^2}\alpha_{33}}{\sqrt{\left(r\sqrt{1-r^2}c\phi_2 + r\sqrt{1-r^2}\right)^2 + (1-r^2)s^2\phi_2}} \right) - \beta \right) \% 2\pi, \quad (1.31)$$

where  $\beta = \text{atan2}(s\phi_2, r(c\phi_2 + 1))$ . From the above equation, for each solution of  $\phi_2$ , at most two solutions can be obtained for  $\phi_3$ .

### 1.3.1 Case with $\phi_2 = \pi$

In this case, it was observed that  $\phi_1$  and  $\phi_3$  cannot be solved by pre- and post-multiplying by vectors. Hence, this case is addressed by solving the matrix equation given in Eq. (1.22) directly, which is simplified as

$$R_L(r, \phi_1)R_G(\pi)R_R(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.32)$$

Expanding the LHS,

$$\begin{aligned} & R_L(r, \phi_1)R_G(\pi)R_R(r, \phi_3) \\ &= R_L(r, \phi_1) \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - (1 - c\phi_3)r^2 & -rs\phi_3 & -(1 - c\phi_3)r\sqrt{1-r^2} \\ rs\phi_3 & c\phi_3 & s\phi_3\sqrt{1-r^2} \\ -(1 - c\phi_3)r\sqrt{1-r^2} & -s\phi_3\sqrt{1-r^2} & c\phi_3 + (1 - c\phi_3)r^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 - (1 - c\phi_1)r^2 & -rs\phi_1 & (1 - c\phi_1)r\sqrt{1-r^2} \\ rs\phi_1 & c\phi_1 & -s\phi_1\sqrt{1-r^2} \\ (1 - c\phi_1)r\sqrt{1-r^2} & s\phi_1\sqrt{1-r^2} & c\phi_1 + (1 - c\phi_1)r^2 \end{pmatrix} \begin{pmatrix} -1 + (1 - c\phi_3)r^2 & rs\phi_3 & (1 - c\phi_3)r\sqrt{1-r^2} \\ -rs\phi_3 & -c\phi_3 & -s\phi_3\sqrt{1-r^2} \\ -(1 - c\phi_3)r\sqrt{1-r^2} & -s\phi_3\sqrt{1-r^2} & c\phi_3 + (1 - c\phi_3)r^2 \end{pmatrix} \\ &= \begin{pmatrix} -1 + r^2 - r^2c(\phi_1 + \phi_3) & rs(\phi_1 + \phi_3) & r\sqrt{1-r^2}(1 - c(\phi_1 + \phi_3)) \\ -rs(\phi_1 + \phi_3) & -c(\phi_1 + \phi_3) & -\sqrt{1-r^2}s(\phi_1 + \phi_3) \\ -r\sqrt{1-r^2}(1 - c(\phi_1 + \phi_3)) & -\sqrt{1-r^2}s(\phi_1 + \phi_3) & r^2 + (1 - r^2)c(\phi_1 + \phi_3) \end{pmatrix}. \end{aligned}$$

Therefore,

$$\tan(\phi_1 + \phi_3) = \frac{\alpha_{12}}{-r\alpha_{22}}. \quad (1.33)$$

Noting that infinitely many solutions exist, and the left and right turns have the same cost,  $\phi_3$  can be set to zero without loss of generality, and  $\phi_1$  can be solved from the above equation.

## 1.4 RGL path

An *RGL* path can be constructed using the derived *LGR* path. To this end, assuming without loss of generality that the initial configuration corresponds to the identity matrix, we reflect the final configuration about the *XY* plane, construct the *LGR* path, and use the obtained parameters for the *RGL* path by reflecting it about the *XY* plane. To this end, the final configuration is modified to be

$$R_{f,mod} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & -\alpha_{13} \\ \alpha_{21} & \alpha_{22} & -\alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.34)$$

An *LGR* path is constructed to the modified final configuration. The obtained values for  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  are the parameters of the *RGL* path to attain the initially provided final configuration (before modification).

