Project Report

Comparison of the Euler-Bernoulli beam theory with the Theory of Elastica

ES 221: Mechanics of Solids

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1 Objective

This report aims to compare the deflection shapes and stress distributions predicted by the Theory of Elastica and the Euler-Bernoulli Theorem for a cantilever beam subjected to a point load. By examining the distinct assumptions of each theory, we can understand how they influence beam shapes and stresses.

2 Problem Statement

The problem statement involves a detailed comparison of deflection shapes and stress distributions in cantilever beams as predicted by two fundamental theories in solid mechanics: the Theory of Elastica and the Euler-Bernoulli Theorem. The Theory of Elastica, considers large deflections and often yields different predictions compared to the Euler-Bernoulli beam theory, which assumes small deflections. Additionally, the report will compute and compare stress distributions predicted by the two theories, offering insights into their accuracy and limitations in capturing the complex behavior of cantilever beams under loading.

3 Methodology

3.1 Euler-Bernoulli Theory

$$EI(\frac{\partial^2 y}{\partial x^2}) = M \tag{1}$$

Where;

EI is Flexural rigidity
y is vertical displacement
M is Bending Moment

- **Stress Analysis**
  For this cantilever beam stress is defined as,

  $$\sigma = \frac{yM_b}{I}$$

  where,

  - σ represents the stress at the point
  - M_b is the bending moment at that point,
  - y is the perpendicular distance from the neutral axis to the point
  - I is the moment of inertia of the beam's cross-sectional area about the neutral axis.Stress in the theory of elastica is defined as, $\sigma = \frac{yM_b}{I}$

Calculating stress for P = 3N, EI = 1, Length = 1m, the stress comes out to be,

$$\sigma = \frac{0.0025 \times 3 \times 1}{0.25 \times 10^{-9}}$$

This gives us $\sigma_1 = 3 \times 10^7$ Pa

- **Deflection**
  Using Singularity function for this cantilever beam we get,

  $$M_b = -PL < x{-}0 >^0 + P < x{-}0 >^1$$

  Considering x will be always greater than 0, this equation will be simplified to

  $$M_b = P(x - L)$$

  Therefore,

  $$EI(\frac{d^2y}{dx^2}) = P(x - L)$$

  From this we get,

  $$y = \frac{-PLx^2}{2} + \frac{Px^3}{6}$$

  where, P is the Point Load applied at the end of beam

Figure 1: Cantilever beam under action of point load at its end.

3.2 Theory of Elastica

$$\theta"(s) + Psin(\theta(s) + \alpha) = 0$$

- **Deflection**
  For the cantilever beam boundary conditions for the above equation are defined as,

  $$\theta(0) = 0$$

  $$\theta'(1) = 0$$

- **Stress Analysis**
  - Stress in the theory of elastica is defined as, $\sigma = -Ey\frac{d\theta}{ds}$
  - Where,
    E is Youngs Modulous,
    y is the distance from the centroid,

  - E=4Gpa,
  - The calculated value of $\frac{d\theta}{ds}$ is -0.30284 and distance from centroid is 0.25 mm
    Therefore σ_1=30.2 $\times 10^7$ Pa
    Note: Calculation of $\frac{d\theta}{ds}$ was done with the help of Python to calculate stress.

4 Numerical Implementation

- In this code we have find the deflection of cantilever beam under point load P, applied at the end of beam. The deflection is computed using the Euler-Bernoulli theory.

```
# Function to compute deflection of cantilever beam (Euler-
                                  Bernoulli)
def deflection_euler_bernoulli(load_force, beam_length):
    EI = 1  # Flexural rigidity
    load_position = beam_length  # Force at the end of the beam

    x = np.linspace(0, beam_length, 100)
    deflection = np.zeros_like(x)
    for i, xi in enumerate(x):
      deflection[i] = (load_force * xi**2) / (6 * EI) * (-3 *
                                    load_position + xi)

    return x, deflection
```