## 1.5 LRL path

Consider an *LRL* path, wherein the angle of the first *L* segment is  $\phi_1$ , the angle of the middle *R* is equal to  $\phi_2$ , and the angle of the final *L* segment is  $\phi_3$ . The equation to be solved is given by

$$R_L(r, \phi_1)R_R(r, \phi_2)R_L(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.35)$$

Pre-multiplying Eq. (1.35) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_L$ ,

$$\mathbf{u}_L^T R_L(r, \phi_1)R_R(r, \phi_2)R_L(r, \phi_3)\mathbf{u}_L = \mathbf{u}_L^T R_R(r, \phi_2)\mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \quad (1.36)$$

Both sides of the above equation can be expanded to obtain

$$\left(1 - 2r^2\right)^2 + 4r^2 \left(1 - r^2\right) \cos \phi_2 = \left(1 - r^2\right) \alpha_{11} + r\sqrt{1 - r^2} (\alpha_{13} + \alpha_{31}) + r^2 \alpha_{33}.$$

The above equation can be rearranged to obtain

$$\cos \phi_2 = \frac{(1 - r^2) \alpha_{11} + r\sqrt{1 - r^2} (\alpha_{13} + \alpha_{31}) + r^2 \alpha_{33} - (1 - 2r^2)^2}{4r^2 (1 - r^2)}.$$

At most one solution for  $\phi_2 \in (\pi, 2\pi)$  can be obtained from the above equation.

Now, it is desired to obtain the solution for  $\phi_1$ . Pre-multiplying Eq. (1.35) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_L$ , the equation obtained is given by

$$\mathbf{u}_R^T R_L(r, \phi_1)R_R(r, \phi_2)R_L(r, \phi_3)\mathbf{u}_L = \mathbf{u}_R^T R_L(r, \phi_1)R_R(r, \phi_2)\mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \quad (1.37)$$

Expanding both sides of the above equation, the equation obtained is given by

$$4r^2 \left(1 - r^2\right) \sin(\phi_1) \sin(\phi_2) + 4 \left(2r^2 - 1\right) \left(1 - r^2\right) r^2 (1 - \cos(\phi_2)) \cos(\phi_1) \\ + 8r^6 - 12r^4 + 6r^2 - 1 - 4 \left(2r^6 - 3r^4 + r^2\right) \cos(\phi_2) = \left(r^2 - 1\right) \alpha_{11} + r\sqrt{1 - r^2} (\alpha_{31} - \alpha_{13}) + r^2 \alpha_{33}.$$

The above equation can be rewritten as

$$\left(2r^2 - 1\right) (1 - \cos \phi_2) \cos \phi_1 + \sin \phi_2 \sin \phi_1 \\ = \frac{(r^2 - 1) \alpha_{11} + r\sqrt{1 - r^2} (\alpha_{31} - \alpha_{13}) + r^2 \alpha_{33} - \left(8r^6 - 12r^4 + 6r^2 - 1 - 4 (2r^6 - 3r^4 + r^2) \cos(\phi_2)\right)}{4r^2 (1 - r^2)}.$$

Noting that  $\sin \phi_2 \neq 0$ , the coefficient of  $\sin \phi_1 \neq 0$ . Therefore, at most two solutions can be obtained for  $\phi_1$  from the above equation for each  $\phi_2$ .

Now, it is desired to obtain the solutions for  $\phi_3$ . To this end, pre-multiplying Eq. (1.35) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_R$ , the equation obtained is given by

$$\mathbf{u}_L^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_R = \mathbf{u}_L^T R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \quad (1.38)$$

Both sides of the above equation can be expanded to obtain

$$4r^2 (1 - r^2) \sin(\phi_3) \sin(\phi_2) + 4 (2r^2 - 1) (1 - r^2) r^2 (1 - \cos(\phi_2)) \cos(\phi_3) \\ + 8r^6 - 12r^4 + 6r^2 - 1 - 4 (2r^6 - 3r^4 + r^2) \cos(\phi_2) = (r^2 - 1) \alpha_{11} + r\sqrt{1 - r^2} (\alpha_{13} - \alpha_{31}) + r^2 \alpha_{33}.$$

The above equation can be rewritten as

$$\begin{aligned} & (2r^2 - 1) (1 - \cos \phi_2) \cos \phi_3 + \sin \phi_2 \sin \phi_3 \\ &= \frac{(r^2 - 1) \alpha_{11} + r\sqrt{1 - r^2} (\alpha_{13} - \alpha_{31}) + r^2 \alpha_{33} - (8r^6 - 12r^4 + 6r^2 - 1 - 4 (2r^6 - 3r^4 + r^2) \cos(\phi_2))}{4r^2 (1 - r^2)}. \end{aligned}$$

Noting that  $\sin \phi_2 \neq 0$ , the coefficient of  $\sin \phi_3 \neq 0$ . Therefore, at most two solutions can be obtained for  $\phi_1$  from the above equation for each  $\phi_2$ .

## 1.6 RLR path

An *RLR* path is constructed by swapping the initial and final configurations, and swapping the tangent vector and tangent-normal directions. That is, the initial configuration and final configuration are modified to be

$$R_i = \begin{pmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ \alpha_{21} & -\alpha_{22} & -\alpha_{23} \\ \alpha_{31} & -\alpha_{32} & -\alpha_{33} \end{pmatrix}, \\ R_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

An *LRL* path is then constructed. The obtained parameters  $\phi_1, \phi_2$ , and  $\phi_3$  are the angles of the final *R* segment, the middle *L* segment, and first *R* segment of the *RLR* path.

## 1.7 LR $_{\pi}$ L path

Consider an *LR $_{\pi}$ L* path, wherein the angle of the first *L* segment is  $\phi_1$ , the angle of the middle *R* segment is  $\pi$ , and the angle of the final *L* segment is  $\phi_3$ . The equation to be solved is given by

$$R_L(r, \phi_1) R_R(r, \pi) R_L(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.39)$$

Pre-multiplying Eq. (1.39) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_L$ , the equation obtained is given by

$$\begin{aligned} & \mathbf{u}_R^T R_L(r, \phi_1) R_R(r, \pi) R_L(r, \phi_3) \mathbf{u}_L \\ &= \mathbf{u}_R^T R_L(r, \phi_1) R_R(r, \pi) \mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_1$  is given by

$$-(2r^2 - 1) \left( -8r^4 + 8(r^2 - 1)r^2 \cos \phi_1 + 8r^2 - 1 \right) = \alpha_{11}(r^2 - 1) + r \left( (\alpha_{31} - \alpha_{13}) \sqrt{1 - r^2} + \alpha_{33}r \right).$$

In the case of  $r \neq \frac{1}{\sqrt{2}}$ , which is a special case that will be handled separately, the above equation can be rewritten as

$$\cos \phi_1 = \frac{1}{8(r^2 - 1)r^2} \left( 1 - 8r^2 + 8r^4 + \frac{\alpha_{11}(r^2 - 1) + r \left( (\alpha_{31} - \alpha_{13}) \sqrt{1 - r^2} + \alpha_{33}r \right)}{1 - 2r^2} \right). \quad (1.40)$$

The above equation yields at most two solutions for  $\phi_1$ .

The equation for  $\phi_3$  can be obtained by pre-multiplying Eq. (1.39) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_R$  as

$$\begin{aligned} & \mathbf{u}_L^T R_L(r, \phi_1) R_R(r, \pi) R_L(r, \phi_3) \mathbf{u}_R \\ &= \mathbf{u}_L^T R_R(r, \pi) R_L(r, \phi_3) \mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_3$  is given by

$$-(2r^2 - 1) \left( -8r^4 + 8(r^2 - 1)r^2 \cos \phi_3 + 8r^2 - 1 \right) = \alpha_{11}(r^2 - 1) + r \left( (\alpha_{13} - \alpha_{31}) \sqrt{1 - r^2} + \alpha_{33}r \right).$$

In the case of  $r \neq \frac{1}{\sqrt{2}}$ , which is a special case that will be handled separately, the above equation can be rewritten as

$$\cos \phi_3 = \frac{1}{8(r^2 - 1)r^2} \left( 1 - 8r^2 + 8r^4 + \frac{\alpha_{11}(r^2 - 1) + r \left( (\alpha_{13} - \alpha_{31}) \sqrt{1 - r^2} + \alpha_{33}r \right)}{1 - 2r^2} \right). \quad (1.41)$$

The above equation yields at most two solutions for  $\phi_3$ .

### 1.7.1 Special case of $r = \frac{1}{\sqrt{2}}$

In this case, the net rotation matrix corresponding to the  $LR_\pi L$  path reduces to

$$R_L \left( \frac{1}{\sqrt{2}}, \phi_1 \right) R_R \left( \frac{1}{\sqrt{2}}, \pi \right) R_L \left( \frac{1}{\sqrt{2}}, \phi_3 \right) = \begin{pmatrix} \frac{1}{2}(\cos(\phi_1 - \phi_3) - 1) & \frac{\sin(\phi_1 - \phi_3)}{\sqrt{2}} & -\cos^2 \left( \frac{\phi_1 - \phi_3}{2} \right) \\ \frac{\sin(\phi_1 - \phi_3)}{\sqrt{2}} & -\cos(\phi_1 - \phi_3) & -\frac{\sin(\phi_1 - \phi_3)}{\sqrt{2}} \\ -\cos^2 \left( \frac{\phi_1 - \phi_3}{2} \right) & -\frac{\sin(\phi_1 - \phi_3)}{\sqrt{2}} & \frac{1}{2}(\cos(\phi_1 - \phi_3) - 1) \end{pmatrix}$$

Noting that this matrix must equal the RHS matrix,  $\phi_1 - \phi_3$  can be obtained as

$$\phi_1 - \phi_3 = \text{atan2} \left( \sqrt{2}\alpha_{21}, -\alpha_{22} \right),$$

which yields an angle in  $(-\pi, \pi]$ . If the obtained angle is negative, then  $\phi_1$  is set to zero and  $\phi_3$  is computed. If the obtained angle is positive,  $\phi_3$  is set to zero, and  $\phi_1$  is computed.