- In the below code code we are solving non linear differential equation derived from theory of elastica using python libraries. The point load is denoted by P and s is veried from 0 to 1.

```
    # Assigning values to parameters
    P = -3
    alpha = -np.pi / 2

    # Solving the differential equations
    def equations(s, vars):
        x, y, theta = vars
        dxds = np.cos(theta)
        dyds = np.sin(theta)
        d2thetads2 = -P * (np.sin(theta + alpha))
```

```
        return [dxds, dyds, d2thetads2]

    # Initial conditions
    initial_conditions = [0, 0, 0]

    # Values of s from 0 to 1
    s_values = np.linspace(0, 1, 100)

    # Solve the differential equations with boundary conditions
    solution = solve_ivp(equations, [0, 1], initial_conditions,
                                    t_eval=s_values)

    # Extract the solution
    xSol = solution.y[0]
    ySol = solution.y[1]
    thetaSol = solution.y[2]
```

5 Results and Discussions

- Experimental setup: A beam-like structure was created using MDF sheet with a length of 25 cm and width of 0.5cm
- A point load with a magnitude of 6.1803 N was applied at one end of the beam.
- Experimental deflection: The experimental deflection observed was 3.3 cm. The Euler-Bernoulli theory deflection prediction came out to be 3.22 cm while the Theory of Elastica deflection prediction came out to be 88.85 cm.

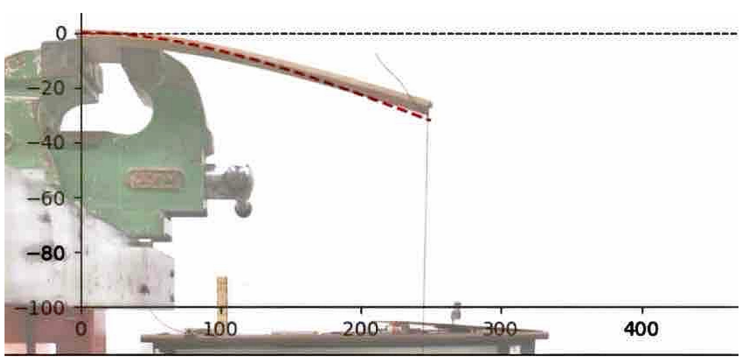

Figure 2: Experimental Deflection vs Theoretical Deflection

- Due to small deflections, the experimental value closely aligned with the prediction of the Euler-Bernoulli theory rather than the Theory of Elastica.

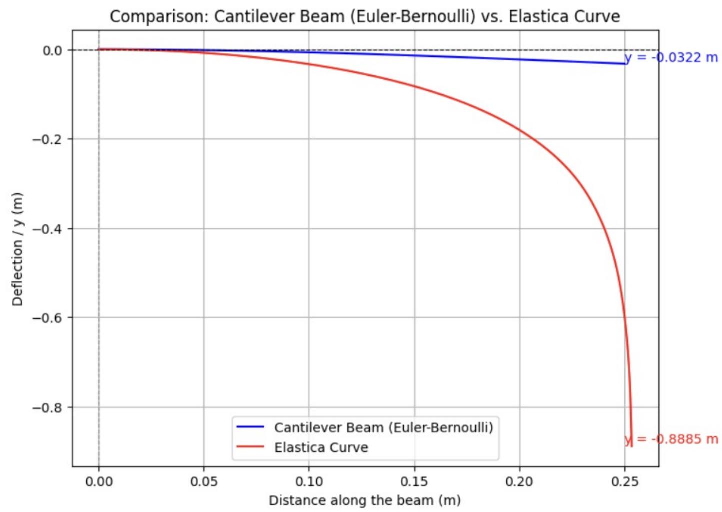

Figure 3: Theoretical Deflection Comparison

6 Learning Outcomes

- Understanding fundamental beam theories: Theory of Elastica and Euler-Bernoulli Theorem and explored the variation in deflected shapes using two different theories
- Applying Euler-Bernoulli Theory for computing small deflections, utilizing Python code and libraries like 'RK45' and 'solve_ivp'.
- Exploring Theory of Elastica, its conditions, and practical applications..

Bibliography

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