## 1.8 $RL_\pi R$ path

The construction of an  $RL_\pi R$  path using the construction of the  $LR_\pi L$  follows similar to the construction of an  $RLR$  path using the  $LRL$  path construction. In particular, the final configuration is reflected about the  $XY$  plane, the  $LR_\pi L$  path is constructed, the parameters of which correspond to the  $RL_\pi R$  path prior to reflection.

## 1.9 LRLR path

Consider an  $LRLR$  path, wherein the angle of the first  $L$  segment is  $\phi_1$ , the angle of the middle  $R$  and  $L$  segments are equal to  $\phi_2$ , and the angle of the final  $R$  segment is  $\phi_3$ . The equation to be solved is given by

$$R_L(r, \phi_1)R_R(r, \phi_2)R_L(r, \phi_2)R_R(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.42)$$

Pre-multiplying Eq. (1.42) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_R$ ,

$$\begin{aligned} & \mathbf{u}_L^T R_L(r, \phi_1)R_R(r, \phi_2)R_L(r, \phi_2)R_R(r, \phi_3)\mathbf{u}_R \\ &= \mathbf{u}_L^T R_R(r, \phi_2)R_L(r, \phi_2)\mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_2$  is given by

$$\begin{aligned} & -1 + 10r^2 - 16r^4 + 8r^6 - 8(r^2 - 3r^4 + 2r^6) \cos \phi_2 + 8r^4(r^2 - 1) \cos^2 \phi_2 \\ &= \alpha_{11}(r^2 - 1) + r(\alpha_{13}\sqrt{1-r^2} - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r). \end{aligned}$$

Noting that the above equation is a quadratic equation in terms of  $\cos \phi_2$ , at most two real solutions can be obtained from  $\cos \phi_2$ . Furthermore, noting that for every value of  $\cos \phi_2$ , two solutions exist, wherein one solution lies in  $[0, \pi]$ , and another solution lies in  $[\pi, 2\pi)$ , at most one solution for  $\phi_2$  is selected since  $\phi_2 \in (\pi, 2\pi)$  for optimality. Therefore, from the obtained equation, at most two solutions for  $\phi_2 \in (\pi, 2\pi)$  can be obtained.

Consider pre-multiplying Eq. (1.42) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_R$ . The equation obtained is given by

$$\begin{aligned} & \mathbf{u}_R^T R_L(r, \phi_1)R_R(r, \phi_2)R_L(r, \phi_2)R_R(r, \phi_3)\mathbf{u}_R \\ &= \mathbf{u}_R^T R_L(r, \phi_1)R_R(r, \phi_2)R_L(r, \phi_2)\mathbf{u}_R = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_1$  is given by

$$\begin{aligned} & (2r^2 - 1) \left( 12r^6 - 20r^4 + 10r^2 + 4(r^2 - 1)r^4 \cos(2\phi_2) - 8(2r^6 - 3r^4 + r^2) \cos(\phi_2) - 1 \right) \\ & - 4r^2(r^2 - 1) \left( 2r^2 \cos(\phi_2) - 2r^2 + 1 \right) \left( \cos(\phi_1) \left( (2r^2 - 1) \cos(\phi_2) - 2r^2 + 2 \right) - \sin(\phi_1) \sin(\phi_2) \right) \\ &= \alpha_{11}(1 - r^2) + r \left( \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r \right). \end{aligned}$$

The above equation can be rewritten as

$$A \cos \phi_1 + B \sin \phi_1 + C = \alpha_{11}(1 - r^2) + r \left( \alpha_{13}(-\sqrt{1-r^2}) - \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r \right), \quad (1.43)$$

where

$$\begin{aligned} A &= 4r^2(1 - r^2) \left( 2r^2 \cos(\phi_2) - 2r^2 + 1 \right) \left( (2r^2 - 1) \cos(\phi_2) - 2r^2 + 2 \right), \\ B &= 4r^2(1 - r^2) \left( 2r^2 \cos(\phi_2) - 2r^2 + 1 \right) (-\sin \phi_2), \\ C &= (2r^2 - 1) \left( 12r^6 - 20r^4 + 10r^2 + 4(r^2 - 1)r^4 \cos(2\phi_2) - 8(2r^6 - 3r^4 + r^2) \cos(\phi_2) - 1 \right). \end{aligned}$$

**In the case that  $2r^2 \cos \phi_2 - 2r^2 + 1 = 0$ ,  $\phi_1$  cannot be uniquely solved from the above equation. This special case will be considered separately.**

Suppose  $2r^2 \cos \phi_2 - 2r^2 + 1 \neq 0$ . It is desired to determine if  $A^2 + B^2 \neq 0$  to obtain a finite number of solutions from the above equation. It should be noted that if  $A^2 + B^2 = 0$ , then  $A = 0, B = 0$ . However, for  $B = 0$ , it is necessary that  $\sin \phi_2 = 0$ . However, for an  $LRLR$  path to be optimal,  $\phi_2 \in (\pi, 2\pi)$ . Hence, since  $B \neq 0$ ,  $A^2 + B^2 \neq 0$ . Therefore, at most two solutions can be obtained for  $\phi_1$  from Eq. (1.43).

Now, it is desired to obtain the expression for  $\phi_3$ . Pre-multiplying Eq. (1.42) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_L$ , the equation obtained is given by

$$\begin{aligned} & \mathbf{u}_L^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_3) \mathbf{u}_L \\ &= \mathbf{u}_L^T R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_3) \mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_3$  is given by

$$A \cos \phi_3 + B \sin \phi_3 + C = (1 - r^2) \alpha_{11} + r \sqrt{1 - r^2} (\alpha_{13} + \alpha_{31}) + r^2 \alpha_{33},$$

where the expressions for  $A, B$ , and  $C$  are the same as obtained for the equation for  $\phi_1$ . At most two solutions can be obtained when  $2r^2 \cos \phi_2 - 2r^2 + 1 \neq 0$  for each  $\phi_2$ .

### 1.9.1 Special case when $\cos \phi_2 = 1 - \frac{1}{2r^2}$

In this case,  $\phi_1$  and  $\phi_3$  cannot be uniquely solved. Noting that using  $\cos \phi_2 = 1 - \frac{1}{2r^2}$ , a solution for  $\phi_2 \in (\pi, 2\pi)$  exists for  $r \in (\frac{1}{2}, 1)$  it is desired to determine whether the path exists or not. Since  $\phi_2 \in (\pi, 2\pi)$ , the corresponding value of  $\sin \phi_2$  can be obtained as  $\sin \phi_2 = -\sqrt{1 - \cos^2 \phi_2} = \frac{-\sqrt{4r^2 - 1}}{2r^2}$ . Using these values of  $\sin \phi_2$  and  $\cos \phi_2$ , the net rotation matrix in Eq. (1.42) reduces to

$$R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_3) = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ -\gamma_{12} & \gamma_{22} & \gamma_{23} \\ -\gamma_{13} & \gamma_{23} & \gamma_{33} \end{pmatrix},$$

where

$$\begin{aligned} \gamma_{11} &= \frac{1}{2} \left( \sqrt{4r^2 - 1} \sin(\phi_1 + \phi_3) + (1 - 2r^2) \cos(\phi_1 + \phi_3) + 2r^2 - 2 \right), \\ \gamma_{12} &= \frac{(2r^2 - 1) \sin(\phi_1 + \phi_3) + \sqrt{4r^2 - 1} \cos(\phi_1 + \phi_3)}{2r}, \\ \gamma_{13} &= \frac{\sqrt{1 - r^2} \left( \sqrt{4r^2 - 1} \sin(\phi_1 + \phi_3) + (1 - 2r^2) \cos(\phi_1 + \phi_3) + 2r^2 \right)}{2r}, \\ \gamma_{22} &= \frac{\sqrt{4r^2 - 1} \sin(\phi_1 + \phi_3) + (1 - 2r^2) \cos(\phi_1 + \phi_3)}{2r^2}, \\ \gamma_{23} &= -\frac{\sqrt{1 - r^2} \left( (2r^2 - 1) \sin(\phi_1 + \phi_3) + \sqrt{4r^2 - 1} \cos(\phi_1 + \phi_3) \right)}{2r^2}, \\ \gamma_{33} &= \frac{2r^4 + (r^2 - 1) \sqrt{4r^2 - 1} \sin(\phi_1 + \phi_3) + (-2r^4 + 3r^2 - 1) \cos(\phi_1 + \phi_3)}{2r^2}. \end{aligned}$$

Clearly, either no solution or infinitely many solutions can be obtained for  $\phi_1$  and  $\phi_3$ , since unique solutions cannot be obtained for  $\phi_1$  and  $\phi_3$ .

If infinitely many solutions are obtained, then,  $\phi_3$  can be set to zero since the left and right turns have the same radius and cost. Then, using  $\gamma_{12}$  and  $\gamma_{22}$ ,

$$\begin{pmatrix} \frac{\sqrt{4r^2 - 1}}{2r} & \frac{2r^2 - 1}{2r} \\ -\frac{2r^2 - 1}{2r^2} & \frac{\sqrt{4r^2 - 1}}{2r^2} \end{pmatrix} \begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1) \end{pmatrix} = \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix}.$$



The determinant of the matrix in the left-hand side is  $r^2$ , which is non-zero. Hence, the expression for  $\cos \phi_1$  and  $\sin \phi_1$  can be obtained as

$$\begin{pmatrix} \cos(\phi_1) \\ \sin(\phi_1) \end{pmatrix} = \frac{1}{r^2} \begin{pmatrix} \frac{\sqrt{4r^2-1}}{2r^2} & -\frac{2r^2-1}{2r} \\ \frac{2r^2-1}{2r^2} & \frac{\sqrt{4r^2-1}}{2r} \end{pmatrix} \begin{pmatrix} \alpha_{12} \\ \alpha_{22} \end{pmatrix}.$$

It should be noted that the obtained expressions for  $\cos \phi_1$  and  $\sin \phi_1$  must be verified to satisfy  $\cos^2 \phi_1 + \sin^2 \phi_1 = 1$ .

## 1.10 RLRL path

The construction of an *RLRL* path can be performed using the construction of the *LRLR* path by first reflecting the final configuration about the *XY* plane (assuming the initial configuration is the identity matrix without loss of generality). The *LRLR* path can then be constructed to attain the modified final configuration. The obtained solutions for  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  will correspond to the arc angles of the first *R* segment, intermediary *L* and *R* segments, and final *L* segment in the *RLRL* path connecting to the initially provided final configuration. To this end, the final configuration is modified as given in Eq. (1.34), using which the *LRLR* path is constructed.

## 1.11 LRLRL path

Consider an *LRLRL* path, wherein the angle of the first *L* segment is  $\phi_1$ , the angle of the middle *R*, *L*, and *R* segments are equal to  $\phi_2$ , and the angle of the final *L* segment is  $\phi_3$ . The equation to be solved is given by

$$R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) R_L(r, \phi_3) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}. \quad (1.44)$$

Pre-multiplying Eq. (1.44) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_L$ ,

$$\begin{aligned} & \mathbf{u}_L^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_L \\ &= \mathbf{u}_L^T R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) \mathbf{u}_L = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_2$  is given by

$$\begin{aligned} & 16r^8 - 48r^6 + 48r^4 - 16r^2 + 1 - 16r^2(1-r^2)^2(3r^2-1)\cos\phi_2 + 16r^4(2-5r^2+3r^4)\cos^2\phi_2 \\ & + 16r^6(1-r^2)\cos^3\phi_2 = \alpha_{11}(1-r^2) + r(\alpha_{13}\sqrt{1-r^2} + \alpha_{31}\sqrt{1-r^2} + \alpha_{33}r). \end{aligned}$$

At most three real solutions can be obtained for  $\cos \phi_2$ . The solutions that lie in  $[-1, 1]$  are selected; for each solution of  $\cos \phi_2$ , at most one solution lies in  $(\pi, 2\pi)$ . It is now desired to obtain the corresponding solutions for  $\phi_1$  and  $\phi_3$ .

Pre-multiplying Eq. (1.44) by  $\mathbf{u}_R^T$  and post-multiplying by  $\mathbf{u}_L$ ,

$$\begin{aligned} & \mathbf{u}_R^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_L \\ &= \mathbf{u}_R^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) \mathbf{u}_L = \mathbf{u}_R^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_L. \end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_1$  is given by

$$A \cos \phi_1 + B \sin \phi_1 + C = (r^2 - 1) \alpha_{11} + r \sqrt{1 - r^2} (\alpha_{31} - \alpha_{13}) + r^2 \alpha_{33},$$

where

$$\begin{aligned}
A &= \gamma \tan\left(\frac{\phi_2}{2}\right) \left(-4r^4 + 2(2r^2 - 1)r^2 \cos(\phi_2) + 6r^2 - 1\right), \\
B &= \gamma \left(1 - 2r^2 + 2r^2 \cos \phi_2\right), \\
C &= (1 - 2r^2) \left[4r^8 \cos(3\phi_2) - 40r^8 - 4r^6 \cos(3\phi_2) + 88r^6 - 64r^4 + 16r^2 - 8(3r^4 - 5r^2 + 2)r^4 \cos(2\phi_2) \right. \\
&\quad \left. + 4(15r^6 - 31r^4 + 20r^2 - 4)r^2 \cos(\phi_2) - 1\right], \\
\gamma &= 8r^2 (1 - r^2) (1 - r^2 + r^2 \cos \phi_2) \sin \phi_2.
\end{aligned}$$

It is claimed that  $A^2 + B^2$  can be zero only when  $\gamma = 0$ . When  $\gamma = 0$ , the corresponding value of  $\phi_2$  must correspond to  $\cos \phi_2 = 1 - \frac{1}{r^2}$ . This is a case that will be considered separately.

Consider  $\gamma \neq 0$ . Then,  $B = 0$  only when  $\cos \phi_2 = 1 - \frac{1}{2r^2}$ , which has a solution in  $(\pi, 2\pi)$  when  $r \in (\frac{1}{2}, 1]$ . Since  $\phi_2 \in (\pi, 2\pi)$ , the corresponding expression for  $\sin \phi_2 = -\sqrt{1 - \cos^2 \phi_2} = \frac{-\sqrt{4r^2 - 1}}{2r^2}$ . Since  $\tan\left(\frac{\phi_2}{2}\right) \neq 0$ ,  $A$  can be zero only if  $-4r^4 + 2(2r^2 - 1)r^2 \cos \phi_2 + 6r^2 - 1$  reduces to zero. Substituting the expression for  $\cos \phi_2$  in the considered expression, we get

$$-4r^4 + 2(2r^2 - 1)r^2 \cos \phi_2 + 6r^2 - 1 = 2r^2,$$

which is non-zero. Hence,  $A^2 + B^2 \neq 0$  if  $\gamma \neq 0$ . Therefore, at most two solutions can be obtained for  $\phi_1$  for each solution of  $\phi_2$ .

**Remark:** For the implementation, the expressions for  $A$  and  $B$  can be alternately written as

$$\begin{aligned}
A &= 16r^2 (r^2 - 1) \sin^2\left(\frac{\phi_2}{2}\right) \left(-6r^6 + 11r^4 - 7r^2 + (r^4 - 2r^6) \cos(2\phi_2) + (8r^4 - 12r^2 + 3)r^2 \cos(\phi_2) + 1\right), \\
B &= 8r^2 (1 - r^2) \sin(\phi_2) \left(r^4 \cos(2\phi_2) + 3r^4 - 3r^2 + (3r^2 - 4r^4) \cos(\phi_2) + 1\right).
\end{aligned}$$

It is now desired to obtain the expression for  $\phi_3$ . Pre-multiplying Eq. (1.44) by  $\mathbf{u}_L^T$  and post-multiplying by  $\mathbf{u}_R$ ,

$$\begin{aligned}
&\mathbf{u}_L^T R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_R \\
&= \mathbf{u}_L^T R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) R_L(r, \phi_3) \mathbf{u}_R = \mathbf{u}_L^T \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \mathbf{u}_R.
\end{aligned}$$

Expanding both sides of the above equation, the equation obtained in terms of  $\phi_3$  is given by

$$A \cos \phi_1 + B \sin \phi_1 + C = (r^2 - 1) \alpha_{11} + r \sqrt{1 - r^2} (\alpha_{13} - \alpha_{31}) + r^2 \alpha_{33},$$

where the expressions for  $A, B$ , and  $C$  are the same as that obtained for the equation for  $\phi_1$ . Hence, at most two solutions can be obtained for  $\phi_3$  for each  $\phi_2$  if  $\gamma \neq 0$ .

### 1.11.1 Case with $\cos \phi_2 = 1 - \frac{1}{r^2}$

In this case, the net rotation matrix corresponding to the  $LRLRL$  path, given by the LHS of Eq. (1.44), reduces to

$$R_L(r, \phi_1) R_R(r, \phi_2) R_L(r, \phi_2) R_R(r, \phi_2) R_L(r, \phi_3) = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ -\delta_{12} & \delta_{22} & \delta_{23} \\ \delta_{13} & -\delta_{23} & \delta_{33} \end{pmatrix},$$

where  $\sin \phi_2 = -\sqrt{1 - \cos^2 \phi_2} = -\frac{\sqrt{2r^2-1}}{r^2}$  was used. Here,

$$\begin{aligned}\delta_{11} &= -2 \sin \left( \frac{\phi_1 + \phi_3}{2} \right) \left( (r^2 - 1) \sin \left( \frac{\phi_1 + \phi_3}{2} \right) + \sqrt{2r^2 - 1} \cos \left( \frac{\phi_1 + \phi_3}{2} \right) \right), \\ \delta_{12} &= -\frac{(r^2 - 1) \sin(\phi_1 + \phi_3) + \sqrt{2r^2 - 1} \cos(\phi_1 + \phi_3)}{r}, \\ \delta_{13} &= \frac{\sqrt{1 - r^2} \left( \sqrt{2r^2 - 1} \sin(\phi_1 + \phi_3) - (r^2 - 1) \cos(\phi_1 + \phi_3) + r^2 \right)}{r}, \\ \delta_{22} &= \frac{(r^2 - 1) \cos(\phi_1 + \phi_3) - \sqrt{2r^2 - 1} \sin(\phi_1 + \phi_3)}{r^2}, \\ \delta_{23} &= -\frac{\sqrt{1 - r^2} \left( (r^2 - 1) \sin(\phi_1 + \phi_3) + \sqrt{2r^2 - 1} \cos(\phi_1 + \phi_3) \right)}{r^2}, \\ \delta_{33} &= \frac{r^4 + (r^2 - 1) \sqrt{2r^2 - 1} \sin(\phi_1 + \phi_3) - (r^2 - 1)^2 \cos(\phi_1 + \phi_3)}{r^2}.\end{aligned}$$

Noting that the net rotation matrix must equal the desired final configuration, given by

$$R_f = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix},$$

the expression for  $\phi_1 + \phi_3$  can be obtained by equation  $\delta_{12}$  to  $\delta_{22}$  to  $\alpha_{12}$  and  $\alpha_{22}$ , respectively, to obtain

$$\begin{pmatrix} \sqrt{2r^2 - 1} & (r^2 - 1) \\ (r^2 - 1) & -\sqrt{2r^2 - 1} \end{pmatrix} \begin{pmatrix} \cos(\phi_1 + \phi_3) \\ \sin(\phi_1 + \phi_3) \end{pmatrix} = \begin{pmatrix} -r\alpha_{12} \\ r^2\alpha_{22} \end{pmatrix}.$$

The determinant of the matrix on the left hand side is  $-r^4$ , which is non-zero. Hence, the matrix on the left hand side can be inverted to obtain

$$\begin{aligned}\begin{pmatrix} \cos(\phi_1 + \phi_3) \\ \sin(\phi_1 + \phi_3) \end{pmatrix} &= \frac{-1}{r^4} \begin{pmatrix} -\sqrt{2r^2 - 1} & -(r^2 - 1) \\ -(r^2 - 1) & \sqrt{2r^2 - 1} \end{pmatrix} \begin{pmatrix} -r\alpha_{12} \\ r^2\alpha_{22} \end{pmatrix} \\ &= \frac{-1}{r^4} \begin{pmatrix} \sqrt{2r^2 - 1}r\alpha_{12} - r^2(r^2 - 1)\alpha_{22} \\ r(r^2 - 1)\alpha_{12} + r^2\sqrt{2r^2 - 1}\alpha_{22} \end{pmatrix}.\end{aligned}$$

Setting  $\phi_3 = 0$  without loss of generality, a solution for  $\phi_1$  can be obtained from the above equation. It should be noted that the obtained expressions for  $\cos \phi_1$  and  $\sin \phi_1$  must be verified to satisfy  $\cos^2 \phi_1 + \sin^2 \phi_1 = 1$ .

## 1.12 RLRLR path

The *RLRLR* path can be constructed by utilizing the construction of an *LRLRL* path. To this end, the final configuration is reflected about the *XY* plane, and the *LRLRL* path is constructed to the reflected final configuration. The modified final configuration is given in Eq. (1.34). The obtained solutions for  $\phi_1$ ,  $\phi_2$ , and  $\phi_3$  correspond to the parameters for the *RLRLR* path.

## References

- [1] D. P. Kumar, S. Darbha, S. G. Manyam, and D. Casbeer, “A new approach to motion planning in 3d for a dubins vehicle: Special case on a sphere,” 2025. [Online]. Available: <https://arxiv.org/abs/2504.01215>